Why yet another course in physics?

Answer to this question is vital to justify yet another course (book) on the subject, specially when, there exists brilliant books on the shelves, successfully meeting the requirement of schools. Understandably, each of these books / courses has been developed through a rigorous process, conforming to a very high level of standards prescribed by state education boards. Why then yet another course (book)? Matter of fact, this question had been uppermost in my mind before I undertook the commitment to take up this project. A good part of the reason lies in the basic nature of creative urge involved in writing and shaping a book. Besides, as an author, I had the strong conviction like others that a subject matter can always be treated in yet another way, which may be a shade different and may be a shade better than earlier efforts. This belief probably clinched my initiation into this project. Further :

1 : It is no wonder that books have been published regularly - many of which have contributed significantly to the understanding of the nature and natural events. Also, there is no doubt that there has been a general improvement in the breath and depth of the material and style of presentation in the new books, leading to a better appreciation among the readers about the powerful theories, propounded by great human minds of all time. However, one book differs to other in content, treatment, emphasis and presentation. This book is different on this count.

2 : Fundamental laws of physics are simple in construct. Take the example of Newton’s second law : $\mathbf{F} = m\mathbf{a}$. This could not have been simpler. Yet, it takes great deal of insight and practice to get to the best of mechanics – a branch of physics, which is largely described by this simple construct. The simplicity of fundamental laws, matter of fact, is one of the greatest wonders of nature. Difficulty arises, mostly, from the complexity of the context of natural phenomena, which are generally culmination of a series of smaller events interwoven in various ways. The challenge here is to resolve complex natural phenomena into simpler components, which can then be subjected to the theories of physics. Resolution of complex natural phenomena into simpler components is an important consideration in

physics. This book keeps this aspect of physics central to its treatment of the subject matter.

3 : Overwhelming and awesome reach of theories in physics, inadvertently, introduces a sense of finality and there is a tendency to take an approach towards the study of physics, which is serene and cautious – short of ‘do_not_fool_around’ kind of approach. This book takes calculated risk to play around with the hypotheses and theories to initiate readers to think deeper and appreciate physics with all its nuances. The book is structured and developed from the perspective of inquisitive young minds and not from the perspective of a matured mind, tending to accept theory at its face value. This shift in approach is the cornerstone of subject treatment in this book.

4 : Mathematics fine tunes physics laws and gives it a quantitative stature. Most of the extension of physical laws into the realm of application is possible with the intelligent use of mathematical tools at our disposal. Further, adaptation of physical laws in mathematical form is concise and accurate. Consider the magnetic force on a moving charge given by :

$$\mathbf{F} = q(\mathbf{v} \text{ X } \mathbf{B})$$

The mathematical expression is complete and accurate. It tells us about both magnitude and direction of the magnetic force on the moving charge. Matter of fact, direction of force in relation to velocity of charge and magnetic field is difficult to predict without this formula. Either, we rely on additional rules like Fleming’s left hand rule or interpret the vector quantities on the right hand side of the equation in accordance with rules concerning cross product of two vectors. The choice of mathematical vector interpretation is found to avoid confusion as Fleming’s rule requires that we memorize direction of each of the vectors by a specific finger from the set of three fingers stretched in mutually perpendicular directions. There is great deal of uncertainty involved. You may forget to remember the correct hand (left or right), correct fingers (first, middle or thumb)and what each of them represents. On the other hand, vector interpretation has no element to memorize to predict the direction of magnetic force! Such is the power of mathematical notation of physical law.

In this sense, mathematics is a powerful tool and preferred language of expression in physics. Separate modules are devoted to describe mathematics relevant to physics in order to prime readers before these tools are used in the context of physics.

5 : The fundamental laws/ theories of physics are universal and result of great insight into the realm of physical proceedings. New constructs and principles are difficult to come by. The last defining moment in physics was development of the quantum physics by Erwin Schrodinger and Werner Heisenberg in the year 1925-26. Since then, there had been advancement in the particle physics and electronics, but no further aggregation of new theories of fundamental nature. Physics, however, has progressed a great deal in its application to other spheres of science, including engineering, medicine and information technology. This book assigns due emphasis to this aspect of applied physics.

6 : Various Boards of State Education prescribe well thought out framework and standards for development of physics text book for classroom teaching. This book emphasizes these standards and goals set up by the Boards.

What is physics?

The word **physics** is derived from Greek word fusis, meaning nature or natural things. As such, physics is defined as that branch of science, which studies natural phenomena in terms of basic laws and physical quantities. The study is generally structured to satisfy queries, arising from the observed events occurring around our world. In this sense, Physics answers questions about universe and the way elements of universe interact to compose natural phenomena.

The underlying principles in physics are simple and general, but defining (basic) in nature. Elements and quantities used to describe natural phenomena are also general and basic. The whole of universe, as a matter of fact, can be considered to be comprising of two basic quantities : (i) matter and (ii) energy. For this reason, some physicists rightly define physics as the study of matter and energy.

Domain of physics

The domain of physics extends from the infinitesimal to the infinite and is largely undefined. At one end of the scale, there are quarks composing nucleons (neutrons and protons) and on the other end, there are galaxies, with sun-like stars as its constituents and a universe that we do not know much about.

In physics, domains are also defined in terms of various important attributes like speed, temperature and other physical quantities. In the domain defined by speed, we study both stationary objects and objects moving at very high speed, perhaps three – fourths of the speed of light. It is thanks to the extraordinary efforts of scientists in the last two centuries that we now know some of the important bounds of nature. For example, the upper limit of speed is the speed of light in a vacuum. Similarly, the lower limit of temperature is 0 K. These are some of the highlights of the development of our basic understanding of nature and its extent.

The uncertainty about the domain of physics stems from the fact that new experiments and discoveries continuously break the bounds (limits) set

before. An example: for many years, the charge on the electron was considered the smallest amount of charge, but today after the discovery of quarks, we know that these carry lesser amounts of charge than that carried by electrons. Thus, the extent of physics is actually changing as we learn more and more about nature.

Generality

Theories of physics are extremely general, being the underlying governing principles of natural events extending to the whole universe. This aspect contrasts physics from other streams of science, which are often specific and sometimes localized. Generality of physics and its theories render physical laws to form the basic scientific framework upon which other branches of science are developed. Take the example of charged molecules called "ions" - a subject of investigation in Chemistry. Oppositely charged ions are glued together under the influence of an electrostatic force irrespective of the nature and type of ions and the atoms or molecules involved. The magnitude of this electrostatic force is secular in that its magnitude is determined by an inverse square law – whatever be the context and location.

Simplification and unification

Simplification and unification have emerged as the basic trait of physics. There are only a few laws to define a wide variety of natural phenomena – a fact that underlines the simplification of governing laws in physics. On the other hand, unification of physical quantities and concepts is also prevalent. Take the example of matter and energy. They are now considered equivalent. The Special Theory of Relativity establishes the equivalence of these two quantities as $E = mc^2$. Further, this dual nature of matter highlights the wave (E energy) nature of particles (m mass), which underlies the concept of mass-energy equivalence. Similarly, the treatment of magnetism in terms of electrical charge is an example of unification of physical concepts.

Simplification can also be seen in the laws governing gravitation and electrostatics. Gravitational and electrical forces are conservative forces, determined by inverse square laws. The similarity of the forms of mathematical expressions is no coincidence, but a sure indication of the underlying nature of the universe, which emphasizes simplification :

$$F_{\mathrm{G}} = \frac{Gm_1m_2}{4\pi r^2} \quad \text{.......... Gravitation Force}$$

$$F_E = \frac{q_1q_2}{4\pi\varepsilon_0 r^2} \quad \text{.......... Electrostatic Force}$$

Simplification and unification of physical quantities, concepts and laws are remarkable, suggesting more such cases – which are yet to be discovered. Consider the physical quantities: “mass” and “charge”. There is as yet no relationship connecting these two fundamental quantities of physics. Similarly, the two major categories of forces known as nuclear and weak forces are not yet fully understood. Scientists are working to examine these unknown territories.

Scientific validation and experimental verification

The fundamental laws of physics are set against either too big or too small quantities, presenting a peculiar problem in establishing direct validation of basic theories in physics. Even today, there is not a single experimental set up which could directly verify Einstein’s theory of relativity. For example, mass of an electron moving at two – third of the speed of light can not be measured directly. As we do not see the atom and its constituents, theories based on them are also not directly verifiable. We can not even verify Newton’s first law of motion, which states that an object in the absence of net external force shall keep moving! We have seen all objects come to rest in the earth's frame of reference, when left unaided. This peculiarity, however, does not mean that these laws have not been validated as required for scientific studies.

It should be amply clear that scientific method for validation also includes inferences based on indirect measurements. In that sense, Einstein’s special theory of relativity has been tested and verified by results obtained from the experiments involving motions of charged particles at great speed.

Surprising is the exactness and accuracy of the results obtained. In the same context, accuracy of predications involving solar systems, satellites etc. have validated laws of gravitation.

Recasting and revalidating laws in the light of new revelations

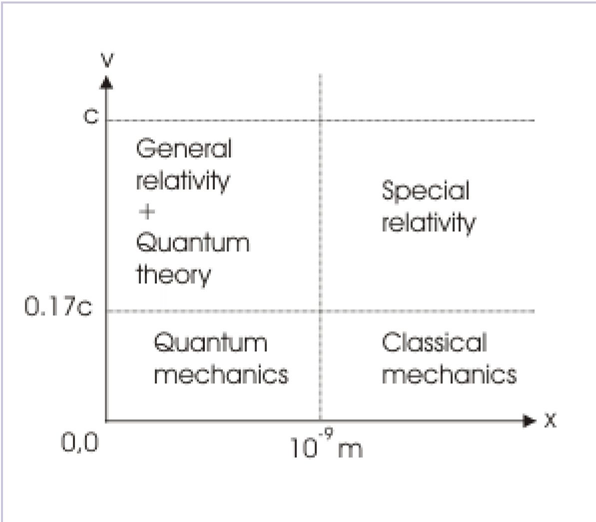
Studies and experiments continue to bring new details and dimensions to our understanding of natural phenomena. New revelations recast old facts, hypotheses and theories. The relativity theory propounded by Albert Einstein, for example, revealed that Newton's laws of motions are basically a subset of more general theory. Similarly, Newton's hypothesis regarding velocity of sound was recast by Laplace, arguing that propagation of sound is adiabatic and not isothermal process as considered by Newton. His assertion was based on the experimental result and was correct. There are many such occasions when an incomplete or erroneous understanding of natural event is recast or validated when new facts are analyzed.

Domains of physical laws

There is an irresistible perception that physical phenomena are governed by a universal law - a fundamental law, which is valid at all dimensions and at all speeds. As against this historically evasive natural conjecture, our understanding and formulation are limited to domains of applicability. Newton's law works fine in our world, where dimensions are bigger than atomic size and speed is not exceeding 0.17c (speed of light, "c"). If the speed of an object exceeds this limit, the relativistic effects can not be ignored.

Consequently, we are currently left with a set of laws, which are domain specific. One law resigns in favor of other as we switch from one domain to another. The plot below approximately defines the domains of four major physical laws in terms of dimension and speed. Though, there are further subdivisions proposed, but this broader classification of applicability of natural laws is a good approximation of our current understanding about natural phenomena.

Domains of physical laws



Approximation of domains in terms of speed and dimension.

The reduction of the special theory of relativity into classical mechanics at smaller speeds gives an indication that there may be a law which is the most general and, therefore, universal. At present, however, there is no such clear cut “deducibility” among other domain specific laws, which involves “quantum mechanics” or “general relativity”. Unquestionably, unification of physical laws is the most fundamental question eluding all scientific investigation to this date.

There is a long way to go

Our knowledge about nature is improving progressively with every passing day. Yet our so called understanding at any point of time is subject to new revelation and meaning. Most of the time, we come to realize that our understanding is mostly limited to our surrounding and the context of application. It may sound bizarre but it is a fact that we have yet not fully understood even the basic concepts like distance, mass and time. Theory of relativity attached new meaning to these terms. It may not be totally brazen to think that even relativistic improvisations be ultimately incomplete and inaccurate. Who knows ?

Units of measurement
Measurement of a quantity involves its comparison with a standard quantity called unit.

The study of science, including physics, is quantitative in nature. We study natural phenomena and events in terms of quantities, which can be measured. A measurement is basically observations to estimate a physical quantity. Its basic objective is to reduce uncertainty and to give definitive stature to the quantities being described.

The measurement of quantity is done by comparing it with some standard called “unit”. A unit, therefore, is any division of quantity, which is accepted as one unit of that quantity. A quantity (Q) is expressed as the product of a number (number of times in comparison to the standard) and the name given to the unit or standard.

$$Q = nX\text{name of unit}$$

$$Q = nu$$

We can have a look at some of the physical quantities that we use in our everyday life: 5 kg of sugar, 110 Volt of electric potential, 35 Horse Power of an engine and so on. The pattern of all these quantitative expression follows the same construct as defined above.

System of units

Our earlier units have been human based (in the context of what we use in our daily life) and, therefore, varied from country to country and even from society to society. We had measure of length (foot) in terms of the length of a foot step as unit.

Relating units to immediate physical world is not wrong; rather it is desirable. What is wrong is that there are many units for the same quantity with no relative merit over each other. We are led to a situation, where we have different units for the same quantity, based on experiences in different parts of the world. These different units of the same quantity do not bear

any logical relation amongst themselves. We, therefore, need to have uniform unit system across the world.

Further, it is seen that there are scores of physical quantities. If we assemble all quantities, which are referred in the study of physics, then the list will have more than 100 entries. Fortunately, however, most physical quantities are “dependent” quantities, which can be expressed as combination of other quantities. This fact leads us to classify quantities in two groups :

1. Basic or fundamental quantities
2. Derived quantities

Basic or fundamental units are a set of units for physical quantities from which other units can be derived. This classification i.e. existence of basic quantities has a great simplifying effect. We are limited to study few of basic units; others (derived) are derived from them.

Here, we should also strike difference between “basic” quantity of a system of measurement to that of quantity of basic nature. The presence of length, mass and time in the basic category may give impression that all members are basic in nature. Neither it is required nor it is so. We can have a system of measurement with quantities, which are not basic in nature.

In modern SI unit system, for example, the electric current is included in the list of “basic” quantities. We, though, know that it is equal to time rate of charge. A quantity of basic nature in the universe is not derivable from other quantities, but current is. As such, current is not basic in nature. We could have included “charge” as the basic quantity instead. But, then there are other requirements of a basic unit like reproducibility and ease of measurement etc, which need to be taken into consideration.

As pointed out earlier, the basic units should be correlated to our immediate context. For example, a meter represents a length that we are able to correlate and visualize with the physical entities in our world. For example, we say that height of the room is 2.1 m – not something like $2.1X10^{-11}$ m.

Nature presents a kind of continuum, which ranges from very small to very large. Consider the dimensions of a nucleus ($\sim 10^{-15}$ m) and distance of

the sun ($\sim 10^{11}$ m). The system of units, therefore, needs to have a scheme to express wide variations often seen in physical quantities.

Finally, the advancement in scientific studies has expanded scope of studies much beyond human physical existence. We study atoms at one hand and galaxies on the other. The quantities involved are ether so small or so big that the physical comparison with a real time measuring device may not be possible. For example, we can not think of going inside an atom and measure its radius with a scale. Inferred (indirect) measurements are, therefore, allowed and accepted in such situations.

Features of fundamental units

Following are the features/ characteristics of fundamental units :

- They are not deducible from each other.
- They are invariant in time and place (in classical context).
- They can be accurately reproduced.
- They describe human physical world.

International system of units

Its short name is SI system, which is an abbreviated form of French equivalent “Systeme Internationale d’ units”. As study of science became more and more definitive and universal, it was felt to have a system of units, which can be refereed internationally. The rationales for adopting SI system as international system are two fold. First, this system is based on the powers of 10. Second, there is a well structured prefixes to represent range of measurements associated with a physical quantity.

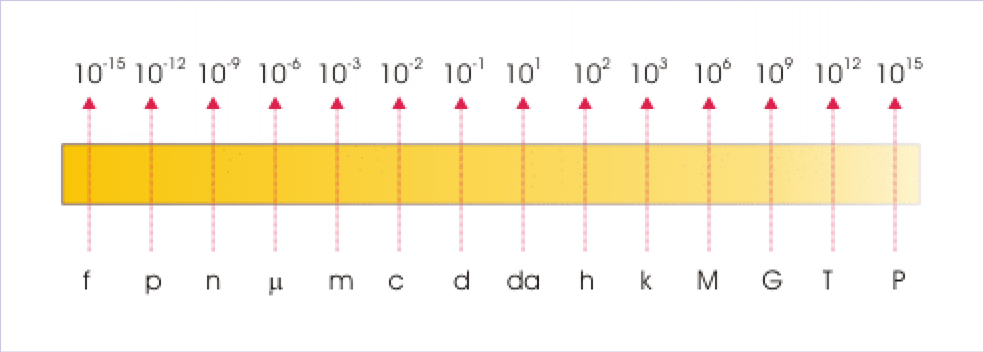
The “power of 10” makes it easy to change smaller to bigger unit and vice versa. A mere shift of “decimal” does the job.

12.0 mm (smaller) = 1.20 cm (bigger)

Equivalently, we multiply the given quantity with 10 raised to positive integer to obtain the measurement in terms of smaller unit; and divide it with 10 raised to positive integer to obtain the measurement in terms of bigger quantity.

Finally, SI system has a set of prefixes for a given unit to represent smaller or bigger quantities. This set of “prefixed” represents a predefined factor in terms of the power of 10. We should remind ourselves that all of these prefixes are applicable uniformly to all quantities.

Prefix factors



The factors are powers of 10.

Femto(10^{-15}), pico(10^{-12}), nano(10^{-9}), micro(10^{-6}), milli(10^{-3}), centi(10^{-2}), deci (10^{-1}), deka (10^{1}), hector(10^{2}), kilo(10^{3}), mega(10^{6}), giga(10^{9}),tera(10^{12}),peta(10^{15})

Note that except for few prefixes in the middle, the powers of the factor differs by “3” or “-3”.

Basic units

The seven basic quantities included in SI system of measurement are :

1. Length
2. Mass

3. Time
4. Current
5. Temperature
6. Amount of substance (mole)
7. Luminous intensity

The corresponding seven basic units with their symbols are defined here (as officially defined):

1: meter (m) : It is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

2: kilogram (kg) : It is equal to the mass of the international prototype of the kilogram. The prototype is a platinum-iridium cylinder kept at International Bureau of Weights and Measures, at Severes, near Paris, France.

3: time (t) : It is the duration of 9, 192, 631, 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium - 133 atom.

4: ampere (A) : It is is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 m apart in vacuum, would produce between these conductors a force equal to $2X10^{-7}$ newton per meter of length.

5: kelvin (K) : It is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.

6: mole (mol) : It is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12.

7: candela (cd) : It is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency $540X10^{12}$ hertz and that has a radiant intensity in that direction of 1/683 watt per steradian (measure of solid angle).

MKS system of units

MKS is an abbreviation of **M**eter, **K**ilogram and **S**econd. These three quantities form the basic set of units in MKS system. Clearly, it is a subset of SI system of units. As we can realize, mechanics - a branch of physics - involves only length, mass and time. Therefore, MKS system is adequate to represent quantities used in mechanics.

This distinction between mechanics and rest of physics is hardly made in recent time. We can, therefore, completely do away with MKS nomenclature in favor of SI system.

Conversion of units

Despite endeavor on world level for adoption of SI unit, there are, as a matter of fact, wide spread variation in the selection of unit system. Engineering world is full of inconsistencies with respect to the use of unit system. We often need to have skill to convert one unit into another. We take a simple example here to illustrate how it is done for the case of basic quantity like mass.

Let us consider a mass of 10 kg, which is required to be converted into gram - the mass unit in cgs unit (Gaussian system). Let the measurements in two systems are " $n_1 u_1$ " and " $n_2 u_2$ " respectively. But, the quantity, "Q", is "10 kg" and is same irrespective of the system of units employed. As such,

$$Q = n_1 u_1 = n_2 u_2$$

$$\Rightarrow n_2 = \frac{u_1}{u_2} n_1$$

$$\Rightarrow n_2 = \frac{1 \text{ kg}}{1 \text{ gm}} n_1$$

$$\Rightarrow n_2 = \frac{10^3 \text{ gm}}{1 \text{ gm}} 10$$

$$n_2 = 10^4$$

$$Q = n_2 u_2 = 10^4 \text{ gm}$$

The process of conversion with respect to basic quantities is straight forward. The conversion of derived quantities, however, would involve dimensions of the derived quantities. We shall discuss conversion of derived quantities in a separate module.

Dimensional analysis
Seven basic quantities are the seven dimensions of physical quantities.

We are familiar with dimensions of motion and motion related quantities. Often, we specify the description of physical process by numbers of coordinates involved – one, two or three. It indicates the context of motion in space. The dimension of physical quantities follows the same philosophy and indicates the nature of the constitution of quantities. In other words, dimension of a physical quantity indicates how it relates to one of the seven basic/ fundamental quantities. Basic quantities are the seven dimensions of the physical quantities.

Dimensions of a physical quantity are the powers with which basic quantities are raised to represent it. The dimension of a physical quantity in an individual basic quantity is the power with which that basic quantity is raised in the dimensional representation of physical quantity. We should be clear here that the dimension is not merely a power, but a combination of basic quantity and its power. Both are taken together and hence represented together. We may keep in mind that units follow dimensional constitution. Speed, for example, has dimension of 1 in length and dimension of -1 in time and hence its unit is m/s.

A pair of square bracket is used to represent the dimension of individual basic quantity with its symbol enclosed within the bracket. There is a convention in using symbol of basic quantities. The dimensions of seven basic quantities are represented as :

1. Mass : [M]
2. Length : [L]
3. Time : [T]
4. Current : [A]
5. Temperature : [K]
6. Amount of substance : [mol]
7. Luminous Intensity : [cd]

We can see here that there is no pattern. Sometimes we use initial letter of the basic physical quantity like "M", sometimes we use initial letter of basic

unit like "A" and we even use abbreviated name of the basic unit like "mol".

Dimensions of derived quantities

The dimensions of derived quantities may include few or all dimensions in individual basic quantities. In order to understand the technique to write dimensions of a derived quantity, we consider the case of force. The force is defined as :

$$F = ma$$

Thus, dimensions of force is :

$$\Rightarrow [F] = [M][a]$$

The dimension of acceleration, represented as [a], is itself a derived quantity being the ratio of velocity and time. In turn, velocity is also a derived quantity, being ratio of length and time.

$$\Rightarrow \left[F\right] = \left[M\right]\left[a\right] = \left[M\right]\left[vT^{-1}\right] = \left[M\right]\left[LT^{-1}T^{-1}\right] = \left[MLT^{-2}\right]$$

We read the dimension of force as : it has “1” dimension in mass, “1” dimension in length and “-2” dimension in time. This reading emphasizes the fact that the dimension is not merely a power, but a combination of basic quantity and its power.

Dimensional formula

The expression of dimensional representation is also called “dimensional formula” of the given physical quantity. For brevity, we do not include basic dimensions, which are not part of derived quantity, in the dimensional formula.

$$\text{Force, } \left[F\right] = \left[MLT^{-2}\right]$$

$$\text{Velocity, } \left[v\right] = \left[LT^{-1}\right]$$

$$\text{Charge, } [q] = [AT]$$

$$\text{Specific heat, } [s] = \left[L^2T^2K^{-1}\right]$$

$$\text{Gas constant, } [R] = \left[ML^2T^{-2}K^{-1}mol^{-1}\right]$$

It is conventional to omit power of “1”. Further, in mechanics, we can optionally use all three symbols corresponding to three basic quantities, which may or may not be involved. Even if one or two of them are not present, it is considered conventional to report absence of the dimension in a particular basic quantity. It is done by raising the symbol to zero as :

$$[v] = \left[M^0LT^{-1}\right]$$

It should be noted here that dimensional representation of a physical quantity does not include “magnitude (n)” of the physical quantity. Further, dimensional formula does not distinguish nature of quantity. For example, the nature of force (or types of force) has no bearing on its dimensional representation. It, then, also follows that variants of a physical quantity bears the same dimensional representation. For example, velocity – whether instantaneous, average, relative – has the same dimensional representation $\left[LT^{-1}\right]$. Moreover, dimensional representation of a vector and its scalar counterpart is same. For example, the dimensions of velocity and speed are same. Further, dimensions of the difference of a physical quantity are same as that of physical quantity itself. For this reason, dimensions of velocity and difference of two velocities are same.

A physical quantity need not have dimensions in any of the basic quantities. Such is the case, where physical quantities are equal to the ratio of quantities having same dimensions. Take the case of an angle, which is a ratio of arc(length) and radius(length). The dimensions of such physical quantity are zero in each of the basic quantities.

$$\text{Angles, } [\theta] = \left[M^0L^0T^0\right]$$

Similar is the case with Reynold’s number, used in fluid mechanics. It also does not have dimensions in basic quantities. Thus, a physical quantity can

be either “dimensional” or “dimensionless”.

On the other hand, there are numerical constants like trigonometric function, pi etc, which are not dependent on the basic physical quantities. The dimensions of such a constant are zero in each of the basic quantities. They are, therefore, called “dimensionless” constants.

$$\text{pi}, \left[\pi\right] = \left[M^0 L^0 T^0\right]$$

However, there are physical constants, which appear as the constant of proportionality in physical formula. These constants have dimensions in basic quantities. Such constant like Gravitational constant, Boltzmann constant, Planck's constant etc. are, therefore, “dimensional” constant.

In the nutshell, we categorize variables and constants in following categories :

- Dimensional variable : speed, force, current etc.
- Dimensionless variable : angle, reynold’s number, trigonometric, logarithmic functions etc.
- Dimensional constant : universal gas constant, permittivity, permeability
- Dimensionless constant : numerical constants, mathematical constants, trigonometric, logarithmic functions etc.

Dimensional equation

Dimensional equation is obtained by equating dimension of a given physical quantity with its dimensional formula. Hence, followings constitute a dimensional equation :

$$\left[F\right] = \left[MLT^{-2}\right]$$

$$\left[v\right] = \left[M^0 LT^{-1}\right]$$

Dimensions and unit

Dimensions of a derived quantities can always be expressed in accordance with their dimensional constitutions. Consider the example of force. Its dimensional formula is MLT^{-2} . Its SI unit, then, can be "kg-m/s2" $\text{kg-m}/s^2$. Instead of using dimensional unit, we give a single name to it like Newton to honour his contribution in understanding force. Such examples are prevalent in physics.

A dimensionless physical quantity, on the other hand, has no dimensional unit as no dimension is involved. Consider the case of reynold's number. It is a number without unit. However, there are exceptions. Consider measure of angle. It is a dimensionless quantity, but has radian as unit.

Basic dimensions and their group

Each of the basic dimensions form a group in dimensional formula. Consider a physical quantity work. The work is product of force and displacement. The basic dimension "length" is present in both force and displacement. Dimension in length, therefore, forms a group – one from force and one from displacement. The dimensions in the group is operated by multiplication following rule of indices $x^n x^m = x^{(m+n)}$.

Determining dimensional formula

The basic approach to determine dimensional formula of a physical quantity is to use its defining equation. A defining equation like that of force in terms of mass and acceleration allows us to determine dimensional formula by further decomposing dependent quantities involved. This is the standard way. However, this approach is cumbersome in certain cases when dependencies are complicated. In such cases, we can take advantage of the fact that a physical quantity may appear in simpler relation in some other context. Let us consider the case of magnetic field strength. We can determine its dimensional expression, using Biot-Savart's law. But, we

know that Magnetic field strength appears in the force calculation on a wire carrying current in a magnetic field. The force per unit length :

$$\frac{F}{l} = BIt$$

$$\Rightarrow [B] = \frac{[F]}{[ALT]} = \frac{\left[MLT^{-2}\right]}{[ALT]} = \left[MT^{-2}A^{-1}\right]$$

Example:
Problem : Find the dimensional formula of thermal conductivity.
Solution : The thermal heat of conduction (Q) is given as :

$$Q = \frac{kA(\theta_2 - \theta_1)t}{d}$$

where "k" is thermal conductivity, "Q" is heat, "A" is area of cross-section, "θ" is temperature and "t" is time.

$$\Rightarrow \left[k\right] = \frac{[Qd]}{[A(\theta_2 - \theta_1)t]}$$

We make use of two facts (i) heat is energy and (ii) dimension of difference of temperature is same as that of temperature.

$$\Rightarrow \left[k\right] = \frac{\left[ML^2T^{-2}\right][L]}{[L^2KT]}$$

$$[k] = \left[MLT^{-3}K^{-1}\right]$$

Example:
Problem : Find the dimension of expression,

$$\frac{CV}{\varepsilon_0 \rho}$$

where “C” is capacitance, “V” is potential difference, “ ε_0 ” is permittivity of vacuum and “ρ” is specific resistance.

Solution :

By inspection of expression, we observe that capacitance of parallel plate capacitor is given by the expression containing “ ε_0 ”. This may enable us to cancel “ ε_0 ” from the ratio. The capacitance is given as :

$$C = \frac{\varepsilon_0 A}{d}$$

On the other hand, specific resistance,

$$\rho = \frac{RA}{L}$$

Hence,

$$\Rightarrow \left[\frac{CV}{\varepsilon_0 \rho}\right] = \frac{[\varepsilon_0 ALV]}{[\varepsilon_0 dRA]}$$

Applying Ohm’s law,

$$\Rightarrow \left[\frac{CV}{\varepsilon_0 \rho}\right] = \frac{[\varepsilon_0 ALV]}{[\varepsilon_0 LRA]} = \frac{[V]}{[R]}$$

Applying Ohm's law,

$$\Rightarrow \left[\frac{CV}{\varepsilon_0 \rho}\right] = \frac{[V]}{[R]} = \left[A\right]$$

Example:

Problem : Find the dimension of term given as :

$$\frac{E^2}{\mu_0}$$

where “E” is electric field and “ μ_0 ” is absolute permeability of vacuum.
Solution : We can approach the problem by evaluating each of the quantities in the expression terms of basic quantities. Alternatively, however, we can evaluate the dimension of the given ratio by expressing the given expression in terms of physical quantities, whose dimensions can be easily found out. For illustration purpose, we shall use the second approach.
Here we can write given expression in the following manner :

$$\frac{E^2}{\mu_0} = \frac{\varepsilon_0 E^2}{\varepsilon_0 \mu_0}$$

where “ε_0 ” is absolute permittivity of vacuum.
However, we know that electric energy density of a medium is $\frac{1}{2}\varepsilon_0 E^2$. It means that numerator has the same dimension as “energy/volume”. On the other hand, speed of light in vacuum is given as :

$$c = \frac{1}{\sqrt{(\varepsilon_0 \mu_0)}}$$

Thus, dimension of denominator is same as that of $\frac{1}{c^2}$.
Therefore,

$$\Rightarrow \frac{E^2}{\mu_0} = \frac{[\text{energy/volume}]}{\left[\frac{1}{c^2}\right]}$$

$$\Rightarrow \frac{E^2}{\mu_0} = \left[\frac{ML^2T^{-2}}{L^3}\right]\left[LT^{-1}\right]^2 = \left[MLT^{-4}\right]$$

Conversion from one system to another

We note that same physical quantity is expressed in different units. Clearly, conversion of unit is associated with single physical quantity. We know that different units for physical quantities have evolved through human progress in different parts of the world and have prevailed despite efforts towards uniform system. Secondly, a physical quantity has different scales of measurement - small, large etc. Whatever be the reason of different units being used, we need to have a “conversion factor” which coverts one unit to another. The conversion related to magnitude of unit is defined by the system of measurement itself. For example, 1 kilogram is equal to 1000 gm. Here, we shall derive a process to affect the conversion of one unit of one kind to another kind. In another words, we shall derive a “dimensionless constant” as conversion factor.

The underlying principle of process involved here is that a given physical quantity has same dimensional constitution – whatever be the system or unit. This fact is underlined by the fact that a given quantity of a given physical entity remains same. For, example, whether we call 1 inch or 2.54 cm, the length remains same. Now, physical quantity is product of “measurement” and “unit. Hence, We know that :

$$Q = n_1 u_1 = n_2 u_2$$

Since unit represents a standard division of the quantity, the dimensional formula of the unit quantity is same as that of the quantity in question. Let dimensions of units in two systems are :

$$u_1 = \left[M_1^a L_1^b T_1^c\right]$$

$$u_2 = \left[M_2^a L_2^b T_2^c\right]$$

Combining equations, we have :

$$\Rightarrow n_1 \left[M_1^a L_1^b T_1^c\right] = n_2 \left[M_2^a L_2^b T_2^c\right]$$

$$\Rightarrow n_2 = \frac{\left[M_1^a L_1^b T_1^c\right]}{\left[M_2^a L_2^b T_2^c\right]} n_1$$

This is the formula used to convert a quantity from one system of units to another. In case, only 1 unit of the quantity in first system is involved, then we can put $n_1 = 1$ in the above equation.

Example:

Problem : Convert 1 Joule in SI system into ergs in “cgs” system

Solution : Putting $n_1 = 1$ and applying formula, we have :

$$\Rightarrow n_2 = \frac{\left[M_1^a L_1^b T_1^c\right]}{\left[M_2^a L_2^b T_2^c\right]}$$

Now, the dimensional formula of energy is $\left[ML^2T^{-2}\right]$. Hence, a = 1, b = 2 and c = -2. Putting values in the equation and rearranging,

$$\Rightarrow n_2 = \left[\frac{1\,kg}{1\,gm}\right]^1 X \left[\frac{1\,m}{1\,cm}\right]^2 X \left[\frac{1\,s}{1\,s}\right]^{-2}$$

The basic units of SI system is related to corresponding basic units in “cgs” system as :

$$1\,Kg = 10^3\,gm$$

$$1 \quad m = 10^2 \quad cm$$

$$1\,s = 1\,s$$

Putting in the conversion formula, we have :

$$\Rightarrow n_2 = \left[\frac{10^3\,gm}{1\,gm}\right]^1 X \left[\frac{10^2\,cm}{1\,cm}\right]^2 X \left[\frac{1\,s}{1\,s}\right]^{-2}$$

$$\Rightarrow n_2 = 10^3 X 10^4 = 10^7$$

Hence,

$$1\,J = 10^7 \quad ergs$$

Dimensional analysis

Dimensional analysis is based on the simple understanding of the fact that only similar quantities can be added, subtracted and equated. Can we add 2 mangoes and 3 oranges? This requirement of similar terms is the underlying principle of dimensional analysis. This principle is known as principle of homogeneity.

Dimensional analysis gives an insight into the composition of a physical quantity. Take the case of frequency. Its SI unit is Hertz. What is its true relation with basic quantities? Notwithstanding the exact definition, its dimensional formula tells us that it is actually inverse of time (T). It has the dimension of "-1" in time.

In this section, we shall discuss three important applications of dimensional analysis involving : (i) Transcendental functions (ii) checking of dimensional formula and (iii) derivation of a formula.

An elaborate form of homogeneity principle is known as Buckingham π theorem. This theorem states that an expression of physical quantity having n variables can be expressed as an expression of n-m dimensionless parameters, where m is the numbers of dimensions. This principle is widely used in fluid mechanics to establish equations of physical quantities. A full treatment of this priciple is beyond the context of physics being covered in this course. Therefore, we shall use the simplified concept of homogeinity that requires that only dimensionally similar quantities can be added, subtracted and equated.

One interesting aspect of Buckingham π theorem is formation of dimensionless groups of variables. It helps us to understand relation among variables (physical quantities) and nature of dependence. Consider a simple example of time period of simple pendulum (T). Other relevant parameters affecting the phenomenon of oscillation are mass of pendulum bob (m), length of pendulum (l), acceleration due to gravity (g). The respective

dimensional formula for time period is [T], mass of bob is [M], length of pendulum is [L] and acceleartion due to gravity is [LT^{-2}].

We can have only one unique dimension-less group of these variables (quantities). Let us consider few combinations that aim to obtain one dimensionless group.

$$\frac{T^2 g}{ML} = \frac{\left[T^2 LT^{-2}\right]}{ML} = \left[M^{-1}\right]$$

This grouping indicates that we can not form a dimensionless group incorporating "mass of the bob". A dimensionless group here is :

$$\left[\frac{T^2 g}{L}\right] = \left[M^0 L^0 T^0\right] = \text{a constant}$$

Rearranging,

$$T = k\sqrt{\frac{l}{g}}$$

Thus, grouping of dimensionless parameter helps us to (i) determine the form of relation and (ii) determine dependencies. In this case, time period is independent of mass of bob. Formation of dimensionless parameter is one of the important considerations involved in the dimensional analysis based on Buckingham π theorem.

Transcendental functions

Many physical quantities are expressed in terms of transcendental functions like trigonometric, exponential or logarithmic functions etc. An immediate fall out of homogeneity principle is that we can only have "dimensionless groups" as the argument of these transcendental functions. These functions are expansions involving power terms :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots.$$

$$\sin x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots.$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} \ldots\ldots.$$

If x is dimensional quantity, then each term on right hand sides have different dimensions as x is raised to different powers . This violates principle of homogeneity. For example, we know that trigonometric function has angle as its argument. Angle being dimensionless is permissible here as no dimension is involved. But, we can not have a dimensional argument like time to trigonometric functions. However, we know of the expressions involving physical quantities have transcendental terms. In accordance with homogeneity principle, argument to the functions needs to be dimensionless. Consider for example the wave equation given by :

$$\text{y} = A\sin(\text{kx} - \omega t)$$

Here, both terms “kx” and “ωt” needs to be dimensionless.

Example:
Problem : Disintegration of radioactive takes place in accordance with relation given as :

$$\text{N(t)} = N_0 e^{-\text{kt}}$$

What is dimension of disintegration constant ?
Solution : Since argument of transcendental function is dimensionless,

$$[\text{kt}] = \left[M^0 L^0 T^0\right]$$

$$[\mathrm{k}] = \frac{[M^0 L^0 T^0]}{[T]} = \left[T^{-1}\right]$$

Example:

Problem : Find the value of “x” in the equation :

$$\int \frac{dx}{\sqrt{(2ax - x^2)}} = a^x \sin\left(\frac{x}{a} - 1\right)$$

Solution : It appears that this question is out of the context of dimensional analysis. Actually, however, we can find the solution of "x" for the unique situation by applying principle of homogeneity. We know that the argument of a trigonometric function is an angle, which does not have dimension. In other words, the argument of trigonometric function is a dimensionless variable. Hence, “ $\frac{x}{a} - 1$ ” is dimensionless. For this,

$$[a] = [x] = [L]$$

With this information, let us, now, find the dimension of the expression enclosed in the integral on the left hand side (LHS).

$$\Rightarrow \left[LHS\right] = \frac{[dx]}{\sqrt{([2ax - x^2])}}$$

We know that dimension of difference is same as that of the quantity. Hence, $[dx] = [L]$. Putting dimensions,

$$\Rightarrow \left[LHS\right] = \frac{[L]}{\sqrt{([L^2] - [L^2])}} = \frac{L}{L} = L^0$$

We see that LHS is dimensionless. Hence, RHS should also be dimensionless. The trigonometric function on the right is also

dimensionless. It means that factor “ a^x ” should also be dimensionless. But, the base “a” has a dimension that of length. The factor can be dimensionless, only if it evaluates to a constant. An exponential with variable power like “ a^x ” is constant if it evaluates to 1 for x=0. Hence,

$$\Rightarrow x = 0$$

Checking a formula

The terms of an equation connecting different physical quantities are dimensionally compatible. In accordance with “principle of homogeneity of dimensions”, each term of the equation has dimensions.

Hence, a formula not having dimensionally compatible terms are incorrect. In plain words, it means that (i) dimensions on two sides of an equation are equal and (ii) terms connected with plus or minus sign in the expression have the same dimensions. As a matter of fact, this deduction of the homogeneity principle can be used to determine the nature of constants/ variables and their units as illustrated in the example here.

Example:
Problem : The Vander Wall’s equation is given as :

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

where “P” is pressure, “V” is volume, “T” is temperature and “R” is gas constant. What are the units of constants “a” and “b”?
Solution : Applying principle of homogeneity of dimensions, we realize that dimensions of the terms connected with “plus” or “minus” signs are equal. Hence,

$$\Rightarrow \left[\frac{a}{V^2}\right] = \left[P\right] = \left[\frac{F}{A}\right] = \left[\frac{MLT^{-2}}{L^2}\right] = \left[ML^{-1}T^{-2}\right]$$

$$\Rightarrow [a] = [ML^{-1}T^{-2}][V^2] = [ML^{-1}T^{-2}][L^3]^2$$

$$\Rightarrow [a] = [ML^5T^{-2}]$$

The unit of “a” is “ $\text{kg} - m^5/s^2$ ”.
Also,

$$\Rightarrow [b] = [V] = [L^3]$$

The unit of “b” is same as that of volume i.e. " m^3" .

Deriving a formula

This is a curious aspect of the application of dimensional analysis. If we could have the general capability to construct formula in this manner, then we would have known all the secrets of nature without much difficulty. Clearly, derivation of formula by dimensional analysis is possible under certain limited circumstance only. Even then, significance of deriving formula is important as it has contributed quite remarkably in fluid mechanics and other branches of science.

We shall use dimensional analysis to derive Stoke’ law for viscous force in the example given here. We should appreciate, however, that dimensional analysis can not determine constant of a formula. As such, we shall work the example given here with this particular limitation.

Example

Problem 3: Stoke observed that viscous drag force, “F”, depends on (i) coefficient of viscosity, “η” (ii) velocity of the spherical mass, “v” and (iii) radius of the spherical body, “r”.

If the dimensional formula of coefficient of viscosity is $[ML^{-1}T^{-1}]$, drive Stoke’s law for viscous force on a small spherical body in motion through a

static fluid medium.

Solution : We first write the observations in mathematical form as :

$$F \propto \eta^a; F \propto v^b; F \propto r^c$$

Combining, we have :

$$F = K\eta^a v^b r^c$$

where “K” is the constant of proportionality and is dimensionless. Applying principle of homogeneity of dimensions, the dimensions on the left and right side of the equation should be same. Hence,

$$\left[F\right] = \left[K\eta^a v^b r^c\right]$$

Neglecting "K" as it is dimensionless and putting dimensional formulas of each quantity, we have :

$$\left[MLT^{-2}\right] = \left[ML^{-1}T^{-1}\right]^a X\left[LT^{-1}\right]^b X\left[L\right]^c$$

Applying indices law and rearranging, we have :

$$\left[MLT^{-2}\right] = \left[M^a L^{-a+b+c} T^{-a-b}\right]$$

For dimensional consistency as required by the principle, the powers of individual basic quantities should be equal. Hence,

$$\Rightarrow a = 1$$

$$\Rightarrow -a + b + c = 1$$

$$\Rightarrow -a - b = -2$$

Combining first and second equations, we have :

$$\Rightarrow -1 + b + c = 1$$

$$\Rightarrow b + c = 2$$

Combining first and third equations, we have :

$$\Rightarrow -1-b=-2$$

$$\Rightarrow b=1$$

Hence, $a=1, c=1$. Now putting values in the equation for the force,

$$F=K\eta vr$$

Note : It is evident that we can not determine the constant, “K”, using dimensional analysis. It is found experimentally to be equal to “6π”. Hence, Stoke’s law for viscous drag is written as :

$$\Rightarrow F=6\pi\eta vr$$

This example highlights the limitations of dimensional analysis. First, we notice that it is not possible to evaluate constant by dimensional analysis. Further, numbers of basic quantities involved in the dimensional formula determine the numbers of equations available for simultaneous evaluation. As such, it is required that the numbers of variables (quantities) involved should be less than or equal to numbers of basic quantities involved in the dimensional formula. If the quantity pertains to mechanics, then numbers of variable (quantities) should be limited to "3" at the most. In the example of viscous force, had there been fourth variable (quantity), then we would not be able to solve simultaneous equations.

It can also be seen that dimensional analysis can **not** be applied in situation, where there are more than one term like in the case of $x=ut+\frac{1}{2}at^2$. We can not derive this equation as dimensional analysis will yield one of two terms - not both simultaneously.

Exercises

Exercise:

Problem: Determine the dimension of expression,

$$\frac{e^2}{\varepsilon_0 hc}$$

where "e" is electronic charge, " ε_0 " is absolute permittivity of vacuum, "c" is the speed of light.

Solution:

We can break up the ratio as :

$$\Rightarrow \frac{e^2}{\varepsilon_0 hc} = \frac{e^2}{\varepsilon_0} X \frac{1}{hc}$$

It is advantageous to use the expression of potential energy of two electrons system to evaluate the first term. The potential energy of two electrons system is :

$$U = \frac{e^2}{4\pi\varepsilon_0 r}$$

$$\Rightarrow \frac{[e^2]}{[\varepsilon_0]} = \left[4\pi Ur\right]$$

Neglecting dimensionless constants,

$$\Rightarrow \frac{[e^2]}{[\varepsilon_0]} = \left[Ur\right] = \left[ML^2T^{-2}L\right] = \left[ML^3T^{-2}\right]$$

The expression "hc" is related to energy of a photon as :

$$E = \frac{hc}{\lambda}$$

$$\Rightarrow [hc] = [E][L] = [ML^2T^{-2}][L] = [ML^3T^{-2}]$$

Putting dimensions as calculated in the original expression,

$$\Rightarrow \frac{\left[e^2\right]}{\left[\varepsilon_0 hc\right]} = \frac{\left[ML^3T^{-2}\right]}{\left[ML^3T^{-2}\right]} = \left[M^0L^0T^0\right]$$

Thus, given expression is dimensionless.

Exercise:

Problem: Pressure in a gas filled container is given by the equation :

$$P = \frac{a + t^2}{bh}$$

where "a" and "b" are constants, "t" is temperature and "h" is the height. Find the dimension of ratio " $\frac{a}{b}$ ".

Solution:

We can find the dimension of the constant "a" by applying principle of homogeneity, appearing in the numerator of the given expression. Hence,

$$\left[a\right] = \left[K^2\right]$$

The dimension of the sum of quantities in numerator is equal to the dimension of individual term. It means that :

$$\Rightarrow \left[a + t^2\right] = \left[K^2\right]$$

Now,

$$\left[P\right] = \frac{\left[a + t^2\right]}{\left[bh\right]}$$

$$\left[b\right] = \frac{\left[a + t^2\right]}{\left[Ph\right]}$$

$$\left[b\right] = \frac{\left[K^2\right]}{\left[\frac{F}{A}h\right]} = \frac{\left[K^2\right]}{\left[\frac{MLT^{-2}}{L^2}XL\right]}$$

$$\Rightarrow \left[b\right] = \left[M^{-1}T^2K^2\right]$$

The dimension of the ratio “ $\frac{a}{b}$ ” is :

$$\Rightarrow \left[\frac{a}{b}\right] = \frac{\left[K^2\right]}{\left[M^{-1}T^2K^2\right]} = \left[MT^{-2}\right]$$

Exercise:

Problem:

If velocity, time and force were chosen as basic quantities, then find the dimension of mass, length and work.

Solution:

Here, the basic quantities of the system are assumed to be different than that of SI system. We see here that mass can be linked to force, using equation of motion,

$$\text{Force} = \text{mass X acceleration} = \frac{\text{mass X velocity}}{\text{time}}$$

$$\Rightarrow \text{mass} = \frac{\text{force X time}}{\text{velocity}}$$

Each of the quantities on the right hand is the basic quantity of new system. Let us represent new basic quantities by first letter of the quantities in capital form as :

$$\Rightarrow \left[\text{mass}\right] = \left[V^{-1}TF\right]$$

Similarly, we can dimensionally link length to basic quantities, using definition of velocity.

$$\text{velocity} = \frac{\text{displacement}}{\text{time}} = \frac{\text{length}}{\text{time}}$$

$$\Rightarrow \text{length} = \text{velocity X time}$$

$$\Rightarrow [\text{length}] = [VT]$$

Now, work is given as :

$$\text{work} = \text{force X displacement}$$

Dimensionally,

$$\Rightarrow \text{work} = \text{force X length}$$

$$\Rightarrow [\text{work}] = [F][VT] = [VTF]$$

Exercise:

Problem:

If the unit of force were kilo-newton, that of time milli-second and that of power kilo-watt, then find the units of mass and length.

Solution:

We are required to find the units of mass and length in terms of the newly defined units.

We need to first find dimensional expression of the quantity as required in terms of units defined in the question. We see that dimensional formula of physical quantities as basic units are :

$$[F] = [MLT^{-2}]$$

$$[T] = [T]$$

$$\left[P\right] = \left[ML^2T^{-3}\right]$$

Dividing dimensional formula of “P” by “F”, we have :

$$\frac{\left[P\right]}{\left[F\right]} = \frac{\left[ML^2T^{-3}\right]}{\left[MLT^{-2}\right]} = \left[\frac{L}{T}\right]$$

As we are required to know the units of length, its dimension in new units are :

$$\Rightarrow \left[L\right] = \frac{\left[PT\right]}{\left[F\right]}$$

In order to find the unit of “L” in terms of new system, we have :

$$n_2 = \frac{\left[P_1\right]\left[T_1\right]\left[F_2\right]}{\left[P_2\right]\left[T_2\right]\left[F_1\right]}$$

We should note here that subscript “1” denotes new system of units, which comprises of force, time and power as basic dimensions. Hence,

$$\Rightarrow n_2 = \frac{\left[10^3W\right]\left[s\right]\left[1N\right]}{\left[1W\right]\left[1s\right]\left[10^3N\right]} = 10^{-3}$$

Hence, length has the unit of length,

$$\Rightarrow L = 10^{-3} \quad m$$

From the dimensional formula of force, we have :

$$\left[F\right] = \left[MLT^{-2}\right]$$

$$\Rightarrow \left[M\right] = \frac{\left[F\right]}{\left[LT^{-2}\right]} = \frac{\left[10^3N\right]}{\left[\left(10^{-3}m\right)\left(10^{-3}\right)^{-2}\right]} = \left[1 \quad kg\right]$$

Hence, unit of mass is "1 kg".

Errors in measurement

Measurement is the basis of scientific study. All measurements are, however, approximate values (not true values) within the limitation of measuring device, measuring environment, process of measurement and human error. We seek to minimize uncertainty and hence error to the extent possible.

Further, there is important aspect of reporting measurement. It should be consistent, systematic and revealing in the context of accuracy and precision. We must understand that an error in basic quantities propagate through mathematical formula leading to compounding of errors and misrepresentation of quantities.

Errors are broadly classified in two categories :

- Systematic error
- Random error

A systematic error impacts “accuracy” of the measurement. Accuracy means how close is the measurement with respect to “true” value. A “true” value of a quantity is a measurement, when errors on all accounts are minimized. We should distinguish “accuracy” of measurement with “precision” of measurement, which is related to the ability of an instrument to measure values with greater details (divisions).

The measurement of a weight on a scale with marking in kg is 79 kg, whereas measurement of the same weight on a different scale having further divisions in hectogram is 79.3 kg. The later weighing scale is more precise. The precision of measurement of an instrument, therefore, is a function of the ability of an instrument to read smaller divisions of a quantity.

In the nutshell,

1. True value of a quantity is an “unknown”. We can not know the true value of a quantity, even if we have measured it by chance as we do not know the exact value of error in measurement. We can only approximate true value with greater accuracy and precision.

2. An accepted “true” measurement of a quantity is a measurement, when errors on all accounts are minimized.
3. “Accuracy” means how close is the measurement with respect to “true” measurement. It is associated with systematic error.
4. “Precision” of measurement is related to the ability of an instrument to measure values in greater details. It is associated with random error.

Systematic error

A systematic error results due to faulty measurement practices. The error of this category is characterized by deviation in one direction from the true value. What it means that the error is introduced, which is either less than or greater than the true value. Systematic error impacts the accuracy of measurement – not the precision of the measurement.

Systematic error results from :

1. faulty instrument
2. faulty measuring process and
3. personal bias

Clearly, this type of error can not be minimized or reduced by repeated measurements. A faulty machine, for example, will not improve accuracy of measurement by repeating measurements.

Instrument error

A zero error, for example, is an instrument error, which is introduced in the measurement consistently in one direction. A zero error results when the zero mark of the scale does not match with pointer. We can realize this with the weighing instrument we use at our home. Often, the pointer is off the zero mark of the scale. Moreover, the scale may in itself be not uniformly marked or may not be properly calibrated. In vernier calipers, the nine divisions of main scale should be exactly equal to ten divisions of vernier scale. In a nutshell, we can say that the instrument error occurs due to faulty

design of the instrument. We can minimize this error by replacing the instrument or by making a change in the design of the instrument.

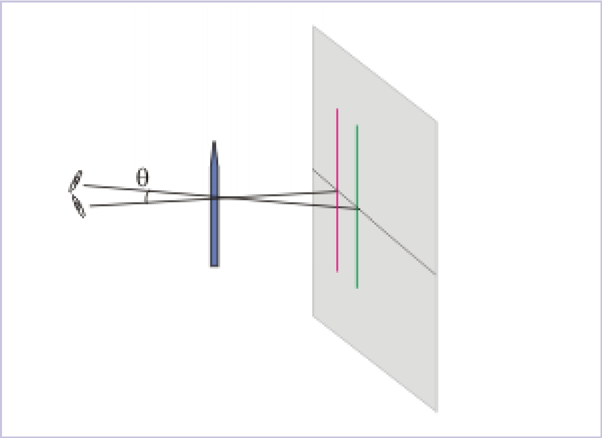
Procedural error

A faulty measuring process may include inappropriate physical environment, procedural mistakes and lack of understanding of the process of measurement. For example, if we are studying magnetic effect of current, then it would be erroneous to conduct the experiment in a place where strong currents are flowing nearby. Similarly, while taking temperature of human body, it is important to know which of the human parts is more representative of body temperature.

This error type can be minimized by periodic assessment of measurement process and improvising the system in consultation with subject expert or simply conducting an audit of the measuring process in the light of new facts and advancements.

Personal bias

A personal bias is introduced by human habits, which are not conducive for accurate measurement. Consider for example, the reading habit of a person. He or she may have the habit of reading scales from an inappropriate distance and from an oblique direction. The measurement, therefore, includes error on account of parallax.
Parallax

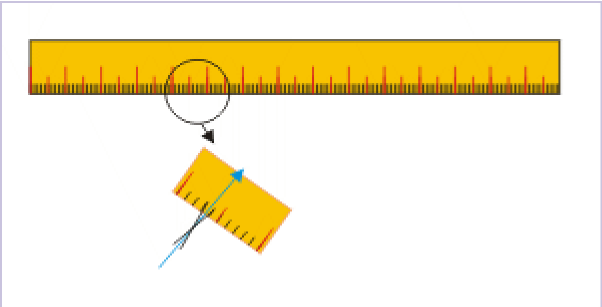


The position of pencil changes with respect to a mark on the background.

We can appreciate the importance of parallax by just holding a finger (pencil) in the hand, which is stretched horizontally. We keep the finger in front of our eyes against some reference marking in the back ground. Now, we look at the finger by closing one eye at a time and note the relative displacement of the finger with respect to the mark in the static background. We can do this experiment any time as shown in the figure above. The parallax results due to the angle at which we look at the object.

It is important that we read position of a pointer or a needle on a scale normally to avoid error on account of parallax.

Parallax



Parallax error is introduced as we may read values at an angle.

Random errors

Random error unlike systematic error is not unidirectional. Some of the measured values are greater than true value; some are less than true value. The errors introduced are sometimes positive and sometimes negative with respect to true value. It is possible to minimize this type of error by repeating measurements and applying statistical technique to get closer value to the true value.

Another distinguishing aspect of random error is that it is not biased. It is there because of the limitation of the instrument in hand and the limitation on the part of human ability. No human being can repeat an action in exactly the same manner. Hence, it is likely that same person reports different values with the same instrument, which measures the quantity correctly.

Least count error

Least count error results due to the inadequacy of resolution of the instrument. We can understand this in the context of least count of a measuring device. The least count of a device is equal to the smallest division on the scale. Consider the meter scale that we use. What is its least count? Its smallest division is in millimeter (mm). Hence, its least count is 1 mm i.e. 10^{-3} m i.e. 0.001 m. Clearly, this meter scale can be used to measure length from 10^{-3} m to 1 m. It is worth to know that least count of a vernier scale is 10^{-4} m and that of screw gauge and spherometer 10^{-5} m.

Returning to the meter scale, we have the dilemma of limiting ourselves to the exact measurement up to the precision of marking or should be limited to a step before. For example, let us read the measurement of a piece of a given rod. One end of the rod exactly matches with the zero of scale. Other end lies at the smallest markings at 0.477 m (= 47.7 cm = 477 mm). We

may argue that measurement should be limited to the marking which can be definitely relied. If so, then we would report the length as 0.47 m, because we may not be definite about millimeter reading.

This is, however, unacceptable as we are sure that length consists of some additional length – only thing that we may err as the reading might be 0.476 m or 0.478 m instead of 0.477 m. There is a definite chance of error due to limitation in reading such small divisions. We would, however, be more precise and accurate by reporting measurement as 0.477 ± some agreed level of anticipated error. Generally, the accepted level of error in reading the smallest division is considered half the least count. Hence, the reading would be :

$$x = 0.477 \pm \frac{0.001}{2} \quad m$$

$$\Rightarrow x = 0.477 \pm 0.0005 m$$

If we report the measurement in centimeter,

$$\Rightarrow x = 47.7 \pm 0.05 cm$$

If we report the measurement in millimeter,

$$\Rightarrow x = 477 \pm 0.5 mm$$

Mean value of measurements

It has been pointed out that random error, including that of least count error, can be minimized by repeating measurements. It is so because errors are not unidirectional. If we take average of the measurements from the repeated measurements, it is likely that we minimize error by canceling out errors in opposite directions.

Here, we are implicitly assuming that measurement is free of “systematic errors”. The averaging of the repeated measurements, therefore, gives the best estimate of “true” value. As such, average or mean value (a_m) of the measurements (excluding "off beat" measurements) is the notional “true”

value of the quantity being measured. As a matter of fact, it is reported as true value, being our best estimate.

$$a_m = \frac{a_1 + a_2 + \ldots\ldots\ldots\ldots + a_n}{n}$$

$$a_m = \sum_{0}^{n} \frac{a_i}{n}$$

Working with errors

In this module, we shall introduce some statistical analysis techniques to improve our understanding about error and enable reporting of error in the measurement of a quantity. There are basically three related approaches, which involves measurement of :

- Absolute error
- Relative error
- Percentage error

Absolute error

The absolute error is the magnitude of error as determined from the difference of measured value from the mean value of the quantity. The important thing to note here is that absolute error is concerned with the magnitude of error – not the direction of error. For a particular n^{th} measurement,

$$|\Delta x_n| = |x_m - x_n|$$

where " x_m " is the mean or average value of measurements and " x_n " is the n^{th} instant of measurement.

In order to calculate few absolute values, we consider a set of measured data for the length of a given rod. Note that we are reporting measurements in centimeter.

$x_1 = 47.7\,cm$, $x_2 = 47.5\,cm$, $x_3 = 47.8\,cm$, $x_4 = 47.4\,cm$, $x_5 = 47.7\,cm$

The means value of length is :

$$\Rightarrow x_m = \frac{47.7 + 47.5 + 47.8 + 47.4 + 47.7}{5} = \frac{238.1}{5} = 47.62$$

It is evident from the individual values that the least count of the scale (smallest division) is 0.001 m = 0.1 cm. For this reason, we limit mean

value to the first decimal place. Hence, we round off the last but one digit as :

$$x_m = 47.6 \quad cm$$

This is the mean or true value of the length of the rod. Now, absolute error of each of the five measurements are :

$$|\Delta x_1| = |x_m - x_1| = |47.6 - 47.7| = |-0.1| = 0.1$$

$$|\Delta x_2| = |x_m - x_2| = |47.6 - 47.5| = |0.1| = 0.1$$

$$|\Delta x_3| = |x_m - x_3| = |47.6 - 47.8| = |-0.2| = 0.2$$

$$|\Delta x_4| = |x_m - x_4| = |47.6 - 47.4| = |0.2| = 0.2$$

$$|\Delta x_5| = |x_m - x_5| = |47.6 - 47.7| = |-0.1| = 0.1$$

Mean absolute error

Earlier, it was stated that a quantity is measured with a range of error specified by half the least count. This is a generally accepted range of error. Here, we shall work to calculate the range of the error, based on the actual measurements and not go by any predefined range of error as that of generally accepted range of error. This means that we want to determine the range of error, which is based on the deviations in the reading from the mean value.

Absolute error associated with each measurement tells us how far the measurement can be off the mean value. The absolute errors so calculated, however, may be different. Now the question is : which of the absolute error be taken for our consideration? We take the average of the absolute error :

$$\Delta x_m = \frac{\Delta x_1 + \Delta x_2 + \ldots\ldots\ldots\ldots + \Delta x_n}{n}$$

$$\Rightarrow \Delta x_m = \sum_0^n \frac{\Delta xi}{n}$$

The value of measurement, now, will be reported with the range of error as :

$$x = x_m \pm \mathit{\Delta} x_m$$

Extending this concept of defining range to the earlier example, we have :

$$\Rightarrow \mathit{\Delta} x_m = \frac{0.1 + 0.1 + 0.2 + 0.2 + 0.1}{5} = \frac{0.7}{5} = 0.14 = 0.1 \quad cm$$

We should note here that we have rounded the result to reflect that the error value has same precision as that of measured value. The value of measurement with the range of error, then, is :

$$\Rightarrow x = 47.6 \pm 0.1 \quad cm$$

What we convey by writing in terms of the range of possible error. A plain reading of above expression is “the length of rod lies in between 47.5 cm and 47.7 cm”. For all practical purpose, we shall use the value of x = 47.6 cm with the caution in mind that this quantity involves an error of the magnitude of “0.1 cm” in either direction.

Relative error

We can report range of error as the ratio of the mean absolute error to the mean value of the quantity. This ratio is known as relative error. Mathematically,

$$\mathit{\Delta} x_r = \frac{\mathit{\Delta} x_m}{x_m}$$

As we use a ratio, this expression of error is also known as “fractional error”. Applying this concept to earlier example, we have :

$$\Rightarrow \mathit{\Delta} x_r = \frac{\mathit{\Delta} x_m}{x_m} = \frac{0.1}{47.6} = 0.0021$$

This is the amount of error which is possible for every “centimeter of length measured”. This is what is the meaning of a ratio. Hence, if there are 47.6

cm of total length, then the amount of error possible is 47.6 X 0.0021 = 0.1 cm. Two error range in the absolute form and relative form, therefore, are equivalent and specify the same range of errors involved with the measurement of a quantity.

Percentage error

Percentage error is equal to relative error expressed in percentage. It is given as :

$$\Delta x_p = \frac{\Delta x_m}{x_m} X100 = \Delta x_r X100$$

Applying this concept to earlier example, we have :

$$\Delta x_p = \Delta x_r X100 = 0.0021 X100 = 0.21$$

Combination of errors

Measurement of a quantity is used in a formula in various combinations to calculate other physical quantities. The mathematical operations in the working of a formula involve arithmetic operations like addition, subtraction, multiplication and division. We need to evaluate the implication of such operations on the error estimates and what is the resulting error in the quantities derived from mathematical operations. For example, let us consider simple example of density. This involves measurement of basic quantities like mass and volume.

Clearly, we need to estimate error in density which is based on the measurements of mass and volume with certain errors themselves. Similarly, there are more complex cases, which may involve different mathematical operations. We shall consider following basic mathematical operations in this section :

- sum or difference
- product or division
- quantity raised to a power

Errors in a sum or difference

We consider two quantities whose values are measured with certain range of errors as :

$$a = a \pm \mathit{\Delta} a$$

$$b = b \pm \mathit{\Delta} b$$

Sum of the two quantities is :

$$\Rightarrow a + b = a \pm \mathit{\Delta} a + b \pm \mathit{\Delta} b = a + b + (\pm \mathit{\Delta} a \pm \mathit{\Delta} b)$$

Two absolute errors can combine in four possible ways. The corresponding possible errors in “a+b” are (Δa + Δb), -(Δa + Δb), (Δa - Δb) and (-Δa + Δb). The maximum absolute error in “a-b”, therefore, is “Δa+ Δb”.

Difference of the two quantities is :

$$\Rightarrow a - b = a \pm \mathit{\Delta} a - b - (\pm \mathit{\Delta} b) = a - b + \{\pm \mathit{\Delta} a - (\pm \mathit{\Delta} b)\}$$

Two absolute errors can combine in four possible ways. The corresponding possible errors in “a+b” are again (Δa + Δb), -(Δa + Δb), (Δa - Δb) and (-Δa + Δb). The maximum absolute error in “a-b”, therefore, is “Δa+ Δb”.

Let “Δc “ be the absolute error of the arithmetic operation of addition or subtraction. Then, in either case, the maximum value of absolute error in the sum or difference is :

$$\mathit{\Delta} c = \mathit{\Delta} a + \mathit{\Delta} b$$

We see here that the absolute error in the sum or difference of two quantities is equal to the sum of the absolute values of errors in the individual quantities. We can write the resulting value as :

For addition as :

$$\Rightarrow c = (a + b) \pm (\mathit{\Delta} a + \mathit{\Delta} b)$$

For subtraction as :

$$\Rightarrow c = (a - b) \pm (\mathit{\Delta} a + \mathit{\Delta} b)$$

Example

Problem 1: The values of two capacitors are measured as :

$$C_1 = 1.2 \pm 0.1 \quad \mu F$$

$$C_2 = 2.3 \pm 0.2 \quad \mu F$$

Two capacitors are connected in parallel. What is the equivalent capacity of two given capacitors? Indicate error in percentage.

Solution : The equivalent capacitance of two capacitors in parallel is given by the sum of the capacitance of individual capacitors :

$$C = C_1 + C_2 = 1.2 + 2.3 = 3.5 \quad \mu F$$

For addition of two quantities, the absolute error in the equivalent capacitance is given by :

$$\mathit{\Delta} C = \mathit{\Delta} C_1 + \mathit{\Delta} C_2$$

$$\Rightarrow \mathit{\Delta} C = 0.1 + 0.2 = 0.3 \quad \mu F$$

Percentage error in the equivalent capacitance is given by :

$$\Rightarrow \mathit{\Delta} Cp = \frac{\mathit{\Delta} C}{C} X100$$

$$\Rightarrow \mathit{\Delta} Cp = \frac{0.3}{3.5} X100 = 8.6$$

Hence, equivalent capacitance, “C”, with percentage error is :

$$\Rightarrow C = 3.5 \mu F \pm 8.6\%$$

Errors in product or division

Error in the product and division of two measured quantities can be similarly worked. For brevity, we shall work out the implication of error for the operation of product only. We shall simply extend the result obtained for the product to division. Let the two measured quantities be :

$$a = a \pm \Delta a$$

$$b = b \pm \Delta b$$

Let "Δc" be the absolute error in the product. Then, product of the two quantities is :

$$(c \pm \Delta c) = (a \pm \Delta a)(b \pm \Delta b)$$

$$\Rightarrow c\left(1 \pm \frac{\Delta c}{c}\right) = ab\left(1 \pm \frac{\Delta a}{a}\right)\left(1 \pm \frac{\Delta b}{b}\right)$$

But, c = ab. Hence, expanding terms, we have :

$$\Rightarrow 1 \pm \frac{\Delta c}{c} = 1 \pm \frac{\Delta a}{a} \pm \frac{\Delta b}{b} \pm \frac{\Delta a \Delta b}{ab}$$

$$\Rightarrow \pm \frac{\Delta c}{c} = \pm \frac{\Delta a}{a} \pm \frac{\Delta b}{b}$$

The terms " $\frac{\Delta a \Delta b}{ab}$ " is negligibly small and as such can be discarded :

$$\Rightarrow \pm \frac{\Delta c}{c} = \pm \left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$$

We see here that the relative error in the product of two quantities is equal to the sum of the relative errors in the individual quantities. This manifestation of individual errors in product is also true for division. Hence, we can broaden our observation that the relative error in the product or division of two quantities is equal to the sum of the relative errors in the individual quantities.

Example

Problem 2: The mass and the volume of a uniform body is given as :

$$m = 9.5 \pm 0.1 \quad kg$$

$$V = 3.1 \pm 0.2 \quad m^3$$

Determine density of the body with error limits.

Solution : Let us first calculate the value of the density, maintaining the decimal place in the result. It is given by :

$$\rho = \frac{m}{v} = \frac{9.5}{3.1} = 3.1 \quad kg/m^3$$

We have rounded the result for precision of density is limited to the minimum of precision of the quantities involved. The relative error in the division of mass by volume is :

$$\frac{\Delta\rho}{\rho} = \left(\frac{\Delta m}{m} + \frac{\Delta V}{V}\right)$$

$$\Rightarrow \frac{\Delta\rho}{\rho} = \left(\frac{0.1}{9.5} + \frac{0.2}{3.1}\right)$$

$$\Rightarrow \frac{\Delta\rho}{\rho} = 0.01 + 0.06 = 0.07$$

The absolute error in the density is :

$$\Rightarrow \Delta\rho = 0.07\rho = 0.07X3.1 = 0.2$$

Hence, density is given as :

$$\Rightarrow \rho = 3.1 \pm 0.2 \quad kg/m^3$$

Errors in raising a quantity to a power

In order to estimate error involving raising of a quantity to some power, we consider two measured quantities. Let us consider that they are related as :

$$x = \frac{a^n}{b^m}$$

Taking logarithm on either side of the equation, we have :

$$\Rightarrow \ln x = n \ln a - m \ln b$$

Differentiating on both sides, we have :

$$\Rightarrow \frac{dx}{x} = n\frac{da}{a} - m\frac{db}{b}$$

We can exchange each of the differential term with corresponding relative error terms.

$$\Rightarrow \pm\frac{\Delta x}{c} = \pm n\frac{\Delta a}{a} - m\left(\pm\frac{\Delta b}{b}\right)$$

Again there are four possible combinations of values for the relative error. The maximum being,

$$\Rightarrow \frac{\Delta x}{c} = n\frac{\Delta a}{a} + m\frac{\Delta b}{b}$$

Thus, relative error in the result is :

$$\Rightarrow \Delta x_r = (n\Delta a_r + m\Delta b_r)$$

We see here that error in the result is equal to power times the relative error, irrespective of whether the power is positive or negative. It leads to an important deduction that quantities with greater powers in an expression should be measured with highest accuracy to minimize error in the "derived" quantity.

Example

Problem 3: The torque required to produce a twist in solid bar is given by :

$$\tau = \frac{\Pi \eta r^4}{2L}$$

If percentage error in the measurement of η, r and L are 1%, 4% and 1% respectively. Find the percentage error in the value of torque.

Solution : We know that relative error in the resultant quantity i.e torque is :

$$\Delta t_r = (n_1 \Delta \eta_r + n_2 \Delta r_r + n_3 \Delta L_r)$$

where n_1 , n_2 and n_3 are the powers of three quantities η, r and L respectively. The constants of the equations are not considered as they are not measured. Now percentage is obtained by just multiplying relative error by 100, we can write,

$$\Rightarrow \Delta t_p = (n_1 \Delta \eta_p + n_2 \Delta r_p + n_3 \Delta L_p)$$

Here, $n_1 = 1, \quad n_2 = 4 \quad \text{and} \quad n_3 = 1$. Note that position of a quantity either in numerator or denominator does not make difference in error combination. Putting values,

$$\Rightarrow \Delta t_p = (1X1 + 4X4 + 1X1) = 18$$

This means that the maximum error estimate in torque would be 18%. This is quite a large amount of uncertainty. Note the role played by the error in "r". Most of error (16 %) is due to 4% error in this quantity as it is raised to a power of 4. This result substantiates the observation that quantity with highest power should be measured with most accuracy.

Significant figures

We have already discussed different types of errors and how to handle them. Surprisingly, however, we hardly ever mention about error while assigning values to different physical quantities. It is the general case. As a matter of fact, we should communicate error appropriately, as there is provision to link error with the values we write.

We can convey existence of error with the last significant digit of the numerical values that we assign. Implicitly, we assume certain acceptable level of error with the last significant digit. If we need to express the actual range of error, based on individual set of observations, then we should write specific range of error explicitly as explained in earlier module.

We are not always aware that while writing values, we are conveying the precision of measurement as well. Remember that random error is linked with the precision of measurement; and, therefore, to be precise we should follow rules that retain the precision of measurement through the mathematical operations that we carry out with the values.

Context of values

Before we go in details of the scheme, rules and such other aspects of writing values to quantities, we need to clarify the context of writing values.

We write values of a quantity on the assumption that there is no systematic error involved. This assumption is, though not realized fully in practice, but is required; as otherwise how can we write value, if we are not sure of its accuracy. If we have doubt on this count, there is no alternative other than to improve measurement quality by eliminating reasons for systematic error. Once, we are satisfied with the measurement, we are only limited to reporting the extent of random error.

Another question that needs to be answered is that why to entertain “uncertain” data (error) at all. Why not we ignore doubtful digit altogether? We have seen that eliminating doubtful digit results in greater inaccuracy (refer module on “errors in measurement”). Measuring value with suspect

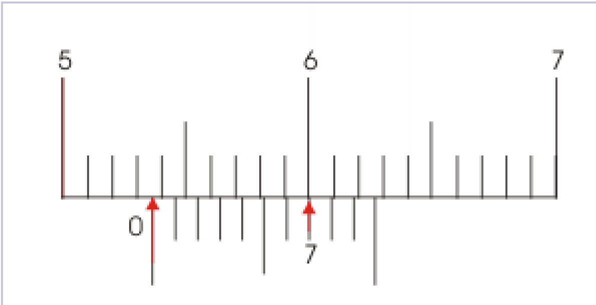
digit is more “accurate” even though it carries the notion of error. This is the reason why we prefer to live with error rather than without it.

We should also realize that error is associated with the smallest division of the scale i.e. its least count. Error is about reading the smallest division – not about estimating value between two consecutive markings of smallest divisions.

Significant figures

Significant figures comprises of digits, which are known reliably and one last digit in the sequence, which is not known reliably. We take an example of the measurement of length by a vernier scale. The measurement of a piece of rod is reported as 5.37 cm. This value comprises of three digits, “5”, “3” and “7”. All three digits are significant as the same are measured by the instrument. The value indicates, however, that last digit is “uncertain”. We know that least count of vernier scale is 10^{-4} m i.e 10^{-2} cm i.e 0.01 cm. There is a possibility of error, which is equal to half the least count i.e. 0.005 cm. The reported value may, therefore, lie between 5.365 cm and 5.375 cm.

In the figure, elements of measurement of a vernier scale are shown. The reading on the main scale (upper scale in the figure) is taken for the zero of vernier scale. The same is shown with an arrow on the left. This reading is "5.3". We can see that zero is between "5.3" and "5.4". In order to read the value between this interval, we look for the division of vernier scale, which exactly matches with the division mark on the main scale. Since 10 divisions of vernier scale is equal to 9 divisions on main scale, it ensures that one pair of marks will match. In this case, the seventh (7) reading on the vernier scale is the best match. Hence, the final reading is "5.37".
Measurement by Vernier calipers



Ten (10) smaller divisions on vernier scale is equal to nine (9) smaller divisions on main scale.

In this example, the last measurement constitutes the suspect reading. On repeated attempts, we may measure different values like "5.35" or "5.38".

Rules to identify significant figures

There are certain rules to identify significant figures in the reported value :

Rule 1 : In order to formulate this rule, we consider the value of measured length as "5.02 cm". Can we drop any of the non-zero digits? No. This will change the magnitude of length. The rule number 1 : All non-zero digits are significant figures.

Rule 2 : Now, can we drop “0” lying in between non- zeros “5” and “2” in the value considered above? Dropping “0” will change the value as measured. Hence, we can not drop "0". Does the decimal matter? No. Here, “0” and “decimal" both fall between non-zeros. It does not change the fact that "0" is part of the reported magnitude of the quantity. The rule number 2 : All zeros between any two non-zeros are significant, irrespective of the placement of decimal point.

Rule 3 : Let us, now, express the given value in micrometer. The value would be 0.000502 micrometer. Should expressing a value in different unit change significant figures. Changing significant figures will amount to changing precision and changing list count of the measuring instrument. We

can not change least count of an instrument – a physical reality - by mathematical manipulation. Therefore, rule number 3 : if the value is less than 1, then zeros between decimal point and first non-zero digit are **not** significant.

Rule 4 : We shall change the example value again to illustrate other rule for identifying significant figures. Let the length measured be 12.3 m. It is equal to 123 decimeter or 1230 cm or 12300 millimeter. Look closely. We have introduced one zero, while expressing the value in centimeter and two zeros, while expressing the value in millimeter. If we consider the trailing zero as significant, then it will again amount to changing precision, which is not possible. The value of 1230 cm, therefore has only three significant figures as originally measured. Therefore, Rule number 4 : The trailing zeros in a non-decimal number are not treated as significant numbers.

Rule 5 : We shall again change the example value to illustrate yet another characteristic of significant number. Let the measurement be exactly 50 cm. We need to distinguish this trailing “0”, which is the result of measurement - from the “0” in earlier case, which was introduced as a result of unit conversion. We need to have a mechanism to distinguish between two types of trailing zeros. Therefore, this rule and the one earlier i.e. 4 are rather a convention - not rules. Trailing zeros appearing due to measurement are reported with decimal point and treated as significant numbers. The rule number 5 is : The trailing zeros in a decimal number are significant.

The question, now, is how to write a measurement of 50 cm in accordance with rule 5, so that it has decimal point to indicate that zeros are significant. We make use of scientific notation, which expresses a value in the powers of 10. Hence, we write different experiment values as given here,

$$50 \quad cm = 5.0X10^1 \quad cm$$

This representation shows that the value has two significant figures. Similarly, consider measurements of 500 cm and 3240 cm as measured by an instrument. Our representation is required to reflect that these values have "3" and "4" significant figures respectively. We do this by representing them in scientific notation as :

$$500 \quad cm = 5.00X10^2 \quad cm$$

$$3240 \quad cm = 3.240X10^2 \quad cm$$

In this manner, we maintain the number of significant numbers, in case measurement value involves trailing zeros.

Features of significant figures

From the discussion above, we observe following important aspects of significant figures :

1. Changing units do not change significant figures.
2. Representation of a value in scientific form, having power of 10, does not change significant figures of the value.
3. We should not append zeros unnecessarily as the same would destroy the meaning of the value with respect to error involved in the measurement.

Mathematical operations and significant numbers

A physical quantity is generally dependent on other quantities. Evaluation of such derived physical quantity involves mathematical operations on measured quantities. Here, we shall investigate the implication of mathematical operations on the numbers of significant digits and hence on error estimate associated with last significant digit.

For example, let us consider calculation of a current in a piece of electrical conductor of resistance 1.23 (as measured). The conductor is connected to a battery of 1.2 V (as measured). Now, The current is given by Ohm’s law as :

$$I = \frac{V}{R} = \frac{1.2}{1.23}$$

The numerical division yields the value as rounded to third decimal place is :

$$\Rightarrow I = 0.976 \quad A$$

How many significant numbers should there be in the value of current? The guiding principle, here, is that the accuracy of final or resulting value after mathematical operation can not be greater than that of the operand (measured value), having least numbers of significant digits. Following this dictum, the significant numbers in the value of current should be limited to "2", as it is the numbers of significant digits in the value of "V". This is the minimum of significant numbers in the measured quantities. As such, we should write the calculated value of current, after rounding off, as:

$$\Rightarrow I = 0.98 \quad A$$

Multiplication or division

We have already dealt the case of division. We take another example of multiplication. Let density of a uniform spherical ball is 3.201 gm/cm^3 and its volume 5.2 $gm/c\,\mathrm{cm}^3$. We can calculate its mass as :

The density is :

$$m = \rho V = 3.201X5.2 = 16.6452 \quad gm$$

In accordance with the guiding principle as stated earlier, we apply the rule that the result of multiplication or division should have same numbers of significant numbers as that of the measured value with least significant numbers.

The measured value of volume has the least "2" numbers of significant figures. In accordance with the rule, the result of multiplication is, therefore, limited to two significant digits. The value after rounding off is :

$$\Rightarrow m = 17 \quad gm$$

Addition or subtraction

In the case of addition or subtraction also, a different version of guiding principle applies. Idea is to maintain least precision of the measured value in the result of mathematical operation. To understand this, let us work out the sum of three masses “23.123 gm”, “120.1 gm” and “80.2 gm”. The arithmetic sum of masses is “223.423 gm”.

Here, we shall first apply earlier rule in order to show that we need to have a different version of rule in this case. We see that third measured value of “80.4 gm” has the least “3” numbers of significant figures. In accordance with the rule for significant figures, the result of sum should be “223”. It can be seen that application of the rule results in loosing the least precision of 1 decimal point in the measured quantities.

Clearly, we need to modify this rule. The correct rule for addition and subtraction, therefore, is that result of addition or subtraction should retain as many decimal places as are there is in the measured value, having least decimal places.

Therefore, the result of addition in the example given above is “223.4 gm”.

Rounding off

The result of mathematical operation can be any rational value with different decimal places. In addition, there can be multiple steps of mathematical operations. How would we maintain the significant numbers and precision as required in such situations.

In the previous section, we learnt that result of multiplication/division operation should be limited to the significant figures to the numbers of least significant figures in the operands. Similarly, the result of addition/subtraction operation should be limited to the decimal places as in the operand, having least decimal places. On the other hand, we have seen that the arithmetic operation results in values with large numbers of decimal places. This requires that we drop digits, which are more than as required by these laws.

We, therefore, follow certain rules to uniformly apply “rounding off” wherever it is required due to application of rules pertaining to mathematical operations :

Rule 1 : The preceding digit (uncertain digit of the significant figures) is raised by 1, if the digit following it is greater than 5. For example, a value of 2.578 is rounded as “2.58” to have three significant figures or to have two decimal places.

Rule 2 : The preceding digit (uncertain digit of the significant figures) is left unchanged, if the digit following it is less than 5. For example, a value of 2.574 is rounded as “2.57” to have three significant figures or to have two decimal places.

Rule 3 : The “odd” preceding digit (uncertain digit of the significant figures) is raised by 1, if the digit following it is 5. For example, a value of 2.535 is rounded as “2.54” to have three significant figures or to have two decimal places.

Rule 4 : The “even” preceding digit (uncertain digit of the significant figures) is left unchanged, if the digit following it is 5. For example, a value of 2.525 is rounded as “2.52” to have three significant figures or to have two decimal places.

Rule 5 : If mathematical operation involves intermediate steps, then we retain one digit more than as specified by the rules of mathematical operation. We do not carry out “rounding off” in the intermediate steps, but only to the final result.

Rule 6 : In the case of physical constants, like value of speed of light, gravitational constant etc. or in the case of mathematical constants like “π”, we take values with the precision of the operand having maximum precision i.e. maximum significant numbers or maximum decimal places. Hence, depending on the requirement in hand, the speed of light having value of “299792458 m/s” can be written as :

$$c = 3X10^8 \quad m/s (1 \text{ significant number})$$

$$c = 3.0X10^8 m/s (2 \text{ significant number})$$

$$c = 3.00X10^8 m/s (3 \text{ significant number})$$

$$c = 2.998X10^8 m/s (4 \text{ significant number})$$

Note that digit "9" appearing in the value is rounded off in the first three examples. In fourth, the digit "7" is rounded to "8".

Scientific notation

Scientific notation uses representation in terms of powers of 10. The representation follows the simple construct as given here :

$$x = a10^b$$

where “a” falls between “1” and “10” and “b” is positive or negative integer. The range of “a” as specified ensures that there is only one digit to the left of decimal. For example, the value of 1 standard atmospheric pressure is :

$$1 \quad atm = 1.013X10^6 \quad \text{Pascal}$$

Similarly, mass of earth in scientific notation is :

$$M = 5.98X10^{24} \quad Kg$$

Order of magnitude

We come across values of quantities and constants, which ranges from very small to very large. It is not always possible to remember significant figures of so many quantities. At the same time we need to have a general appreciation of the values involved. For example, consider the statement that dimension of hydrogen atom has a magnitude of the order of “-10”. This means that diameter of hydrogen atom is approximately "10" raised to the power of "-10" i.e. 10^{-10} .

$$d \sim 10^{-10}$$

Scientific notation helps to estimate order of magnitude in a consistent manner, if we follow certain rule. Using scientific notation, we have :

$$x = a10^b$$

We follow the rule as given here : If “a” is less than or equal to “5”, then we reduce the value of "a" to “1”. If “a” is greater than “5” and less than “10”, then we increase the value of “a” to "10".

In order to understand the operation, let us compare the order of magnitude of the diameter of a hydrogen atom and Sun. The diameters of Sun is :

$$d_S = 6.96X10^8 \quad m$$

Following the rule, approximate size of the sun is :

$$d_{S'} = 10X10^8 = 10^9 \quad m$$

Thus, order of magnitude of Sun is “9”. On the other hand, the order of magnitude of hydrogen atom as given earlier is “-10”. The ratio of two sizes is about equal to the difference of two orders of magnitude. This ratio can be obtained by deducting smaller order from bigger order. For example, the relative order of magnitude of sun with respect to hydrogen atom, therefore, is 9 – (-10) = 19.

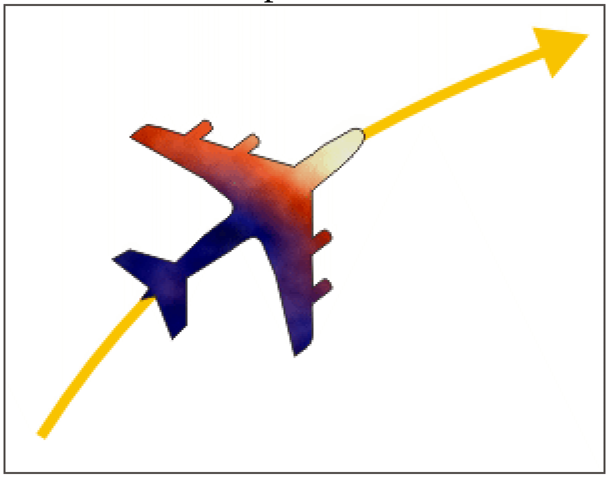
This means that diameter of sun is about 10^{19} greater than that of a hydrogen atom. In general, we should have some idea of the order of magnitudes of natural entities like particle, atom, planets and stars with respect to basic quantities like length, mass and time.

Motion

Motion is a universal attribute of all objects. Our current understanding estimates that everything in this universe is constantly moving. Contrary to our common belief based on our existence on earth, rest is simply a perception in a given frame of reference.

Motion is a state, which indicates change of position. Surprisingly, everything in this world is constantly moving and nothing is stationary. The apparent state of rest, as we shall learn, is a notional experience confined to a particular system of reference. A building, for example, is at rest in Earth's reference, but it is a moving body for other moving systems like train, motor, airplane, moon, sun etc.

Motion of an airplane



The position of plane with respect to the earth keeps changing with time.

Motion

Motion of a body refers to the change in its position with respect to time in a given frame of reference.

A frame of reference is a mechanism to describe space from the perspective of an observer. In other words, it is a system of measurement for locating positions of the bodies in space with respect to an observer (reference).

Since, frame of reference is a system of measurement of positions in space as measured by the observer, frame of reference is said to be attached to the observer. For this reason, terms “frame of reference” and “observer” are interchangeably used to describe motion.

In our daily life, we recognize motion of an object with respect to ourselves and other stationary objects. If the object maintains its position with respect to the stationary objects, we say that the object is at rest; else the object is moving with respect to the stationary objects. Here, we conceive all objects moving with earth without changing their positions on earth surface as stationary objects in the earth’s frame of reference. Evidently, all bodies not changing position with respect to a specific observer is stationary in the frame of reference attached with the observer.

We require an observer to identify motion

Motion has no meaning without a reference system.

An object or a body under motion, as a matter of fact, is incapable of identifying its own motion. It would be surprising for some to know that we live on this earth in a so called stationary state without ever being aware that we are moving around sun at a very high speed - at a speed faster than the fastest airplane that the man kind has developed. The earth is moving around sun at a speed of about 30 km/s (≈ 30000 m/s ≈ 100000 km/hr) – a speed about 1000 times greater than the motoring speed and 100 times greater than the aircraft’s speed.

Likewise, when we travel on aircraft, we are hardly aware of the speed of the aircraft. The state of fellow passengers and parts of the aircraft are all moving at the same speed, giving the impression that passengers are simply sitting in a stationary cabin. The turbulence that the passengers experience occasionally is a consequence of external force and is not indicative of the motion of the aircraft.

It is the external objects and entities which indicate that aircraft is actually moving. It is the passing clouds and changing landscape below, which make us think that aircraft is actually moving. The very fact that we land at

geographically distant location at the end of travel in a short time, confirms that aircraft was actually cruising at a very high speed.

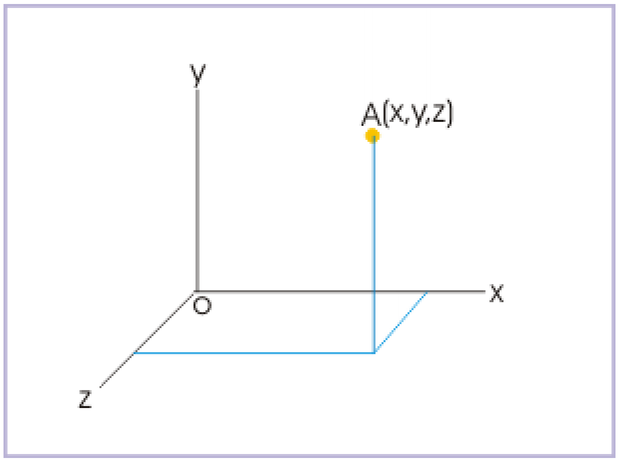
The requirement of an observer in both identifying and quantifying motion brings about new dimensions to the understanding of motion. Notably, the motion of a body and its measurement is found to be influenced by the state of motion of the observer itself and hence by the state of motion of the attached frame of reference. As such, a given motion is evaluated differently by different observers (system of references).

Two observers in the same state of motion, such as two persons standing on the platform, perceive the motion of a passing train in exactly same manner. On the other hand, the passenger in a speeding train finds that the other train crossing it on the parallel track in opposite direction has the combined speed of the two trains $(v_1 + v_1)$. The observer on the ground, however, find them running at their individual speeds v_1 and v_2.

From the discussion above, it is clear that motion of an object is an attribute, which can **not** be stated in absolute term; but it is a kind of attribute that results from the interaction of the motions of the both object and observer (frame of reference).

Frame of reference and observer

Frame of reference is a mathematical construct to specify position or location of a point object in space. Basically, frame of reference is a coordinate system. There are plenty of coordinate systems in use, but the Cartesian coordinate system, comprising of three mutually perpendicular axes, is most common. A point in three dimensional space is defined by three values of coordinates i.e. x,y and z in the Cartesian system as shown in the figure below. We shall learn about few more useful coordinate systems in next module titled " Coordinate systems in physics ".
Frame of reference

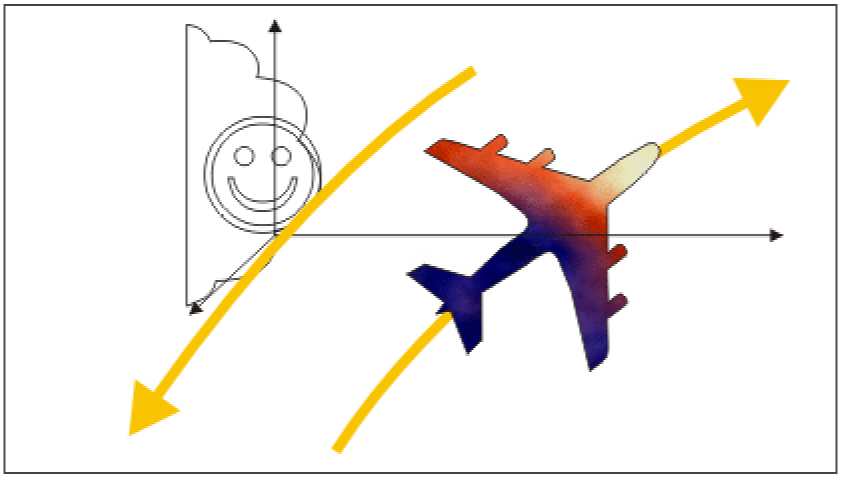


A point in three dimensional space is defined by three values of coordinates

We need to be specific in our understanding of the role of the observer and its relation with frame of reference. Observation of motion is considered an human endeavor. But motion of an object is described in reference of both human and non-human bodies like clouds, rivers, poles, moon etc. From the point of view of the study of motion, we treat reference bodies capable to make observations, which is essentially a human like function. As such, it is helpful to imagine ourselves attached to the reference system, making observations. It is essentially a notional endeavor to consider that the measurements are what an observer in that frame of reference would make, had the observer with the capability to measure was actually present there.

Earth is our natural abode and we identify all non-moving ground observers equivalent and at rest with the earth. For other moving systems, we need to specify position and determine motion by virtually (in imagination) transposing ourselves to the frame of reference we are considering.

Measurement from a moving reference

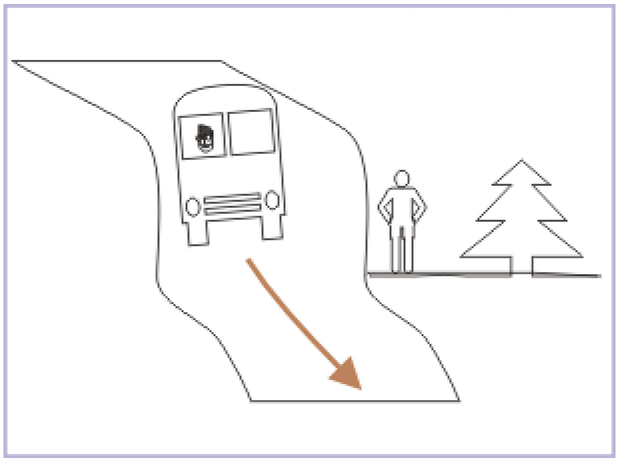


Take the case of observations about the motion of an aircraft made by two observers one at a ground and another attached to the cloud moving at certain speed. For the observer on the ground, the aircraft is moving at a speed of ,say, 1000 km/hr.

Further, let the reference system attached to the cloud itself is moving, say, at the speed of 50 km/hr, in a direction opposite to that of the aircraft as seen by the person on the ground. Now, locating ourselves in the frame of reference of the cloud, we can visualize that the aircraft is changing its position more rapidly than as observed by the observer on the ground i.e. at the combined speed and would be seen flying by the observer on the cloud at the speed of 1000 + 50 = 1050 km/hr.

We need to change our mind set

The scientific measurement requires that we change our mindset about perceiving motion and its scientific meaning. To our trained mind, it is difficult to accept that a stationary building standing at a place for the last 20 years is actually moving for an observer, who is moving towards it. Going by the definition of motion, the position of the building in the coordinate system of an approaching observer is changing with time. Actually, the building is moving for all moving bodies. What it means that the study of motion requires a new scientific approach about perceiving motion. It also means that the scientific meaning of motion is not limited to its interpretation from the perspective of earth or an observer attached to it.
Motion of a tree



Motion of the tree as seen by the person driving the truck

Consider the motion of a tree as seen from a person driving a truck (See Figure above) . The tree is undeniably stationary for a person standing on the ground. The coordinates of the tree in the frame of reference attached to the truck, however, is changing with time. As the truck moves ahead, the coordinates of the tree is increasing in the opposite direction to that of the truck. The tree, thus, is moving backwards for the truck driver – though we may find it hard to believe as the tree has not changed its position on the ground and is stationary. This deep rooted perception negating scientific hard fact is the outcome of our conventional mindset based on our life long perception of the bodies grounded to the earth.

Is there an absolute frame of reference?

Let us consider following :

In nature, we find that smaller entities are contained within bigger entities, which themselves are moving. For example, a passenger is part of a train, which in turn is part of the earth, which in turn is part of the solar system and so on. This aspect of containership of an object in another moving object is chained from smaller to bigger bodies. We simply do not know

which one of these is the ultimate container and the one, which is not moving.

These aspects of motion as described in the above paragraph leads to the following conclusions about frame of reference :

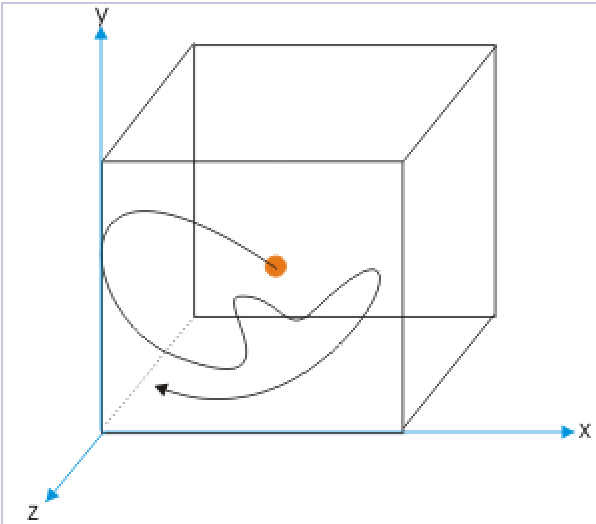
"There is no such thing like a “mother of all frames of reference” or the ultimate container, which can be considered at rest. As such, no measurement of motion can be considered absolute. All measurements of motion are, therefore, relative."

Motion types

Nature displays motions of many types. Bodies move in a truly complex manner. Real time motion is mostly complex as bodies are subjected to various forces. These motions are not simple straight line motions. Consider a bird’s flight for example. Its motion is neither in the straight line nor in a plane. The bird flies in a three dimensional space with all sorts of variations involving direction and speed. A boat crossing a river, on the other hand, roughly moves in the plane of water surface.

Motion in one dimension is rare. This is surprising, because the natural tendency of all bodies is to maintain its motion in both magnitude and direction. This is what Newton’s first law of motion tells us. Logically all bodies should move along a straight line at a constant speed unless it is acted upon by an external force. The fact of life is that objects are subjected to verities of forces during their motion and hence either they deviate from straight line motion or change speed.

Since, real time bodies are mostly non-linear or varying in speed or varying in both speed and direction, we may conclude that bodies are always acted upon by some force. The most common and omnipresent forces in our daily life are the gravitation and friction (electrical) forces. Since force is not the subject of discussion here, we shall skip any further elaboration on the role of force. But, the point is made : bodies generally move in complex manner as they are subjected to different forces.
Real time motion

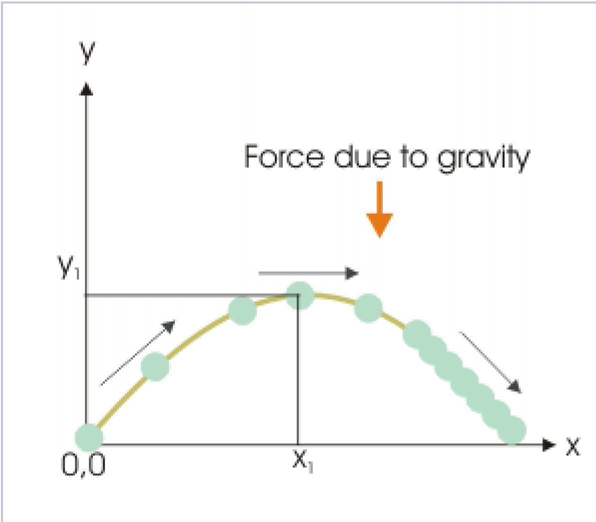


A gas molecule in a container moves randomly under electrostatic interaction with other molecules

Nevertheless, study of motion in one dimension is basic to the understanding of more complex scenarios of motion. The very nature of physical laws relating to motion allows us to study motion by treating motions in different directions separately and then combining the motions in accordance with vector rules to get the overall picture.

A general classification of motion is done in the context of the dimensions of the motion. A motion in space, comprising of three dimensions, is called three dimensional motion. In this case, all three coordinates are changing as the time passes by. While, in two dimensional motion, any two of the three dimensions of the position are changing with time. The parabolic path described by a ball thrown at certain angle to the horizon is an example of the two dimensional motion (See Figure). A ball thrown at an angle with horizon is described in terms of two coordinates x and y.

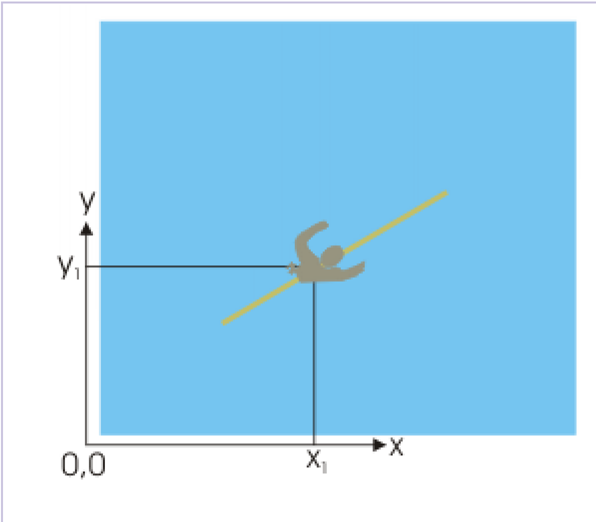
Two dimensional motion



A ball thrown at an angle with horizon is described in terms of two coordinates x and y

One dimensional motion, on the other hand, is described using any one of the three coordinates; remaining two coordinates remain constant through out the motion. Generally, we believe that one dimensional motion is equivalent to linear motion. This is not further from truth either. A linear motion in a given frame of reference, however, need not always be one dimensional. Consider the motion of a person swimming along a straight line on a calm water surface. Note here that position of the person at any given instant in the coordinate system is actually given by a pair of coordinate (x,y) values (See Figure below).

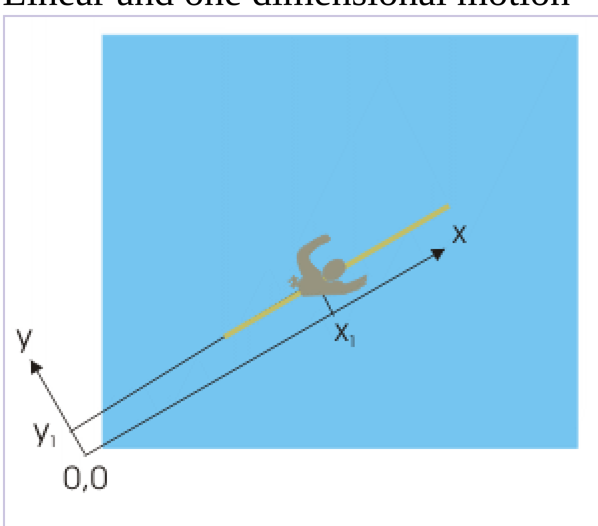
Linear motion



Description of motion requires two coordinate values

There is a caveat though. We can always rotate the pair of axes such that one of it lies parallel to the path of motion as shown in the figure. One of the coordinates, yy_1 is constant through out the motion. Only the x-coordinate is changing and as such motion can be described in terms of x-coordinate alone. It follows then that all linear motion can essentially be treated as one dimensional motion.

Linear and one dimensional motion



Choice of appropriate coordinate system renders linear motion as one dimensional motion.

Kinematics

Kinematics refers to the study of motion of natural bodies. The bodies that we see and deal with in real life are three dimensional objects and essentially not a point object.

A point object would occupy a point (without any dimension) in space. The real bodies, on the other hand, are entities with dimensions, having length, breadth and height. This introduces certain amount of complexity in so far as describing motion. First of all, a real body can not be specified by a single set of coordinates. This is one aspect of the problem. The second equally important aspect is that different parts of the bodies may have path trajectories different to each other.

When a body moves **with** rotation (rolling while moving), the path trajectories of different parts of the bodies are different; on the other hand, when the body moves **without** rotation (slipping/ sliding), the path trajectories of the different parts of the bodies are parallel to each other.

In the second case, the motion of all points within the body is equivalent as far as translational motion of the body is concerned and hence, such bodies may be said to move like a point object. It is, therefore, possible to treat the body under consideration to be equivalent to a point so long rotation is not involved.

For this reason, study of kinematics consists of studies of :

1. Translational kinematics
2. Rotational kinematics

A motion can be pure translational or pure rotational or a combination of the two types of motion.

The translational motion allows us to treat a real time body as a point object. Hence, we freely refer to bodies, objects and particles in one and the same sense that all of them are point entities, whose position can be represented by a single set of coordinates. We should keep this in mind while studying translational motion of a body and treating the same as point.

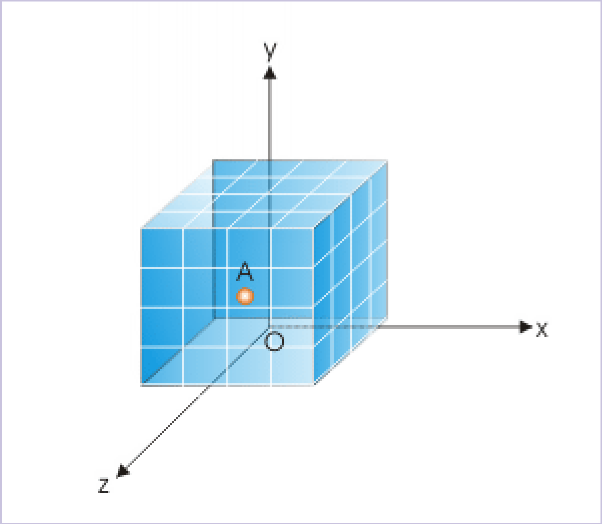
Coordinate systems in physics

Coordinate system is a mathematical construct to measure distance and direction in relation to a system of rigid bodies. The connection to rigid bodies is a crucial consideration for drawing a coordinate system, with out which coordinate system has no meaning in physics.

Coordinate system is a system of measurement of distance and direction with respect to rigid bodies. Structurally, it comprises of coordinates and a reference point, usually the origin of the coordinate system. The coordinates primarily serve the purpose of reference for the direction of motion, while origin serves the purpose of reference for the magnitude of motion.

Measurements of magnitude and direction allow us to locate a position of a point in terms of measurable quantities like linear distances or angles or their combinations. With these measurements, it is possible to locate a point in the spatial extent of the coordinate system. The point may lie anywhere in the spatial (volumetric) extent defined by the rectangular axes as shown in the figure. (Note : The point, in the figure, is shown as small sphere for visual emphasis only)

A point in the coordinate system



A distance in the coordinate system is measured with a standard rigid linear length like that of a “meter” or a “foot”. A distance of 5 meters, for example, is 5 times the length of the standard length of a meter. On the other hand, an angle is defined as a ratio of lengths and is dimensional-less.

Hence, measurement of direction is indirectly equivalent to the measurement of distances only.

The coordinate system represents the system of rigid body like earth, which is embodied by an observer, making measurements. Since measurements are implemented by the observer, they (the measurements in the coordinate system) represent distance and direction as seen by the observer. It is, therefore, clearly implied that measurements in the coordinates system are specific to the state of motion of the coordinate system.

In a plane language, we can say that the description of motion is specific to a system of rigid bodies, which involves measurement of distance and direction. The measurements are done, using standards of length, by an observer, who is at rest with the system of rigid bodies. The observer makes use of a coordinate system attached to the system of rigid bodies and uses the same as reference to make measurements.

It is apparent that the terms “system of rigid bodies”, “observer” and “coordinate system” etc. are similar in meaning; all of which conveys a system of reference for carrying out measurements to describe motion. We sum up the discussion thus far as :

1. Measurements of distance, direction and location in a coordinate system are specific to the system of rigid bodies, which serve as reference for both magnitude and direction.
2. Like point, distance and other aspects of motion, the concept of space is specific to the reference represented by coordinate system. It is, therefore, suggested that use of word “space” independent of coordinate system should be avoided and if used it must be kept in mind that it represents volumetric extent of a specific coordinate system. The concept of space, if used without caution, leads to an inaccurate understanding of the laws of nature.
3. Once the meanings of terms are clear, “the system of reference” or “frame of reference” or “rigid body system” or “observer” or “coordinate system” may be used interchangeably to denote an unique system for determination of motional quantities and the representation of a motion.

Coordinate system types

Coordinate system types determine position of a point with measurements of distance or angle or combination of them. A spatial point requires three measurements in each of these coordinate types. It must, however, be noted that the descriptions of a point in any of these systems are equivalent. Different coordinate types are mere convenience of appropriateness for a given situation. Three major coordinate systems used in the study of physics are :

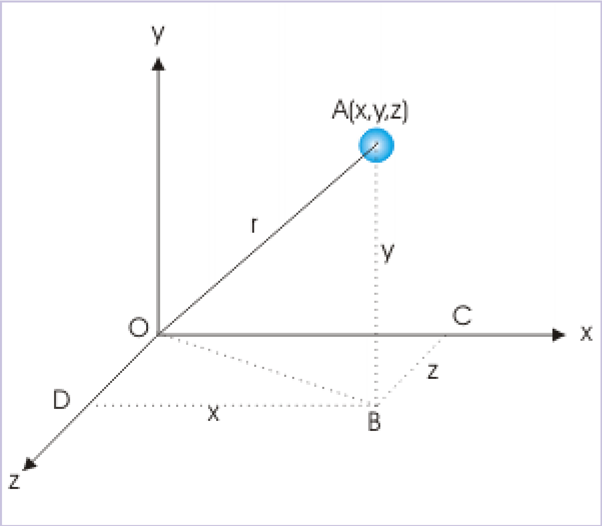
- Rectangular (Cartesian)
- Spherical
- Cylindrical

Rectangular (Cartesian) coordinate system is the most convenient as it is easy to visualize and associate with our perception of motion in daily life. Spherical and cylindrical systems are specifically designed to describe motions, which follow spherical or cylindrical curvatures.

Rectangular (Cartesian) coordinate system

The measurements of distances along three mutually perpendicular directions, designated as x,y and z, completely define a point A (x,y,z).

A point in rectangular coordinate system



A point is specified with three coordinate values

From geometric consideration of triangle OAB,

$$r = \sqrt{\mathrm{OB}^2 + \mathrm{AB}^2}$$

From geometric consideration of triangle OBD,

$$\mathrm{OB}^2 = \sqrt{\mathrm{BD}^2 + \mathrm{OD}^2}$$

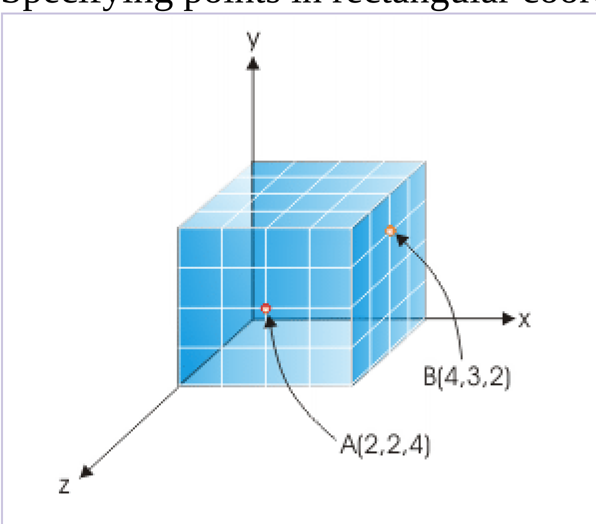
Combining above two relations, we have :

$$\Rightarrow r = \sqrt{\mathrm{BD}^2 + \mathrm{OD}^2 + \mathrm{AB}^2}$$

$$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

The numbers are assigned to a point in the sequence x, y, z as shown for the points A and B.

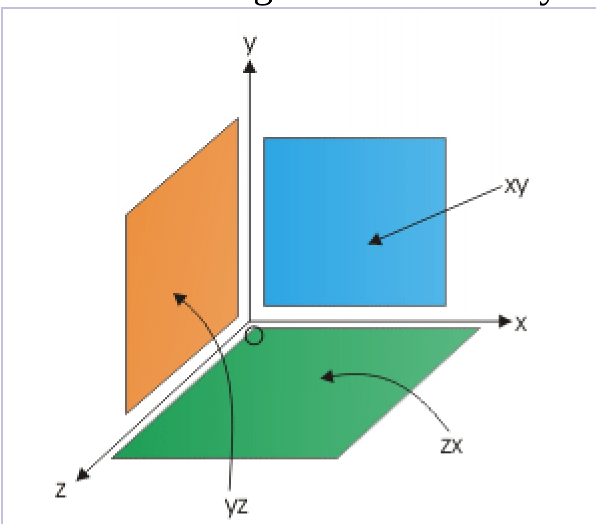
Specifying points in rectangular coordinate system



A point is specified with coordinate values

Rectangular coordinate system can also be viewed as volume composed of three rectangular surfaces. The three surfaces are designated as a pair of axial designations like “xy” plane. We may infer that the “xy” plane is defined by two lines (x and y axes) at right angle. Thus, there are “xy”, “yz” and “zx” rectangular planes that make up the space (volumetric extent) of the coordinate system (See figure).

Planes in rectangular coordinate system



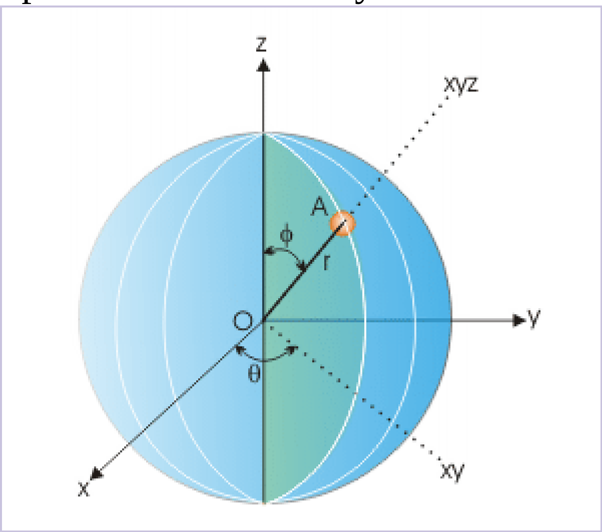
Three mutually perpendicular planes define domain of rectangular system

The motion need not be extended in all three directions, but may be limited to two or one dimensions. A circular motion, for example, can be represented in any of the three planes, whereby only two axes with an origin will be required to describe motion. A linear motion, on the other hand, will require representation in one dimension only.

Spherical coordinate system

A three dimensional point “A” in spherical coordinate system is considered to be located on a sphere of a radius “r”. The point lies on a particular cross section (or plane) containing origin of the coordinate system. This cross section makes an angle “θ” from the “zx” plane (also known as longitude angle). Once the plane is identified, the angle, φ, that the line joining origin O to the point A, makes with “z” axis, uniquely defines the point A (r, θ, φ).

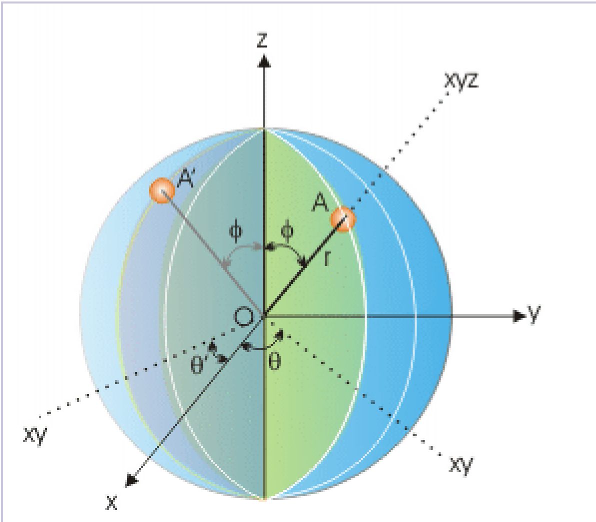
Spherical coordinate system



A point is specified with three coordinate values

It must be realized here that we need to designate three values r, θ and φ to uniquely define the point A. If we do not specify θ, the point could then lie any of the infinite numbers of possible cross section through the sphere like A'(See Figure below).

Spherical coordinate system



A point is specified with three coordinate values

From geometric consideration of spherical coordinate system :

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$x = r \sin \varphi \cos \theta$$
$$y = r \sin \varphi \sin \theta$$
$$z = r \cos \varphi$$
$$\tan \varphi = \frac{\sqrt{x^2+y^2}}{z}$$
$$\tan \theta = \frac{y}{z}$$

These relations can be easily obtained, if we know to determine projection of a directional quantity like position vector. For example, the projection of "r" in "xy" plane is "r sinφ". In turn, projection of "r sinφ" along x-axis is ""r sinφ cosθ". Hence,

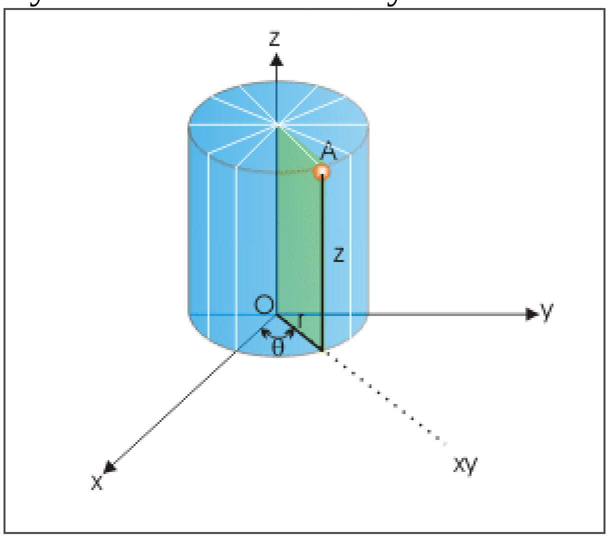
$$x = r \sin \varphi \cos \theta$$

In the similar fashion, we can determine other relations.

Cylindrical coordinate system

A three dimensional point “A” in cylindrical coordinate system is considered to be located on a cylinder of a radius “r”. The point lies on a particular cross section (or plane) containing origin of the coordinate system. This cross section makes an angle “θ” from the “zx” plane. Once the plane is identified, the height, z, parallel to vertical axis “z” uniquely defines the point A(r, θ, z)

Cylindrical coordinate system



A point is specified with three coordinate values

$$r = \sqrt{x^2 + y^2}$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$\tan\theta = \frac{y}{z}$$

Distance

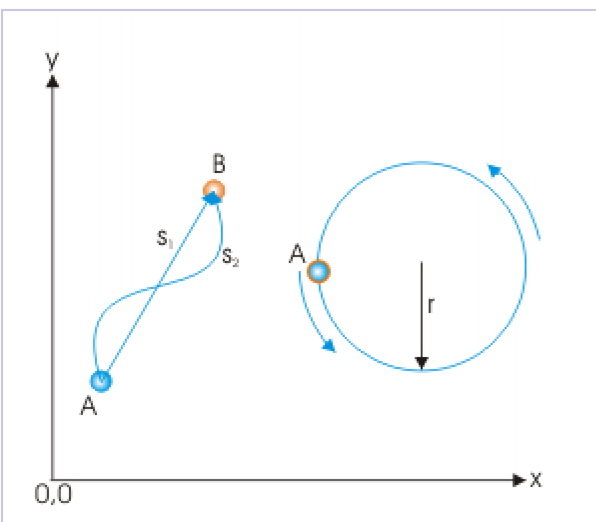
Motion involves two types of measurements : one, which depends on the end points (displacement) and the other, which depends on all points (distance) of motion.

Distance represents the magnitude of motion in terms of the "length" of the path, covered by an object during its motion. The terms "distance" and "distance covered" are interchangeably used to represent the same length along the path of motion and are considered equivalent terms. Initial and final positions of the object are mere start and end points of measurement and are not sufficient to determine distance. It must be understood that the distance is measured by the length covered, which may not necessarily be along the straight line joining initial and final positions. The path of the motion between two positions is an important consideration for determining distance. One of the paths between two points is the shortest path, which may or may not be followed during the motion.

Distance

Distance is the length of path followed during a motion.

Distance



Distance depends on the choice of path between two points

In the diagram shown above, s_1 , represents the shortest distance between points A and B. Evidently,

$$s_2 \geq s_1$$

The concept of distance is associated with the magnitude of movement of an object during the motion. It does not matter if the object goes further away or suddenly moves in a different direction or reverses its path. The magnitude of movement keeps adding up so long the object moves. This notion of distance implies that distance is not linked with any directional attribute. The distance is, thus, a scalar quantity of motion, which is cumulative in nature.

An object may even return to its original position over a period of time without any “net” change in position; the distance, however, will not be zero. To understand this aspect of distance, let us consider a point object that follows a circular path starting from point A and returns to the initial position as shown in the figure above. Though, there is no change in the position over the period of motion; but the object, in the meantime, covers a circular path, whose length is equal to its perimeter i.e. 2πr.

Generally, we choose the symbol 's' to denote distance. A distance is also represented in the form of “Δs” as the distance covered in a given time interval Δt. The symbol “Δ” pronounced as “del” signifies the change in the quantity before which it appears.

Distance is a scalar quantity but with a special feature. It does not take negative value unlike some other scalar quantities like “charge”, which can assume both positive and negative values. The very fact that the distance keeps increasing regardless of the direction, implies that distance for a body in motion is always positive. Mathematically :

$$s > 0$$

Since distance is the measurement of length, its dimensional formula is [L] and its SI measurement unit is “meter”.

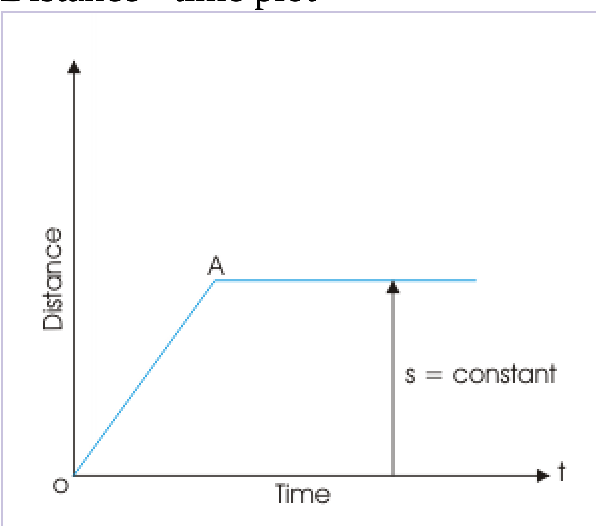
Distance – time plot

Distance – time plot is a simple plot of two scalar quantities along two axes. However, the nature of distance imposes certain restrictions, which characterize "distance - time" plot.

The nature of "distance – time" plot, with reference to its characteristics, is summarized here :

1. Distance is a positive scalar quantity. As such, "the distance – time" plot is a curve in the first quadrant of the two dimensional plot.
2. As distance keeps increasing during a motion, the slope of the curve is always positive.
3. When the object undergoing motion stops, then the plot becomes straight line parallel to time axis so that distance is constant as shown in the figure here.

Distance - time plot



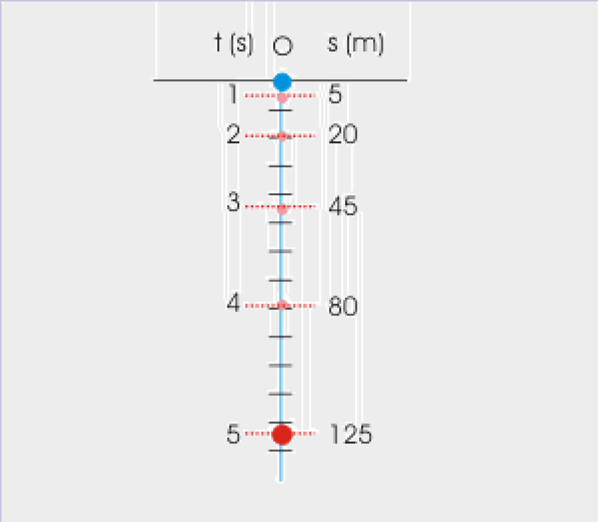
One important implication of the positive slope of the "distance - time" plot is that the curve never drops below a level at any moment of time. Besides, it must be noted that "distance - time" plot is handy in determining "instantaneous speed", but we choose to conclude the discussion of "distance - time" plot as these aspects are separately covered in subsequent module.

Example:

Distance – time plot

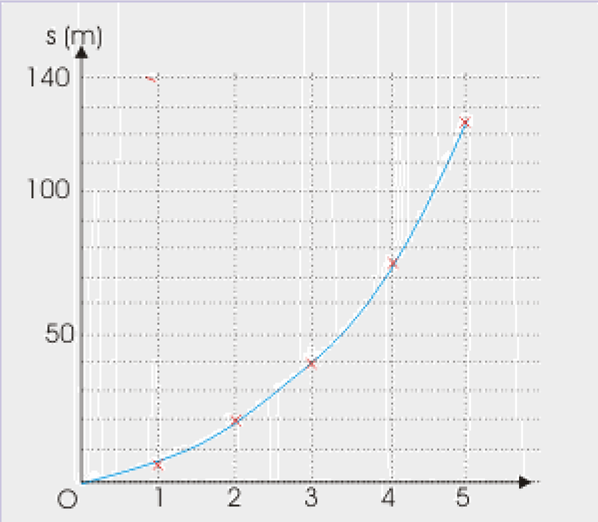
Question : A ball falling from an height 'h' strikes the ground. The distance covered during the fall at the end of each second is shown in the figure for the first 5 seconds. Draw distance – time plot for the motion during this period. Also, discuss the nature of the curve.

Motion of a falling ball



Solution : We have experienced that a free falling object falls with increasing speed under the influence of gravity. The distance covered in successive time intervals increases with time. The magnitudes of distance covered in successive seconds given in the plot illustrate this point. In the plot between distance and time as shown, the origin of the reference (coordinate system) is chosen to coincide with initial point of the motion.

Distance – time plot



From the plot, it is clear that the ball covers more distance as it nears the ground. The "distance- time" curve during fall is, thus, flatter near start point and steeper near earth surface. Can you guess the nature of plot when a ball is thrown up against gravity?

Exercise:

Problem:

A ball falling from an height 'h' strikes a hard horizontal surface with increasing speed. On each rebound, the height reached by the ball is half of the height it fell from. Draw "distance – time" plot for the motion covering two consecutive strikes, emphasizing the nature of curve (ignore actual calculation). Also determine the total distance covered during the motion.

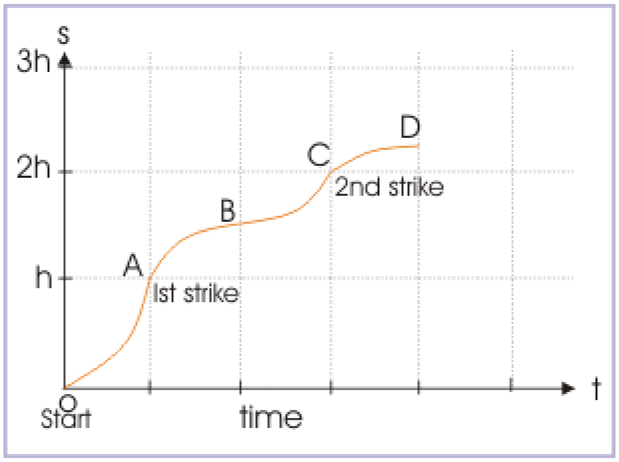
Solution:

Here we first estimate the manner in which distance is covered under gravity as the ball falls or rises.

The distance- time curve during fall is flatter near start point and steeper near earth surface. On the other hand, we can estimate that the distance- time curve, during rise, is steeper near the earth surface (covers more distance due to greater speed) and flatter as it reaches the maximum height, when speed of the ball becomes zero.

The "distance – time" plot of the motion of the ball, showing the nature of curve during motion, is :

Distance – time plot

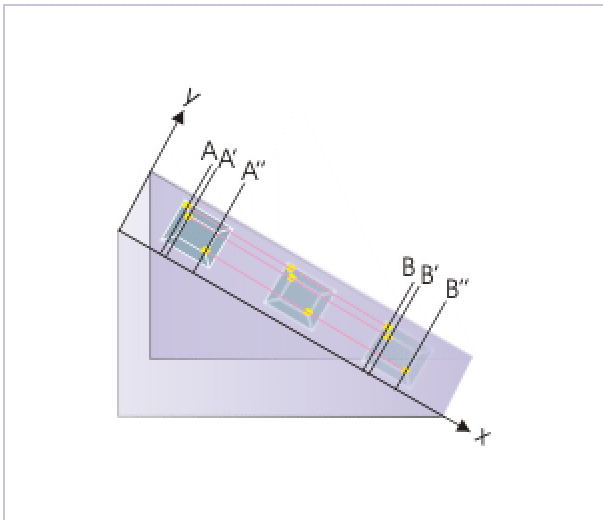


The origin of plot (O) coincides with the initial position of the ball (t = 0). Before striking the surface for the first time (A), it travels a distance of ‘h’. On rebound, it rises to a height of ‘h/2’ (B on plot). Total distance is ‘h + h/2 = 3h/2’. Again falling from a height of ‘h/2’, it strikes the surface, covering a distance of ‘h/2’. The total distance from the start to the second strike (C on plot) is ‘3h/2 + h/2 = 2h’.

Position

Coordinate system enables us to specify a point in its defined volumetric space. We must recognize that a point is a concept without dimensions; whereas the objects or bodies under motion themselves are not points. The real bodies, however, approximates a point in translational motion, when paths followed by the particles, composing the body are parallel to each other (See Figure). As we are concerned with the geometry of the path of motion in kinematics, it is, therefore, reasonable to treat real bodies as “point like” mass for description of translational motion.

Translational motion



Particles follow parallel paths

We conceptualize a particle in order to facilitate the geometric description of motion. A particle is considered to be dimensionless, but having a mass. This hypothetical construct provides the basis for the logical correspondence of point with the position occupied by a particle.

Without any loss of purpose, we can designate motion to begin at A or A’ or A’’ corresponding to final positions B or B’ or B’’ respectively as shown in the figure above.

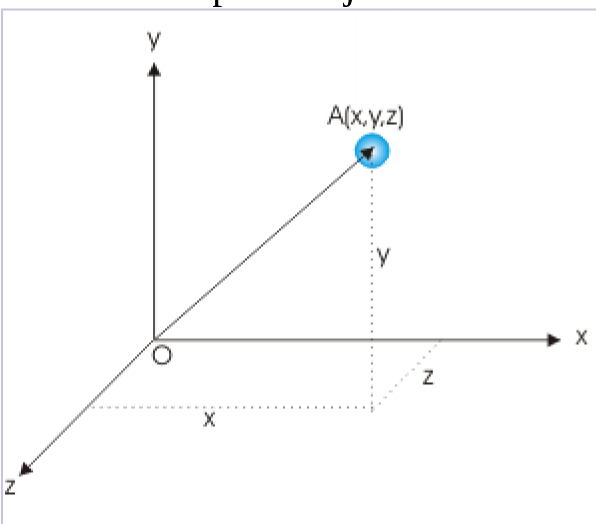
For the reasons as outlined above, we shall freely use the terms “body” or “object” or “particle” in one and the same way as far as description of translational motion is concerned. Here, pure translation conveys the meaning that the object is under motion without rotation, like sliding of a block on a smooth inclined plane.

Position

Position

> The position of a particle is a point in the defined volumetric space of the coordinate system.

Position of a point object



Position of a point is specified by three coordinate values

The position of a point like object, in three dimensional coordinate space, is defined by three values of coordinates i.e. x, y and z in Cartesian coordinate system as shown in the figure above.

It is evident that the relative position of a point with respect to a fixed point such as the origin of the system “O” has directional property. The position

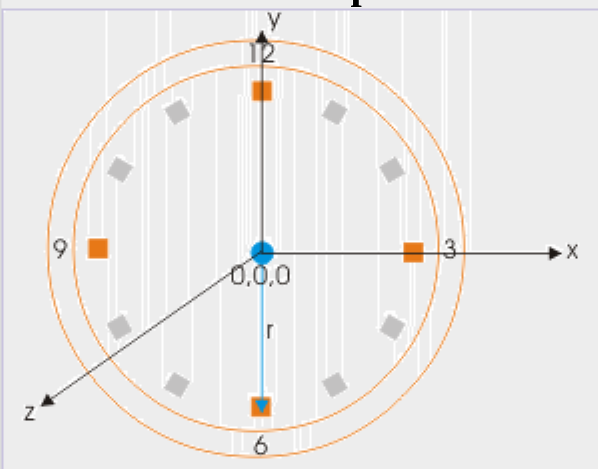
of the object, for example, can lie either to the left or to the right of the origin or at a certain angle from the positive x - direction. As such the position of an object is associated with directional attribute with respect to a frame of reference (coordinate system).

Example:

Coordinates

Problem : The length of the second's hand of a round wall clock is 'r' meters. Specify the coordinates of the tip of the second's hand corresponding to the markings 3,6,9 and 12 (Consider the center of the clock as the origin of the coordinate system.).

Coordinates of the tip of the second's hand



The origin coincides with the center

Solution : The coordinates of the tip of the second's hand is given by the coordinates :

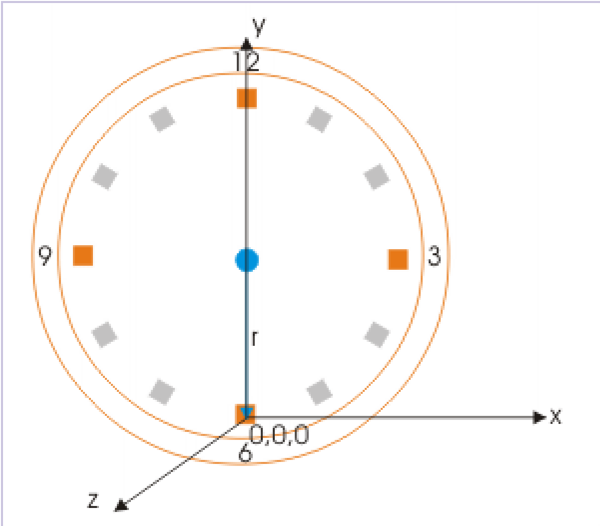
`3 : r, 0, 0 6 : 0, -r, 0 9 : -r, 0, 0 12 : 0, r, 0`

Exercise:

Problem:

What would be the coordinates of the markings 3,6,9 and 12 in the earlier example, if the origin coincides with the marking 6 on the clock ?

Coordinates of the tip of the second's hand



Origin coincides with the marking 6 O' clock

Solution:

The coordinates of the tip of the second's hand is given by the coordinates :

`3 : r, r, 0 6 : 0, 0, 0 9 : -r, r, 0 12 : 0, 2r, 0`

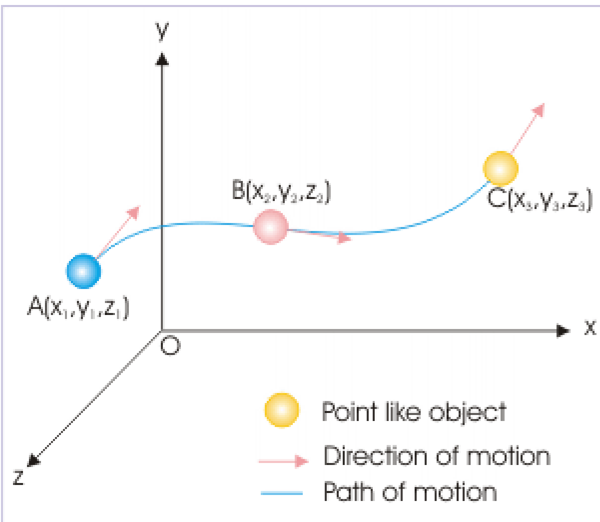
The above exercises point to an interesting feature of the frame of reference: that the specification of position of the object (values of coordinates) depends on the choice of origin of the given frame of reference. We have already seen that description of motion depends on the state of observer i.e. the attached system of reference. This additional

dependence on the choice of origin of the reference would have further complicated the issue, but for the linear distance between any two points in a given system of reference, is found to be independent of the choice of the origin. For example, the linear distance between the markings 6 and 12 is ‘2r’, irrespective of the choice of the origin.

Plotting motion

Position of a point in the volumetric space is a three dimensional description. A plot showing positions of an object during a motion is an actual description of the motion in so far as the curve shows the path of the motion and its length gives the distance covered. A typical three dimensional motion is depicted as in the figure below :

Motion in three dimension



The plot is the path followed by the object during motion.

In the figure, the point like object is deliberately shown not as a point, but with finite dimensions. This has been done in order to emphasize that an object of finite dimensions can be treated as point when the motion is purely translational.

The three dimensional description of positions of an object during motion is reduced to be two or one dimensional description for the planar and linear motions respectively. In two or one dimensional motion, the remaining coordinates are constant. In all cases, however, the plot of the positions is meaningful in following two respects :

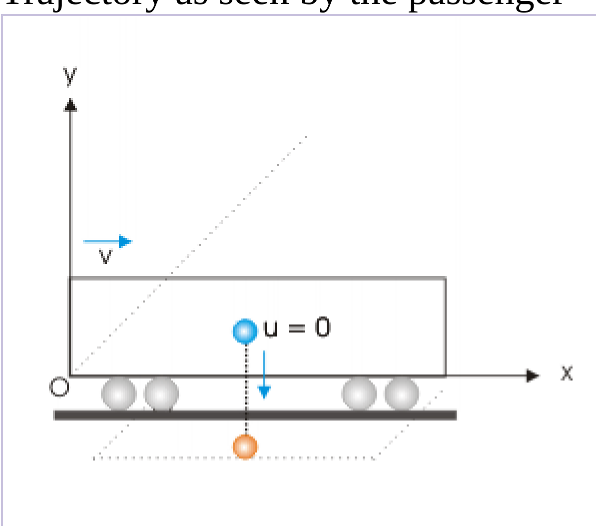
- The length of the curve (i.e. plot) is equal to the distance.
- A tangent in forward direction at a point on the curve gives the direction of motion at that point

Description of motion

Position is the basic element used to describe motion. Scalar properties of motion like distance and speed are expressed in terms of position as a function of time. As the time passes, the positions of the motion follow a path, known as the trajectory of the motion. It must be emphasized here that the path of motion (trajectory) is unique to a frame of reference and so is the description of the motion.

To illustrate the point, let us consider that a person is traveling on a train, which is moving with the velocity **v** along a straight track. At a particular moment, the person releases a small pebble. The pebble drops to the ground along the vertical direction as seen by the person.

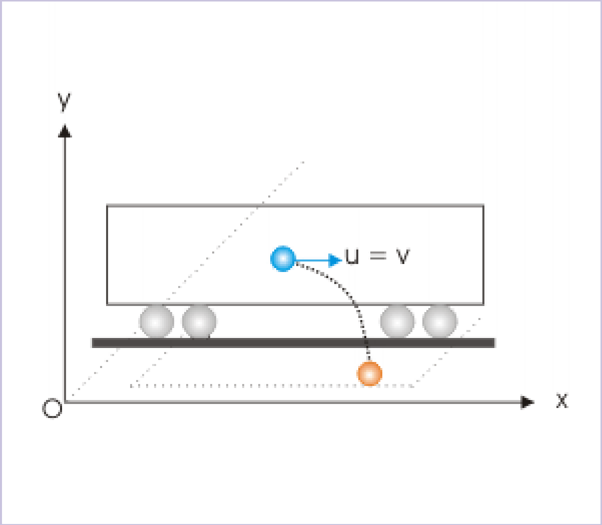
Trajectory as seen by the passenger



Trajectory is a straight line.

The same incident, however, is seen by an observer on the ground as if the pebble followed a parabolic path (See Figure blow). It emerges then that the path or the trajectory of the motion is also a relative attribute, like other attributes of the motion (speed and velocity). The coordinate system of the passenger in the train is moving with the velocity of train (**v**) with respect to the earth and the path of the pebble is a straight line. For the person on the ground, however, the coordinate system is stationary with respect to earth. In this frame, the pebble has a horizontal velocity, which results in a parabolic trajectory.

Trajectory as seen by the person on ground



Trajectory is a parabolic curve.

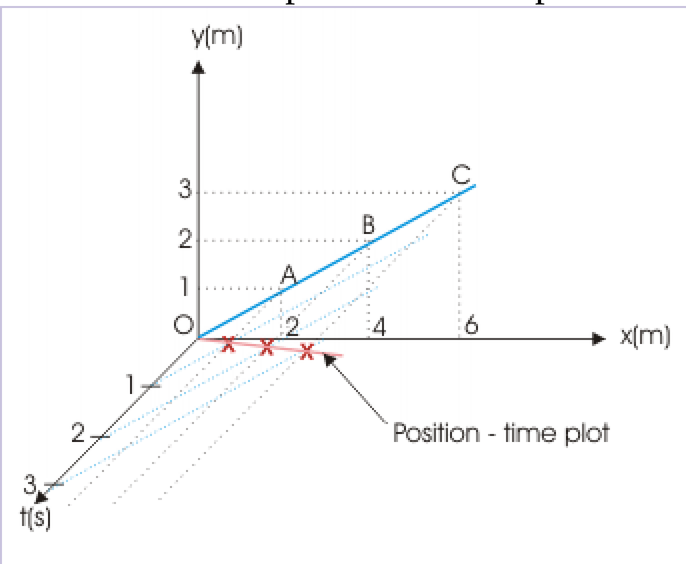
Without overemphasizing, we must acknowledge that the concept of path or trajectory is essentially specific to the frame of reference or the coordinate system attached to it. Interestingly, we must be aware that this particular observation happens to be the starting point for the development of special theory of relativity by Einstein (see his original transcript on the subject of relativity).

Position – time plot

The position in three dimensional motion involves specification in terms of three coordinates. This requirement poses a serious problem, when we want to investigate positions of the object with respect to time. In order to draw such a graph, we would need three axes for describing position and one axis for plotting time. This means that a position – time plot for a three dimensional motion would need four (4) axes !

A two dimensional position – time plot, however, is a possibility, but its drawing is highly complicated for representation on a two dimensional paper or screen. A simple example consisting of a linear motion in the x-y plane is plotted against time on z – axis (See Figure).

Two dimensional position – time plot



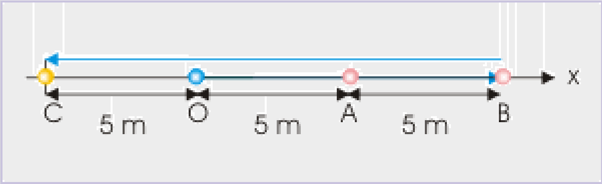
As a matter of fact, it is only the one dimensional motion, whose position – time plot can be plotted conveniently on a plane. In one dimensional motion, the point object can either be to the left or to the right of the origin in the direction of reference line. Thus, drawing position against time is a straight forward exercise as it involves plotting positions with appropriate sign.

Example:

Coordinates

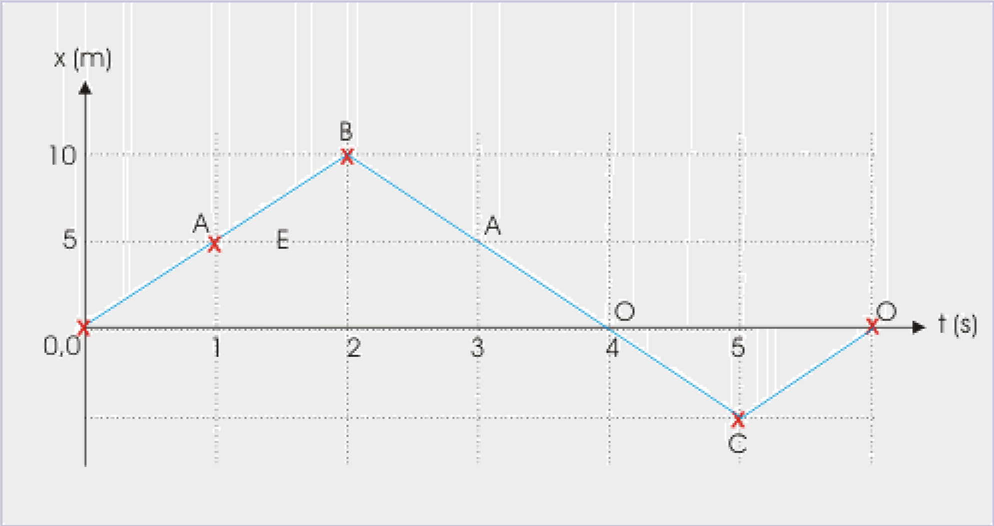
Problem : A ball moves along a straight line from O to A to B to C to O along x-axis as shown in the figure. The ball covers each of the distance of 5 m in one second. Plot the position – time graph.

Motion along a straight line



Solution : The coordinates of the ball are 0,5,10, -5 and 0 at points O, A, B, C and O (on return) respectively. The position – time plot of the motion is as given below :

Position – time plot in one dimension



Exercise:

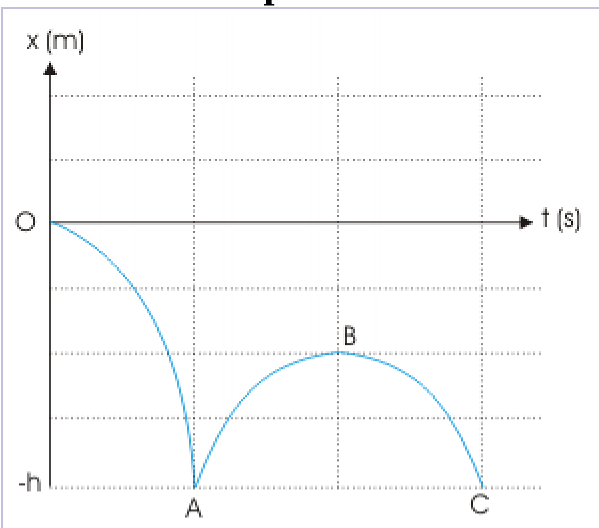
Problem:

A ball falling from a height ‘h’ strikes a hard horizontal surface with increasing speed. On each rebound, the height reached by the ball is half of the height it fell from. Draw position – time plot for the motion covering two consecutive strikes, emphasizing the nature of curve (ignore actual calculation).

Solution:

Now, as the ball falls towards the surface, it covers path at a quicker pace. As such, the position changes more rapidly as the ball approaches the surface. The curve (i.e. plot) is, therefore, steeper towards the surface. On the return upward journey, the ball covers lesser distance as it reaches the maximum height. Hence, the position – time curve (i.e. plot) is flatter towards the point of maximum height.

Position – time plot in one dimension

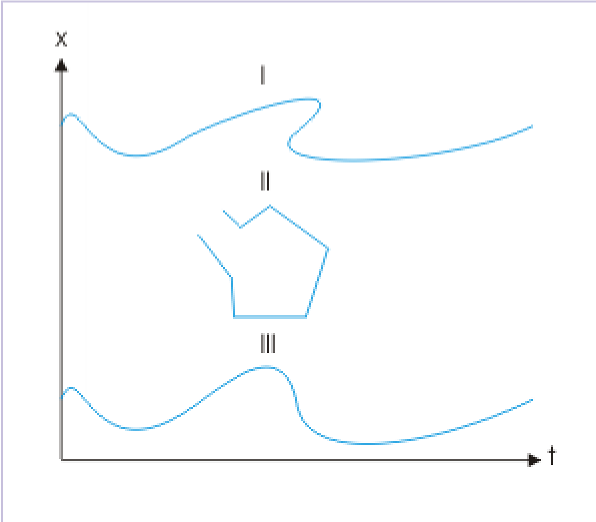


Exercise:

Problem:

The figure below shows three position – time plots of a motion of a particle along x-axis. Giving reasons, identify the valid plot(s) among them. For the valid plot(s), determine following :

Position – time plots in one dimension

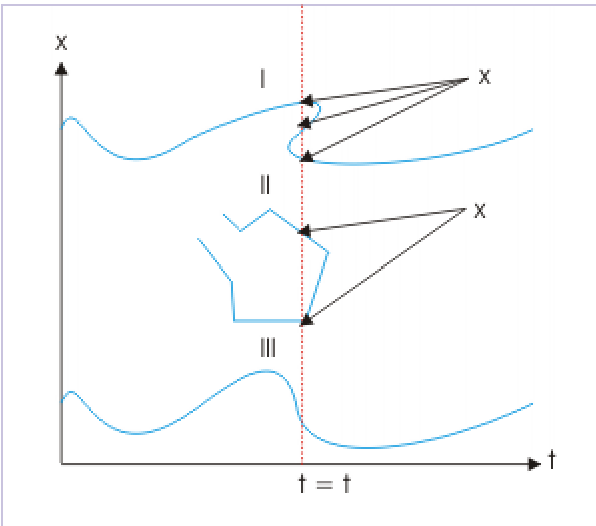


1. How many times the particle has come to rest?
2. Does the particle reverse its direction during motion?

Solution:

Validity of plots : In the portion of plot I, we can draw a vertical line that intersects the curve at three points. It means that the particle is present at three positions simultaneously, which is not possible. Plot II is also not valid for the same reason. Besides, it consists of a vertical portion, which would mean presence of the particle at infinite numbers of positions at the same instant. Plot III, on the other hand, is free from these anomalies and is the only valid curve representing motion of a particle along x – axis (See Figure).

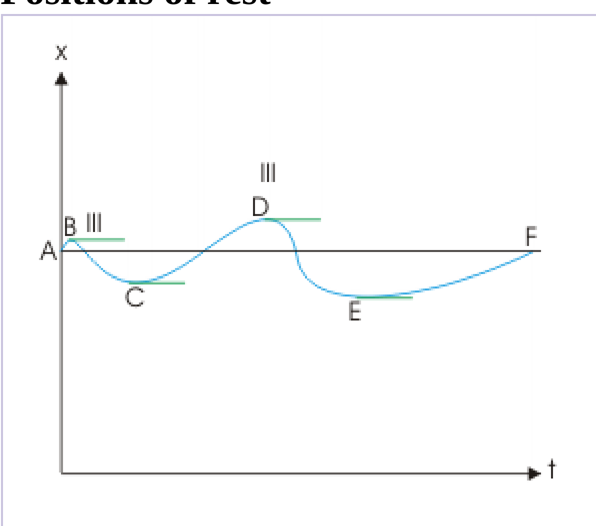
Valid position – time plot in one dimension



A particle can not be present at more than one position at a given instant.

When the particle comes to rest, there is no change in the value of “x”. This, in turn, means that tangent to the curve at points of rest is parallel to x-axis. By inspection, we find that tangent to the curve is parallel to x-axis at four points (B,C,D and E) on the curve shown in the figure below. Hence, the particle comes to rest four times during the motion.

Positions of rest



At rest, the tangent to the path is parallel to time axis.

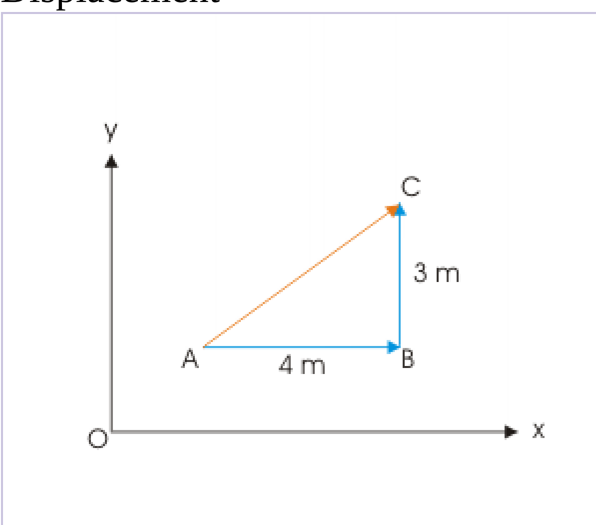
Vectors

A number of key fundamental physical concepts relate to quantities, which display directional property. Scalar algebra is not suited to deal with such quantities. The mathematical construct called vector is designed to represent quantities with directional property. A vector, as we shall see, encapsulates the idea of “direction” together with “magnitude”.

In order to elucidate directional aspect of a vector, let us consider a simple example of the motion of a person from point A to point B and from point B to point C, covering a distance of 4 and 3 meters respectively as shown in the Figure . Evidently, AC represents the linear distance between the initial and the final positions. This linear distance, however, is not equal to the sum of the linear distances of individual motion represented by segments AB and BC (4 + 3 = 7 m) i.e.

$$\mathrm{AC} \neq \mathrm{AB} + \mathrm{BC}$$

Displacement



Scalar inequality

However, we need to express the end result of the movement appropriately as the sum of two individual movements. The inequality of the scalar

equation as above is basically due to the fact that the motion represented by these two segments also possess directional attributes; the first segment is directed along the positive x – axis, where as the second segment of motion is directed along the positive y –axis. Combining their magnitudes is not sufficient as the two motions are perpendicular to each other. We require a mechanism to combine directions as well.

The solution of the problem lies in treating individual distance with a new term "displacement" – a vector quantity, which is equal to “linear distance plus direction”. Such a conceptualization of a directional quantity allows us to express the final displacement as the sum of two individual displacements in vector form :

$$\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$$

The magnitude of displacement is obtained by applying Pythagoras theorem :

$$\mathrm{AC} \sqrt{\left(\mathrm{AB}^2 + \mathrm{BC}^2\right)} = \sqrt{\left(4^2 + 3^2\right)} = 5\ \mathrm{m}$$

It is clear from the example above that vector construct is actually devised in a manner so that physical reality having directional property is appropriately described. This "fit to requirement" aspect of vector construct for physical phenomena having direction is core consideration in defining vectors and laying down rules for vector operation.

A classical example, illustrating the “fit to requirement” aspect of vector, is the product of two vectors. A product, in general, should evaluate in one manner to yield one value. However, there are natural quantities, which are product of two vectors, but evaluate to either scalar (example : work) or vector (example : torque) quantities. Thus, we need to define the product of vectors in two ways : one that yields scalar value and the other that yields vector value. For this reason product of two vectors is either defined as dot product to give a scalar value or defined as cross product to give vector value. This scheme enables us to appropriately handle the situations as the case may be.

$$W = \mathbf{F}.\Delta\mathbf{r} \quad \text{.........}\ \text{Scalar dot product}$$
$$\boldsymbol{\tau} = \mathbf{r} \text{ x } \mathbf{F} \quad \text{.........}\ \text{Vector cross product}$$

Mathematical concept of vector is basically secular in nature and general in application. This means that mathematical treatment of vectors is without reference to any specific physical quantity or phenomena. In other words, we can employ vector and its methods to all quantities, which possess directional attribute, in a uniform and consistent manner. For example two vectors would be added in accordance with vector addition rule irrespective of whether vectors involved represent displacement, force, torque or some other vector quantities.

The moot point of discussion here is that vector has been devised to suit the requirement of natural process and not the other way around that natural process suits vector construct as defined in vector mathematics.

What is a vector?

Vector

Vector is a physical quantity, which has both magnitude and direction.

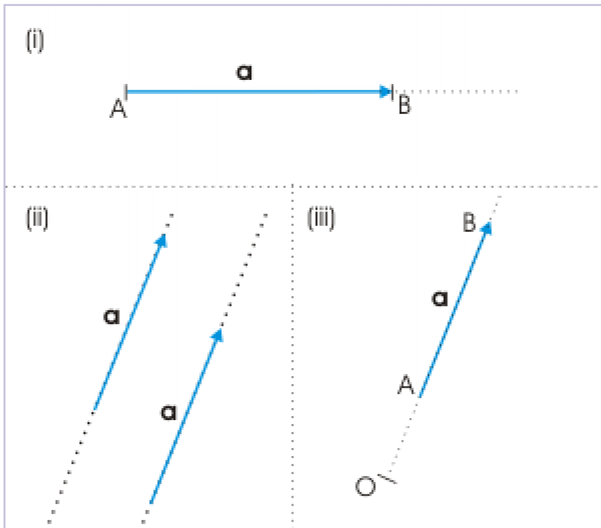
A vector is represented graphically by an arrow drawn on a scale as shown Figure i. In order to process vectors using graphical methods, we need to draw all vectors on the same scale. The arrow head point in the direction of the vector.

A vector is notionally represented in a characteristic style. It is denoted as bold face type like "$\mathbf{a}$ " as shown Figure (i) or with a small arrow over the symbol like "$\overrightarrow{a}$" or with a small bar as in "$\bar{a}$ ". The magnitude of a vector quantity is referred by simple identifier like "a" or as the absolute value of the vector as "$|\mathbf{a}|$ " .

Two vectors of equal magnitude and direction are equal vectors (Figure (ii)). As such, a vector can be laterally shifted as long as its direction remains same (Figure (ii)). Also, vectors can be shifted along its line of application represented by dotted line (Figure (iii)). The flexibility by virtue of

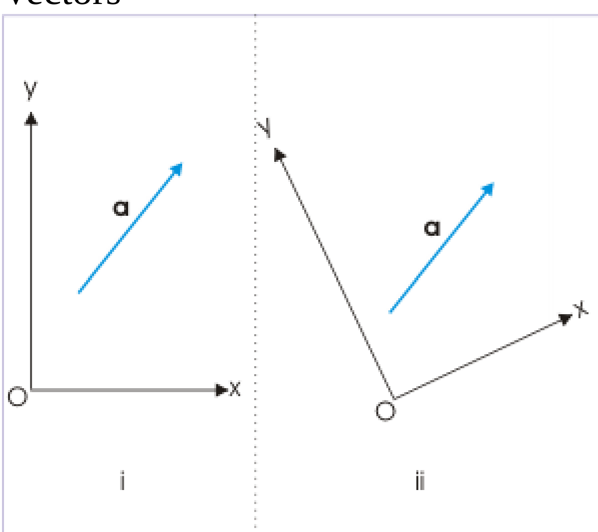
shifting vector allows a great deal of ease in determining vector’s interaction with other scalar or vector quantities.

Vectors



It should be noted that graphical representation of vector is independent of the origin or axes of coordinate system except for few vectors like position vector (called localized vector), which is tied to the origin or a reference point by definition. With the exception of localized vector, a change in origin or orientation of axes or both does not affect vectors and vector operations like addition or multiplication (see figure below).

Vectors



The vector is not affected, when the coordinate is rotated or displaced as shown in the figure above. Both the orientation and positioning of origin i.e

reference point do not alter the vector representation. It remains what it is. This feature of vector operation is an added value as the study of physics in terms of vectors is simplified, being independent of the choice of coordinate system in a given reference.

Vector algebra

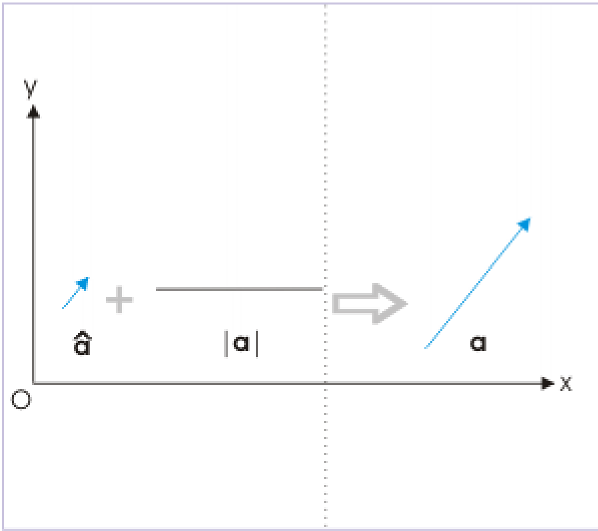
Graphical method is slightly meticulous and error prone as it involves drawing of vectors on scale and measurement of angles. In addition, it does not allow algebraic manipulation that otherwise would give a simple solution as in the case of scalar algebra. We can, however, extend algebraic techniques to vectors, provided vectors are represented on a rectangular coordinate system. The representation of a vector on a coordinate system uses the concept of unit vectors and scalar magnitudes. We shall discuss these aspects in a separate module titled Components of a vector . Here, we briefly describe the concept of unit vector and technique to represent a vector in a particular direction.

Unit vector

Unit vector has a magnitude of one and is directed in a particular direction. It does not have dimension or unit like most other physical quantities. Thus, multiplying a scalar by unit vector converts the scalar quantity into a vector without changing its magnitude, but assigning it a direction (Figure).

$$\mathbf{a} = a\widehat{a}$$

Vector representation with unit vector



This is an important relation as it allows determination of unit vector in the direction of any vector "**a** as :

$$\widehat{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Conventionally, unit vectors along the rectangular axes is represented with bold type face symbols like : $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$, or with a cap heads like $\hat{i}, \hat{j}$ and $\widehat{k}$. The unit vector along the axis denotes the direction of individual axis.

Using the concept of unit vector, we can denote a vector by multiplying the magnitude of the vector with unit vector in its direction.

$$\mathbf{a} = a\widehat{a}$$

Following this technique, we can represent a vector along any axis in terms of scalar magnitude and axial unit vector like (for x-direction) :

$$\mathbf{a} = a\mathbf{i}$$

Other important vector terms

Null vector

Null vector is conceptualized for completing the development of vector algebra. We may encounter situations in which two equal but opposite vectors are added. What would be the result? Would it be a zero real number or a zero vector? It is expected that result of algebraic operation should be compatible with the requirement of vector. In order to meet this requirement, we define null vector, which has neither magnitude nor direction. In other words, we say that null vector is a vector whose all components in rectangular coordinate system are zero.

Strictly, we should denote null vector like other vectors using a bold faced letter or a letter with an overhead arrow. However, it may generally not be done. We take the exception to denote null vector by number “0” as this representation does not contradicts the defining requirement of null vector.

$$\mathbf{a} + \mathbf{b} = 0$$

Negative vector

Negative vector

> A negative vector of a given vector is defined as the vector having same magnitude, but applied in the opposite direction to that of the given vector.

It follows that if $\mathbf{b}$ is the negative of vector $\mathbf{a}$, then

$$\begin{aligned} &\mathbf{a} = -\mathbf{b} \\ &\Rightarrow \mathbf{a} + \mathbf{b} = 0 \\ &\text{and } \ |\mathbf{a}| = |\mathbf{b}| \end{aligned}$$

There is a subtle point to be made about negative scalar and vector quantities. A negative scalar quantity, sometimes, conveys the meaning of lesser value. For example, the temperature -5 K is a smaller temperature than any positive value. Also, a greater negative like – 100 K is less than the smaller negative like -50 K. However, a scalar like charge conveys different meaning. A negative charge of -10 μC is a bigger negative charge than – 5 μC. The interpretation of negative scalar is, thus, situational.

On the other hand, negative vector always indicates the sense of opposite direction. Also like charge, a greater negative vector is larger than smaller negative vector or a smaller positive vector. The magnitude of force -10 **i** N, for example is greater than 5 **i** N, but directed in the opposite direction to that of the unit vector **i**. In any case, negative vector does not convey the meaning of lesser or greater magnitude like the meaning of a scalar quantity in some cases.

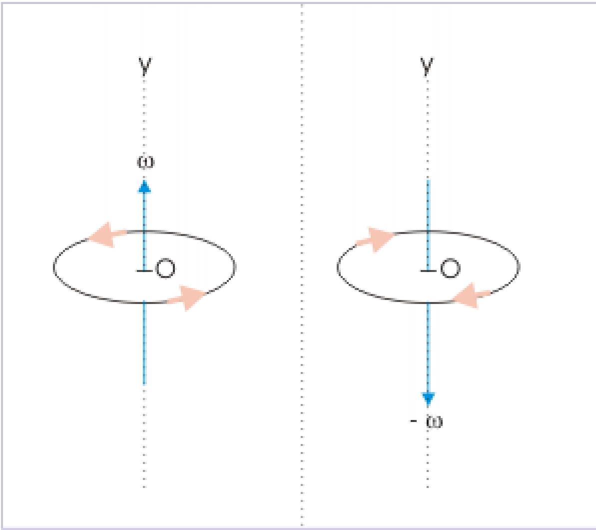
Co-planar vectors

A pair of vectors determines an unique plane. The pair of vectors defining the plane and other vectors in that plane are called coplanar vectors.

Axial vector

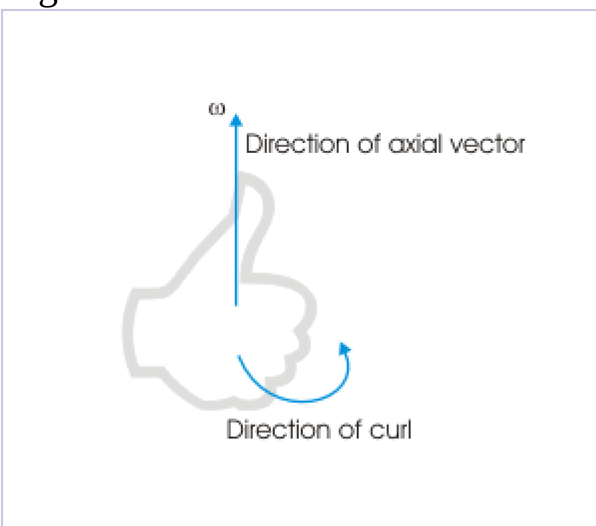
Motion has two basic types : translational and rotational motions. The vector and scalar quantities, describing them are inherently different. Accordingly, there are two types of vectors to deal with quantities having direction. The system of vectors that we have referred so far is suitable for describing translational motion and such vectors are called “rectangular” or "polar" vectors.

A different type of vector called axial vector is used to describe rotational motion. Its graphical representation is same as that of rectangular vector, but its interpretation is different. What it means that the axial vector is represented by a straight line with an arrow head as in the case of polar vector; but the physical interpretation of axial vector differs. An axial vector, say $\boldsymbol{\omega}$, is interpreted to act along the positive direction of the axis of rotation, while rotating anti –clockwise. A negative axial vector like, $-\boldsymbol{\omega}$, is interpreted to act along the negative direction of axis of rotation, while rotating clockwise.
Axial vector



The figure above captures the concept of axial vector. It should be noted that the direction of the axial vector is essentially tied with the sense of rotation (clockwise or anti-clockwise). This linking of directions is stated with "Right hand (screw) rule". According to this rule (see figure below), if the stretched thumb of right hand points in the direction of axial vector, then the curl of the fist gives the direction of rotation. Its inverse is also true i.e if the curl of the right hand fist is placed in a manner to follow the direction of rotation, then the stretched thumb points in the direction of axial vector.

Righ hand rule



Axial vector is generally shown to be perpendicular to a plane. In such cases, we use a shortened symbol to represent axial or even other vectors,

which are normal to the plane, by a "dot" or "cross" inscribed within a small circle. A "dot" inscribed within the circle indicates that the vector is pointing towards the viewer of the plane and a "cross" inscribed within the circle indicates that the vector is pointing away from the viewer of the plane.

Axial vector are also known as "pseudovectors". It is because axial vectors do not follow transformation of rectangular coordinate system. Vectors which follow coordinate transformation are called "true" or "polar" vectors. One important test to distinguish these two types of vector is that axial vector has a mirror image with negative sign unlike true vectors. Also, we shall learn about vector or cross product subsequently. This operation represent many important physical phenomena such as rotation and magnetic interaction etc. We should know that the vector resulting from cross product of true vectors is always axial i.e. pseudovectors vector like magnetic field, magnetic force, angular velocity, torque etc.

Why should we study vectors?

The basic concepts in physics – particularly the branch of mechanics - have a direct and inherently characterizing relationship with the concept of vector. The reason lies in the directional attribute of quantities, which is used to describe dynamical aspect of natural phenomena. Many of the physical terms and concepts are simply vectors like position vector, displacement vector etc. They are as a matter of fact defined directly in terms of vector like “it is a vector ……………”.

The basic concept of “cause and effect” in mechanics (comprising of kinematics and dynamics), is predominantly based on the interpretation of direction in addition to magnitude. Thus, there is no way that we could accurately express these quantities and their relationship without vectors. There is, however, a general tendency (particular in the treatment designed for junior classes) to try to evade vectors and look around ways to deal with these inherently vector based concepts without using vectors! As expected this approach is a poor reflection of the natural process, where basic concepts are simply ingrained with the requirement of handling direction along with magnitude.

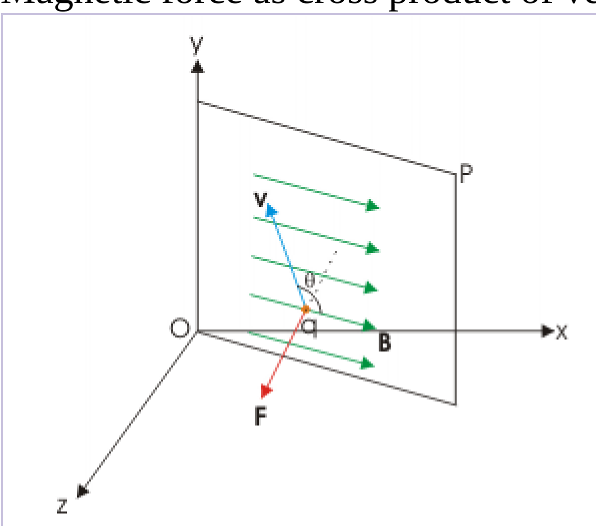
It is, therefore, imperative that we switch over from work around approach to vector approach to study physics as quickly as possible. Many a times, this scalar “work around” inculcates incorrect perception and understanding that may persist for long, unless corrected with an appropriate vector description.

The best approach, therefore, is to study vector in the backdrop of physical phenomena and use it with clarity and advantage in studying nature. For this reasons, our treatment of “vector physics” – so to say - in this course will strive to correlate vectors with appropriate physical quantities and concepts.

The most fundamental reason to study nature in terms of vectors, wherever direction is involved, is that vector representation is concise, explicit and accurate.

To score this point, let us consider an example of the magnetic force experienced by a charge, q, moving with a velocity $\mathbf{v}$ in a magnetic field, “ $\mathbf{B}$. The magnetic force, $\mathbf{F}$, experienced by moving particle, is perpendicular to the plane, P, formed by the the velocity and the magnetic field vectors as shown in the figure .

Magnetic force as cross product of vectors



The force is given in the vector form as :

$$\mathbf{F} = q(\mathbf{v} \text{ x } \mathbf{B})$$

This equation does not only define the magnetic force but also outlines the intricacies about the roles of the each of the constituent vectors. As per vector rule, we can infer from the vector equation that :

- The magnetic force (**F**) is perpendicular to the plane defined by vectors **v** and **B**.
- The direction of magnetic force i.e. which side of plane.
- The magnitude of magnetic force is "qvB sinθ", where θ is the smaller angle enclosed between the vectors **v** and **B**.

This example illustrates the compactness of vector form and completeness of the information it conveys. On the other hand, the equivalent scalar strategy to describe this phenomenon would involve establishing an empirical frame work like Fleming's left hand rule to determine direction. It would be required to visualize vectors along three mutually perpendicular directions represented by three fingers in a particular order and then apply Fleming rule to find the direction of the force. The magnitude of the product, on the other hand, would be given by qvB sinθ as before.

The difference in two approaches is quite remarkable. The vector method provides a paragraph of information about the physical process, whereas a paragraph is to be followed to apply scalar method ! Further, the vector rules are uniform and consistent across vector operations, ensuring correctness of the description of physical process. On the other hand, there are different set of rules like Fleming left and Fleming right rules for two different physical processes.

The last word is that we must master the vectors rather than avoid them - particularly when the fundamentals of vectors to be studied are limited in extent.

Vector addition

Vectors operate with other scalar or vector quantities in a particular manner. Unlike scalar algebraic operation, vector operation draws on graphical representation to incorporate directional aspect.

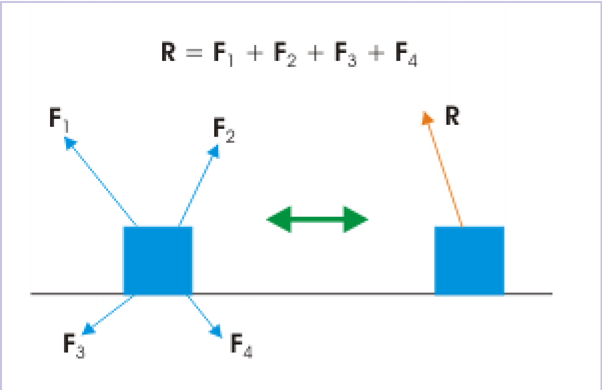
Vector addition is, however, limited to vectors only. We can not add a vector (a directional quantity) to a scalar (a non-directional quantity). Further, vector addition is dealt in three conceptually equivalent ways :

1. graphical methods
2. analytical methods
3. algebraic methods

In this module, we shall discuss first two methods. Third algebraic method will be discussed in a separate module titled [Components of a vector]

The resulting vector after addition is termed as sum or resultant vector. The resultant vector corresponds to the "resultant" or "net" effect of a physical quantities having directional attributes. The effect of a force system on a body, for example, is determined by the resultant force acting on it. The idea of resultant force, in this case, reflects that the resulting force (vector) has the same effect on the body as that of the forces (vectors), which are added.

Resultant force



It is important to emphasize here that vector rule of addition (graphical or algebraic) do not distinguish between vector types (whether displacement or

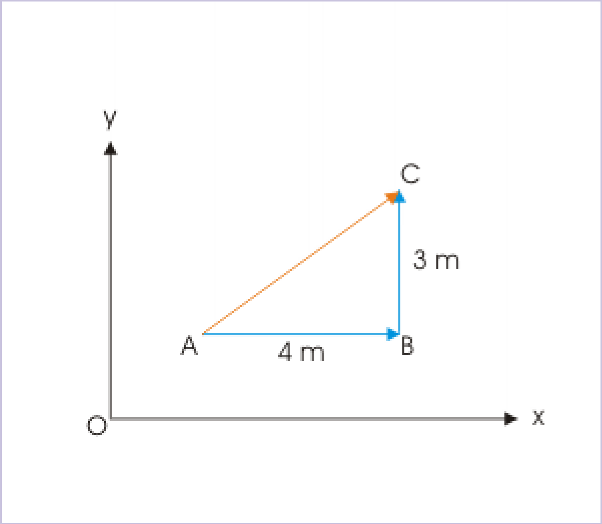
acceleration vector). This means that the rule of vector addition is general for all vector types.

It should be clearly understood that though rule of vector addition is general, which is applicable to all vector types in same manner, but vectors being added should be like vectors only. It is expected also. The requirement is similar to scalar algebra where 2 plus 3 is always 5, but we need to add similar quantity like 2 meters plus 3 meters is 5 meters. But, we can not add, for example, distance and temperature.

Vector addition : graphical method

Let us examine the example of displacement of a person in two different directions. The two displacement vectors, perpendicular to each other, are added to give the “resultant” vector. In this case, the closing side of the right triangle represents the sum (i.e. resultant) of individual displacements **AB** and **BC**.

Displacement



Equation:

$$\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$$

The method used to determine the sum in this particular case (in which, the closing side of the triangle represents the sum of the vectors in both

magnitude and direction) forms the basic consideration for various rules dedicated to implement vector addition.

Triangle law

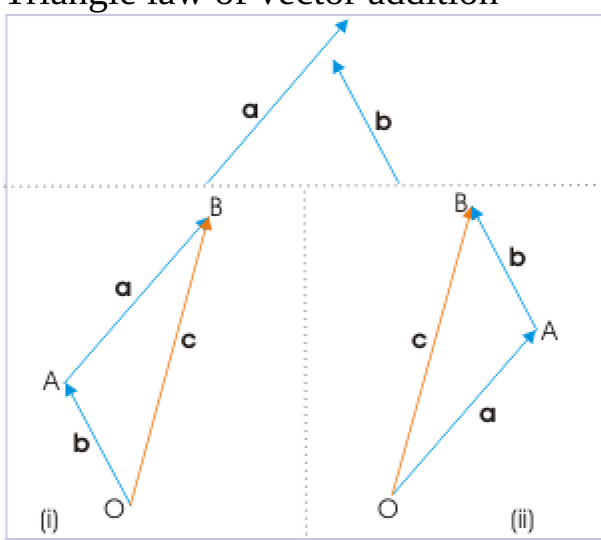
In most of the situations, we are involved with the addition of two vector quantities. Triangle law of vector addition is appropriate to deal with such situation.

Triangle law of vector addition

> If two vectors are represented by two sides of a triangle in sequence, then third closing side of the triangle, in the opposite direction of the sequence, represents the sum (or resultant) of the two vectors in both magnitude and direction.

Here, the term "sequence" means that the vectors are placed such that tail of a vector begins at the arrow head of the vector placed before it.

Triangle law of vector addition



The triangle law does not restrict where to start i.e. with which vector to start. Also, it does not put conditions with regard to any specific direction for the sequence of vectors, like clockwise or anti-clockwise, to be maintained. In figure (i), the law is applied starting with vector,**b**; whereas

the law is applied starting with vector, **a**, in figure (ii). In either case, the resultant vector, **c**, is same in magnitude and direction.

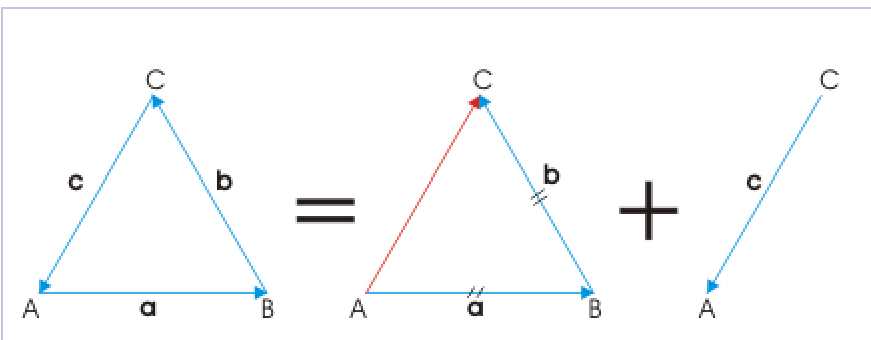
This is an important result as it conveys that vector addition is commutative in nature i.e. the process of vector addition is independent of the order of addition. This characteristic of vector addition is known as "commutative" property of vector addition and is expressed mathematically as :

Equation:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

If three vectors are represented by three sides of a triangle in sequence, then resultant vector is zero. In order to prove this, let us consider any two vectors in sequence like AB and BC as shown in the figure. According to triangle law of vector addition, the resultant vector is represented by the third closing side in the opposite direction. It means that :

Three vectors



Three vectors are represented by three sides in sequence.

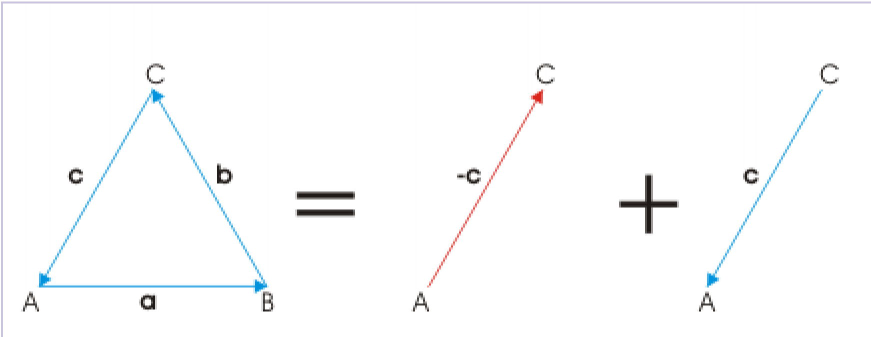
$$\Rightarrow \mathbf{AB} + \mathbf{BC} = \mathbf{AC}$$

Adding vector **CA** on either sides of the equation,

$$\Rightarrow \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = \mathbf{AC} + \mathbf{CA}$$

The right hand side of the equation is vector sum of two equal and opposite vectors, which evaluates to zero. Hence,

Three vectors



The resultant of three vectors represented by three sides is zero.

$$\Rightarrow \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$$

Note : If the vectors represented by the sides of a triangle are force vectors, then resultant force is zero. It means that three forces represented by the sides of a triangle in a sequence is a balanced force system.

Parallelogram law

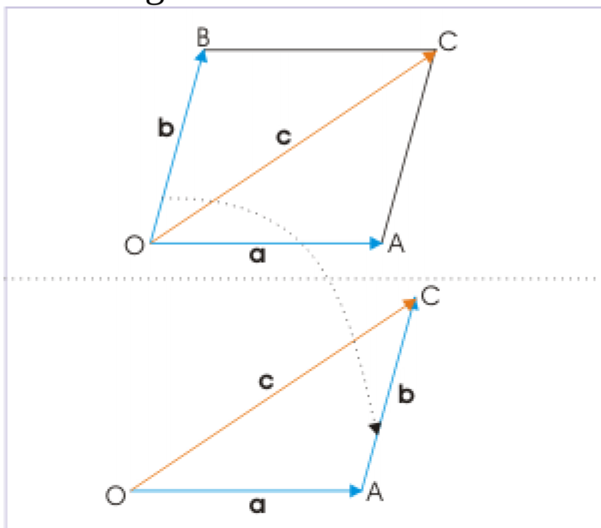
Parallelogram law, like triangle law, is applicable to two vectors.

Parallelogram law

> If two vectors are represented by two adjacent sides of a parallelogram, then the diagonal of parallelogram through the common point represents the sum of the two vectors in both magnitude and direction.

Parallelogram law, as a matter of fact, is an alternate statement of triangle law of vector addition. A graphic representation of the parallelogram law and its interpretation in terms of the triangle is shown in the figure :

Parallelogram law

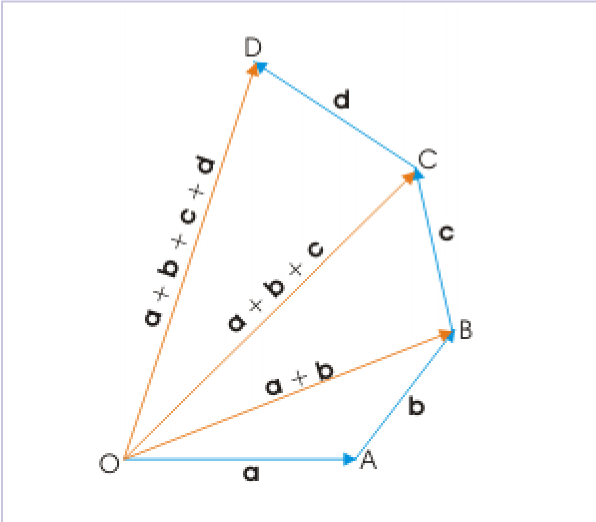


Converting parallelogram sketch to that of triangle law requires shifting vector, **b**, from the position OB to position AC laterally as shown, while maintaining magnitude and direction.

Polygon law

The polygon law is an extension of earlier two laws of vector addition. It is successive application of triangle law to more than two vectors. A pair of vectors (**a**, **b**) is added in accordance with triangle law. The intermediate resultant vector (**a** + **b**) is then added to third vector (c) again, successively till all vectors to be added have been exhausted.

Successive application of triangle law

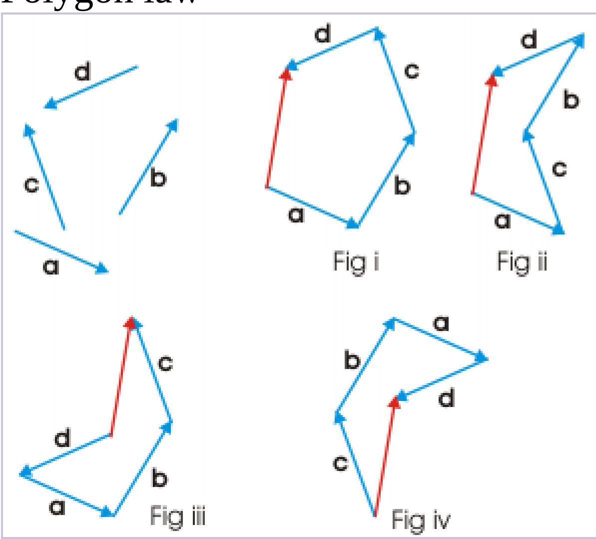


Polygon law

Polygon law of vector addition : If (n-1) numbers of vectors are represented by (n-1) sides of a polygon in sequence, then n^{th} side, closing the polygon in the opposite direction, represents the sum of the vectors in both magnitude and direction.

In the figure shown below, four vectors namely **a**, **b**, **c** and **d** are combined to give their sum. Starting with any vector, we add vectors in a manner that the subsequent vector begins at the arrow end of the preceding vector. The illustrations in figures i, iii and iv begin with vectors **a**, **d** and **c** respectively.

Polygon law



Matter of fact, polygon formation has great deal of flexibility. It may appear that we should elect vectors in increasing or decreasing order of direction (i.e. the angle the vector makes with reference to the direction of the first vector). But, this is not so. This point is demonstrated in figure (i) and (ii), in which the vectors **b** and **c** have simply been exchanged in their positions in the sequence without affecting the end result.

It means that the order of grouping of vectors for addition has no consequence on the result. This characteristic of vector addition is known as “associative” property of vector addition and is expressed mathematically as :

Equation:

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$$

Subtraction

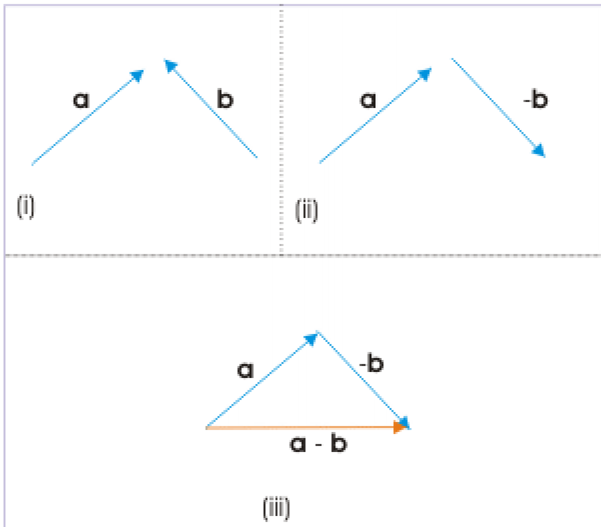
Subtraction is considered an addition process with one modification that the second vector (to be subtracted) is first reversed in direction and is then added to the first vector. To illustrate the process, let us consider the problem of subtracting vector, **b**, from , **a**. Using graphical techniques, we first reverse the direction of vector, **b**, and obtain the sum applying triangle or parallelogram law.

Symbolically,

Equation:

$$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$$

Subtraction

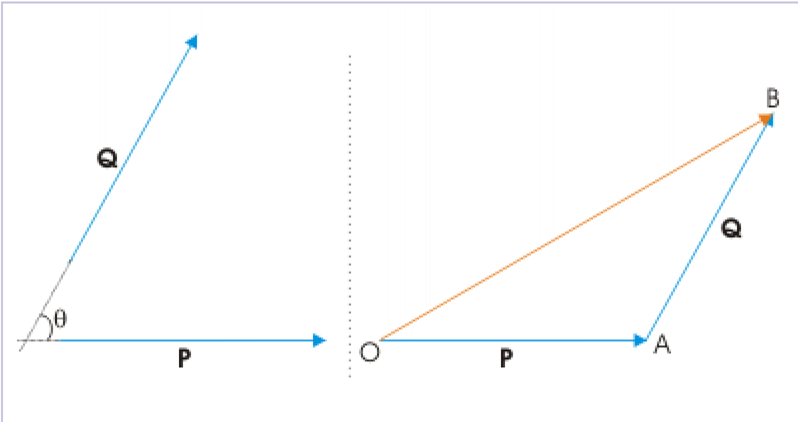


Similarly, we can implement subtraction using algebraic method by reversing sign of the vector being subtracted.

Vector addition : Analytical method

Vector method requires that all vectors be drawn true to the scale of magnitude and direction. The inherent limitation of the medium of drawing and measurement techniques, however, renders graphical method inaccurate. Analytical method, based on geometry, provides a solution in this regard. It allows us to accurately determine the sum or the resultant of the addition, provided accurate values of magnitudes and angles are supplied.

Here, we shall analyze vector addition in the form of triangle law to obtain the magnitude of the sum of the two vectors. Let P and Q be the two vectors to be added, which make an angle θ with each other. We arrange the vectors in such a manner that two adjacent sides OA and AB of the triangle OAB, represent two vectors P and Q respectively as shown in the figure.
Analytical method

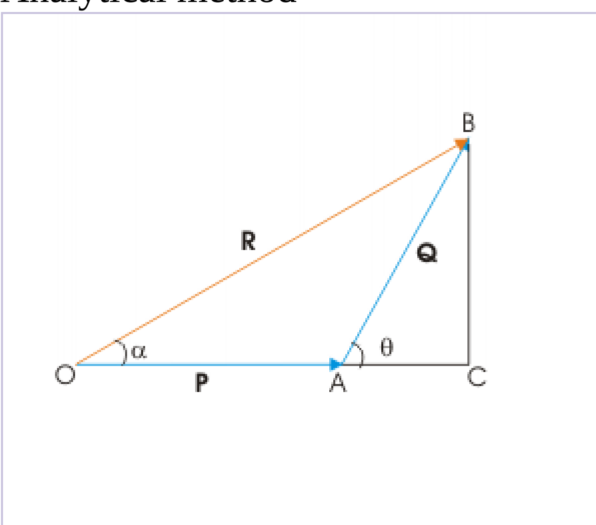


According to triangle law, the closing side OB represent sum of the vectors in both magnitude and direction.

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \mathbf{P} + \mathbf{Q}$$

In order to determine the magnitude, we drop a perpendicular BC on the extended line OC.

Analytical method



In ΔACB,

$$\mathbf{AC} = \mathbf{AB} \cos\theta = \mathbf{Q} \cos\theta$$
$$\mathbf{BC} = \mathbf{AB} \sin\theta = \mathbf{Q} \sin\theta$$

In right ΔOCB, we have :

$$\mathrm{OB} = \sqrt{\left(\mathrm{OC}^2 + \mathrm{BC}^2\right)} = \sqrt{\left\{(\mathrm{OA} + \mathrm{AC})^2 + \mathrm{BC}^2\right\}}$$

Substituting for AC and BC,
Equation:

$$\Rightarrow \mathrm{OB} = \sqrt{\left((P + Q \cos\theta)^2 + Q \sin\theta^2\right)}$$
$$\Rightarrow \mathrm{OB} = \sqrt{\left(P^2 + Q^2 \cos^2\theta + 2PQ \cos\theta + Q^2 \sin^2\theta\right)}$$
$$\Rightarrow R = \mathrm{OB} = \sqrt{\left(P^2 + 2PQ \cos\theta + Q^2\right)}$$

Let "α" be the angle that line OA makes with OC, then

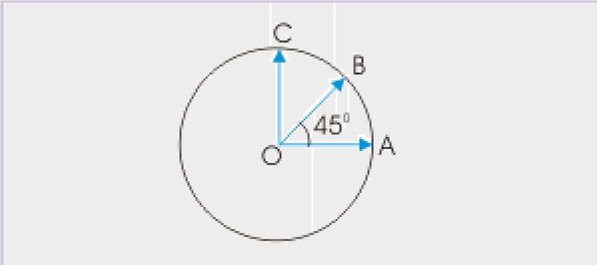
$$\tan\alpha = \frac{\mathrm{BC}}{\mathrm{OC}} = \frac{Q\sin\theta}{P + Q\cos\theta}$$

The equations give the magnitude and direction of the sum of the vectors. The above equation reduces to a simpler form, when two vectors are perpendicular to each other. In that case, θ = 90°; sinθ = sin90° = 1; cosθ = cos90° = 0 and,
Equation:

$$\Rightarrow \mathrm{OB} = \sqrt{\left(P^2 + Q^2\right)}$$
$$\Rightarrow \tan\alpha = \frac{Q}{P}$$

These results for vectors at right angle are exactly same as determined, using Pythagoras theorem.

Example:
Problem : Three radial vectors OA, OB and OC act at the center of a circle of radius “r” as shown in the figure. Find the magnitude of resultant vector.
Sum of three vectors



Three radial vectors OA, OB and OC act at the center of a circle of radius “r”.

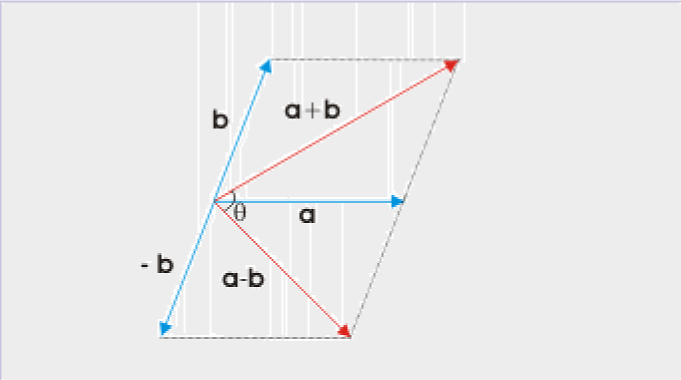
Solution : It is evident that vectors are equal in magnitude and is equal to the radius of the circle. The magnitude of the resultant of horizontal and vertical vectors is :

$$\text{R'} = \sqrt{\left(r^2 + r^2\right)} = \sqrt{2}r$$

The resultant of horizontal and vertical vectors is along the bisector of angle i.e. along the remaining third vector OB. Hence, magnitude of resultant of all three vectors is :

$$\text{R'} = \text{OB} + \text{R'} = r + \sqrt{2}r = (1 + \sqrt{2})r$$

Example:
Problem : At what angle does two vectors **a**+**b** and **a**-**b** act so that the resultant is $\sqrt{\left(3a^2 + b^2\right)}$.
Solution : The magnitude of resultant of two vectors is given by :
Angle



The angle between the sum and difference of vectors.

$$R = \sqrt{\left\{(a+b)^2 + (a-b)^2 + 2\left(a+b\right)\left(a-b\right)\cos\theta\right\}}$$

Substituting the expression for magnitude of resultant as given,

$$\Rightarrow \sqrt{\left(3a^2 + b^2\right)} = \sqrt{\left\{(a+b)^2 + (a-b)^2 + 2\left(a+b\right)\left(a-b\right)\cos\theta\right\}}$$

Squaring on both sides, we have :

$$\Rightarrow \left(3a^2 + b^2\right) = \left\{(a+b)^2 + (a-b)^2 + 2\left(a+b\right)\left(a-b\right)\cos\theta\right\}$$

$$\Rightarrow \cos\theta = \frac{(a^2-b^2)}{2(a^2-b^2)} = \frac{1}{2} = \cos 60^\circ$$

$$\Rightarrow \theta = 60^\circ$$

Nature of vector addition

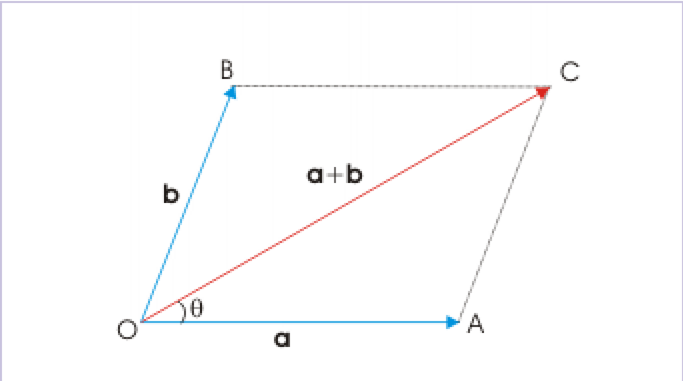
Vector sum and difference

The magnitude of sum of two vectors is either less than or equal to sum of the magnitudes of individual vectors. Symbolically, if **a** and **b** be two vectors, then

$$|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$$

We know that vectors **a**, **b** and their sum **a**+**b** is represented by three side of a triangle OAC. Further we know that a side of triangle is always less than the sum of remaining two sides. It means that :

Two vectors



Sum of two vectors

$$\mathrm{OC} < \mathrm{OA} + \mathrm{AC}$$
$$\mathrm{OC} < \mathrm{OA} + \mathrm{OB}$$
$$\Rightarrow |\mathbf{a}+\mathbf{b}| < |\mathbf{a}| + |\mathbf{b}|$$

There is one possibility, however, that two vectors **a** and **b** are collinear and act in the same direction. In that case, magnitude of their resultant will be "equal to" the sum of the magnitudes of individual vector. This magnitude represents the maximum or greatest magnitude of two vectors being combined.

$$\mathrm{OC} = \mathrm{OA} + \mathrm{OB}$$
$$\Rightarrow |\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$$

Combining two results, we have :

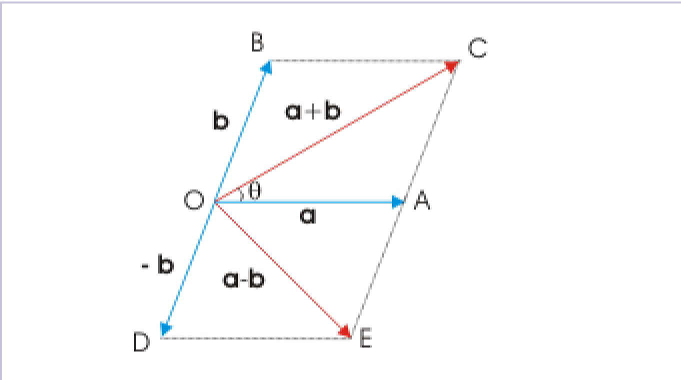
$$|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

On the other hand, the magnitude of difference of two vectors is either greater than or equal to difference of the magnitudes of individual vectors. Symbolically, if **a** and **b** be two vectors, then

$$|\mathbf{a}-\mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$$

We know that vectors **a**, **b** and their difference **a-b** are represented by three side of a triangle OAE. Further we know that a side of triangle is always less than the sum of remaining two sides. It means that sum of two sides is greater than the third side :

Two vectors



Difference of two vectors

$$\mathrm{OE} + \mathrm{AE} > \mathrm{OA}$$
$$\Rightarrow \mathrm{OE} > \mathrm{OA} - \mathrm{AE}$$
$$\Rightarrow |\mathbf{a}-\mathbf{b}| > |\mathbf{a}| - |\mathbf{b}|$$

There is one possibility, however, that two vectors **a** and **b** are collinear and act in the opposite directions. In that case, magnitude of their difference will be equal to the difference of the magnitudes of individual vector. This magnitude represents the minimum or least magnitude of two vectors being combined.

$$\Rightarrow \mathrm{OE} = \mathrm{OA} - \mathrm{AE}$$
$$\Rightarrow |\mathbf{a} - \mathbf{b}| = |\mathbf{a}| - |\mathbf{b}|$$

Combining two results, we have :

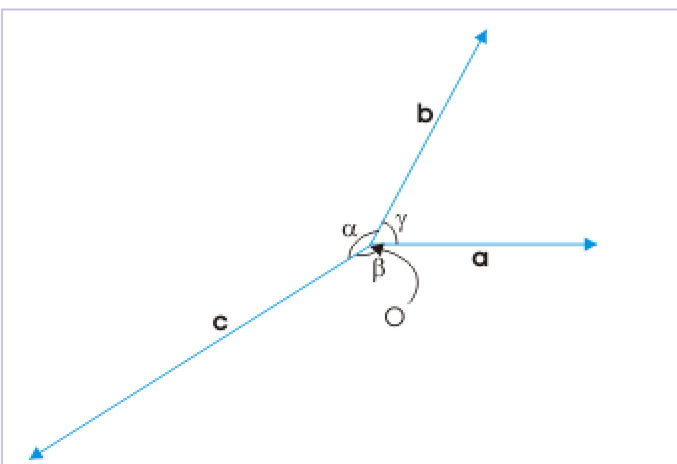
$$|\mathbf{a} - \mathbf{b}| \geq |\mathbf{a}| - |\mathbf{b}|$$

Lami's theorem

Lami's theorem relates magnitude of three non-collinear vectors with the angles enclosed between pair of two vectors, provided resultant of three vectors is zero (null vector). This theorem is a manifestation of triangle law of addition. According to this theorem, if resultant of three vectors **a**, **b** and **c** is zero (null vector), then

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

Three vectors

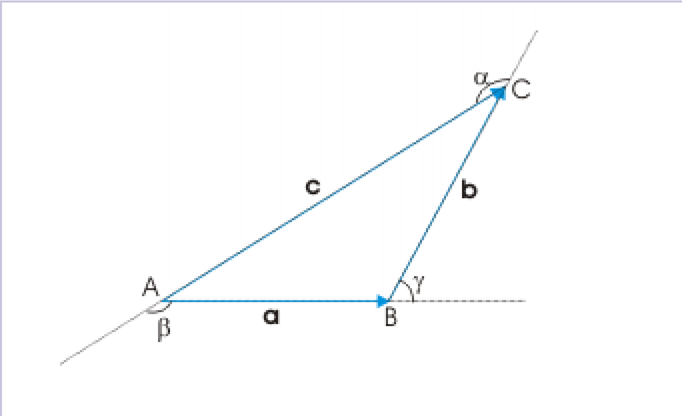


Three non-collinear vectors.

where α, β and γ be the angle between the remaining pairs of vectors.

We know that if the resultant of three vectors is zero, then they are represented by three sides of a triangle in magnitude and direction.

Three vectors



Three vectors are represented by three sides of a triangle.

Considering the magnitude of vectors and applying sine law of triangle, we have :

$$\frac{\mathrm{AB}}{\sin \mathrm{BCA}} = \frac{\mathrm{BC}}{\sin \mathrm{CAB}} = \frac{\mathrm{CA}}{\sin \mathrm{ABC}}$$

$$\Rightarrow \frac{\mathrm{AB}}{\sin(\pi-\alpha)} = \frac{\mathrm{BC}}{\sin(\pi-\beta)} = \frac{\mathrm{CA}}{\sin(\pi-\gamma)}$$

$$\Rightarrow \frac{\mathrm{AB}}{\sin \alpha} = \frac{\mathrm{BC}}{\sin \beta} = \frac{\mathrm{CA}}{\sin \gamma}$$

It is important to note that the ratio involves exterior (outside) angles – not the interior angles of the triangle. Also, the angle associated with the magnitude of a vector in the individual ratio is the included angle between the remaining vectors.

Exercises

Exercise:

Problem:

Two forces of 10 N and 25 N are applied on a body. Find the magnitude of maximum and minimum resultant force.

Solution:

Resultant force is maximum when force vectors act along the same direction. The magnitude of resultant force under this condition is :

$$R_{\text{max}} = 10 + 25 = 35\ N$$
$$R_{\text{min}} = 25 - 10 = 15\ N$$

Exercise:

Problem:

Can a body subjected to three coplanar forces 5 N, 17 N and 9 N be in equilibrium?

Solution:

The resultant force, on a body in equilibrium, is zero. It means that three forces can be represented along three sides of a triangle. However, we know that sum of any two sides is greater than third side. In this case, we see that :

$$5 + 9 < 17$$

Clearly, three given forces can not be represented by three sides of a triangle. Thus, we conclude that the body is not in equilibrium.

Exercise:

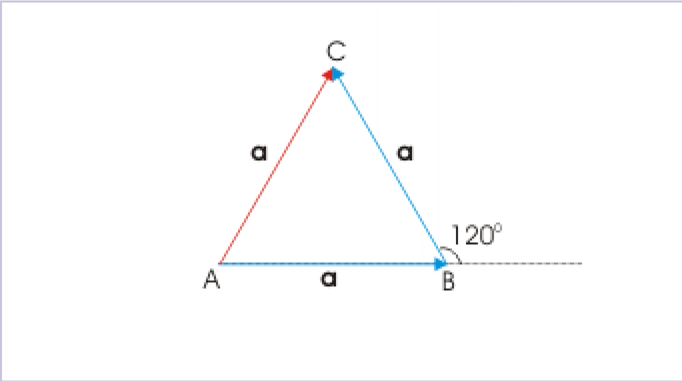
Problem:

Under what condition does the magnitude of the resultant of two vectors of equal magnitude, is equal in magnitude to either of two equal vectors?

Solution:

We know that resultant of two vectors is represented by the closing side of a triangle. If the triangle is equilateral then all three sides are equal. As such magnitude of the resultant of two vectors is equal to the magnitude of either of the two vectors.

Two vectors



Resultant of two vectors

Under this condition, vectors of equal magnitude make an angle of 120° between them.

Components of a vector

The concept of component of a vector is tied to the concept of vector sum. We have seen that the sum of two vectors represented by two sides of a triangle is given by a vector represented by the closing side (third) of the triangle in opposite direction. Importantly, we can analyze this process of summation of two vectors inversely. We can say that a single vector (represented by third side of the triangle) is equivalent to two vectors in two directions (represented by the remaining two sides).

We can generalize this inverse interpretation of summation process. We can say that a vector can always be considered equivalent to a pair of vectors. The law of triangle, therefore, provides a general frame work of resolution of a vector in two components in as many ways as we can draw triangle with one side represented by the vector in question. However, this general framework is not very useful. Resolution of vectors turns out to be meaningful, when we think resolution in terms of vectors at right angles. In that case, associated triangle is a right angle. The vector being resolved into components is represented by the hypotenuse and components are represented by two sides of the right angle triangle.

Resolution of a vector into components is an important concept for two reasons : (i) there are physical situations where we need to consider the effect of a physical vector quantity in specified direction. For example, we consider only the component of weight along an incline to analyze the motion of the block over it and (ii) the concept of components in the directions of rectangular axes, enable us to develop algebraic methods for vectors.

Resolution of a vector in two perpendicular components is an extremely useful technique having extraordinary implication. Mathematically, any vector can be represented by a pair of co-planar vectors in two perpendicular directions. It has, though, a deeper meaning with respect to physical phenomena and hence physical laws. Consider projectile motion for example. The motion of projectile in two dimensional plane is equivalent to two motions – one along the vertical and one along horizontal direction. We can accurately describe motion in any of these two directions independent of motion in the other direction! To some extent, this

independence of two component vector quantities from each other is a statement of physical law – which is currently considered as property of vector quantities and not a law by itself. But indeed, it is a law of great importance which we employ to study more complex physical phenomena and process.

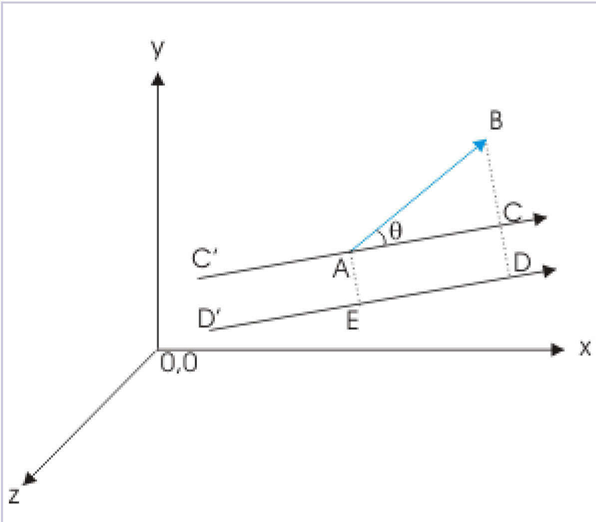
Second important implication of resolution of a vector in two perpendicular directions is that we can represent a vector as two vectors acting along two axes of a rectangular coordinate system. This has far reaching consequence. It is the basis on which algebraic methods for vectors are developed – otherwise vector analysis is tied, by definition, to geometric construct and analysis. The representation of vectors along rectangular axes has the advantage that addition of vector is reduced to simple algebraic addition and subtraction as consideration along an axis becomes one dimensional. It is not difficult to realize the importance of the concept of vector components in two perpendicular directions. Resolution of vector quantities is the way we transform two and three dimensional phenomena into one dimensional phenomena along the axes.

Components of a vector

A scalar component, also known as projection, of a vector **AB** in the positive direction of a straight line C'C is defined as :

$$\mathrm{AC} = |\mathbf{AB}| \cos\theta = \mathrm{AB}\cos\theta$$

Where θ is the angle that vector AB makes with the specified direction C'C as shown in the figure.
Analytical method



It should be understood that the component so determined is not rooted to the common point “A” or any specific straight line like C’C in a given direction. Matter of fact, component of vector AB drawn on any straight lines like D’D parallel to CC’ are same. Note that the component i.e. projection of the vector is graphically obtained by drawing two perpendicular lines from the ends of the vector on the straight line in the specified direction.

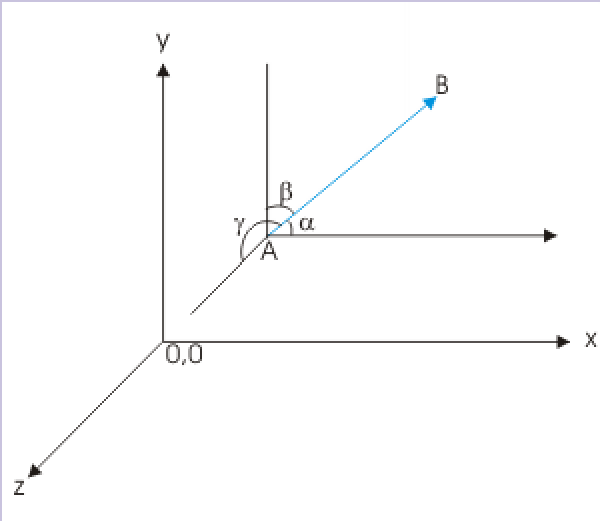
Equation:

$$\mathrm{ED} = \mathrm{AC} = \mathrm{AB} \cos \theta$$

It is clear that scalar component can either be positive or negative depending on the value of angle that the vector makes with the referred direction. The angle lies between the range given by $0 \leq \theta \leq 180°$. This interval means that we should consider the smaller angle between two vectors.

In accordance with the above definition, we resolve a given vector in three components in three mutually perpendicular directions of rectangular coordinate system. Note here that we measure angle with respect to parallel lines to the axes. By convention, we denote components by using the non-bold type face of the vector symbol with a suffix representing direction (x or y or z).

Components of a vector



$$AB_x = \text{AB} \cos \alpha$$
$$AB_y = \text{AB} \cos \beta$$
$$AB_y = \text{AB} \cos \gamma$$

Where α, β and γ are the angles that vector AB makes with the positive directions of x, y and z directions respectively.

As a vector can be resolved in a set of components, the reverse processing of components in a vector is also expected. A vector is composed from vector components in three directions along the axes of rectangular coordinate system by combining three component vectors. The vector component is the vector counterpart of the scalar component, which is obtained by multiplying the scalar component with the unit vector in axial direction. The vector components of a vector ,**a**, are $a_{\mathrm{x}}\mathbf{i}$, $a_{\mathrm{y}}\mathbf{j}$ and $a_{\mathrm{z}}\mathbf{k}$.

A vector,**a**, is equal to the vector sum of component vectors in three mutually perpendicular directions of rectangular coordinate system.
Equation:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the respective directions.
Equation:

$$\mathbf{a} = a\cos\alpha\mathbf{i} + a\cos\beta\mathbf{j} + a\cos\gamma\mathbf{k}$$

Magnitude of the sum of two vectors ($\mathrm{a}_x\mathbf{i} + \mathrm{a}_y\mathbf{j}$) in x and y direction, which are perpendicular to each other, is :

$$\text{a'} = \sqrt{\left(a_x{}^2 + a_y{}^2\right)} = \left(a_x{}^2 + a_y{}^2\right)^{\frac{1}{2}}$$

We observe here that resultant vector lies in the plane formed by the two component vectors being added. The resultant vector is, therefore, perpendicular to third component vector. Thus, the magnitude of the sum of vector ($\mathrm{a}_x\mathbf{i} + \mathrm{a}_y\mathbf{j}$) and third vector , $\mathrm{a}_z\mathbf{k}$, which are perpendicular to each other, is :

Equation:

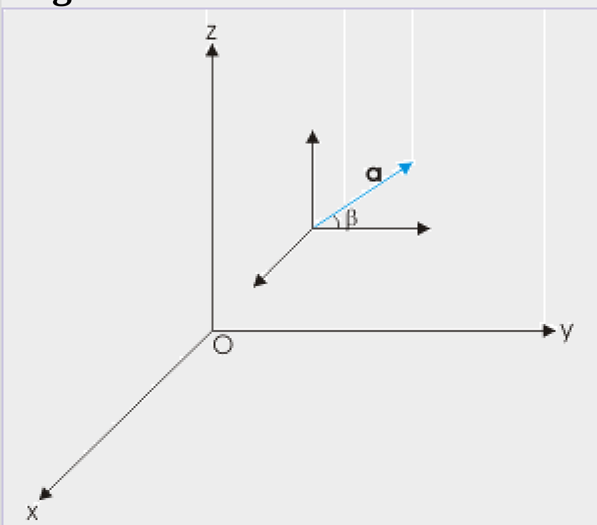
$$\Rightarrow a = \sqrt{\left[\left\{\left(a_x{}^2 + a_y{}^2\right)^{\frac{1}{2}}\right\}^2 + a_z{}^2\right]}$$

$$a = \sqrt{\left(a_x{}^2 + a_y{}^2 + a_z{}^2\right)}$$

Example:

Problem : Find the angle that vector 2**i** + **j** – **k** makes with y-axis.
Solution : We can answer this question with the help of expression for the cosine of angle that a vector makes with a given axis. We know that component along y-axis is :

Angle



Vector makes an angle with y-axis.

$$a_y = a\cos\beta$$
$$\Rightarrow \cos_\beta = \frac{a_y}{a}$$

Here,

$$a_y = 1$$

and

$$a = \sqrt{(2^2 + 1^2 + 1^2)}$$

Hence,

$$\Rightarrow \cos_{\beta} = \frac{a_y}{a} = \frac{1}{\sqrt{6}}$$

$$\Rightarrow \beta = \cos^{-1}\frac{1}{\sqrt{6}}$$

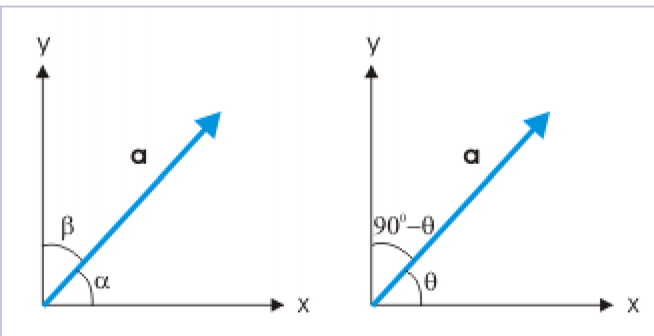
Planar components of a vector

The planar components of a vector lies in the plane of vector. Since there are two perpendicular axes involved with a plane, a vector is resolved in two components which lie in the same plane as that of vector. Clearly, a vector is composed of components in only two directions :

$$\mathbf{a} = a\cos\alpha\mathbf{i} + a\cos\beta\mathbf{j}$$

From the figure depicting a planar coordinate, it is clear that angle “β” is compliment of angle “α”. If α = θ, then

Planar vector



The direction of a planar vector with respect to rectangular axes can be described by a single angle.

$$\alpha = \theta \ \text{ and } \ \beta = 90^\circ - \theta$$

Putting in the expression for the vector,

Equation:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$$
$$\mathbf{a} = a\cos\theta\mathbf{i} + a\cos(90^\circ - \theta)\mathbf{j}$$
$$\mathbf{a} = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j}$$

From graphical representation, the tangent of the angle that vector makes with x-axis is :

Equation:

$$\tan\alpha = \tan\theta = \frac{a\sin\theta}{a\cos\theta} = \frac{a_y}{a_x}$$

Similarly, the tangent of the angle that vector makes with y-axis is :

Equation:

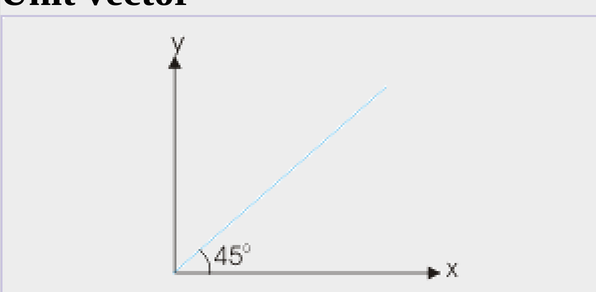
$$\tan\beta = \tan\left(90^\circ - \theta\right) = \cot\theta = \frac{a\cos\theta}{a\sin\theta} = \frac{a_x}{a_y}$$

Example:

Problem : Find the unit vector in the direction of a bisector of the angle between a pair of coordinate axes.

Solution : The unit vector along the direction of a bisector lies in the plane formed by two coordinates. The bisector makes an angle of 45° with either of the axes. Hence, required unit vector is :

Unit vector



Unit vector along the bisector.

$$\mathbf{n} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} = \cos 45^\circ\mathbf{i} + \sin 45^\circ\mathbf{j}$$

$$\Rightarrow \mathbf{n} = \frac{1}{\sqrt{2}} \text{ x } \left(\mathbf{i} + \mathbf{j}\right)$$

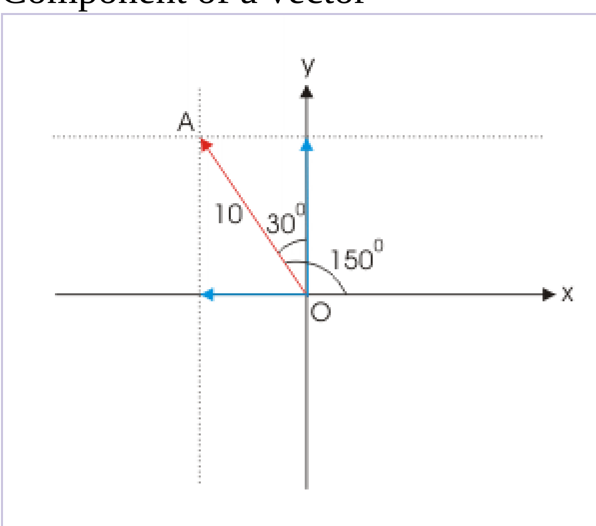
Note : We may check that the magnitude of the unit vector is indeed 1.

$$\left|\mathbf{n}\right| = n = \sqrt{\left\{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right\}} = \sqrt{1} = 1$$

Representation of a vector in component form

The angle involved in determination of components plays important role. There is a bit of ambiguity about angle being used. Actually, there are two ways to write a vector in component form. We shall illustrate these two methods with an illustration. Let us consider an example. Here, we consider a vector **OA** having magnitude of 10 units. The vector makes 150° and 30° angle with x and y axes as shown in the figure.

Component of a vector



Component of a vector

Using definition of components, the vector **OA** is represented as :

$$\mathbf{OA} = 10\cos 150^0 \mathbf{i} + 10\cos 30^0 \mathbf{j}$$

$$\Rightarrow \mathbf{OA} = 10X - \frac{1}{2}\mathbf{i} + 10X\frac{\sqrt{3}}{2}\mathbf{j}$$

$$\Rightarrow \mathbf{OA} = -5\mathbf{i} + 5\sqrt{3}\mathbf{j}$$

Note that we use angles that vector makes with the axes to determine scalar components. We obtain corresponding component vectors by multiplying scalar components with respective unit vectors of the axes involved. We can, however, use another method in which we only consider acute angle – irrespective of directions of axes involved. Here, vector **OA** makes acute angle of 60° with x-axis. While representing vector, we put a negative sign if the component is opposite to the positive directions of axes. We can easily determine this by observing projection of vector on the axes. Following this :

$$\mathbf{OA} = -\cos 60^0 \mathbf{i} + 10\sin 60^0 \mathbf{j}$$

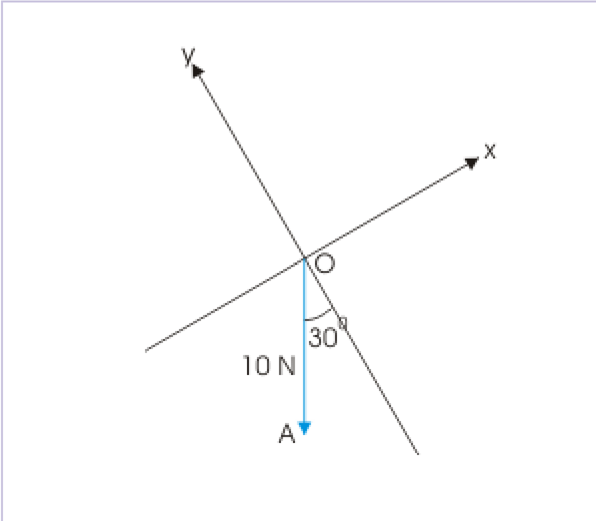
$$\Rightarrow \mathbf{OA} = -5\mathbf{i} + 5\sqrt{3}\mathbf{j}$$

Note that we put a negative sign before component along x-direction as projection of vector on x-axis is in opposite direction with respect to positive direction of x-axis. The y - projection, however, is in the positive direction of y-axis. As such, we do not need to put a negative sign before the component. Generally, people prefer second method as trigonometric functions are positive in first quadrant. We are not worried about the sign of trigonometric function at all.

Exercise:

Problem: Write force **OA** in component form.

Component of a vector



Component of a vector.

Solution:

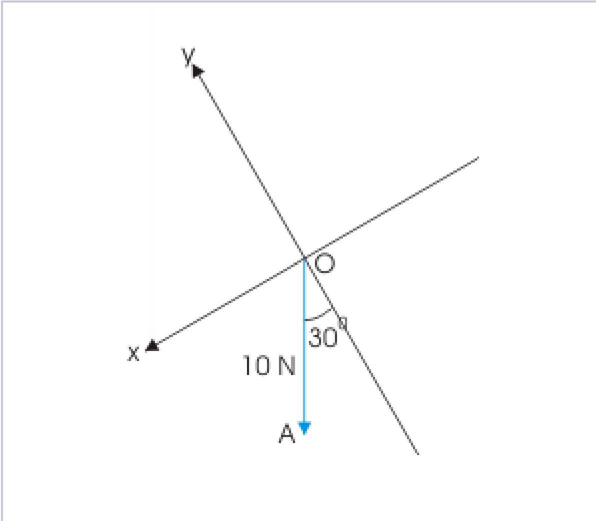
$$F_x = 10\sin 30^0 = 10X\frac{1}{2} = 5 \quad N$$

$$F_y = 10\cos 30^0 = 10X\frac{\sqrt{3}}{2} = 5\sqrt{3} \quad N$$

$$\mathbf{F} = -F_x\mathbf{i} - F_y\mathbf{j} = -5\mathbf{i} - 5\sqrt{3}\mathbf{j} \quad N$$

Exercise:

Problem: Write force **OA** in component form.
Component of a vector



Component of a vector.

Solution:

$$F_x = 10\sin 30^0 = 10X\frac{1}{2} = 5 \quad N$$

$$F_y = 10\cos 30^0 = 10X\frac{\sqrt{3}}{2} = 5\sqrt{3} \quad N$$

$$\mathbf{F} = F_x\mathbf{i} - F_y\mathbf{j} = 5\mathbf{i} - 5\sqrt{3}\mathbf{j} \quad N$$

Vector addition : Algebraic method

Graphical method is meticulous and tedious as it involves drawing of vectors on a scale and measurement of angles. More importantly, it does not allow algebraic operations that otherwise would give a simple solution. We can, however, extend algebraic techniques to vectors, **provided vectors are represented on a rectangular coordinate system**. The representation of a vector on coordinate system uses the concept of unit vector and component.

Now, the stage is set to design a frame work, which allows vector addition with algebraic methods. The frame work for vector addition draws on two important concepts. The first concept is that a vector can be equivalently expressed in terms of three component vectors :

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

The component vector form has important significance. It ensures that component vectors to be added are restricted to three known directions only. This paradigm eliminates the possibility of unknown direction. The second concept is that vectors along a direction can be treated algebraically. If two vectors are along the same line, then resultant is given as :

When θ = 0°, cosθ = cos 0° = 1 and,

$$\Rightarrow R = \sqrt{\left(P^2 + 2PQ + Q^2\right)} = \sqrt{\left\{(P+Q)^2\right\}} = P + Q$$

When θ = 180°, cosθ = cos 180° = -1 and,

$$\Rightarrow R = \sqrt{\left(P^2 - 2PQ + Q^2\right)} = \sqrt{\left\{(P-Q)^2\right\}} = P - Q$$

Thus, we see that the magnitude of the resultant is equal to algebraic sum of the magnitudes of the two vectors.

Using these two concepts, the addition of vectors is affected as outlined here :

1: Represent the vectors (**a** and **b**) to be added in terms of components :

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

2: Group components in a given direction :

$$\mathbf{a} + \mathbf{b} = a_{\mathrm{x}}\mathbf{i} + a_{\mathrm{y}}\mathbf{j} + a_{\mathrm{z}}\mathbf{k} + b_{\mathrm{x}}\mathbf{i} + b_{\mathrm{y}}\mathbf{j} + b_{\mathrm{z}}\mathbf{k}$$
$$\Rightarrow \mathbf{a} + \mathbf{b} = (a_{\mathrm{x}} + b_{\mathrm{x}})\mathbf{i} + (a_{\mathrm{y}} + b_{\mathrm{y}})\mathbf{j} + (a_{\mathrm{z}} + b_{\mathrm{z}})\mathbf{k}$$

3: Find the magnitude and direction of the sum, using analytical method
Equation:

$$\Rightarrow a = \sqrt{\left\{(a_x + b_x)^2 + (a_y + b_y)^2 + (a_z + b_z)^2\right)}$$

Exercises

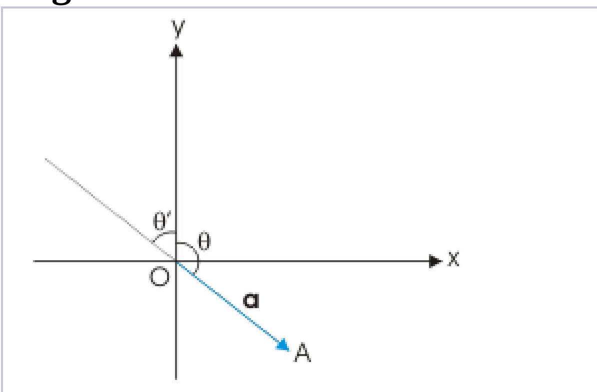
Exercise:

Problem: Find the angle that vector √3**i** - **j** makes with y-axis.

Solution:

From graphical representation, the angle that vector makes with y-axis has following trigonometric function :

Angle



Vector makes an angle with y-axis.

$$\tan\theta = \frac{a_x}{a_y}$$

Now, we apply the formulae to find the angle, say θ, with y-axis,

$$\tan\theta = -\frac{\surd 3}{1} = -\sqrt{3} = \tan 120^\circ$$

$$\Rightarrow \theta = 120^\circ$$

In case, we are only interested to know the magnitude of angle between vector and y-axis, then we can neglect the negative sign,

$$\tan \theta' = \sqrt{3} = \tan 60^\circ$$

$$\Rightarrow \theta' = 60^\circ$$

Exercise:

Problem:

If a vector makes angles α,β and γ with x,y and z axes of a rectangular coordinate system, then prove that :

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

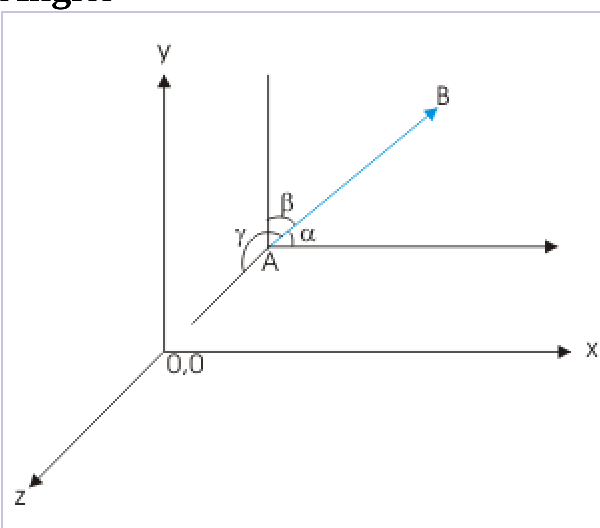
Solution:

The vector can be expressed in terms of its component as :

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

where $a_x = a\cos\alpha;\ \ a_y = a\cos\beta;\ \ a_z = a\cos\gamma$.

Angles



Vector makes different angles with axes..

The magnitude of the vector is given by :

$$a = \sqrt{\left({a_x}^2 + {a_y}^2 + {a_z}^2\right)}$$

Putting expressions of components in the equation,

$$\Rightarrow a = \sqrt{\left(a^2 \cos^2 \alpha + a^2 \cos^2 \beta + a^2 \cos^2 \gamma\right)}$$

$$\Rightarrow \sqrt{\left(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma\right)} = 1$$

Squaring both sides,

$$\Rightarrow \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Exercise:

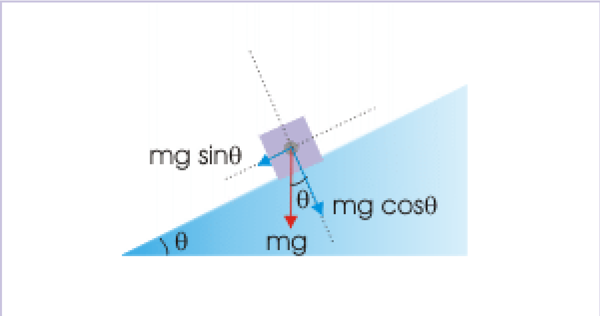
Problem:

Find the components of weight of a block along the incline and perpendicular to the incline.

Solution:

The component of the weight along the incline is :

Weight on an incline



Components of weight along the incline and perpendicualt to the incline.

$$W_x = \text{mg} \ \sin\theta = 100 \ \text{ x } \ \sin 30^\circ = 100 \ \text{ x } \ \frac{1}{2} = 50\,N$$

and the component of weight perpendicular to incline is :

$$W_y = \text{mg} \ \cos\theta = 100 \ \text{x} \ \cos 30^\circ = 100 \ \text{x} \ \frac{\sqrt{3}}{2} = 50\sqrt{3} \ N$$

Exercise:

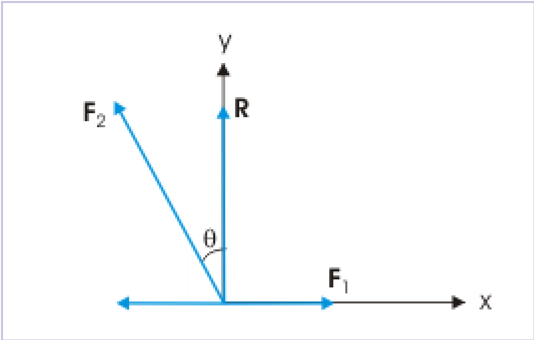
Problem:

The sum of magnitudes of two forces acting at a point is 16 N. If the resultant of the two forces is 8 N and it is normal to the smaller of the two forces, then find the forces.

Solution:

We depict the situation as shown in the figure. The resultant force R is shown normal to small force F_1 . In order that the sum of the forces is equal to R, the component of larger force along the x-direction should be equal to smaller force :

Two forces acting on a point



The resultant force is perpendicular to smaller of the two forces.

$$F_2 \sin\theta = F_1$$

Also, the component of the larger force along y-direction should be equal to the magnitude of resultant,

$$F_2 \cos\theta = R = 8$$

Squaring and adding two equations, we have :

$$F_2^2 = F_1^2 + 64$$

$$\Rightarrow F_2^2 - F_1^2 = 64$$

However, according to the question,

$$\Rightarrow F_1 + F_2 = 16$$

Substituting, we have :

$$\Rightarrow F_2 - F_1 = \frac{64}{16} = 4$$

Solving,

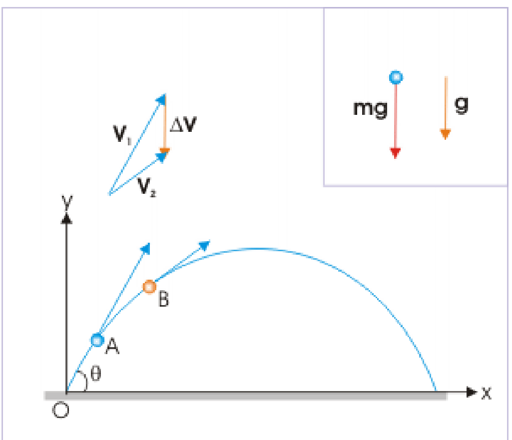
$$\Rightarrow F_1 = 6\,N$$
$$\Rightarrow F_2 = 10\,N$$

More illustrations on the subject are available in the module titled [Resolution of forces]

Scalar (dot) product

In physics, we require to multiply a vector with other scalar and vector quantities. The vector multiplication, however, is not an unique mathematical construct like scalar multiplication. The multiplication depends on the nature of quantities (vector or scalar) and on the physical process, necessitating scalar or vector multiplication.

The rules of vector multiplication have been formulated to encapsulate physical processes in their completeness. This is the core consideration. In order to explore this aspect, let us find out the direction of acceleration in the case of parabolic motion of a particle. There may be two ways to deal with the requirement. We may observe the directions of velocities at two points along the path and find out the direction of the change of velocities. Since we know that the direction of change of velocity is the direction of acceleration, we draw the vector diagram and find out the direction of acceleration. We can see that the direction of acceleration turns out to act in vertically downward direction.
Direction of acceleration



The conceptualization of physical laws in vector form, however, provides us with powerful means to arrive at the result in relatively simpler manner. If we look at the flight of particle in parabolic motion, then we observe that the motion of particle is under the force of gravity, which is acting vertically downward. There is no other force (neglecting air resistance). Now, from second law of motion, we know that :

$$\mathbf{F}_{\text{Resultant}} = m\mathbf{a}$$

This equation reveals that the direction of acceleration is same as that of the resultant force acting on the particle. Thus, acceleration of the particle in parabolic motion is acting vertically downward. We see that this second approach is more elegant of the two methods. We could arrive at the correct answer in a very concise manner, without getting into the details of the motion. It is possible, because Newton's second law in vector form states that net force on the body is product of acceleration vector with scalar mass. As multiplication of scalar with a vector does not change the direction of resultant vector, we conclude that direction of acceleration is same as that of net force acting on the projectile.

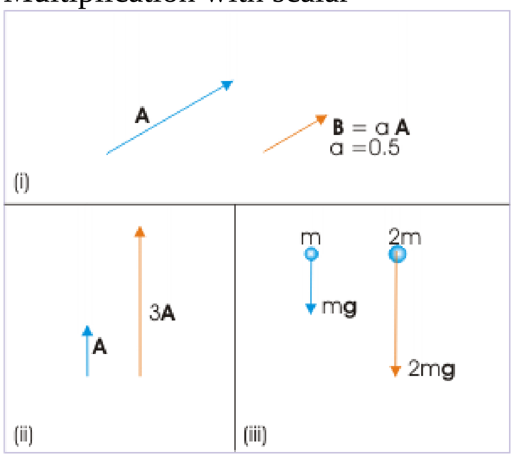
Multiplication with scalar

Multiplication of a vector, **A**, with another scalar quantity, a, results in another vector, **B**. The magnitude of the resulting vector is equal to the product of the magnitude of vector with the scalar quantity. The direction of the resulting vector, however, is same as that of the original vector (See Figures below).

Equation:

$$\mathbf{B} = a\mathbf{A}$$

Multiplication with scalar



We have already made use of this type of multiplication intuitively in expressing a vector in component form.

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

In this vector representation, each component vector is obtained by multiplying the scalar component with the unit vector. As the unit vector has the magnitude of 1 with a specific direction, the resulting component vector retains the magnitude of the scalar component, but acquires the direction of unit vector.

Equation:

$$\mathbf{A}_x = A_x\mathbf{i}$$

Products of vectors

Some physical quantities are themselves a scalar quantity, but are composed from the product of vector quantities. One such example is “work”. On the other hand, there are physical quantities like torque and magnetic force on a moving charge, which are themselves vectors and are also composed from vector quantities.

Thus, products of vectors are defined in two distinct manner – one resulting in a scalar quantity and the other resulting in a vector quantity. The product that results in scalar value is scalar product, also known as dot product as a "dot" (**.**) is the symbol of operator for this product. On the other hand, the product that results in vector value is vector product, also known as cross product as a "cross" (**x**) is the symbol of operator for this product. We shall discuss scalar product only in this module. We shall cover vector product in a separate module.

Scalar product (dot product)

Scalar product of two vectors **a** and **b** is a scalar quantity defined as :
Equation:

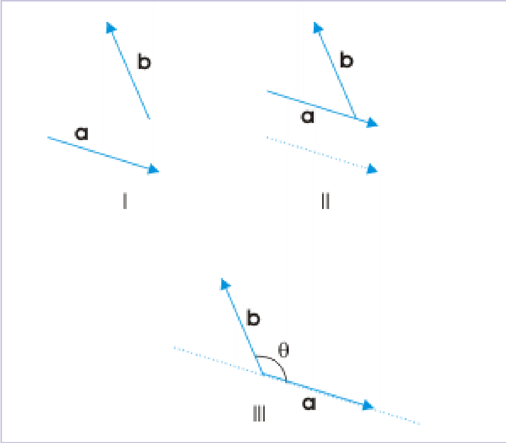
$$\mathbf{a.b} = ab\cos\theta$$

where "a" and "b" are the magnitudes of two vectors and "θ" is the angle between the direction of two vectors. It is important to note that vectors have two angles θ and 2π - θ. We can use either of them as cosine of both "θ" and "2π - θ" are same. However, it is suggested to use the smaller of the enclosed angles to be consistent with cross product in which it is required to use the smaller of the enclosed angles. This approach will maintain consistency with regard to enclosed angle in two types of vector multiplications.

The notation "**a.b**" is important and should be mentally noted to represent a scalar quantity – even though it involves bold faced vectors. It should be noted that the quantity on the right hand side of the equation is a scalar.

Angle between vectors

The angle between vectors is measured with precaution. The direction of vectors may sometimes be misleading. The basic consideration is that it is the angle between vectors at the common point of intersection. This intersection point, however, should be the common tail of vectors. If required, we may be required to shift the vector parallel to it or along its line of action to obtain common point at which tails of vectors meet.
Angle between vectors



Angle between vectors

See the steps shown in the figure. First, we need to shift one of two vectors say, **a** so that it touches the tail of vector **b**. Second, we move vector **a** along its line of action till tails of two vectors meet at the common point. Finally, we measure the angle θ such that $0 \le \theta \le \pi$.

Meaning of scalar product

We can read the definition of scalar product in either of the following manners :

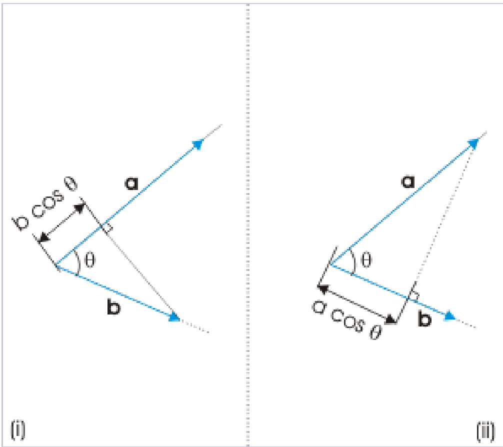
Equation:

$$\mathbf{a.b} = a(b\cos\theta)$$
$$\mathbf{a.b} = b(a\cos\theta)$$

Recall that “bcos θ” is the scalar component of vector **b** along the direction of vector **a** and “a cos θ” is the scalar component of vector **a** along the direction of vector **b**. Thus, we may consider the scalar product of vectors **a** and **b** as the product of the magnitude of one vector and the scalar component of other vector along the first vector.

The figure below shows drawing of scalar components. The scalar component of vector in figure (i) is obtained by drawing perpendicular from the tip of the vector, **b**, on the direction of vector, **a**. Similarly, the scalar component of vector in figure (ii) is obtained by drawing perpendicular from the tip of the vector, **a**, on the direction of vector, **b**.

Scalar product



The two alternate ways of evaluating dot product of two vectors indicate that the product is commutative i.e. independent of the order of two vectors :

Equation:

$$\mathbf{a.b} = \mathbf{b.a}$$

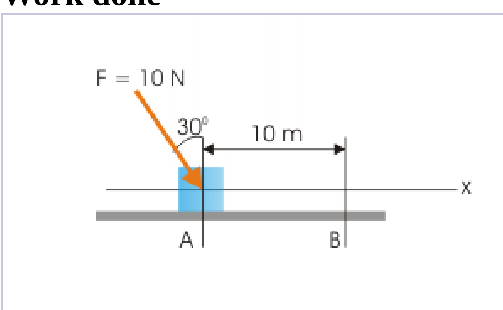
Exercise:

Problem:

A block of mass "m" moves from point A to B along a smooth plane surface under the action of force as shown in the figure. Find the work done if it is defined as :

W = **F.** Δ**x**

Work done

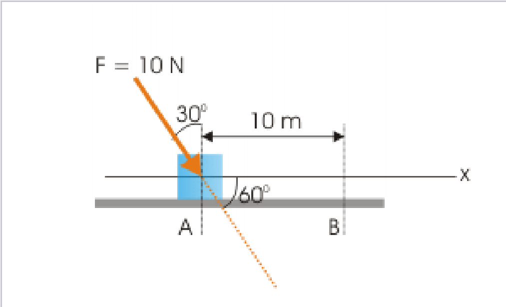


where **F** and Δ**x** are force and displacement vectors.

Solution:

Expanding the expression of work, we have :

Work done



$$W = \mathbf{F}.\,\Delta\mathbf{x} = F\Delta x \cos\theta$$

Here, F = 10 N, Δx = 10 m and cosθ = cos60° = 0.5.

$$\Rightarrow W = 10 \text{ x } 10 \text{ x } 0.5 = 50 \text{ J}$$

Values of scalar product

The value of dot product is maximum for the maximum value of cosθ. Now, the maximum value of cosine is $\cos 0^\circ = 1$. For this value, dot product simply evaluates to the product of the magnitudes of two vectors.

$$(\mathbf{a}.\,\mathbf{b})_{\text{max}} = ab$$

For $\theta = 180^\circ$, $\cos 180^\circ =$ -1 and

$$\mathbf{a}.\,\mathbf{b} = -ab$$

Thus, we see that dot product can evaluate to negative value as well. This is a significant result as many scalar quantities in physics are given negative value. The work done, for example, can be negative, when displacement is in the opposite direction to the component of force along that direction.

The scalar product evaluates to zero for θ = 90° and 270° as cosine of these angles are zero. These results have important implication for unit vectors. The dot products of same unit vector evaluates to 1.

$$\mathbf{i}.\,\mathbf{i} = \mathbf{j}.\,\mathbf{j} = \mathbf{k}.\,\mathbf{k} = 1$$

The dot products of combination of different unit vectors evaluate to zero.

$$\mathbf{i}.\,\mathbf{j} = \mathbf{j}.\,\mathbf{k} = \mathbf{k}.\,\mathbf{i} = 0$$

Example:
Problem : Find the angle between vectors 2**i** + **j** – **k** and **i** – **k**.
Solution : The cosine of the angle between two vectors is given in terms of dot product as :

$$\cos\theta = \frac{\mathbf{a.b}}{\mathrm{ab}}$$

Now,

$$\mathbf{a.\,b} = (2\mathbf{i} + \mathbf{j} - \mathbf{k})\mathbf{.}\,(2\mathbf{i} - \mathbf{k})$$

Ignoring dot products of different unit vectors (they evaluate to zero), we have :

$$\mathbf{a.\,b} = 2\mathbf{i.\,i} + (-\mathbf{k})\mathbf{.}\,(-\mathbf{k}) = 2 + 1 = 3$$
$$a = \sqrt{\left(2^2 + 1^2 + 1^2\right)} = \sqrt{6}$$
$$b = \sqrt{\left(1^2 + 1^2\right)} = \sqrt{2}$$
$$\mathrm{ab} = \sqrt{6}\ \ \mathrm{x}\ \ \sqrt{2} = \sqrt{(12)} = 2\sqrt{3}$$

Putting in the expression of cosine, we have :

$$\cos\theta = \frac{\mathbf{a.b}}{\mathrm{ab}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} = \cos 30^\circ$$
$$\theta = 30^\circ$$

Scalar product in component form

Two vectors in component forms are written as :

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$
$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

In evaluating the product, we make use of the fact that multiplication of the same unit vectors is 1, while multiplication of different unit vectors is zero. The dot product evaluates to scalar terms as :
Equation:

$$\mathbf{a.\,b} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\mathbf{.}\,(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k})$$
$$\Rightarrow \mathbf{a.\,b} = a_x\mathbf{i.}\,b_x\mathbf{i} + a_y\mathbf{j.}\,b_y\mathbf{j} + a_z\mathbf{k.}\,b_z\mathbf{k}$$
$$\Rightarrow \mathbf{a.\,b} = a_xb_x + a_yb_y + a_zb_z$$

Component as scalar (dot) product

A closer look at the expansion of dot product of two vectors reveals that the expression is very similar to the expression for a component of a vector. The expression of the dot product is :

$$\mathbf{a.b} = ab\cos\theta$$

On the other hand, the component of a vector in a given direction is :

$$a_x = a\cos\theta$$

Comparing two equations, we can define component of a vector in a direction given by unit vector "**n**" as :

Equation:

$$a_x = \mathbf{a.n} = a\cos\theta$$

This is a very general and useful relation to determine component of a vector in any direction. Only requirement is that we should know the unit vector in the direction in which component is to be determined.

Example:

Problem : Find the components of vector 2**i** + 3**j** along the direction **i**+**j**.

Solution : The component of a vector "**a**" in a direction, represented by unit vector "**n**" is given by dot product :

$$a_n = \mathbf{a.n}$$

Thus, it is clear that we need to find the unit vector in the direction of **i**+**j**. Now, the unit vector in the direction of the vector is :

$$n = \frac{\mathbf{i+j}}{|\mathbf{i+j}|}$$

Here,

$$\left|\mathbf{i}+\mathbf{j}\right| = \sqrt{\left(1^2+1^2\right)} = \sqrt{2}$$

Hence,

$$n = \frac{1}{\sqrt{2}} \text{ x } \left(\mathbf{i}+\mathbf{j}\right)$$

The component of vector 2**i** + 3**j** in the direction of “**n**” is :

$$a_n = \mathbf{a.\,n} = \left(2\mathbf{i} + 3\mathbf{j}\right)\mathbf{.}\ \frac{1}{\sqrt{2}}\ \text{ x }\ \left(\mathbf{i} + \mathbf{j}\right)$$

$$\Rightarrow a_n = \frac{1}{\sqrt{2}}\ \text{ x }\ \left(2\mathbf{i} + 3\mathbf{j}\right)\mathbf{.}\left(\mathbf{i} + \mathbf{j}\right)$$

$$\Rightarrow a_n = \frac{1}{\sqrt{2}}\ \text{ x }\ \left(2 \text{ x } 1 + 3 \text{ x } 1\right)$$

$$\Rightarrow a_n = \frac{5}{\sqrt{2}}$$

Attributes of scalar (dot) product

In this section, we summarize the properties of dot product as discussed above. Besides, some additional derived attributes are included for reference.

1: Dot product is commutative

This means that the dot product of vectors is not dependent on the sequence of vectors :

$$\mathbf{a.\,b} = \mathbf{b.\,a}$$

We must, however, be careful while writing sequence of dot product. For example, writing a sequence involving three vectors like **a.b.c** is incorrect. For, dot product of any two vectors is a scalar. As dot product is defined for two vectors (not one vector and one scalar), the resulting dot product of a scalar (**a.b**) and that of third vector **c** has no meaning.

2: Distributive property of dot product :

$$\mathbf{a.}\left(\mathbf{b} + \mathbf{c}\right) = \mathbf{a.\,b} + \mathbf{a.\,c}$$

3: The dot product of a vector with itself is equal to the square of the magnitude of the vector.

$$\mathbf{a.\,a} = a\ \text{ x }\ a\cos\theta = a^2 \cos 0^\circ = a^2$$

4: The magnitude of dot product of two vectors can be obtained in either of the following manner :

$$\mathbf{a.\,b} = ab\cos\theta$$
$$\mathbf{a.\,b} = ab\cos\theta = a\ \text{ x }\ \left(b\cos\theta\right) = a\ \text{ x }\ \text{component of } \mathbf{b} \text{ along } \mathbf{a}$$
$$\mathbf{a.\,b} = ab\cos\theta = \left(a\cos\theta\right)\ \text{ x }\ b = b\ \text{ x }\ \text{component of } \mathbf{a} \text{ along } \mathbf{b}$$

The dot product of two vectors is equal to the algebraic product of magnitude of one vector and component of second vector in the direction of first vector.

5: The cosine of the angle between two vectors can be obtained in terms of dot product as :

$$\mathbf{a.b} = ab\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\mathbf{a.b}}{ab}$$

6: The condition of two perpendicular vectors in terms of dot product is given by :

$$\mathbf{a.b} = ab\cos 90^\circ = 0$$

7: Properties of dot product with respect to unit vectors along the axes of rectangular coordinate system are :

$$\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1$$
$$\mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0$$

8: Dot product in component form is :

$$\mathbf{a.b} = a_x b_x + a_y b_y + a_z b_z$$

9: The dot product does not yield to cancellation. For example, if **a.b** = **a.c**, then we can not conclude that **b** = **c**. Rearranging, we have :

$$\mathbf{a.b} - \mathbf{a.c} = 0$$
$$\mathbf{a.}(\mathbf{b} - \mathbf{c}) = 0$$

This means that **a** and (**b - c**) are perpendicular to each other. In turn, this implies that (**b - c**) is not equal to zero (null vector). Hence, **b** is not equal to **c** as we would get after cancellation.

We can understand this difference with respect to cancellation more explicitly by working through the problem given here :

Example:
Problem : Verify vector equality **B** = **C**, if **A.B** = **A.C**.
Solution : The given equality of dot products is :

$$\mathbf{A.B} = \mathbf{A.C}$$

We should understand that dot product is not a simple algebraic product of two numbers (read magnitudes). The angle between two vectors plays a role in determining the magnitude of the dot product. Hence, it is entirely possible that vectors **B** and **C** are different yet their dot products with common vector A are equal. Let θ_1 and θ_2 be the angles for first and second pairs of dot products. Then,

$$\mathbf{A.B} = \mathbf{A.C}$$

$$\mathrm{AB}\ \cos\theta_1 = \mathrm{AC}\ \cos\theta_2$$

If $\theta_1 = \theta_2$, then $B = C$. However, if $\theta_1 \neq \theta_2$, then $B \neq C$.

Law of cosine and dot product

Law of cosine relates sides of a triangle with one included angle. We can determine this relationship using property of a dot product. Let three vectors are represented by sides of the triangle such that closing side is the sum of other two vectors. Then applying triangle law of addition :

Cosine law

[missing_resource: sp1.gif]

Cosine law

$$\mathbf{c} = (\mathbf{a} + \mathbf{b})$$

We know that the dot product of a vector with itself is equal to the square of the magnitude of the vector. Hence,

$$\begin{aligned}
&c^2 = (\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b}) = \mathbf{a.a} + 2\mathbf{a.b} + \mathbf{b.b}\\
&c^2 = a^2 + 2ab\cos\theta + b^2\\
&c^2 = a^2 + 2ab\cos(\pi\text{-}\varphi) + b^2\\
&c^2 = a^2 - 2ab\cos\varphi + b^2
\end{aligned}$$

This is known as cosine law of triangle. Curiously, we may pay attention to first two equations above. As a matter of fact, second equation gives the square of the magnitude of resultant of two vectors **a** and **b**.

Differentiation and dot product

Differentiation of a vector expression yields a vector. Consider a vector expression given as :

$$\mathbf{a} = \left(x^2 + 2x + 3\right)\mathbf{i}$$

The derivative of the vector with respect to x is :

$$\mathbf{a}' = (2x + 2)\mathbf{i}$$

As the derivative is a vector, two vector expressions with dot product is differentiated in a manner so that dot product is retained in the final expression of derivative. For example,

$$\frac{d}{dx}\left(\mathbf{a}.\,\mathbf{b}\right) = \mathbf{a}'\mathbf{b} + \mathbf{a}\mathbf{b}'$$

Exercises

Exercise:

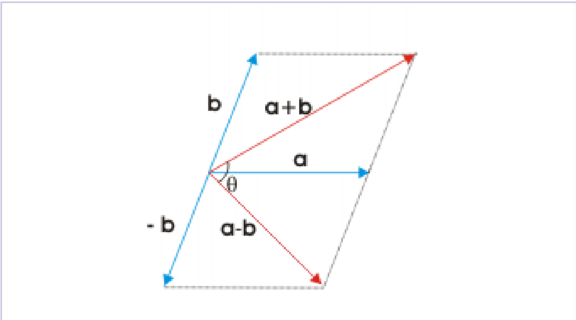
Problem:

Sum and difference of two vectors **a** and **b** are perpendicular to each other. Find the relation between two vectors.

Solution:

The sum **a**+**b** and difference **a-b** are perpendicular to each other. Hence, their dot product should evaluate to zero.

Sum and difference of two vectors



Sum and difference of two vectors are perpendicular to each other.

$$(\mathbf{a} + \mathbf{b}).\,(\mathbf{a} - \mathbf{b}) = 0$$

Using distributive property,

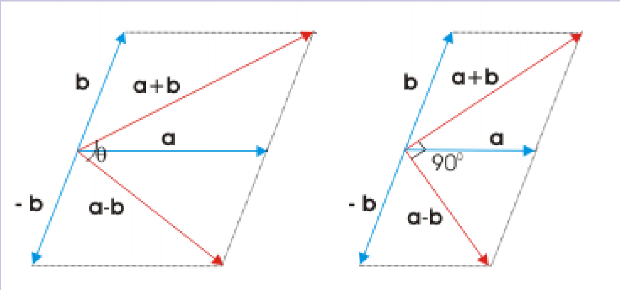
$$\mathbf{a.a} - \mathbf{a.b} + \mathbf{b.a} - \mathbf{b.b} = 0$$

Using commutative property, **a.b** = **b.a**, Hence,

$$\mathbf{a.a} - \mathbf{b.b} = 0$$
$$a^2 - b^2 = 0$$
$$a = b$$

It means that magnitudes of two vectors are equal. See figure below for enclosed angle between vectors, when vectors are equal :

Sum and difference of two vectors



Sum and difference of two vectors are perpendicular to each other, when vectors are equal.

Exercise:

Problem: If |**a** + **b**| = |**a** – **b**|, then find the angle between vectors **a** and **b**.

Solution:

A question that involves modulus or magnitude of vector can be handled in specific manner to find information about the vector (s). The specific identity that is used in this circumstance is :

$$\mathbf{A.A} = A^2$$

We use this identity first with the sum of the vectors (**a**+**b**),

$$\left(\mathbf{a} + \mathbf{b}\right).\left(\mathbf{a} + \mathbf{b}\right) = |\mathbf{a} + \mathbf{b}|^2$$

Using distributive property,

$$\Rightarrow \mathbf{a.a} + \mathbf{b.a} + \mathbf{a.b} + \mathbf{b.b} = a^2 + b^2 + 2ab\cos\theta = |\mathbf{a}+\mathbf{b}|^2$$
$$\Rightarrow |\mathbf{a}+\mathbf{b}|^2 = a^2 + b^2 + 2ab\cos\theta$$

Similarly, using the identity with difference of the vectors (a-b),

$$\Rightarrow |\mathbf{a}-\mathbf{b}|^2 = a^2 + b^2 - 2ab\cos\theta$$

It is, however, given that :

$$\Rightarrow |\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$$

Squaring on either side of the equation,

$$\Rightarrow |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$$

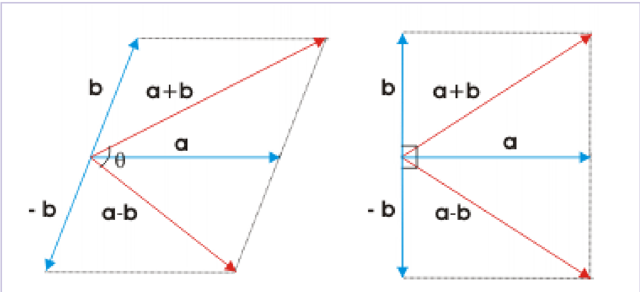
Putting the expressions,

$$\Rightarrow a^2 + b^2 + 2ab\cos\theta = a^2 + b^2 - 2ab\cos\theta$$
$$\Rightarrow 4ab\cos\theta = 0$$
$$\Rightarrow \cos\theta = 0$$
$$\Rightarrow \theta = 90^\circ$$

Note : We can have a mental picture of the significance of this result. As given, the magnitude of sum of two vectors is equal to the magnitude of difference of two vectors. Now, we know that difference of vectors is similar to vector sum with one exception that one of the operand is rendered negative. Graphically, it means that one of the vectors is reversed.

Reversing one of the vectors changes the included angle between two vectors, but do not change the magnitudes of either vector. It is, therefore, only the included angle between the vectors that might change the magnitude of resultant. In order that magnitude of resultant does not change even after reversing direction of one of the vectors, it is required that the included angle between the vectors is not changed. This is only possible, when included angle between vectors is 90°. See figure.

Sum and difference of two vectors



Magnitudes of Sum and difference of two vectors are same when vectors at right angle to each other.

Exercise:

Problem:

If **a** and **b** are two non-collinear unit vectors and |**a** + **b**| = √3, then find the value of expression :

$$(\mathbf{a}-\mathbf{b}).\,(2\mathbf{a}+\mathbf{b})$$

Solution:

The given expression is scalar product of two vector sums. Using distributive property we can expand the expression, which will comprise of scalar product of two vectors **a** and **b**.

$$\left(\mathbf{a}-\mathbf{b}\right).\left(2\mathbf{a}+\mathbf{b}\right)=2\mathbf{a}.\mathbf{a}+\mathbf{a}.\mathbf{b}-\mathbf{b}.2\mathbf{a}+\left(-\mathbf{b}\right).\left(-\mathbf{b}\right)=2a^2-\mathbf{a}.\mathbf{b}-b^2$$

$$\Rightarrow\left(\mathbf{a}-\mathbf{b}\right).\left(2\mathbf{a}+\mathbf{b}\right)=2a^2-b^2-ab\cos\theta$$

We can evaluate this scalar product, if we know the angle between them as magnitudes of unit vectors are each 1. In order to find the angle between the vectors, we use the identity,

$$\mathbf{A}.\mathbf{A}=A^2$$

Now,

$$|\mathbf{a}+\mathbf{b}|^2=\left(\mathbf{a}+\mathbf{b}\right).\left(\mathbf{a}+\mathbf{b}\right)=a^2+b^2+2ab\cos\theta=1+1+2\ \text{x}\ 1\ \text{x}\ 1\ \text{x}\ \cos\theta$$

$$\Rightarrow|\mathbf{a}+\mathbf{b}|^2=2+2\cos\theta$$

It is given that :

$$|\mathbf{a}+\mathbf{b}|^2=(\sqrt{3})^2=3$$

Putting this value,

$$\Rightarrow 2\cos\theta=|\mathbf{a}+\mathbf{b}|^2-2=3-2=1$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$
$$\Rightarrow \theta = 60^\circ$$

Using this value, we now proceed to find the value of given identity,

$$\left(\mathbf{a}-\mathbf{b}\right).\left(2\mathbf{a}+\mathbf{b}\right) = 2a^2 - b^2 - ab\cos\theta = 2 \text{ x } 1^2 - 1^2 - 1 \text{ x } 1 \text{ x } \cos 60^\circ$$

$$\Rightarrow \left(\mathbf{a}-\mathbf{b}\right).\left(2\mathbf{a}+\mathbf{b}\right) = \frac{1}{2}$$

Exercise:

Problem:

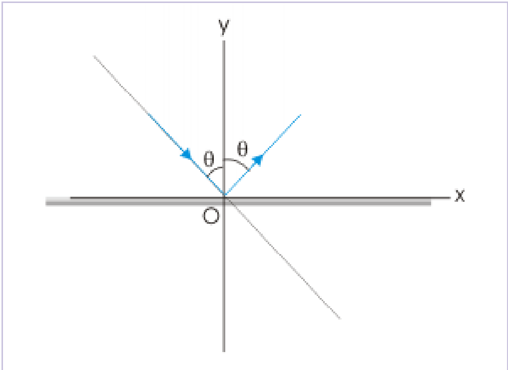
In an experiment of light reflection, if **a**, **b** and **c** are the unit vectors in the direction of incident ray, reflected ray and normal to the reflecting surface, then prove that :

$$\Rightarrow \mathbf{b} = \mathbf{a} - 2(\mathbf{a}.\mathbf{c})\mathbf{c}$$

Solution:

Let us consider vectors in a coordinate system in which “x” and “y” axes of the coordinate system are in the direction of reflecting surface and normal to the reflecting surface respectively as shown in the figure.

Reflection



Angle of incidence is equal to angle of reflection.

We express unit vectors with respect to the incident and reflected as :

$$\mathbf{a} = \sin\theta\mathbf{i} - \cos\theta\mathbf{j}$$
$$\mathbf{b} = \sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

Subtracting first equation from the second equation, we have :

$$\Rightarrow \mathbf{b} - \mathbf{a} = 2\cos\theta\mathbf{j}$$
$$\Rightarrow \mathbf{b} = \mathbf{a} + 2\cos\theta\mathbf{j}$$

Now, we evaluate dot product, involving unit vectors :

$$\mathbf{a.}\,\mathbf{c} = 1 \ \ \mathrm{x} \ \ 1 \ \ \mathrm{x} \ \ \cos(180^\circ - \theta) = -\cos\theta$$

Substituting for cosθ, we have :

$$\Rightarrow \mathbf{b} = \mathbf{a} - 2(\mathbf{a.}\,\mathbf{c})\mathbf{c}$$

Scalar product (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the scalar vector product. For this reason, questions are categorized in terms of the characterizing features of the subject matter :

- Angle between two vectors
- Condition of perpendicular vectors
- Component as scalar product
- Nature of scalar product
- Scalar product of a vector with itself
- Evaluation of dot product

Angle between two vectors

Example:
Problem : Find the angle between vectors 2**i** + **j** – **k** and **i** – **k**.
Solution : The cosine of the angle between two vectors is given in terms of dot product as :

$$\cos\theta = \frac{\mathbf{a.b}}{\mathrm{ab}}$$

Now,

$$\mathbf{a.\,b} = (2\mathbf{i} + \mathbf{j} - \mathbf{k})\mathbf{.}\,(2\mathbf{i} - \mathbf{k})$$

Ignoring dot products of different unit vectors (they evaluate to zero), we have :

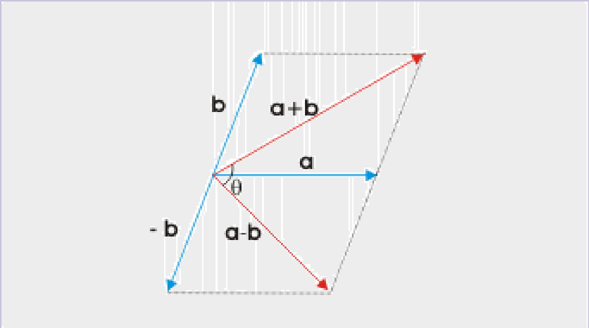
$$\mathbf{a.b} = 2\mathbf{i.i} + (-\mathbf{k}).(-\mathbf{k}) = 2 + 1 = 3$$
$$a = \sqrt{(2^2 + 1^2 + 1^2)} = \sqrt{6}$$
$$b = \sqrt{(1^2 + 1^2)} = \sqrt{2}$$
$$\text{ab} = \sqrt{6}\ \text{ x }\ \sqrt{2} = \sqrt{(12)} = 2\sqrt{3}$$

Putting in the expression of cosine, we have :

$$\cos\theta = \frac{\mathbf{a.b}}{\text{ab}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} = \cos 30^\circ$$
$$\theta = 30^\circ$$

Condition of perpendicular vectors

Example:
Problem : Sum and difference of two vectors **a** and **b** are perpendicular to each other. Find the relation between two vectors.
Solution : The sum **a**+**b** and difference **a-b** are perpendicular to each other. Hence, their dot product should evaluate to zero.
Sum and difference of two vectors



Sum and difference of two vectors are perpendicular to each other.

$$(\mathbf{a} + \mathbf{b}).(\mathbf{a} - \mathbf{b}) = 0$$

Using distributive property,

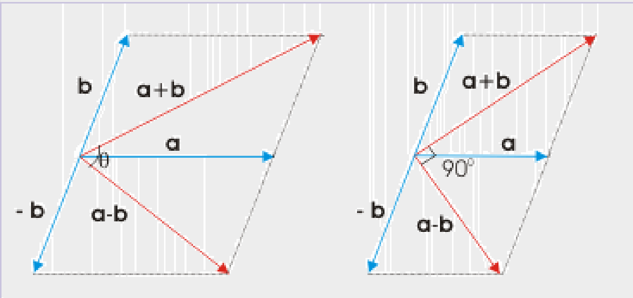
$$\mathbf{a.a} - \mathbf{a.b} + \mathbf{b.a} - \mathbf{b.b} = 0$$

Using commutative property, **a.b** = **b.a**, Hence,

$$\mathbf{a.a} - \mathbf{b.b} = 0$$
$$a^2 - b^2 = 0$$
$$a = b$$

It means that magnitudes of two vectors are equal. See figure below for enclosed angle between vectors, when vectors are equal :

Sum and difference of two vectors



Sum and difference of two vectors are perpendicular to each other, when vectors are equal.

Component as scalar product

Example:
Problem : Find the components of vector 2**i** + 3**j** along the direction **i**+**j**.
Solution : The component of a vector "**a**" in a direction, represented by unit vector "**n**" is given by dot product :

$$a_n = \mathbf{a.n}$$

Thus, it is clear that we need to find the unit vector in the direction of **i**+**j**. Now, the unit vector in the direction of the vector is :

$$n = \frac{\mathbf{i+j}}{|\mathbf{i+j}|}$$

Here,

$$\left|\mathbf{i}+\mathbf{j}\right| = \sqrt{\left(1^2+1^2\right)} = \sqrt{2}$$

Hence,

$$n = \frac{1}{\sqrt{2}} \text{ x } \left(\mathbf{i}+\mathbf{j}\right)$$

The component of vector 2**i** + 3**j** in the direction of “**n**” is :

$$a_n = \mathbf{a.n} = \left(2\mathbf{i}+3\mathbf{j}\right)\mathbf{.} \frac{1}{\sqrt{2}} \text{ x } \left(\mathbf{i}+\mathbf{j}\right)$$

$$\Rightarrow a_n = \frac{1}{\sqrt{2}} \text{ x } \left(2\mathbf{i}+3\mathbf{j}\right)\mathbf{.}\left(\mathbf{i}+\mathbf{j}\right)$$

$$\Rightarrow a_n = \frac{1}{\sqrt{2}} \text{ x } \left(2 \text{ x } 1 + 3 \text{ x } 1\right)$$

$$\Rightarrow a_n = \frac{5}{\sqrt{2}}$$

Nature of scalar product

Example:
Problem : Verify vector equality **B** = **C**, if **A.B** = **A.C**.
Solution : The given equality of dot products is :

$$\mathbf{A.B} = \mathbf{A.C}$$

The equality will result if **B** = **C**. We must, however, understand that dot product is not a simple algebraic product of two numbers (read magnitudes). The angle between two vectors plays a role in determining the magnitude of the dot product. Hence, it is entirely possible that vectors **B** and **C** are different yet their dot products with common vector A are equal.
We can attempt this question mathematically as well. Let θ_1 and θ_2 be the angles for first and second pairs of dot products. Then,

$$\mathbf{A.B} = \mathbf{A.C}$$

$$\text{AB } \cos\theta_1 = \text{AC } \cos\theta_2$$

If $\theta_1 = \theta_2$, then $B = C$. However, if $\theta_1 \neq \theta_2$, then $B \neq C$.

Scalar product of a vector with itself

Example:

Problem : If |**a** + **b**| = |**a** – **b**|, then find the angle between vectors **a** and **b**.

Solution : A question that involves modulus or magnitude of vector can be handled in specific manner to find information about the vector (s). The specific identity that is used in this circumstance is :

$$\mathbf{A}.\,\mathbf{A} = A^2$$

We use this identity first with the sum of the vectors (**a**+**b**),

$$\left(\mathbf{a} + \mathbf{b}\right).\left(\mathbf{a} + \mathbf{b}\right) = |\mathbf{a} + \mathbf{b}|^2$$

Using distributive property,

$$\Rightarrow \mathbf{a}.\,\mathbf{a} + \mathbf{b}.\,\mathbf{a} + \mathbf{a}.\,\mathbf{b} + \mathbf{b}.\,\mathbf{b} = a^2 + b^2 + 2ab\cos\theta = |\mathbf{a} + \mathbf{b}|^2$$
$$\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = a^2 + b^2 + 2ab\cos\theta$$

Similarly, using the identity with difference of the vectors (a-b),

$$\Rightarrow |\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab\cos\theta$$

It is, however, given that :

$$\Rightarrow |\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$$

Squaring on either side of the equation,

$$\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$$

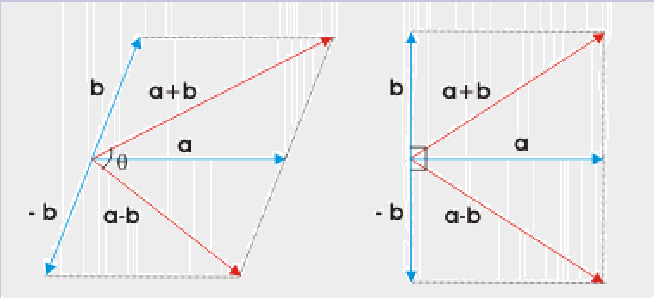
Putting the expressions,

$$\Rightarrow a^2 + b^2 + 2ab\cos\theta = a^2 + b^2 - 2ab\cos\theta$$
$$\Rightarrow 4ab\cos\theta = 0$$
$$\Rightarrow \cos\theta = 0$$
$$\Rightarrow \theta = 90^\circ$$

Note : We can have a mental picture of the significance of this result. As given, the magnitude of sum of two vectors is equal to the magnitude of difference of two vectors. Now, we know that difference of vectors is similar to vector sum with one exception

that one of the operand is rendered negative. Graphically, it means that one of the vectors is reversed.
Reversing one of the vectors changes the included angle between two vectors, but do not change the magnitudes of either vector. It is, therefore, only the included angle between the vectors that might change the magnitude of resultant. In order that magnitude of resultant does not change even after reversing direction of one of the vectors, it is required that the included angle between the vectors is not changed. This is only possible, when included angle between vectors is 90°. See figure.

Sum and difference of two vectors



Magnitudes of Sum and difference of two vectors are same when vectors at right angle to each other.

Example:

Problem : If **a** and **b** are two non-collinear unit vectors and |**a** + **b**| = √3, then find the value of expression :

$$(\mathbf{a}-\mathbf{b}).\,(2\mathbf{a}+\mathbf{b})$$

Solution : The given expression is scalar product of two vector sums. Using distributive property we can expand the expression, which will comprise of scalar product of two vectors **a** and **b**.

$$\left(\mathbf{a}-\mathbf{b}\right).\left(2\mathbf{a}+\mathbf{b}\right)=2\mathbf{a}.\,\mathbf{a}+\mathbf{a}.\,\mathbf{b}-\mathbf{b}.\,2\mathbf{a}+\left(-\mathbf{b}\right).\left(-\mathbf{b}\right)=2a^2-\mathbf{a}.\,\mathbf{b}-b^2$$

$$\Rightarrow\left(\mathbf{a}-\mathbf{b}\right).\left(2\mathbf{a}+\mathbf{b}\right)=2a^2-b^2-ab\cos\theta$$

We can evaluate this scalar product, if we know the angle between them as magnitudes of unit vectors are each 1. In order to find the angle between the vectors, we use the identity,

$$\mathbf{A.A} = A^2$$

Now,

$$|\mathbf{a}+\mathbf{b}|^2 = \left(\mathbf{a}+\mathbf{b}\right).\left(\mathbf{a}+\mathbf{b}\right) = a^2 + b^2 + 2ab\cos\theta = 1 + 1 + 2 \text{ x } 1 \text{ x } 1 \text{ x } \cos\theta$$

$$\Rightarrow |\mathbf{a}+\mathbf{b}|^2 = 2 + 2\cos\theta$$

It is given that :

$$|\mathbf{a}+\mathbf{b}|^2 = (\sqrt{3})^2 = 3$$

Putting this value,

$$\Rightarrow 2\cos\theta = |\mathbf{a}+\mathbf{b}|^2 - 2 = 3 - 2 = 1$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$
$$\Rightarrow \theta = 60^\circ$$

Using this value, we now proceed to find the value of given identity,

$$\left(\mathbf{a}-\mathbf{b}\right).\left(2\mathbf{a}+\mathbf{b}\right) = 2a^2 - b^2 - ab\cos\theta = 2 \text{ x } 1^2 - 1^2 - 1 \text{ x } 1 \text{ x } \cos 60^\circ$$

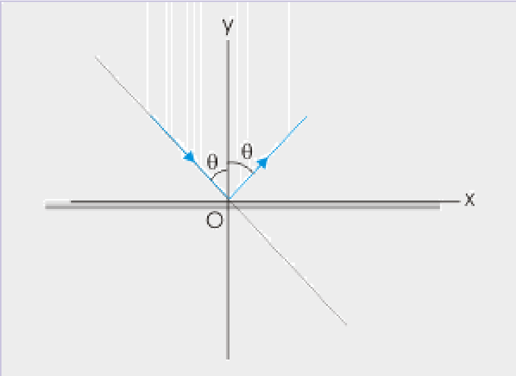
$$\Rightarrow \left(\mathbf{a}-\mathbf{b}\right).\left(2\mathbf{a}+\mathbf{b}\right) = \frac{1}{2}$$

Evaluation of dot product

Example:
Problem : In an experiment of light reflection, if **a**, **b** and **c** are the unit vectors in the direction of incident ray, reflected ray and normal to the reflecting surface, then prove that :

$$\Rightarrow \mathbf{b} = \mathbf{a} - 2(\mathbf{a.c})\mathbf{c}$$

Solution : Let us consider vectors in a coordinate system in which “x” and “y” axes of the coordinate system are in the direction of reflecting surface and normal to the reflecting surface respectively as shown in the figure.
Reflection



Angle of incidence is equal to angle of reflection.

We express unit vectors with respect to the incident and reflected as :

$$\mathbf{a} = \sin\theta\mathbf{i} - \cos\theta\mathbf{j}$$
$$\mathbf{b} = \sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

Subtracting first equation from the second equation, we have :

$$\Rightarrow \mathbf{b} - \mathbf{a} = 2\cos\theta\mathbf{j}$$
$$\Rightarrow \mathbf{b} = \mathbf{a} + 2\cos\theta\mathbf{j}$$

Now, we evaluate dot product, involving unit vectors :

$$\mathbf{a.\,c} = 1 \ \text{x} \ 1 \ \text{x} \ \cos(180^\circ - \theta) = -\cos\theta$$

Substituting for cosθ, we have :

$$\Rightarrow \mathbf{b} = \mathbf{a} - 2(\mathbf{a.\,c})\mathbf{c}$$

Vector (cross) product
Vector multiplication provides concise and accurate representation of natural laws, which involve vectors.

The cross product of two vectors **a** and **b** is a third vector. The magnitude of the vector product is given by the following expression :
Equation:

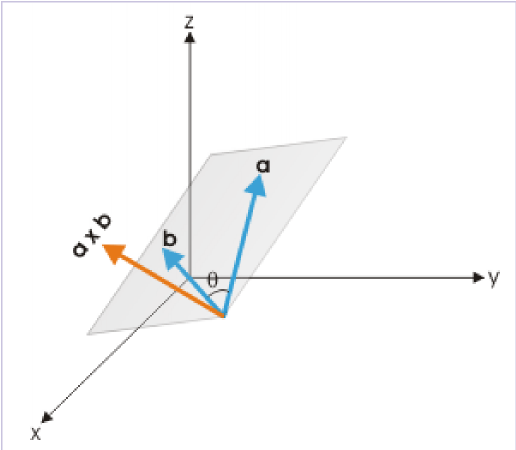
$$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = ab\sin\theta$$

where θ is the smaller of the angles between the two vectors. It is important to note that vectors have two angles θ and 2π - θ. We should use the smaller of the angles as sine of θ and 2π - θ are different.

If "**n**" denotes unit vector in the direction of vector product, then
Equation:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = ab\sin\theta\,\mathbf{n}$$

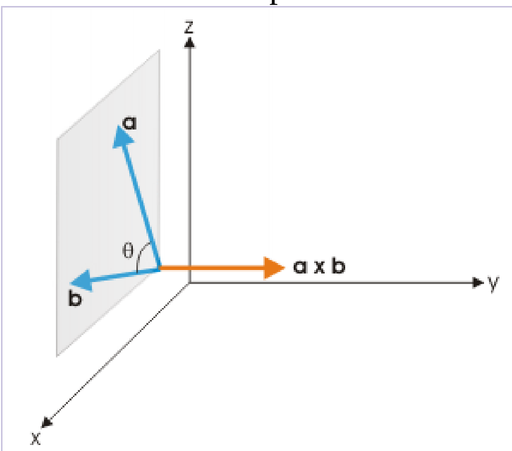
Direction of vector product

The two vectors (**a** and **b**) define an unique plane. The vector product is perpendicular to this plane defined by the vectors as shown in the figure below. The most important aspect of the direction of cross product is that it is independent of the angle, θ, enclosed by the vectors. The enclosed angle , θ, only impacts the magnitude of the cross product and not its direction.
Direction of vector product


Incidentally, the requirement for determining direction suits extremely well with rectangular coordinate system. We know that rectangular coordinate system comprises of

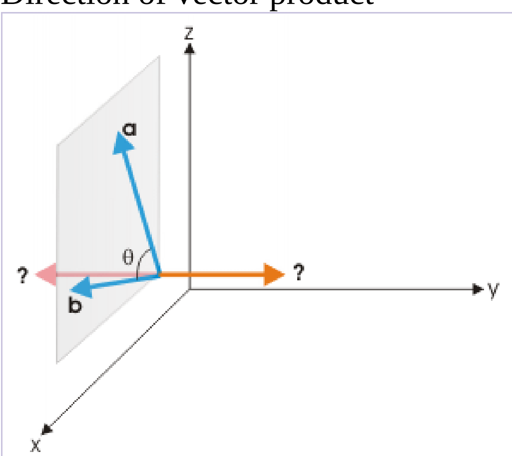
three planes, which are at right angles to each other. It is, therefore, easier if we orient our coordinate system in such a manner that vectors lie in one of the three planes defined by the rectangular coordinate system. The cross product is, then, oriented in the direction of axis, perpendicular to the plane of vectors.

Direction of vector product



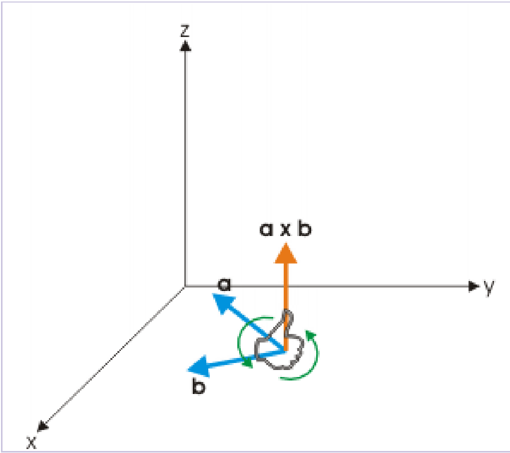
As a matter of fact, the direction of vector product is not yet actually determined. We can draw the vector product perpendicular to the plane on either of the two sides. For example, the product can be drawn either along the positive direction of y – axis or along the negative direction of y-axis (See Figure below).

Direction of vector product



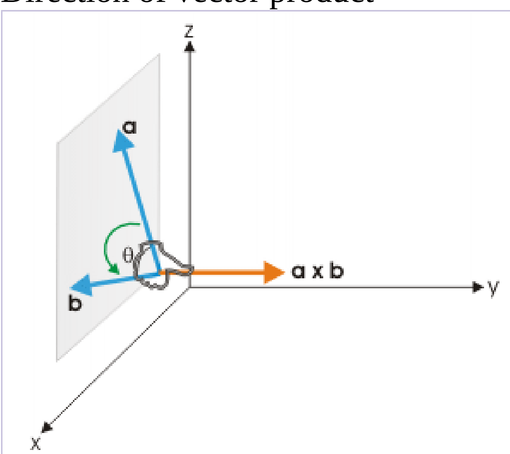
The direction of the vector product, including which side of the plane, is determined by right hand rule for vector products. According to this rule, we place right fist such that the curl of the fist follows as we proceed from the first vector, **a**, to the second vector, **b**. The stretched thumb, then gives the direction of vector product.

Direction of vector product



When we apply this rule to the case discussed earlier, we find that the vector product is in the positive y – direction as shown below :

Direction of vector product



Here, we notice that we move in the anti-clockwise direction as we move from vector, **a**, to vector, **b**, while looking at the plane formed by the vectors. This fact can also be used to determine the direction of the vector product. If the direction of movement is anticlockwise, then the vector product is directed towards us; otherwise the vector product is directed away on the other side of the plane.

It is important to note that the direction of cross product can be on a particular side of the plane, depending upon whether we take the product from **a** to **b** or from **b** to **a**. This implies :

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

Thus, vector product is not commutative like vector addition. It can be inferred from the discussion of direction that change of place of vectors in the sequence of cross product actually changes direction of the product such that :

Equation:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

Values of cross product

The value of vector product is maximum for the maximum value of sinθ. Now, the maximum value of sine is sin 90° = 1. For this value, the vector product evaluates to the product of the magnitude of two vectors multiplied. Thus maximum value of cross product is :

Equation:

$$(\mathbf{a} \times \mathbf{b})_{\text{max}} = ab$$

The vector product evaluates to zero for θ = 0° and 180° as sine of these angles are zero. These results have important implication for unit vectors. The cross product of same unit vector evaluates to 0.

Equation:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

The cross products of combination of different unit vectors evaluate as :

Equation:

$$\begin{gathered}\mathbf{i} \times \mathbf{j} = \mathbf{k}; \mathbf{j} \times \mathbf{k} = \mathbf{i}; \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}; \mathbf{k} \times \mathbf{j} = -\mathbf{i}; \mathbf{i} \times \mathbf{k} = -\mathbf{j}\end{gathered}$$

There is a simple rule to determine the sign of the cross product. We write the unit vectors in sequence **i**,**j**,**k**. Now, we can form pair of vectors as we move from left to right like **i** x **j**, **j** x **k** and right to left at the end like **k** x **i** in cyclic manner. The cross products of these pairs result in the remaining unit vector with positive sign. Cross products of other pairs result in the remaining unit vector with negative sign.

Cross product in component form

Two vectors in component forms are written as :

$$\begin{gathered}\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \\ \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}\end{gathered}$$

In evaluating the product, we make use of the fact that multiplication of the same unit vectors gives the value of 0, while multiplication of two different unit vectors result in

remaining vector with appropriate sign. Finally, the vector product evaluates to vector terms :

Equation:

$$\begin{aligned}&\mathbf{a}\times\mathbf{b}=(a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})\times(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k})\\&\Rightarrow\mathbf{a}\times\mathbf{b}=a_x\mathbf{i}\times b_y\mathbf{j}+a_x\mathbf{i}\times b_z\mathbf{k}+a_y\mathbf{j}\times b_x\mathbf{i}+a_y\mathbf{j}\times b_z\mathbf{k}+a_z\mathbf{k}\times b_x\mathbf{i}+a_z\mathbf{k}\times b_y\mathbf{j}\\&\Rightarrow\mathbf{a}\times\mathbf{b}=a_xb_x\mathbf{k}-a_xb_z\mathbf{j}-a_yb_x\mathbf{k}+a_yb_z\mathbf{i}+a_zb_x\mathbf{j}-a_zb_y\mathbf{i}\\&\Rightarrow\mathbf{a}\times\mathbf{b}=(a_yb_z-a_zb_y)\mathbf{i}+(a_zb_x-a_xb_z)\mathbf{j}+(a_xb_y-a_yb_x)\mathbf{k}\end{aligned}$$

Evidently, it is difficult to remember above expression. If we know to expand determinant, then we can write above expression in determinant form, which is easy to remember.

Equation:

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}$$

Exercise:

Problem: If **a** = 2**i** + 3**j** and **b** = -3**i** – 2**j**, find **A** x **B**.

Solution:

$$\mathbf{a}\times\mathbf{b}=(2\mathbf{i}+3\mathbf{j})\times(-3\mathbf{i}-2\mathbf{j})$$

Neglecting terms involving same unit vectors, we expand the multiplication algebraically as :

$$\begin{aligned}&\mathbf{a}\times\mathbf{b}=(2\mathbf{i})\times(-2\mathbf{j})+(3\mathbf{j})\times(-3\mathbf{i})\\&\Rightarrow\mathbf{a}\times\mathbf{b}=-4\mathbf{k}+9\mathbf{k}=5\mathbf{k}\end{aligned}$$

Exercise:

Problem: Consider the magnetic force given as :

F = q (**v** x **B**)

Given q = 10^{-6} C, **v** = (3**i** + 4**j**) m/s, B = 1**i** Tesla. Find the magnetic force.

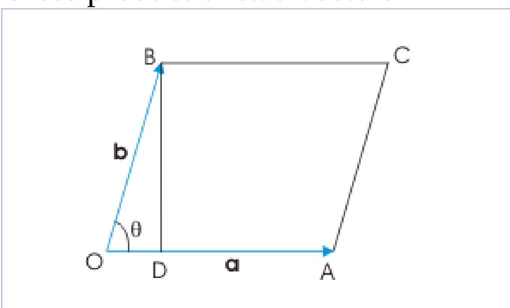
Solution:

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) = 10^{-6}\{(3\mathbf{i} + 4\mathbf{j}) \times (-1\mathbf{i})\}$$
$$\Rightarrow \mathbf{F} = 4 \text{ x } 10^{-6}\mathbf{k}$$

Geometric meaning vector product

In order to interpret the geometric meaning of the cross product, let us draw two vectors by the sides of a parallelogram as shown in the figure. Now, the magnitude of cross product is given by :

Cross product of two vectors



Two vectors are represented by two sides of a parallelogram.

$$|\mathbf{a} \times \mathbf{b}| = ab\sin\theta$$

We drop a perpendicular BD from B on the base line OA as shown in the figure. From ΔOAB,

$$b\sin\theta = \mathrm{OB}\sin\theta = \mathrm{BD}$$

Substituting, we have :

$$|\mathbf{a} \times \mathbf{b}| = \mathrm{OA} \;\text{ x }\; \mathrm{BD} = \text{Base} \;\text{ x }\; \text{Height} = \text{Area of parallelogram}$$

It means that the magnitude of cross product is equal to the area of parallelogram formed by the two vectors. Thus,

Equation:

$$\text{Area of parallelogram} = |\mathbf{a} \times \mathbf{b}|$$

Since area of the triangle OAB is half of the area of the parallelogram, the area of the triangle formed by two vectors is :
Equation:

$$\text{Area of triangle} = \frac{1}{2}\ \text{x}\ \left|\mathbf{a} \times \mathbf{b}\right|$$

Attributes of vector (cross) product

In this section, we summarize the properties of cross product :

1: Vector (cross) product is not commutative

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

A change of sequence of vectors results in the change of direction of the product (vector) :

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

The inequality resulting from change in the order of sequence, denotes "anti-commutative" nature of vector product as against scalar product, which is commutative.

Further, we can extend the sequence to more than two vectors in the case of cross product. This means that vector expressions like **a x b x c** is valid. Ofcourse, the order of vectors in sequence will impact the ultimate product.

2: Distributive property of cross product :

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

3: The magnitude of cross product of two vectors can be obtained in either of the following manner :

$$|\mathbf{a} \times \mathbf{b}| = ab \sin\theta$$

or,

$$|\mathbf{a} \times \mathbf{b}| = a\ \text{ x }\ (b \sin\theta)$$
$$\Rightarrow |\mathbf{a} \times \mathbf{b}| = a\ \text{ x }\ \text{component of } \mathbf{b} \text{ in the direction perpendicular to vector } \mathbf{a}$$

or,

$$|\mathbf{a} \times \mathbf{b}| = b \text{ x } (a \sin\theta)$$
$$\Rightarrow |\mathbf{b} \times \mathbf{a}| = a \text{ x } \text{component of } \mathbf{a} \text{ in the direction perpendicular to vector } \mathbf{b}$$

4: Vector product in component form is :

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

5: Unit vector in the direction of cross product

Let "**n**" be the unit vector in the direction of cross product. Then, cross product of two vectors is given by :

$$\mathbf{a} \times \mathbf{b} = ab\sin\theta\,\mathbf{n}$$

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a} \times \mathbf{b}|\,\mathbf{n}$$

$$\mathbf{n} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

6: The condition of two parallel vectors in terms of cross product is given by :

$$\mathbf{a} \times \mathbf{b} = ab\sin\theta\,\mathbf{n} = ab\sin 0^\circ\,\mathbf{n} = 0$$

If the vectors involved are expressed in component form, then we can write the above condition as :

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = 0$$

Equivalently, this condition can be also said in terms of the ratio of components of two vectors in mutually perpendicular directions :

$$\frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$$

7: Properties of cross product with respect to unit vectors along the axes of rectangular coordinate system are :

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}; \mathbf{j} \times \mathbf{k} = \mathbf{i}; \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}; \mathbf{k} \times \mathbf{j} = -\mathbf{i}; \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Vector product (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the vector product. For this reason, questions are categorized in terms of the characterizing features of the subject matter :

- Condition of parallel vectors
- Unit vector of cross product
- Nature of vector product
- Evaluation of vector product
- Area of parallelogram

Condition of parallel vectors

Example:
Problem : Determine whether vectors 2**i** – **j** + 2**k** and 3**i** – 3**j** + 6**k** are parallel to each other?
Solution : If the two vectors are parallel, then ratios of corresponding components of vectors in three coordinate directions are equal. Here,

$$\frac{a_x}{b_x} = \frac{2}{3}$$
$$\frac{a_y}{b_y} = \frac{1}{3}$$
$$\frac{a_z}{b_z} = \frac{1}{3}$$

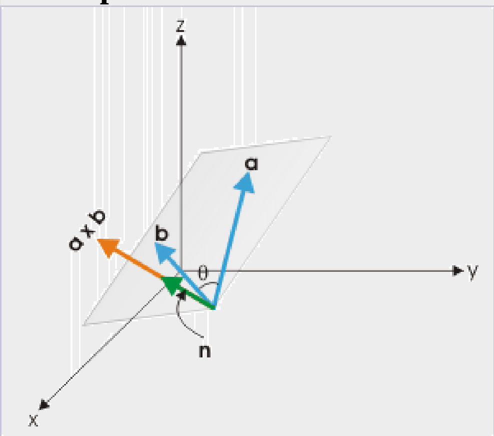
The ratios are, therefore, not equal. Hence, given vectors are not parallel to each other.

Unit vector of cross product

Example:

Problem : Find unit vector in the direction perpendicular to vectors **i** + **j** – 2**k** and 2**i** –**j** + 3**k**.
Solution : We know that cross product of two vectors is perpendicular to each of vectors. Thus, unit vector in the direction of cross product is perpendicular to the given vectors. Now, unit vector of cross product is given by :

Vector product



Unit vector in the direction of vector product.

$$\mathbf{n} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

Here,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{vmatrix}$$

$\Rightarrow \mathbf{a} \times \mathbf{b} = \{1 \text{ x } 3 - (-2 \text{ x } -1)\}\mathbf{i} + \{(2 \text{ x } -2) - 1 \text{ x } 3\}\mathbf{j} + \{(1 \text{ x } -1) - 1 \text{ x } 2\}\mathbf{k}$
$\Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{i} - 7\mathbf{j} - 3\mathbf{k}$

$$\Rightarrow \left|\mathbf{a} \times \mathbf{b}\right| = \sqrt{\left\{1^2 + (-7)^2 + (-3)^2\right\}}$$

$$\Rightarrow \mathbf{n} = \frac{1}{\sqrt{(59)}} \text{ x } \left(\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}\right)$$

Nature of vector product

Example:
Problem : Verify vector equality **B** = **C**, if **AxB** = **AxC**.
Solution : Let θ_1 and θ_2 be the angles for first and second pairs of cross products. Then,

$$\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$$
$$\Rightarrow \mathrm{AB} \sin \theta_1 \mathbf{n}_1 = \mathrm{AC} \sin \theta_2 \mathbf{n}_2$$
$$\Rightarrow B \sin \theta_1 \mathbf{n}_1 = C \sin \theta_2 \mathbf{n}_2$$

It is clear that **B**=**C** is true only when $\sin \theta_1 \mathbf{n}_1 = \sin \theta_2 \mathbf{n}_2$. It is always possible that the angles involved or the directions of cross products are different. Thus, we can conclude that **B** need not be equal to **C**.

Evaluation of vector product

Example:
Problem : If **a.b** = |**axb**| for unit vectors **a** and **b**, then find the angle between unit vectors.
Solution : According to question,

$$\mathbf{a.\,b} = |\mathbf{a} \times \mathbf{b}|$$
$$\Rightarrow \mathrm{ab} \cos \theta = \mathrm{ab} \sin \theta$$
$$\Rightarrow \tan \theta = 1 \;\; \mathrm{x} \;\; 1 \;\; \mathrm{x} \;\; \tan 45^\circ$$
$$\Rightarrow \theta = 45^\circ$$

Example:
Problem : Prove that :

$$|\mathbf{a.\,b}|^2 - |\mathbf{a} \times \mathbf{b}|^2 = a^2 \;\; \mathrm{x} \;\; b^2 \;\; \mathrm{x} \;\; \cos 2\theta$$

Solution : Expanding LHS, we have :

$$|\mathbf{a.\,b}|^2 - |\mathbf{a} \times \mathbf{b}|^2 = (\mathrm{ab} \cos \theta)^2 - (\mathrm{ab} \sin \theta)^2 = a^2 b^2 \left(\cos^2\theta - \sin^2\theta\right)$$
$$\Rightarrow |\mathbf{a.\,b}|^2 - |\mathbf{a} \times \mathbf{b}|^2 == a^2 b^2 \cos 2\theta$$

Area of parallelogram

Example:

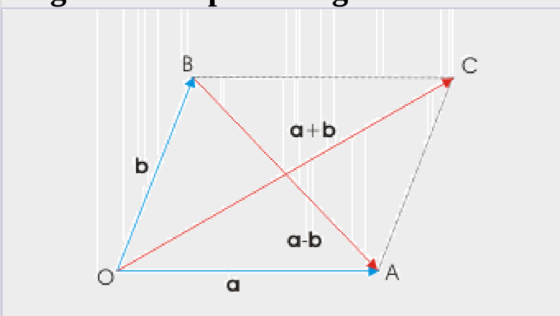
Problem : The diagonals of a parallelogram are represented by vectors 3**i**+**j**+**k** and **i**-**j**-**k**. Find the area of parallelogram.

Solution : The area of parallelogram whose sides are formed by vectors **a** and **b**, is given by :

$$\text{Area} = |\mathbf{a} \times \mathbf{b}|$$

However, we are given in question vectors representing diagonals – not the sides. But, we know that the diagonals are sum and difference of vectors representing sides of a parallelogram. It means that :

Diagonals of a parallelogram



The vectors along diagonals are sum and difference of two vectors representing the sides.

$$\mathbf{a} + \mathbf{b} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

and

$$\mathbf{a} - \mathbf{b} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

Now the vector product of vectors representing diagonals is :

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = \mathbf{a} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} + \mathbf{b} \times (-\mathbf{b}) = -\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a}$$

Using anti-commutative property of vector product,

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = -2\mathbf{a} \times \mathbf{b}$$

Thus,

$$\mathbf{a} \times \mathbf{b} = -\tfrac{1}{2} \text{ x } \left(\mathbf{a} + \mathbf{b}\right) \times \left(\mathbf{a} - \mathbf{b}\right)$$

$$\mathbf{a} \times \mathbf{b} = -\frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$$

$\mathbf{a} \times \mathbf{b} = -\frac{1}{2}\ \text{x}\ \{(1\ \text{x}\ -1 - 1\ \text{x}\ -1)\mathbf{i} + (1\ \text{x}\ 1 - 3\ \text{x}\ -1)\mathbf{j} + (3\ \text{x}\ -1 - 1\ \text{x}\ 1)\mathbf{k}\}$
$\mathbf{a} \times \mathbf{b} = -\frac{1}{2}\ \text{x}\ (4\mathbf{j} - 4\mathbf{k})$
$\mathbf{a} \times \mathbf{b} = -2\mathbf{j} + 2\mathbf{k}$

The volume of the parallelogram is :

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{\left\{(-2)^2 + 2^2\right\}} = 2\sqrt{2}\ \text{units}$$

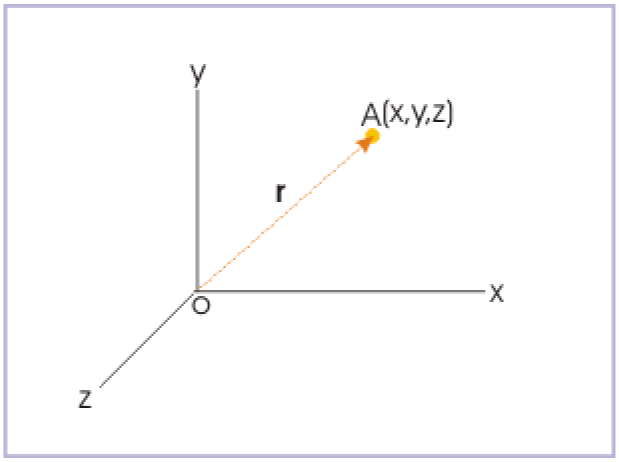
Position vector

Position vector is a convenient mathematical construct to encapsulate the twin ideas of magnitude (how far?) and direction (in which direction?) of the position, occupied by an object.

Position vector

> Position vector is a vector that extends from the reference point to the position of the particle.

Position vector



Position vector is represented by a vector, joining origin to the position of point object

Generally, we take origin of the coordinate system as the reference point.

It is easy to realize that vector representation of position is appropriate, where directional properties of the motion are investigated. As a matter of fact, three important directional attributes of motion, namely displacement, velocity and acceleration are defined in terms of position vectors.

Consider the definitions : the “displacement” is equal to the change in position vector; the “velocity” is equal to the rate of change of position vector with respect to time; and “acceleration” is equal to the rate of change

of velocity with respect to time, which, in turn, is the rate of change of position vector. Thus, all directional attributes of motion is based on the processing of position vectors.

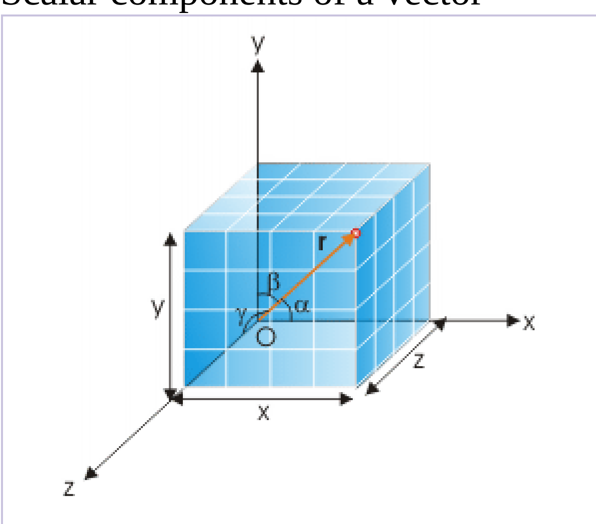
Position Vector in component form

One of the important characteristics of position vector is that it is rooted to the origin of the coordinate system. We shall find that most other vectors associated with physical quantities, having directional properties, are floating vectors and not rooted to a point of the coordinate system like position vector.

Recall that scalar components are graphically obtained by dropping two perpendiculars from the ends of the vector to the axes. In the case of position vector, one of the end is the origin itself. As position vector is rooted to the origin, the scalar components of position vectors in three mutually perpendicular directions of the coordinate system are equal to the coordinates themselves. The scalar components of position vector, **r**, by definition in the designated directions of the rectangular axes are :

$$x = r\cos\alpha$$
$$y = r\cos\beta$$
$$z = r\cos\gamma$$

Scalar components of a vector



Scalar components are equal to
coordiantes of the position

and position vector in terms of components (coordinates) is :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in x, y and z directions.

The magnitude of position vector is given by :

$$r = |\mathbf{r}| = \sqrt{(x^2 + y^2 + z^2)}$$

Example:
Position and distance
Problem : Position (in meters) of a moving particle as a function of time (in seconds) is given by :

$$\mathbf{r} = (3t^2 - 3)\mathbf{i} + (4 - 7t)\mathbf{j} + (-t^3)\mathbf{k}$$

Find the coordinates of the positions of the particle at the start of the motion and at time t = 2 s. Also, determine the linear distances of the positions of the particle from the origin of the coordinate system at these time instants.
Solution : The coordinates of the position are projection of position vector on three mutually perpendicular axes. Whereas linear distance of the position of the particle from the origin of the coordinate system is equal to the magnitude of the position vector. Now,
When t = 0 (start of the motion)

$$\mathbf{r} = (3 \text{ x } 0 - 3)\mathbf{i} + (4 - 7 \text{ x } 0)\mathbf{j} + (-0)\mathbf{k}$$

The coordinates are :

$$x = -3m \text{ and } y = 4m$$

and the linear distance from the origin is :

$$r = \left|\mathbf{r}\right| = \sqrt{\left(\left(-3\right)^2 + 4^2\right)} = \sqrt{25} = 5m$$

When t = 2 s

$$\mathbf{r} = \left(3 \text{ x } 2^2 - 3\right)\mathbf{i} + \left(4 - 7 \text{ x } 2\right)\mathbf{j} + \left(-2^3\right)\mathbf{k} = 9\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$$

The coordinates are :

$$x = 9\text{m}, y = -10m \text{ and } z = -8\text{m}.$$

and the linear distance from the origin is :

$$r = \left|\mathbf{r}\right| = \sqrt{\left(9^2 + \left(-10\right)^2 + \left(-8\right)^2\right)} = \sqrt{245} = 15.65m$$

Motion types and position vector

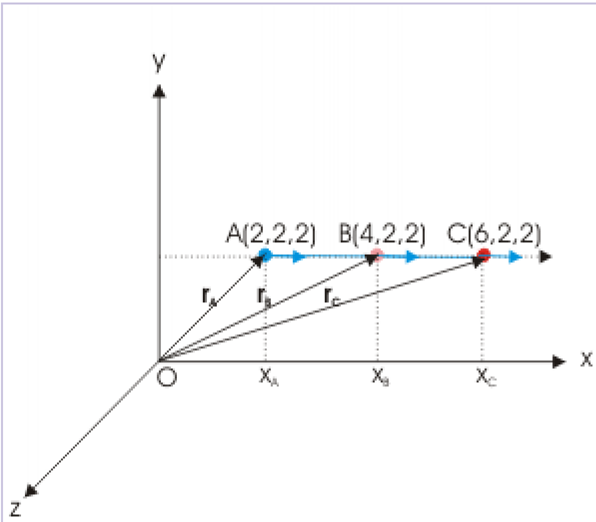
Position is a three dimensional concept, requiring three coordinate values to specify it. Motion of a particle, however, can take place in one (linear) and two (planar) dimensions as well.

In two dimensional motion, two of the three coordinates change with time. The remaining third coordinate is constant. By appropriately choosing the coordinate system, we can eliminate the need of specifying the third coordinate.

In one dimensional motion, only one of the three coordinates is changing with time. Other two coordinates are constant through out the motion. As such, it would be suffice to describe positions of the particle with the values of changing coordinate and neglecting the remaining coordinates.

A motion along x –axis or parallel to x – axis is, thus, described by x - component of the position vector i.e. x – coordinate of the position as shown in the figure. It is only the x-coordinate that changes with time; other two coordinates remain same.

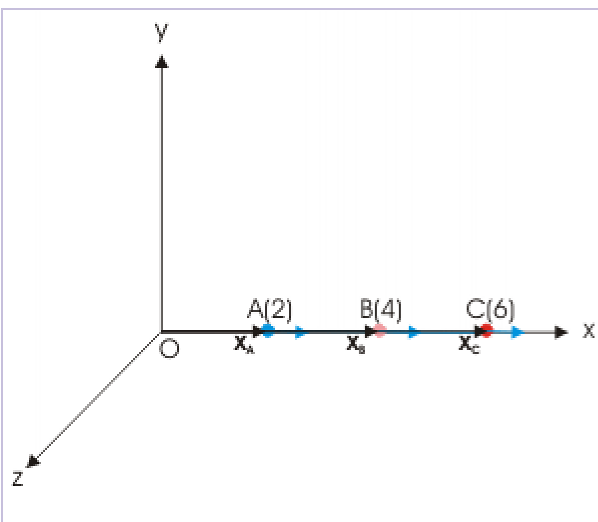
Motion in one dimesnion



The description of one dimensional motion is further simplified by shifting axis to the path of motion as shown below. In this case, other coordinates are individually equal to zero.

$$x = x; y = 0; z = 0$$

Motion in one dimesnion



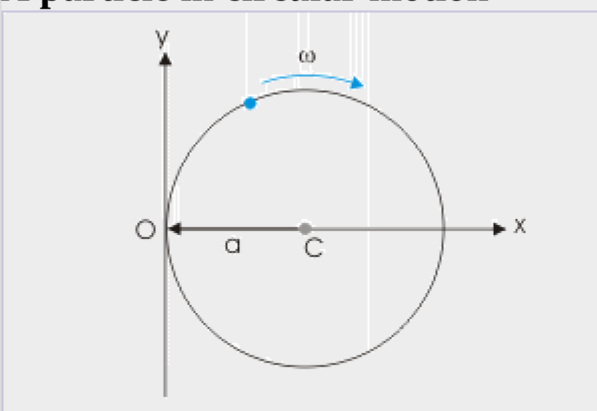
In this case, position vector itself is along x – axis and, therefore, its magnitude is equal to x – coordinate.

Examples

Example:

Problem : A particle is executing motion along a circle of radius “a” with a constant angular speed “ω” as shown in the figure. If the particle is at “O” at t = 0, then determine the position vector of the particle at an instant in xy - plane with "O" as the origin of the coordinate system.

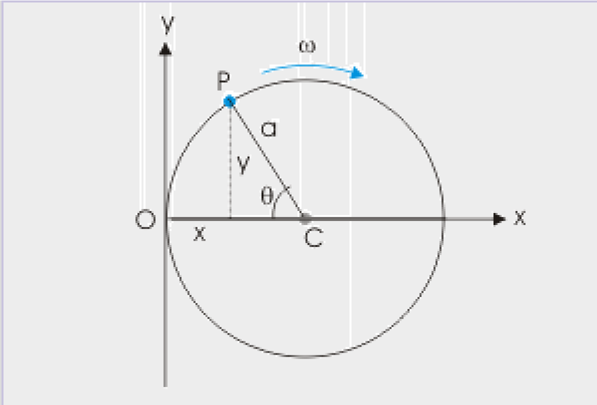
A particle in circular motion



The particle moves with a constant angular velocity.

Solution : Let the particle be at position “P” at a given time “t”. Then the position vector of the particle is :

A particle in circular motion



The particle moves with a constant angular velocity starting from “O” at t = 0.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

Note that "x" and "y" components of position vector is measured from the origin "O". From the figure,

$$y = a\sin\theta = a\sin\omega t$$

It is important to check that as particle moves in clockwise direction, y-coordinate increase in first quarter starting from origin, decreases in second quarter and so on. Similarly, x-coordinate is given by the expression :

$$x = a - a\cos\omega t = a(1 - \cos\omega t)$$

Thus, position vector of the particle in circular motion is :

$$\mathbf{r} = a(1 - \cos\omega t)\mathbf{i} + a\sin\omega t\mathbf{j}$$

Displacement
Motion involves two types of measurements : one which depends on the end points (displacement) and the other which depends on all points(distance) of motion.

Displacement is a measurement of change in position of the particle in motion. Its magnitude and direction are measured by the length and direction of the straight line joining initial and final positions of the particle. Obviously, the length of the straight line between the positions is the shortest distance between the points.

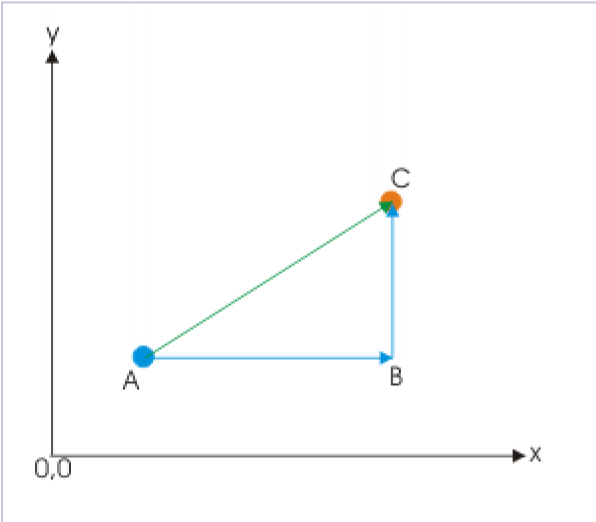
Displacement

> Displacement is the vector extending from initial to final positions of the particle in motion during an interval.

From physical view point, displacement conveys the meaning of shortest distance plus direction of the motion between two time instants or corresponding two positions. Initial and final positions of the point object are the only important consideration for measuring magnitude of displacement. Actual path between two positions has no consequence in so far as displacement is concerned.

The quantum of displacement is measured by the length of the straight line joining two ends of motion. If there is no change in the position at the end of a motion, the displacement is zero.

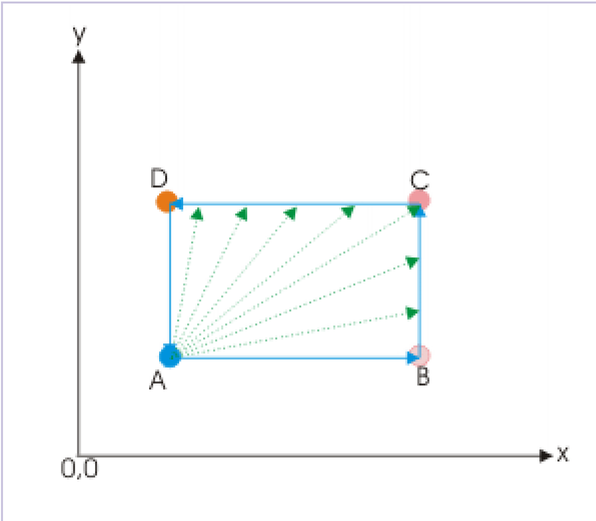
In order to illustrate the underlying concept of displacement, let us consider the motion of a particle from A to B to C. The displacement vector is represented by the vector AC and its magnitude by the length of AC.
Displacement



Once motion has begun, magnitude of displacement may increase or decrease (at a slow, fast or constant rate) or may even be zero, if the object returns to its initial position. Since a body under motion can take any arbitrary path, it is always possible that the end point of the motion may come closer or may go farther away from the initial point. Thus, displacement, unlike distance, may decrease from a given level.

In order to understand the variations in displacement with the progress of motion, let us consider another example of the motion of a particle along the rectangular path from A to B to C to D to A. Magnitude of displacement, shown by dotted vectors, is increasing during motion from A to B to C. Whereas magnitude of displacement is decreasing as the particle moves from C to D to A, eventually being equal to zero, when the particle returns to A.

Displacement



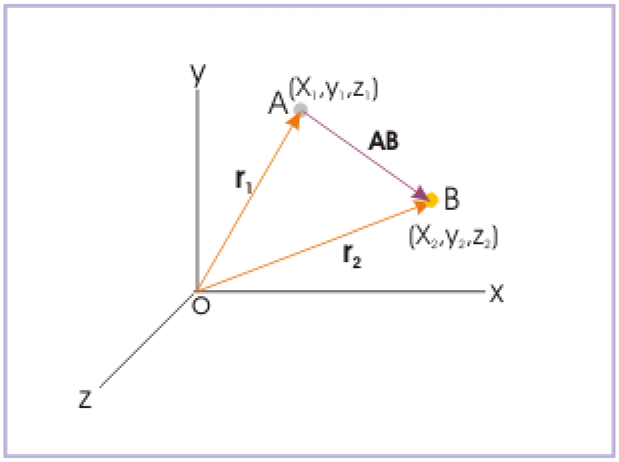
However, displacement is essentially a measurement of length combined with direction. As direction has no dimension, its dimensional formula is also [L] like that of distance; and likewise, its SI measurement unit is ‘meter’.

Displacement and Position vector

We have the liberty to describe displacement vector as an independent vector (**AB**) or in terms of position vectors ($\mathbf{r}_1$ and $\mathbf{r}_2$). The choice depends on the problem in hand. The description, however, is equivalent.

Let us consider that a point object moves from point A (represented by position vector $\mathbf{r}_1$) to point B (represented by position vector $\mathbf{r}_2$) as shown in the figure. Now, using triangle law (moving from O to A to B to O), we have :

Displacement in terms of position vectors



Displacement is equal to the difference between final and initial position vectors

Equation:

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB}$$
$$\mathbf{r}_1 + \mathbf{AB} = \mathbf{r}_2$$
$$\Rightarrow \mathbf{AB} = \mathbf{r}_2 - \mathbf{r}_1 = \Delta\mathbf{r}$$

Thus, displacement is equal to the difference between final and initial position vectors. It is important to note that we obtain the difference between final and initial position vectors by drawing a third vector starting from the tip of the initial position vector and ending at the tip of the final position vector. This approach helps us to quickly draw the vector representing the difference of vectors and is a helpful procedural technique that can be used without any ambiguity. Equivalently, we can express displacement vector in terms of components of position vectors as :

Equation:

$$\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$
$$\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$
$$\Rightarrow \mathbf{AB} = \Delta\mathbf{r} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$
$$\Rightarrow \mathbf{AB} = \Delta\mathbf{r} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta x\mathbf{k}$$

We must emphasize here that position vectors and displacement vector are different vectors quantities. We need to investigate the relation between position vectors and displacement vector - a bit more closely. It is very important to mentally note that the difference of position vectors i.e. displacement,$\Delta\mathbf{r}$, has different directional property to that of the position vectors themselves ($\mathbf{r}_1$ and $\mathbf{r}_2$).

In the figure above, the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are directed along OA and OB respectively, while displacement vector, $\Delta\mathbf{r}$, is directed along AB. This means that the direction of displacement vector need not be same as that of either of the position vectors. Now, what would be the situation, when the motion begins from origin O instead of A? In that case, initial position vector is zero (null vector). Now, let the final position vector be denoted as $\mathbf{r}$. Then

$$\mathbf{r}_1 = 0$$
$$\mathbf{r}_2 = \mathbf{r}$$

and displacement is :
Equation:

$$\Rightarrow \mathbf{AB} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r} - 0 = \mathbf{r}$$

This is a special case, when final position vector itself is equal to the displacement. For this reason, when motion is studied from the origin of reference or origin of reference is chosen to coincide with initial position, then displacement and final position vectors are same and denoted by the symbol, "$\mathbf{r}$".

In general, however, it is the difference between final and initial position vectors, which is equal to the displacement and we refer displacement in terms of change in position vector and use the symbol Δ**r** to represent displacement.

Magnitude of displacement is equal to the absolute value of the displacement vector. In physical sense, the magnitude of displacement is equal to the linear distance between initial and final positions along the straight line joining two positions i.e. the shortest distance between initial and final positions. This value may or may not be equal to the distance along the actual path of motion. In other words, magnitude of displacement represents the minimum value of distance between any two positions.

Example:
Displacement
Question : Consider a person walking from point A to B to C as shown in the figure. Find distance, displacement and magnitude of displacement.

Characteristics of motion : Two dimensional
Solution : The distance covered, s, during the motion from A to C is to the sum of the lengths AB and BC.

$$s = 4 + 3 = 7m$$

Displacement, **AC**, is given as :

$$\mathbf{AC} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$
$$\mathbf{AC} = \Delta\mathbf{r} = (6-2)\mathbf{i} + (5-2)\mathbf{j} = 4\mathbf{i} + 3\mathbf{j}$$

The displacement vector makes an angle with the x – axis given by :

$$\theta = \tan^{-1}\left(3/4\right)$$

and the magnitude of the displacement is :

$$|\mathbf{AB}| = |\mathbf{r}_2 - \mathbf{r}_1|$$
$$\Rightarrow |\mathbf{AB}| = |\Delta\mathbf{r}| = |4\mathbf{i} + 3\mathbf{j}| = \surd\left(4^2 + 3^2\right) = 5 \text{ m}$$

The example above brings out nuances associated with terms used in describing motion. In particular, we see that distance and magnitude of displacement are not equal. This inequality arises due to the path of motion, which may be other than the shortest linear path between initial and final positions.

This means that distance and magnitude of displacement may not be equal. They are equal as a limiting case when particle moves in one direction without reversing direction; otherwise, distance is greater than the magnitude of displacement in most of the real time situation.
Equation:

$$s \geq |\Delta\mathbf{r}|$$

This inequality is important. It implies that displacement is not **distance plus direction** as may loosely be considered. As a matter of fact, displacement is **shortest distance plus direction** . For this reason, we need to avoid representing displacement by the symbol **“s”** as a vector counterpart of scalar distance, represented by “s”. In vector algebra, modulus of a vector, **A**, is represented by its non bold type face letter “A”. Going by this convention, if “**s**” and “s” represent displacement and distance respectively, then s = |**s**|, which is incorrect.

When a body moves in a straight line maintaining its direction (unidirectional linear motion), then magnitude of displacement, |Δ**r**| is equal to distance, "s". Often, this situational equality gives the impression that two quantities are always equal, which is not so. For this reason, we would be careful to write magnitude of displacement by the modulus |Δ**r**| or in terms of displacement vector like |**AB**| and not by "|**s**|".

Example:
Displacement
Question : Position (in meters) of a moving particle as a function of time (in second) is given by :

$$\mathbf{r} = \left(3t^2 - 3\right)\mathbf{i} + \left(4 - 7t\right)\mathbf{j} + \left(-t^3\right)\mathbf{k}$$

Find the displacement in first 2 seconds.
Characteristics of motion : Three dimensional, variable velocity, variable speed
Solution : The position vector at t = 0 and 2 seconds are calculated to identify initial and final positions. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors at t = 0 and t = 2 s.
When t = 0 (start of the motion)

$$\mathbf{r}_1 = (3 \text{ x } 0 - 3)\mathbf{i} + (4 - 7 \text{ x } 0)\mathbf{j} + (-0)\mathbf{k} = -3\mathbf{i} + 4\mathbf{j}$$

When t = 2 s,

$$\mathbf{r}_2 = \left(3 \text{ x } 2^2 - 3\right)\mathbf{i} + \left(4 - 7 \text{ x } 2\right)\mathbf{j} + \left(-2^3\right)\mathbf{k} = 9\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$$

The displacement, Δ**r**, is given by :

$$\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (9\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}) - (-3\mathbf{i} + 4\mathbf{j})$$

$$\Rightarrow \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (9\mathbf{i} - 10\mathbf{j} - 8\mathbf{k} + 3\mathbf{i} - 4\mathbf{j}) = 12\mathbf{i} - 14\mathbf{j} - 8\mathbf{k}$$

Magnitude of displacement is given by :

$$\Rightarrow\left|\mathbf{r}\right| = \sqrt{\left\{12^2 + (-14)^2 + (-8)^2\right\}} = \sqrt{404} = 20.1m$$

Displacement and dimension of motion

We have so far discussed displacement as a general case in three dimensions. The treatment of displacement in one or two dimensions is relatively simplified. The expression for displacement in component form for these cases are given here :

1. Motion in two dimension : Let the motion takes place in the plane determined by x and y axes, then :

$$\Delta\mathbf{r} = \Delta x\mathbf{i} + \Delta y\mathbf{j};\, \Delta z = 0$$

If the initial position of the particle coincides with the origin of reference system, then :

$$\Delta\mathbf{r} = \mathbf{r} = x\mathbf{i} + y\mathbf{j};\, z = 0$$

2. Motion in one dimension : Let the motion takes place along the straight line parallel to x - axis, then :

$$\Delta\mathbf{r} = \Delta x\mathbf{i};\, \Delta y = \Delta z = 0$$

If the initial position of the particle coincides with the origin of reference system, then :

$$\Delta\mathbf{r} = \mathbf{r} = x\mathbf{i};\, y = z = 0$$

Displacement – time plot

Plotting displacement vector requires three axes. Displacement – time plot will, therefore, need a fourth axis for representing time. As such, displacement – time plot can not be represented on a three dimensional

Cartesian coordinate system. Even plotting two dimensional displacement with time is complicated.

One dimensional motion, having only two directions – along or opposite to the positive direction of axis, allows plotting displacement – time graph. One dimensional motion involves only one way of changing direction i.e. the particle under motion can reverse its direction of motion. Any other change of direction is not possible; otherwise the motion would not remain one dimensional motion.

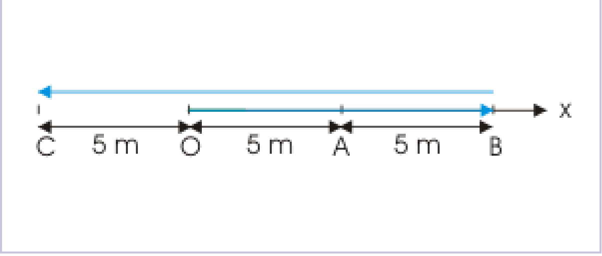
The simplication, in the case of one directional motion, allows us to do away with the need to use vector notation . Instead, the vectors are treated simply as scalars with one qualification that vectors in the direction of chosen reference is considered positive and vectors in the opposite direction to chosen reference is considered negative.

Representation of a displacement vector as a scalar quantity uses following construct :

1. Assign an axis along the motion
2. Assign the origin with the start of motion; It is, however, a matter of convenience and is not a requirement of the construct.
3. Consider displacement in the direction of axis as positive
4. Consider displacement in the opposite direction of axis as negative

To illustrate the construct, let us consider a motion of a ball which transverses from O to A to B to C to O along x-axis as shown in the figure.

Rectilinear motion



A particle moves along a straight line.

The magnitude of displacements (in meters) at various points of motion are :

$$x_1 = \mathrm{OA} = 5$$
$$x_2 = \mathrm{OB} = 10$$
$$x_3 = \mathrm{OC} = -5$$

It is important to note from above data that when origin is chosen to coincide with initial position of the particle, then displacement and position vectors are equal.

The data for displacement as obtained above also reveals that by assigning proper sign to a scalar value, we can represent directional attribute of a vector quantity. In other words, plotting magnitude of displacement in one dimension with appropriate sign would completely represent the displacement vector in both magnitude and direction.

Interpreting change of position

There are different difference terms like Δr, Δ|**r**|, |Δ**r**| and Δ**r**, which denote different aspects of change in position. It might appear trivial, but it is the understanding and ability to distinguish these quantities that will enable us to treat and describe motion appropriately in different dimensions. From the discussion so far, we have realized that the symbol "Δ**r**" denotes displacement, which is equal to the difference in position vectors between "final" and "initial" positions. The meaning of the symbol "|Δ**r**|" follows from the meaning of "Δ**r**" as magnitude of displacement.

As a matter of fact, it is the symbol "Δr", which creates certain "unexpected" confusion or ambiguity – if not handled appropriately. Going by the conventional understanding, we may be tempted to say that "Δr" represents magnitude of displacement and is equal to the absolute value of displacement i.e. |Δ**r**|. The point that we want to emphasize here is that it is not so.

Equation:

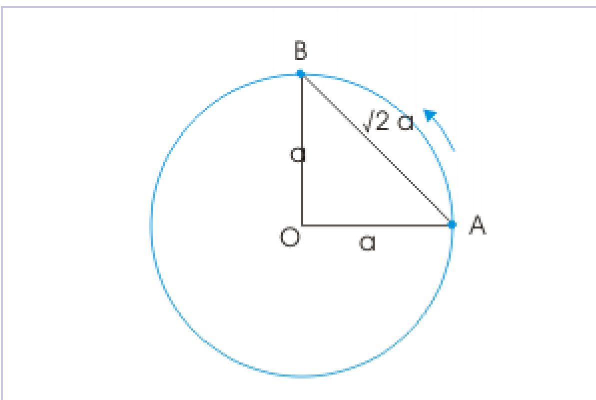
$$\Delta r \neq |\mathbf{\Delta r}|$$

Going by the plain meaning of the symbol, "Δ", we can interpret two terms as :

- Δr = change in the magnitude of position vector, "r"
- |Δ**r**| = magnitude of change in the position vector, "**r**"

Therefore, what we mean by the inequality emphasized earlier is that change in the magnitude of position vector, "r”, is not equal to the magnitude of change in the position vector. In order to appreciate the point, we can consider the case of two dimensional circular motion. Let us consider the motion from point "A" to "B" along a circle of radius "a", as shown in the figure.

Circular motion



A particle transverses a quarter of circle from A to B.

Since radius of the circle remains same, the change in the magnitude of position vector,"r", is zero during the motion. Hence,

$$\Delta r = 0$$

However, the magnitude of displacement during the motion is :

$$\mathrm{AB} = \surd\left(a^2 + a^2\right) = \surd 2a$$

Hence, the magnitude of change in the position vector is :

$$|\Delta\mathbf{r}| = \mathrm{AB} = \surd 2a$$

Clearly,

$$\Delta r \neq |\Delta\mathbf{r}|$$

Further, $|\mathbf{r}|$ represents the magnitude of position vector and is equal to “r” by conventional meaning. Hence,

$$\Delta r = \Delta|\mathbf{r}|$$

Therefore,

$$\Delta|\mathbf{r}| \neq |\Delta\mathbf{r}|$$

This completes the discussion on similarities and differences among the four symbols. But, the question remains why there are differences in the first place. The answer lies in the fact that position vector is a vector quantity with directional property. It means that it can change in either of the following manner :

1. change in magnitude
2. change in direction
3. change in both magnitude and direction

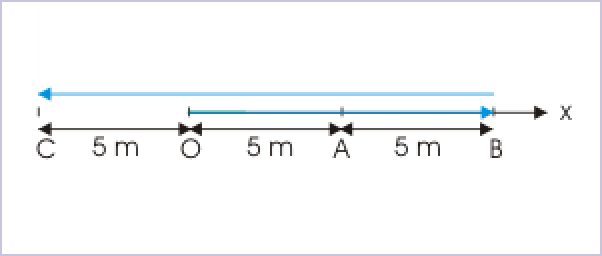
Thus we see that it is entirely possible, as in the case of circular motion, that change in position vector is attributed to the change in direction alone (not the magnitude). In that case, Δr and $|\Delta\mathbf{r}|$ are not same. We can see that such difference in meaning arises due to the consideration of direction. Will this difference persist even in one dimensional motion?

In one dimensional motion, representations of change in position vector and displacement are done with equivalent scalar system. Let us examine the meaning of equivalent scalar terms. Here, the change in the magnitude of position vector "r" is equivalent to the change in the magnitude of position

vector, represented by scalar equivalent "|x|". Also, the magnitude of change in the position vector "**r**" is equivalent to the magnitude of change in the position vector, "x" (note that signed scalar "x" denotes position vector).

Let us consider the case of a rectilinear motion (motion along a straight line) taking place from from A to B to C. The magnitude of change in the position vector, "x", considering "O" as the origin is :

Rectilinear motion



A particle moves along a straight line.

$$|\Delta x| = |x_2 - x_1| = |-5 - 5| = 10\,m$$

Now, the change in the magnitude of position vector, "|x|", is :

$$\Delta |x| = |x_2| - |x_1| = 5 - 5 = 0$$

Thus we see that difference in two terms exist even in one dimensional motion. This is expected also as one dimensional motion can involve reversal of direction as in the case considered above. Hence,

Equation:

$$\Delta |x| \neq |\Delta x|$$

Let us now consider unidirectional one dimensional motion like uniform motion in which velocity is constant and particle moves in only one direction. In this case, this difference disappears and

Equation:

$$\Delta|x| = |\Delta x|$$

The discussion here on this subtle difference is very important as this becomes an important consideration subsequently with velocity and acceleration as well, which are defined in terms of position vector.

Example

Example:
Problem : The displacement (x) of a particle is given by :

$$x = A\sin(\omega t + \theta)$$

At what time from the start of motion is the displacement maximum?
Solution : The displacement (x) depends on the value of sine function. It will be maximum for maximum value of sin(wt + θ). The maximum value of sine function is 1. Hence,

$$\sin\left(\omega t + \theta\right) = 1 = \sin\left(\frac{\pi}{2}\right)$$
$$\Rightarrow \omega t + \theta = \pi 2$$
$$\Rightarrow t = \frac{\pi}{2\omega} - \frac{\theta}{\omega}$$

The motion is oscillating as expression for displacement is sinusoidal. The particle will attain maximum displacement at regular intervals.

Speed

Motion is the change of position with respect to time. Speed quantifies this change in position, but notably without direction. It tells us exactly : how rapidly this change is taking place with respect to time.

Motion

> **Speed is the rate of change of distance with respect to time and is expressed as distance covered in unit time.**

Equation:

$$v = \frac{\Delta s}{\Delta t}$$

Equation:

$$\Rightarrow \Delta \mathrm{s} = v \Delta \mathrm{t}$$

Evaluation of ratio of distance and time for finite time interval is called “average” speed, where as evaluation of the ratio for infinitesimally small time interval, when Δt-->0, is called instantaneous speed. In order to distinguish between average and instantaneous speed, we denote them with symbols v_a and v respectively.

Determination of speed allows us to compare motions of different objects. An aircraft, for example, travels much faster than a motor car. This is an established fact. But, we simply do not know how fast the aircraft is in comparison to the motor car. We need to measure speeds of each of them to state the difference in quantitative terms.

Speed is defined in terms of distance and time, both of which are scalar quantities. It follows that speed is a scalar quantity, having only magnitude and no sense of direction. When we say that a person is pacing at a speed of 3 km/hr, then we simply mean that the person covers 3 km in 1 hour. It is not known, however, where the person is actually heading and in which direction.

Dimension of speed is $\mathbf{LT}_{-1}$ and its SI unit is meter/second (m/s).

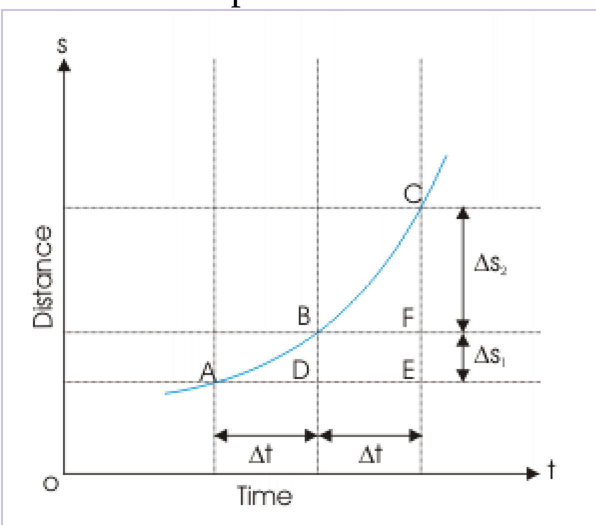
Some values of speed

- Light : 3×10^8 m/s
- Sound : 330 m/s
- Continental drift : 10^{-9} m/s

Distance .vs. time plots

Motion of an object over a period of time may vary. These variations are conveniently represented on a distance - time plot as shown in the figure.

Distance time plot



Distance is given by the vertical segment parallel to the axis representing distance.

The figure above displays distance covered in two equal time intervals. The vertical segment DB and FC parallel to the axis represents distances covered in the two equal time intervals. The distance covered in two equal time Δt intervals may not be equal as average speeds of the object in the two equal time intervals may be different.

$$s_1 = v_1 \Delta \text{ t} = \text{DB}$$

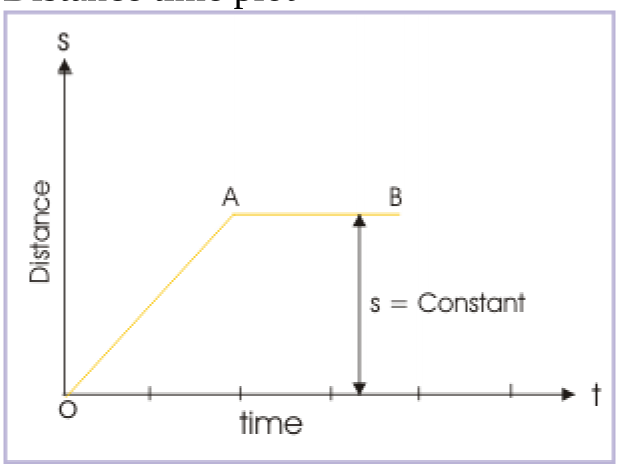
$$s_2 = v_2 \Delta \text{ t} = \text{FC}$$

and

$$\text{DB} \ne \text{FC}$$

The distance - time plot characterizes the nature of distance. We see that the plot is always drawn in the first quadrant as distance can not be negative. Further, distance – time plot is ever increasing during the motion. It means that the plot can not decrease from any level at a given instant. When the object is at rest, the distance becomes constant and plot is a horizontal line parallel to time axis. Note that the portion of plot with constant speed does not add to the distance and the vertical segment representing distance remains constant during the motion.

Distance time plot



Static condition is represented by a horizontal section on distance - time plot.

Average speed

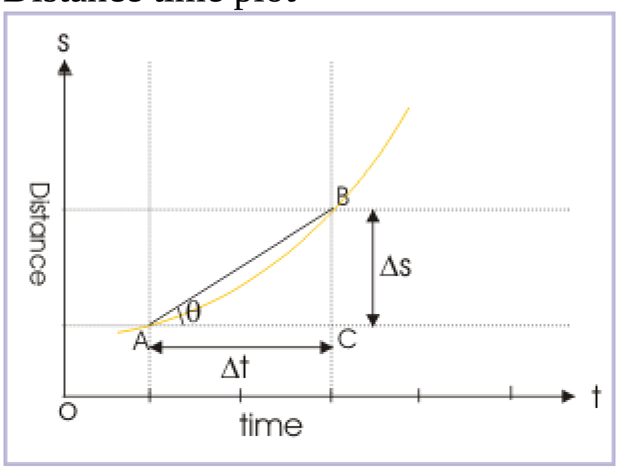
Average speed, as the name suggests, gives the overall view of the motion. It does not, however, give the details of motion. Let us take the example of the school bus. Ignoring the actual, let us consider that the average speed of the journey is 50 km/ hour. This piece of information about speed is very useful in planning the schedule, but the information is not complete as far as the motion is concerned. The school bus could have stopped at predetermined stoppages and crossings, besides traveling at different speeds for variety of reasons. Mathematically,

Equation:

$$v_a = \frac{\Delta s}{\Delta t}$$

On a distance time plot, average speed is equal to the slope of the the straight line, joining the end points of the motion, makes with the time axis. Note that average speed is equal to the slope of the chord (AB) and **not** that of the tangent to the curve.

Distance time plot



Distance is equal to the tangent of the angle that the chord of motion makes with time axis.

$$v_a = \tan\theta = \frac{\Delta s}{\Delta t} \quad = \frac{\text{BC}}{\text{AC}}$$

Example:
Average speed
Problem : The object is moving with two different speeds v_1 and v_2 in two equal time intervals. Find the average speed.
Solution : The average speed is given by :

$$v_a = \frac{\Delta s}{\Delta t}$$

Let the duration of each time interval is t. Now, total distance is :

$$\Delta s = v_1 t + v_2 t$$

The total time is :

$$\Delta t = t + t = \text{2t}$$

$$\Rightarrow \quad v_a = \quad \frac{v_1 t + v_2 t}{\text{2t}}$$

$$\Rightarrow \quad v_a = \quad \frac{v_1 + v_2}{2}$$

The average speed is equal to the arithmetic mean of two speeds.

Exercise:

Problem:

The object is moving with two different speeds v_1 and v_1 in two equal intervals of distances. Find the average speed.

Solution:

Solution :
The average speed is given by :

$$v = \frac{\Delta s}{\Delta t}$$

Let "s" be distance covered in each time interval t_1 and t_2. Now,

$$\Delta s = s + s = 2\mathrm{s}$$

$$\Delta t = t_1 + t_2 = \quad \frac{s}{v_1} + \quad \frac{s}{v_2}$$

$$\Rightarrow \quad v_a = \quad \frac{2v_1 v_2}{v_1 + v_2}$$

The average speed is equal to the harmonic mean of two velocities.

Instantaneous speed (v)

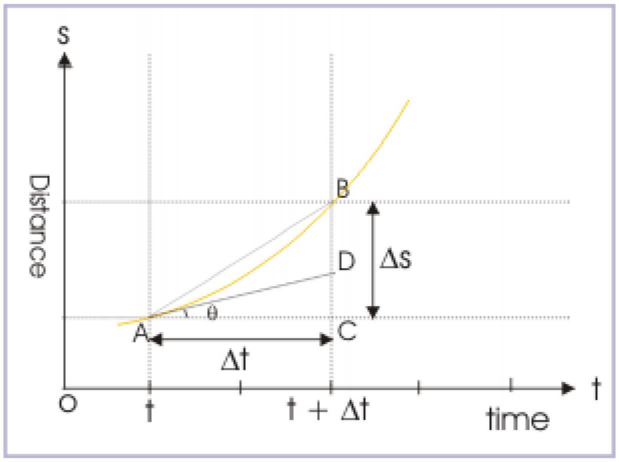
Instantaneous speed is also defined exactly like average speed i.e. it is equal to the ratio of total distance and time interval, but with one qualification that time interval is extremely (infinitesimally) small. This qualification of average speed has important bearing on the value and meaning of speed. The instantaneous speed is the speed at a particular instant of time and may have entirely different value than that of average speed. Mathematically,
Equation:

$$v = \quad \underset{\Delta s \to 0 \Delta s \to}{\text{limit}} \quad \frac{\Delta s}{\Delta t} \quad = \quad \frac{ds}{dt}$$

Where Δs is the distance traveled in time Δt.

As Δt tends to zero, the ratio defining speed becomes finite and equals to the first derivative of the distance. The speed at the moment ‘t’ is called the instantaneous speed at time ‘t’.

On the distance - time plot, the speed is equal to the slope of the tangent to the curve at the time instant ‘t’. Let A and B points on the plot corresponds to the time t and $t + \Delta t$ during the motion. As Δt approaches zero, the chord AB becomes the tangent AC at A. The slope of the tangent equals ds/dt, which is equal to the instantaneous speed at 't'.
Speed time plot



Instantaneous speed is equal to the slope of the tangent at given instant.

$$v = \tan\theta \quad = \quad \frac{\mathrm{DC}}{\mathrm{AC}} \quad = \quad \frac{\mathrm{ds}}{\mathrm{dt}}$$

Speed - time plot

The distance covered in the small time period dt is given by :

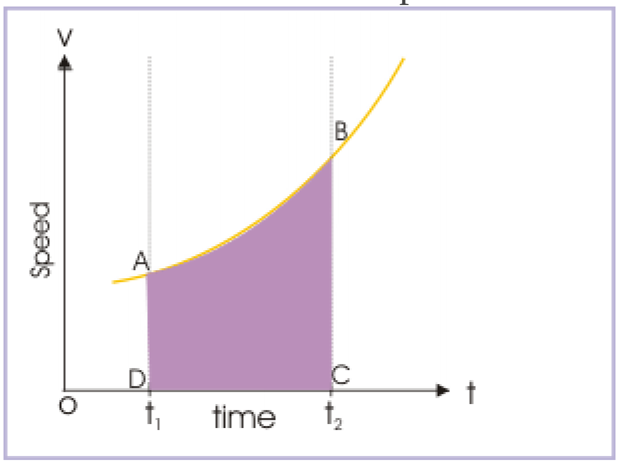
$\mathrm{ds} = v\ \mathrm{dt}$

Integrating on both sides between time intervals t_1 and t_2 ,
Equation:

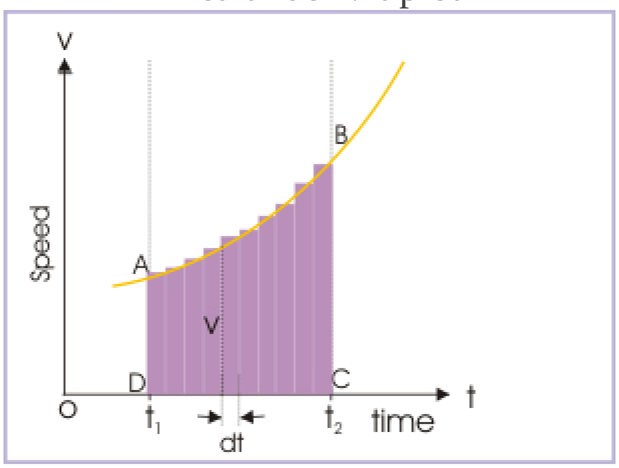
$$s = \int_{t_1}^{t_2} v \,\mathrm{d}\, t$$

The right hand side of the integral represents an area on a plot drawn between two variables, speed (v) and time (t). The area is bounded by (i) v-t curve (ii) two time ordinates t_1 and t_2 and (iii) time (t) axis as shown by the shaded region on the plot.

Area under v-t plot



Area under v-t plot



Area under v-t plot gives the distance covered by the object in a given time interval.

Alternatively, we can consider the integral as the sum of areas of small strip of rectangular regions (vxdt), each of which represents the distance covered (ds) in the small time interval (dt). As such, the area under speed - time plot gives the total distance covered in a given time interval.

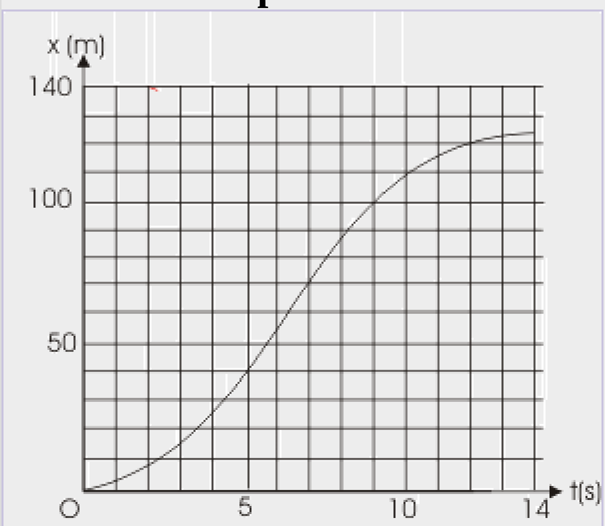
Position - time plot

The position – time plot is similar to distance – time plot in one dimensional motion. In this case, we can assess distance and speed from a position - time plot. Particularly, if the motion is unidirectional i.e. without any reversal of direction, then we can substitute distance by position variable. The example here illustrates this aspect of one-dimensional motion.

Example:

Problem The position – time plot of a particle's motion is shown below.

Position - time plot



Determine :

1. average speed in the first 10 seconds
2. Instantaneous speed at 5 seconds
3. Maximum speed speed
4. The time instant(s), when average speed measured from the beginning of motion equals instantaneous speed

Solution

1. Average speed in the first 10 seconds

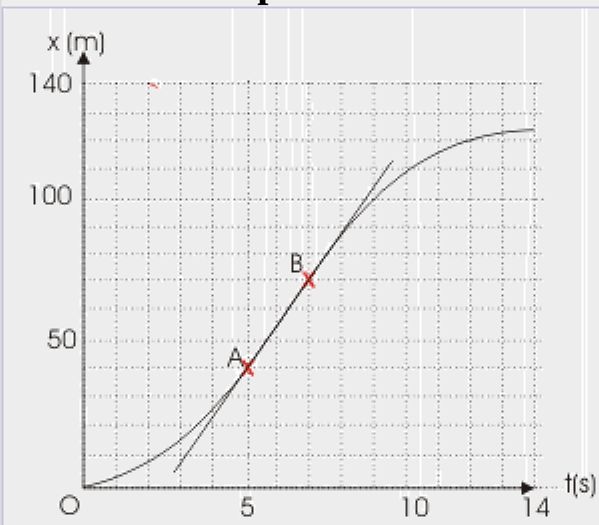
The particle covers a distance of 110 m. Thus, average speed in the first 10 seconds is :

$$v_a = \frac{110}{10} = 11 \text{ m/s}$$

2. Instantaneous speed at 5 second

We draw the tangent at t = 5 seconds as shown in the figure. The tangent coincides the curve between t = 5 s to 7 s. Now,

Position - time plot



$$v = \text{slope of the tangent} = \frac{30}{2} = 15 \text{ m/s}$$

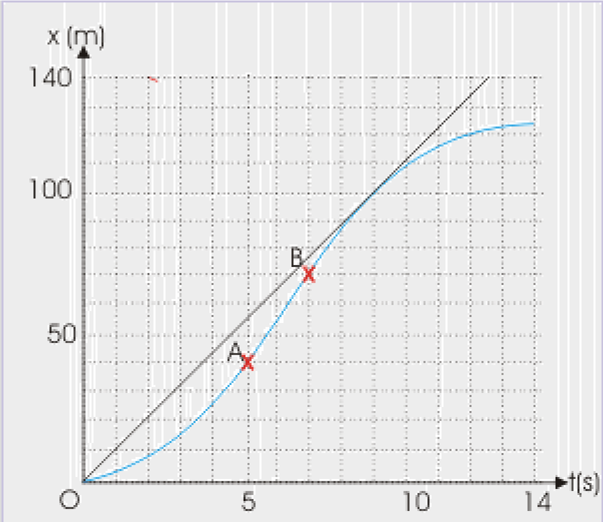
3. Maximum speed

We observe that the slope of the tangent to the curve first increase to become constant between A and B. The slope of the tangent after point B decreases to become almost flat at the end of the motion. It means the maximum speed corresponds to the constant slope between A and B, which is is 15 m/s

4. Time instant(s) when average speed equals instantaneous speed

Average speed is equal to slope of chord between one point to another. On the other hand, speed is equal to slope of tangent at point on the plot. This means that slope of chord between two times is equal to tangent at the end point of motion. Now, as time is measured from the start of motion i.e. from origin, we draw a straight line from the origin, which is tangent to the curve. The only such tangent is shown in the figure below. Now, this straight line is the chord between origin and a point. This line is also tangent to the curve at that point. Thus, average speed equals instantaneous speeds at t = 9 s.

Position - time plot

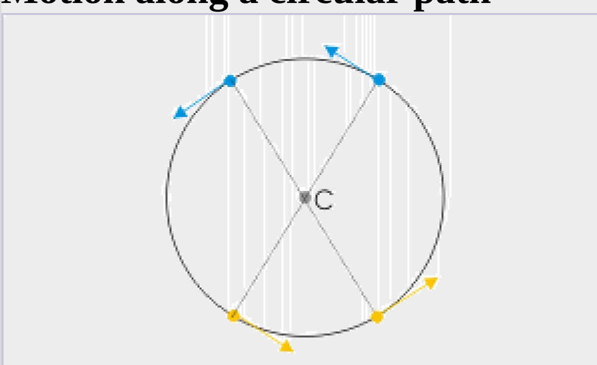


Example:

Problem : Two particles are moving with the same constant speed, but in opposite direction. Under what circumstance will the separation between two remains constant?

Solution : The condition of motion as stated in the question is possible, if particles are at diametrically opposite positions on a circular path. Two particles are always separated by the diameter of the circular path. See the figure below to evaluate the motion and separation between the particles.

Motion along a circular path



Two particles are always separated by the diameter of the circle transversed by the particles.

Velocity

Velocity is the measure of rapidity with which a particle covers shortest distance between initial and final positions, irrespective of the actual path. It also indicates the direction of motion as against speed, which is devoid of this information.

Velocity

> **Velocity is the rate of change of displacement with respect to time and is expressed as the ratio of displacement and time.**

Equation:

$$\mathbf{v} = \frac{\text{Displacement}}{\Delta\text{t}}$$

$$\Rightarrow \quad \text{Displacement} = \mathbf{\Delta v}t$$

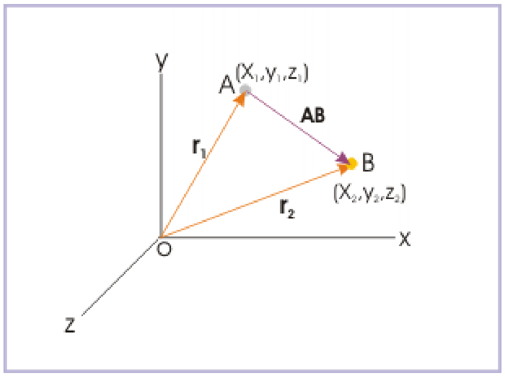
If the ratio of displacement and time is evaluated for finite time interval, we call the ratio "average" velocity, whereas if the ratio is evaluated for infinitesimally small time interval($\Delta t \rightarrow 0$) , then we call the ratio "instantaneous" velocity. Conventionally, we denote average and instantaneous velocities as $\mathbf{v_a}$ and $\mathbf{v}$ respectively to differentiate between the two concepts of velocity.

As against speed, which is defined in terms of distance, velocity is defined in terms of displacement. Velocity amounts to be equal to the multiplication of a scalar (1/Δt) with a vector (displacement). As scalar multiplication of a vector is another vector, velocity is a vector quantity, having both magnitude and direction. The direction of velocity is same as that of displacement and the magnitude of velocity is numerically equal to the absolute value of the velocity vector, denoted by the corresponding non bold face counterpart of the symbol.

Dimension of velocity is LT^{-1} and its SI unit is meter/second (m/s).

Position vector and velocity

The displacement is equal to the difference of position vectors between initial and final positions. As such, velocity can be conveniently expressed in terms of position vectors.
Displacement in terms of position vector



Displacement is equal to the change in position vector

Let us consider that $\mathbf{r_1}$ and $\mathbf{r_2}$ be the position vectors corresponding to the object positions at time instants t_1 and t_2 . Then, displacement is given by :
Equation:

$$\mathbf{v} = \frac{\mathbf{r_2} - \mathbf{r_1}}{t_2 - t_1} = \frac{\Delta\mathbf{r}}{\Delta t}$$

Velocity

Velocity is the rate of change of position vector with respect to time and is expressed as the ratio of change in position vector and time.

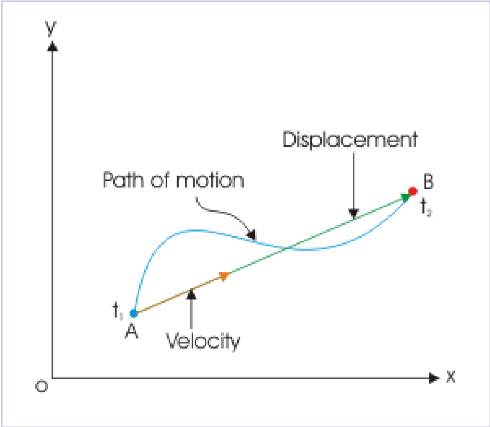
The expression of velocity in terms of position vectors is generally considered more intuitive and basic to the one expressed in terms of displacement. This follows from the fact that displacement vector itself is equal to the difference in position vectors between final and initial positions.

Average velocity

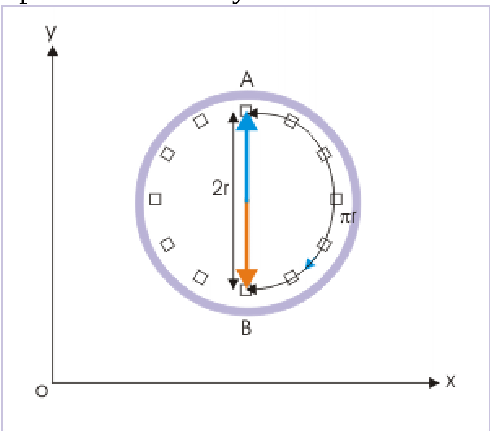
Average velocity is defined as the ratio of total displacement and time interval.
Equation:

$$\mathbf{v}_a = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r_2} - \mathbf{r_1}}{t_2 - t_1}$$

Average velocity gives the overall picture about the motion. The magnitude of the average velocity tells us the rapidity with which the object approaches final point along the straight line – not the rapidity along the actual path of motion. It is important to notice here that the magnitude of average velocity does not depend on the actual path as in the case of speed, but depends on the shortest path between two points represented by the straight line joining the

two ends. Further, the direction of average velocity is from the initial to final position along the straight line (See Figure).
Direction of velocity


Average velocity may be different to instantaneous velocities in between the motion in either magnitude or direction or both. Consider the example of the tip of the second's hand of a wall clock. It moves along a circular path of 2π r in 60 seconds. The magnitude of average velocity is zero in this period (60 seconds) as the second's hand reaches the initial position. This is the overall picture. However, the tip of the second's hand has actually traveled the path of 2π r, indicating that intermediate instantaneous velocities during the motion were not zero.

Also, the magnitude of average velocity may be entirely different than that of average speed. We know that distance is either greater than or equal to the magnitude of displacement. It follows then that average speed is either greater than or equal to the magnitude of average velocity. For the movement along semi-circle as shown in the figure below, the magnitude of velocity is 2r/30 m/s, where as average speed is π r/30 = 3.14 r/30 m/s. Clearly, the average speed is greater than the magnitude of average velocity.
Speed and velocity


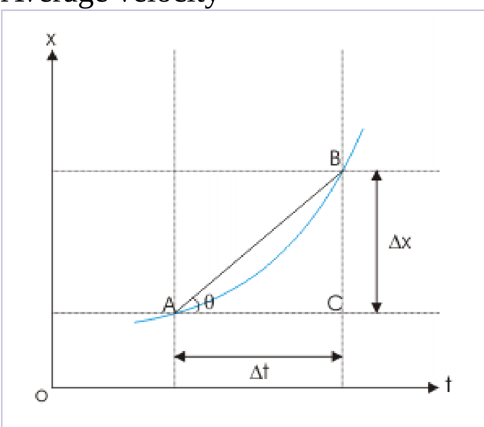
Position – time plot and average velocity

Position – time plot is a convenient technique to interpret velocity of a motion. The limitation here is that we can plot position – time graph only for one and two dimensional motions. As a matter of fact, it is only one dimensional (linear or rectilinear) motion, which renders itself for convenient drawing.

On the plot, positions are plotted with appropriate sign against time. A positive value of position indicates that particle is lying on the positive side of the origin, whereas negative value of position indicates that the particle is lying on the opposite side of the origin. It must, therefore, be realized that a position – time plot may extend to two quadrants of a two dimensional coordinate system as the value of x can be negative.

On a position – time plot, the vertical intercept parallel to position axis is the measure of displacement, whereas horizontal intercept is the measure of time interval (See Figure). On the other hand, the slope of the chord is equal to the ratio of two intercepts and hence equal to the magnitude of average velocity.

Average velocity



Equation:

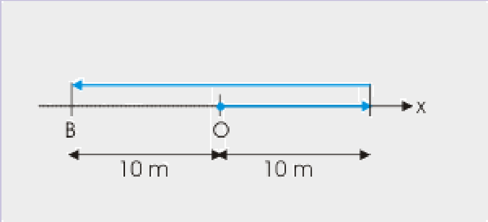
$$\left|\mathbf{v_a}\right| = \frac{\text{BC}}{\text{AC}} \quad = \frac{\Delta x}{\Delta t}$$

Example:

Average velocity

Problem : A particle completes a motion in two parts. It covers a straight distance of 10 m in 1 s in the first part along the positive x - direction and 20 m in 5 s in the second part along negative x- direction (See Figure). Find average speed and velocity.

Motion in straight line



Characteristics of motion : One dimensional

Solution : In order to find the average speed, we need to find distance and time. Here total time is 1 + 5 = 6 s and total distance covered is 10 + 20 = 30 m. Hence,

$$v_a = \frac{30}{6} = 5 \text{ m/s}$$

The displacement is equal to the linear distance between initial and final positions. The initial and final positions are at a linear distance = -10 m. The value is taken as negative as final position falls on the opposite side of the origin. Hence,

$$v_a = \frac{-10}{6} = \text{-}1.66 \text{ m/s}$$

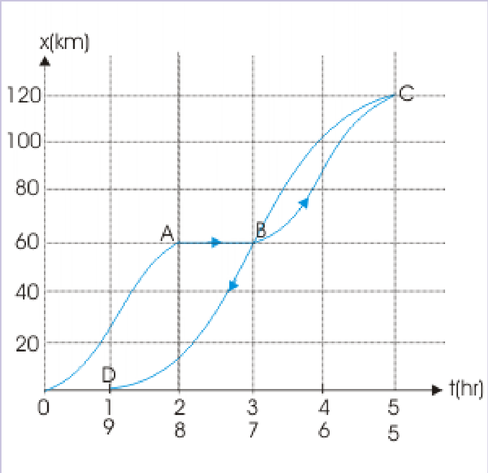
The negative value indicates that the average velocity is directed in the opposite direction to that of the positive reference direction. We have discussed earlier that one dimensional motion consists of only two direction and as such an one dimensional velocity can be equivalently represented by scalar value with appropriate sign scheme. Though, the symbol for average velocity is shown to be like a scalar symbol (not bold), but its value represents direction as well (the direction is opposite to reference direction).

Also significantly, we may note that average speed is not equal to the magnitude of average velocity.

Exercise:

Problem: Consider position – time plot as shown below showing a trip by a motor car.

Motion in straight line



Determine :

1. Total distance
2. Displacement
3. Average speed and average velocity for the round trip
4. Average speed and average velocity during motion from O to C
5. The parts of motion for which magnitudes of average velocity are equal in each direction.
6. Compare speeds in the portion OB and BD

Solution:

Characteristics of motion : One dimensional, variable speed

(i) Total distance in the round trip = 120 + 120 = 240 Km

(ii) Displacement = 120 – 120 = 0

(iii) Average speed for the round trip

$v_{\mathrm{OD}} = \frac{240}{9} = 26.67 \text{ km/hr} \quad \text{and average velocity for the round trip :} \quad v_{\mathrm{OD}} = \frac{0}{9} = 0$

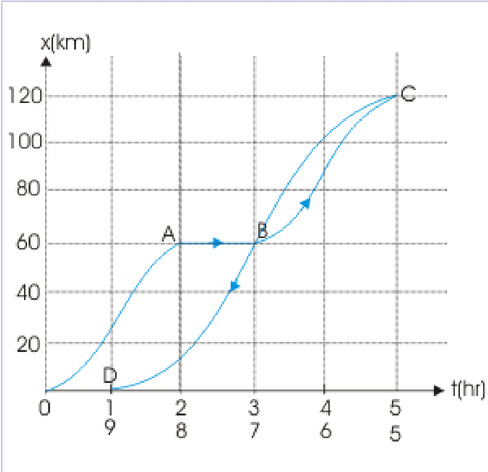
(iv) Average speed during motion from O to C

$$v_{\mathrm{OC}} = \frac{120}{5} = 24 \text{ km/hr} \quad \text{and average velocity :} \quad v_{\mathrm{OC}} = \frac{120}{5} = 24 \text{ km/hr}$$

As magnitude of average velocity is positive, the direction of velocity is in the positive x – direction. It is important to note for motion in one dimension and in one direction (unidirectional), distance is equal to the magnitude of displacement and average speed is equal to the magnitude of average velocity. Such is the case for this portion of motion.

(v) The part of motion for which average velocity is equal in each direction.

By inspection of the plots, we see that time interval is same for motion from B to C and from C to B (on return). Also, the displacements in these two segments of motion are equal. Hence magnitudes of velocities in two segments are equal.



(vi) Compare speeds in the portion OB and BD.

By inspection of the plots, we see that the motor car travels equal distances of 60 m. We see that distances in each direction is covered in equal times i.e. 2 hrs. But, the car actually stops for 1 hour in the forward journey and as such average speed is smaller in this case.

$$v_{\text{a,OB}} = \frac{60}{3} = 20\ \text{km/hr} \quad \text{and} \quad v_{\text{a,BD}} = \frac{60}{2} = 30\ \text{km/hr}$$

Instantaneous velocity

velocity

> **Instantaneous velocity is equal to the rate of change of position vector i.e displacement with respect to time at a given time and is equal to the first differential of position vector.**

Instantaneous velocity is defined exactly like speed. It is equal to the ratio of total displacement and time interval, but with one qualification that time interval is extremely (infinitesimally) small. Thus, instantaneous velocity can be termed as the average velocity at a particular instant of time when Δt tends to zero and may have entirely different value than that of average velocity. Mathematically,

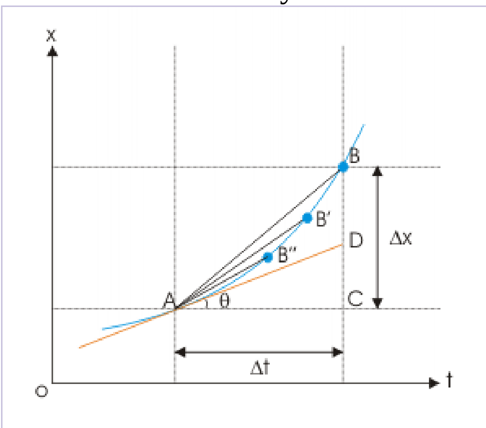
$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta \text{t}} = \frac{d\mathbf{r}}{dt}$$

As Δt tends to zero, the ratio defining velocity becomes finite and equals to the first derivative of the position vector. The velocity at the moment ‘t’ is called the instantaneous velocity or simply velocity at time ‘t’.

Instantaneous velocity and position - time plot

Position - time plot provides for calculation of the magnitude of velocity, which is equal to speed. The discussion of position – time plot in the context of velocity, however, differs in one important respect that we can also estimate the direction of motion.

Instantaneous Velocity



In the figure above, as we proceed from point B to A through intermediate points B’ and B’’, the time interval becomes smaller and smaller and the chord becomes tangent to the curve at point A as $\Delta t \rightarrow 0$. The magnitude of instantaneous velocity (speed) at A is given by the slope of the curve.

$$\left|\mathbf{v}\right| = \frac{\mathrm{DC}}{\mathrm{AC}} \quad = \frac{dx}{dt}$$

There is one important difference between average velocity and instantaneous velocity. The magnitude of average velocity $|\mathbf{v}_{\mathrm{avg}}|$ and average speed v_{avg} may not be equal, but magnitude of instantaneous velocity $|\mathbf{v}|$ is always equal to instantaneous speed v.

We have discussed that magnitude of displacement and distance are different quantities. The magnitude of displacement is a measure of linear shortest distance, whereas distance is measure of actual path. As such, magnitude of average velocity $|\mathbf{v}_{\mathrm{avg}}|$ and average speed v_{avg} are not be equal. However, if the motion is along a straight line and without any change in direction (i.e unidirectional), then distance and displacement are equal and so magnitude of average velocity and average speed are equal. In the case of instantaneous velocity, the time interval is infinitesimally small for which displacement and distance are infinitesimally small. In such situation, both displacement and distance are same. Hence, magnitude of instantaneous velocity $|\mathbf{v}|$ is always equal to instantaneous speed v.

Components of velocity

Velocity in a three dimensional space is defined as the ratio of displacement (change in position vector) and time. The object in motion undergoes a displacement, which has components in three mutually perpendicular directions in Cartesian coordinate system.

$$\Delta\mathbf{r} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}$$

It follows from the component form of displacement that a velocity in three dimensional coordinate space is the vector sum of component velocities in three mutually perpendicular directions. For a small time interval when Δ t → 0,

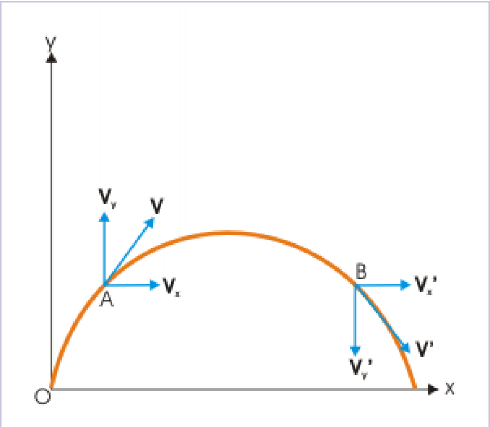
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$
$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

For the sake of clarity, it must be understood that components of velocity is a conceptual construct for examining a physical situation. It is so because it is impossible for an object to have two velocities at a given time. If we have information about the variations of position along three mutually perpendicular directions, then we can find out component velocities along the axes leading to determination of resultant velocity. The resultant velocity is calculated using following relation :

$$v = \left|\mathbf{v}\right| = \sqrt{\left(v_x^2 + v_y^2 + v_z^2\right)}$$

The component of velocity is a powerful concept that makes it possible to treat a three or two dimensional motion as composition of component straight line motions. To illustrate the point, consider the case of two dimensional parabolic motions. Here, the velocity of the body is resolved in two mutually perpendicular directions; treating motion in each direction independently and then combining the component directional attributes by using rules of vector addition

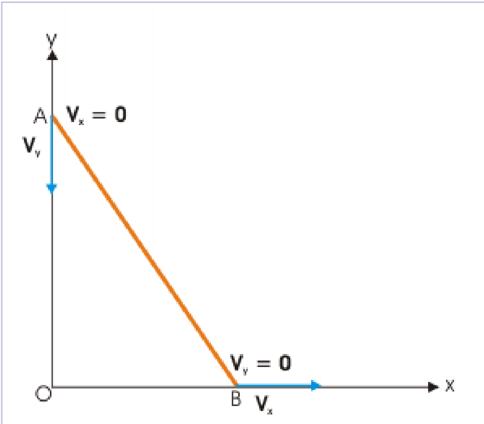
Parabolic motion



Motion is treated separately in two perpendicular directions

Similarly, the concept of component velocity is useful when motion is constrained. We may take the case of the motion of the edge of a pole as shown in the figure here. The motion of the ends of the pole is constrained in one direction, whereas other component of velocity is zero.

Constrained motion



The motion in space is determined by the component velocities in three mutually perpendicular directions. In two dimensional or planar motion, one of three components is zero. 10. The velocity of the object is determined by two relevant components of velocities in the plane. For example, motion in x and y direction yields :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$
$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$$
$$v = |\mathbf{v}| = \sqrt{\left(v_x^2 + v_y^2\right)}$$

Similarly, one dimensional motion (For example : x – direction) is described by one of the components of velocity.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i}$$
$$\mathbf{v} = v_x\mathbf{i}$$
$$v = |\mathbf{v}| = v_x$$

Few words of caution

Study of kinematics usually brings about closely related concepts, terms and symbols. It is always desirable to be precise and specific in using these terms and symbols. Following list of the terms along with their meaning are given here to work as reminder :

1: Position vector : **r** : a vector specifying position and drawn from origin to the point occupied by point object

2: Distance : s : length of actual path : not treated as the magnitude of displacement

3: Displacement : **AB** or Δ**r** : a vector along the straight line joining end points A and B of the path : its magnitude, |**AB**| or |Δ**r** | is not equal to distance, s.

4: Difference of position vector : Δ**r** : equal to displacement, **AB**. Direction of Δ**r** is not same as that of position vector (**r**).

5: Magnitude of displacement : |**AB**| or |Δ**r** |: length of shortest path.

6: Average speed : v_a : ratio of distance and time interval : not treated as the magnitude of average velocity

7: Speed : v : first differential of distance with respect to time : equal to the magnitude of velocity, |**v**|

8: Average velocity : $\mathbf{v_a}$: ratio of displacement and time interval : its magnitude, $|\mathbf{v_a}|$ is not equal to average speed, v_a .

9: Velocity : **v** : first differential of displacement or position vector with respect to time

Summary

The paragraphs here are presented to highlight the similarities and differences between the two important concepts of speed and velocity with a view to summarize the discussion held so far.

1: Speed is measured without direction, whereas velocity is measured with direction. Speed and velocity both are calculated at a position or time instant. As such, both of them are independent of actual path. Most physical measurements, like speedometer of cars, determine instantaneous speed. Evidently, speed is the magnitude of velocity,

$$v = |\mathbf{v}|$$

2: Since, speed is a scalar quantity, it can be plotted on a single axis. For this reason, tangent to distance – time curve gives the speed at that point of the motion. As $ds = v \text{ X } dt$, the area under speed – time plot gives distance covered between two time instants.

3: On the other hand, velocity requires three axes to be represented on a plot. It means that a velocity – time plot would need 4 dimensions to be plotted, which is not possible on three dimensional Cartesian coordinate system. A two dimensional velocity and time plot is possible, but is highly complicated to be drawn.

4: One dimensional velocity can be treated as a scalar magnitude with appropriate sign to represent direction. It is, therefore, possible to draw one dimension velocity – time plot.

5: Average speed involves the length of path (distance), whereas average velocity involves shortest distance (displacement). As distance is either greater than or equal to the magnitude of displacement,

$$s \geq |\mathbf{\Delta r}| \text{ and } v_a \geq |\mathbf{v_a}|$$

Exercises

Exercise:

Problem: The position vector of a particle (in meters) is given as a function of time as :

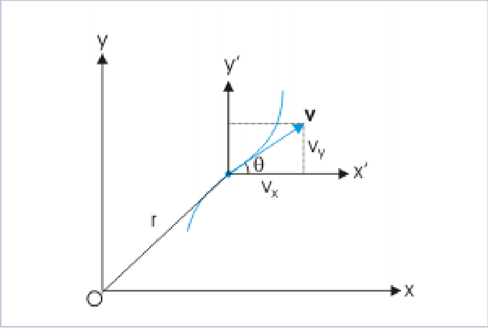
$$\mathbf{r} = 2t\mathbf{i} + 2t^2\mathbf{j}$$

Determine the time rate of change of the angle “θ” made by the velocity vector with positive x-axis at time, t = 2 s.

Solution:

Solution : It is a two dimensional motion. The figure below shows how velocity vector makes an angle "θ" with x-axis of the coordinate system. In order to find the time rate of change of this angle "θ", we need to express trigonometric ratio of the angle in terms of the components of velocity vector. From the figure :

Velocity of a particle in two dimensions



The velocity has two components.

$$\tan\theta = \frac{v_y}{v_x}$$

As given by the expression of position vector, its component in coordinate directions are :

$$x = 2t \text{ and } y = 2t^2$$

We obtain expression of the components of velocity in two directions by differentiating "x" and "y" components of position vector with respect to time :

$$v_x = 2 \text{ and } v_y = 4t$$

Putting in the trigonometric function, we have :

$$\tan\theta = \frac{v_y}{v_x} = \frac{4t}{2} = 2t$$

Since we are required to know the time rate of the angle, we differentiate the above trigonometric ratio with respect to time as,

$$\sec^2\theta \frac{d\theta}{dt} = 2$$

$$\Rightarrow \left(1+\tan^2\theta\right)\frac{d\theta}{dt} = 2$$
$$\Rightarrow \left(1+4t^2\right)\frac{d\theta}{dt} = 2$$
$$\Rightarrow \frac{d\theta}{dt} = \frac{2}{(1+4t^2)}$$

At t = 2 s,

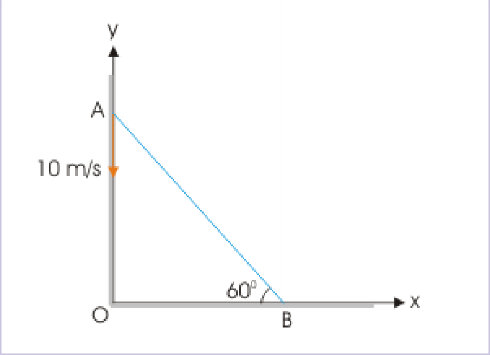
$$\Rightarrow \frac{d\theta}{dt} = \frac{2}{(1+4 \text{ x } 2^2)} = \frac{2}{17}\text{ rad}/s$$

Exercise:

Problem:

Two particles A and B are connected by a rigid rod AB. The rod slides along perpendicular rails as shown here. The velocity of A moving down is 10 m/s. What is the velocity of B when angle θ = 60° ?

Motion of a leaning rod



One end of the ros is moving with a speed 10 m/s in vertically downward direction.

Solution:

Solution : The velocity of B is not an independent velocity. It is tied to the velocity of the particle “A” as two particles are connected through a rigid rod. The relationship between two velocities is governed by the inter-particles separation, which is equal to the length of rod.

The length of the rod, in turn, is linked to the positions of particles “A” and “B” . From figure,

$$x = \sqrt{(L^2 - y^2)}$$

Differentiatiting, with respect to time :

$$\Rightarrow v_x = \frac{dx}{dt} = -\frac{2y}{2\sqrt{(L^2 - y^2)}} X \frac{dy}{dt} = -\frac{y v_y}{\sqrt{(L^2 - y^2)}} = -v_y \tan\theta$$

Considering magnitude only,

$$\Rightarrow v_x = v_y \tan\theta = 10 \tan 60^0 = 10\sqrt{3} \quad \frac{m}{s}$$

Exercise:

Problem: Problem : The position vector of a particle is :

$$\mathbf{r} = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$$

where “a” is a constant. Show that velocity vector is perpendicular to position vector.

Solution:

Solution : In order to prove as required, we shall use the fact that scalar (dot) product of two perpendicular vectors is zero. Now, we need to find the expression of velocity to evaluate the dot product as intended. We can obtain the same by differentiating the expression of position vector with respect to time as :

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -a\omega\sin\omega t\mathbf{i} + a\omega\cos\omega t\mathbf{j}$$

To check whether velocity is perpendicular to the position vector, we evalaute the scalar product of **r** and **v**, which should be equal to zero.

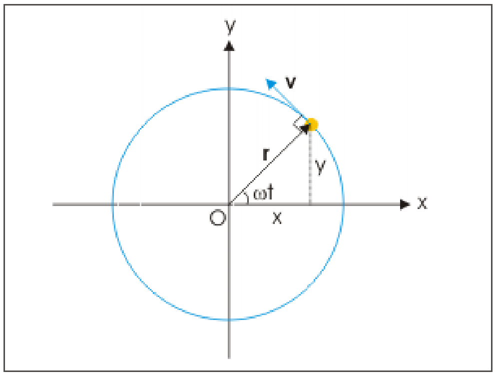
$$\mathbf{r.\,v} = 0$$

In this case,

$$\Rightarrow \mathbf{r.\,v} = (a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j})\,\mathbf{.}\,(-a\omega\sin\omega t\mathbf{i} + a\omega\cos\omega t\mathbf{j})$$
$$\Rightarrow -a^2\omega\sin\omega t\cos\omega t + a^2\omega\sin\omega t\cos\omega t = 0$$

This means that the angle between position vector and velocity are at right angle to each other. Hence, velocity is perpendicular to position vector. It is pertinent to mention here that this result can also be inferred from the plot of motion. An inspection of position vector reveals that it represents uniform circular motion as shown in the figure here.

Circular motion



The particle describes a circular path.

The position vector is always directed radially, whereas velocity vector is always tangential to the circular path. These two vectors are, therefore, perpendicular to each other.

Exercise:

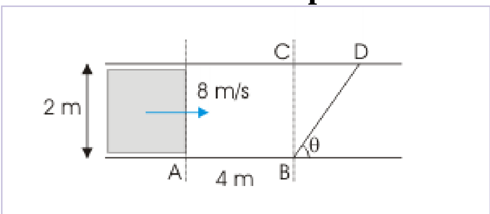
Problem:

Problem : A car of width 2 m is approaching a crossing at a velocity of 8 m/s. A pedestrian at a distance of 4 m wishes to cross the road safely. What should be the minimum speed of pedestrian so that he/she crosses the road safely?

Solution:

Solution : We draw the figure to illustrate the situation. Here, car travels the linear distance (AB + CD) along the direction in which it moves, by which time the pedestrian travels the linear distance BD. Let pedestrian travels at a speed "v" along BD, which makes an angle "θ" with the direction of car.

Motion of a car and a pedestrian



The pedestrian crosses the road at angle with direction of car.

We must understand here that there may be number of combination of angle and speed for which pedestrian will be able to safely cross before car reaches. However, we are required to find the minimum speed. This speed should, then, correspond to a particular value of θ.

We can also observe that pedestrian should move obliquely. In doing so he/she gains extra time to cross the road.

From triangle BCD,

$$\tan\left(90-\theta\right) = \cot\theta = \frac{\mathrm{CD}}{\mathrm{BC}} = \frac{\mathrm{CD}}{2}$$
$$\Rightarrow \mathrm{CD} = 2\cot\theta$$

Also,

$$\cos\left(90-\theta\right) = \sin\theta = \frac{\mathrm{BC}}{\mathrm{BD}} = \frac{2}{\mathrm{BD}}$$
$$\Rightarrow \mathrm{BD} = \frac{2}{\sin\theta}$$

According to the condition given in the question, the time taken by car and pedestrian should be equal for the situation outlined above :

$$t = \frac{4+2\cot\theta}{8} = \frac{\frac{2}{\sin\theta}}{v}$$

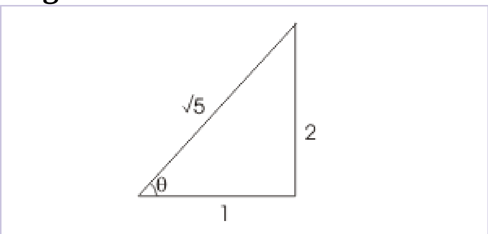
$$v = \frac{8}{2\sin\theta+\cos\theta}$$

For minimum value of speed, $\frac{\mathrm{d}v}{\mathrm{d}\theta} = 0$,

$$\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}\theta} = \frac{-8\text{ x }(2\cos\theta-\sin\theta)}{(2\sin\theta+\cos\theta)^2} = 0$$
$$\Rightarrow \left(2\cos\theta - \sin\theta\right) = 0$$
$$\Rightarrow \tan\theta = 2$$

In order to evaluate the expression of velocity with trigonometric ratios, we take the help of right angle triangle as shown in the figure, which is consistent with the above result.

Trigonometric ratio



The tangent of angle is equal to 2.

From the triangle, defining angle “θ”, we have :

$$\sin\theta = 2\sqrt{5}$$

and

$$\cos\theta = \frac{1}{\sqrt{5}}$$

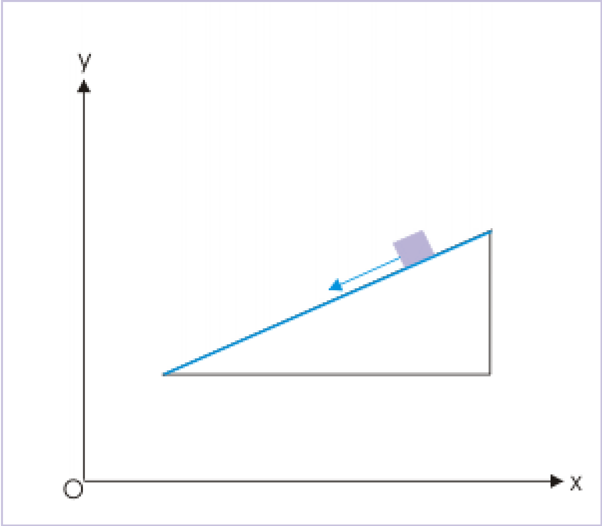
The minimum velocity is :

$$v = \frac{8}{2\text{ x }2\sqrt{5}+\frac{1}{\sqrt{5}}} = \frac{8}{\sqrt{5}} = 3.57\,m/s$$

Rectilinear motion

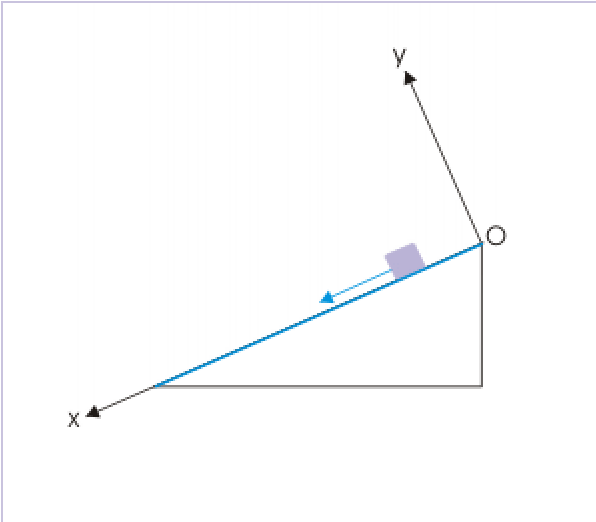
A motion along straight line is called rectilinear motion. In general, it need not be one – dimensional; it can take place in a two dimensional plane or in three dimensional space. But, it is always possible that rectilinear motion be treated as one dimensional motion, by suitably orienting axes of the coordinate system. This fact is illustrated here for motion along an inclined plane. The figure below depicts a rectilinear motion of the block as it slides down the incline. In this particular case, the description of motion in the coordinate system, as shown, involves two coordinates (x and y).

Motion along inclined plane



The reorientation of the coordinate system renders two dimensional description (requiring x and y values) of the motion to one dimensional (requiring only x value). A proper selection, most of the time, results in simplification of measurement associated with motion. In the case of the motion of the block, we may choose the orientation such that the progress of motion is along the positive x – direction as shown in the figure. A proper orientation of the coordinates results in positive values of quantities like displacement and velocity. It must be emphasized here that we have complete freedom in choosing the orientation of the coordinate system. The description of the rectilinear motion in independent of the orientation of axes.

Motion along inclined plane



In rectilinear motion, we are confined to the measurement of movement of body in only one direction. This simplifies expressions of quantities used to describe motion. In the following section, we discuss (also recollect from earlier discussion) the simplification resulting from motion in one dimension (say in x –direction).

Position vector in rectilinear motion

Position still requires three coordinates for specification. But, only one of them changes during the motion; remaining two coordinates remain constant. In practice, we choose one dimensional reference line to coincide with the path of the motion. It follows then that position of the particle under motion is equal to the value of x – coordinate (others being zero).

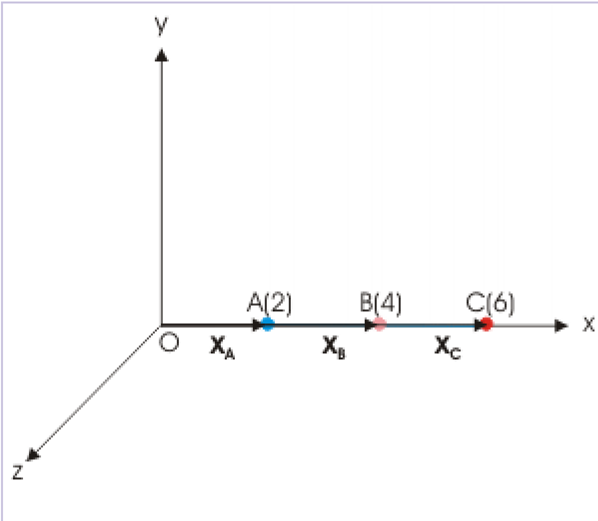
Corresponding position vector also remains a three dimensional quantity. However, if the path of motion coincides with the reference direction and origin of the reference coincides with origin, then position vector is simply equal to component vector in x – direction i.e

Equation:

$$\mathbf{r} = x\mathbf{i}$$

The position vectors corresponding to points A, B and C as shown in the figure are 2 **i**, 4 **i** and 6 **i**. units respectively.

Position vectors



As displacement is equal to change in position vector, the displacement for the indicated positions are given as :

$$\mathbf{AB} = (x_2 - x_1)\mathbf{i} = \Delta x\mathbf{i} = (4-2)\mathbf{i} = 2\mathbf{i}$$
$$\mathbf{BC} = (x_2 - x_1)\mathbf{i} = \Delta x\mathbf{i} = (6-4)\mathbf{i} = 2\mathbf{i}$$
$$\mathbf{AC} = (x_2 - x_1)\mathbf{i} = \Delta x\mathbf{i} = (6-2)\mathbf{i} = 4\mathbf{i}$$

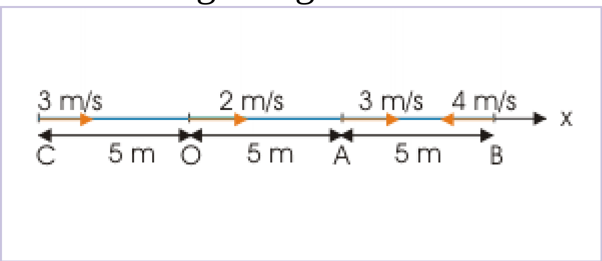
Vector interpretation and equivalent system of scalars

Rectilinear motion involves motion along straight line and thus is described usually in one dimension. Further, rectilinear motion involves only one way of changing direction i.e. the particle under motion can only reverse motion. The particle can move either in positive x-direction or in negative x-direction. There is no other possible direction valid in rectilinear motion.

This attribute of rectilinear motion allows us to do away with the need to use vector notation and vector algebra for quantities with directional attributes like position vector, displacement and velocity. Instead, the vectors are treated simply as scalars with one qualification that vectors in the direction of chosen reference is considered positive and vectors in the opposite direction to the chosen reference is considered negative.

The most important aspect of the sign convention is that a vector like velocity can be expressed by a scalar value say, 5 m/s. Though stated without any aid for specifying direction like using unit vector, the direction of the velocity is indicated, which is in the positive x-direction. If the velocity of motion is -5 m/s, then the velocity is in the direction opposite to the direction of reference.

To illustrate the construct, let us consider a motion of a ball which transverses from O to A to B to C to O along x-axis as shown in the figure.
Motion along straight line



The velocities at various points of motion in m/s (vector form) are :

$$\mathbf{v_O} = 2\mathbf{i}; \ \ \mathbf{v_A} = 3\mathbf{i}; \ \ \mathbf{v_B} = -4\mathbf{i}; \ \ \mathbf{v_C} = 3\mathbf{i}$$

Going by the scalar construct, we can altogether drop use of unit vector like "**i**" in describing all vector quantities used to describe motion in one dimension. The velocities at various points of motion in m/s (in equivalent scalar form) can be simply stated in scalar values for rectilinear motion as :

$$v_O = 2; \ \ v_A = 3; \ \ v_B = -4; \ \ v_C = 3$$

Similarly, we can represent position vector simply by the component in one direction, say x, in meters, as :

$$x_O = 0; \ \ x_A = 5; \ \ x_B = 10; \ \ x_C = -5$$

Also, the displacement vector (in meters) is represented with scalar symbol and value as :

$$\mathrm{OA} = 5; \ \ \mathrm{OB} = 10; \ \ \mathrm{OC} = 5$$

Following the same convention, we can proceed to write defining equations of speed and velocity in rectilinear motion as :

Equation:

$$\left|v\right| = \left|\frac{đx}{đt}\right|$$

and

Equation:

$$v = \frac{đx}{đt}$$

Example:

Rectilinear motion

Problem : If the position of a particle along x – axis varies in time as :

$$x = 2t^2 - 3t + 1$$

Then :

1. What is the velocity at t = 0 ?
2. When does velocity become zero?
3. What is the velocity at the origin ?
4. Plot position – time plot. Discuss the plot to support the results obtained for the questions above.

Solution : We first need to find out an expression for velocity by differentiating the given function of position with respect to time as :

$$v = \frac{đ}{đt}\left(2t^2 - 3t + 1\right) = 4t - 3$$

(i) The velocity at t = 0,

$$v = 4 \text{ x } 0 - 3 = -3 \text{m/s}$$

(ii) When velocity becomes zero :
For v = 0,

$$4t - 3 = 0$$
$$\Rightarrow t = \frac{3}{4} = 0.75 \text{ s.}$$

(iii) The velocity at the origin :
At origin, x = 0,

$$x = 2\mathbf{t}^2 - 3t + 1 = 0$$
$$\Rightarrow 2\mathbf{t}^2 - 2t - t + 1 = 0$$
$$\Rightarrow 2t(t - 1) - (t - 1) = 0$$
$$\Rightarrow t = 0.5 \text{ s, } 1 \text{ s.}$$

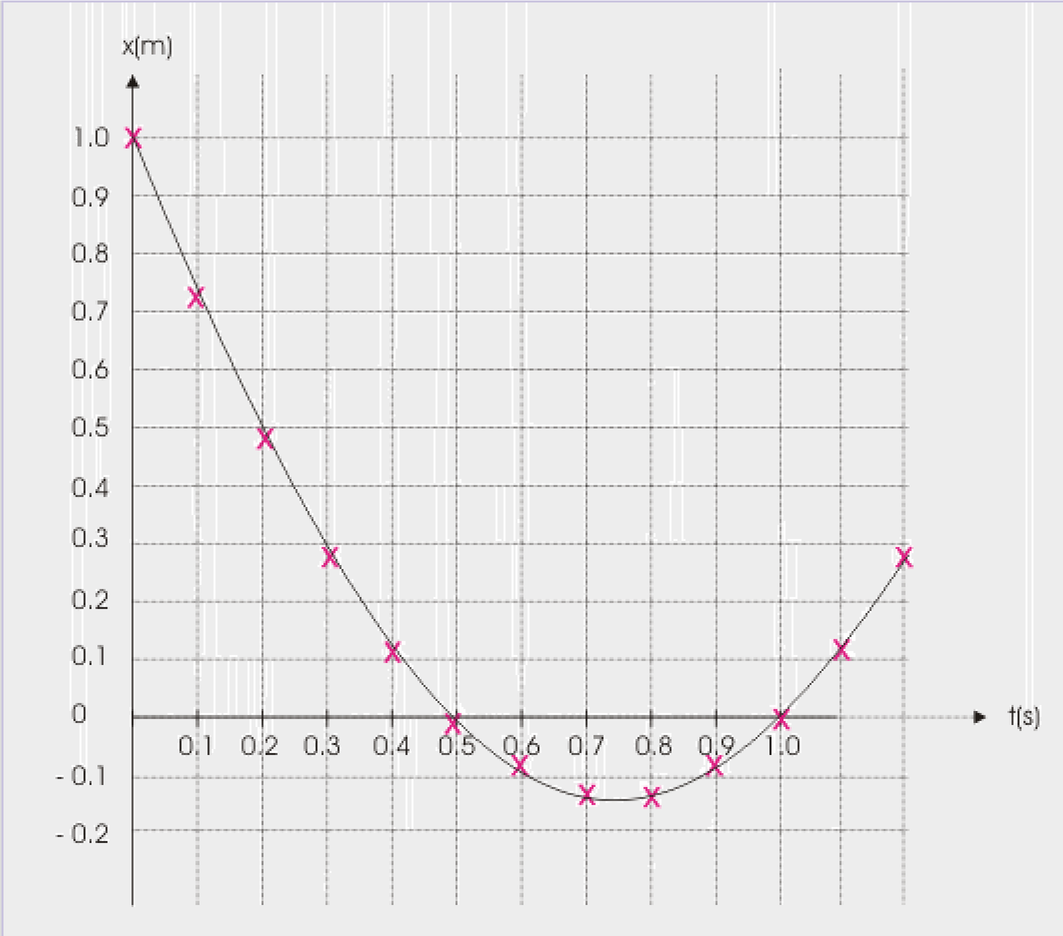
This means that particle is twice at the origin at t = 0.5 s and t = 1 s. Now,

$$v(t = 0.5 \text{ s}) = 4t{-}3 = 4 \text{ x } 0.5 - 3 = \text{-1 m/s.}$$

Negative sign indicates that velocity is directed in the negative x – direction.

$$v(t = 1 \text{ s}) = 4t{-}3 = 4 \text{ x } 1 - 3 = 1 \text{ m/s.}$$

Position – time plot



We observe that slope of the curve from t = 0 s to t < 0.75 s is negative, zero for t = 0.75 and positive for t > 0.75 s. The velocity at t = 0, thus, is negative. We can realize here that the slope of the tangent to the curve at t = 0.75 is zero. Hence, velocity is zero at t = 0.75 s.

The particle arrives at x = 0 for t = 0.5 s and t = 1 s. The velocity at first arrival is negative as the position falls on the part of the curve having negative slope, whereas the velocity at second arrival is positive as the position falls on the part of the curve having positive slope.

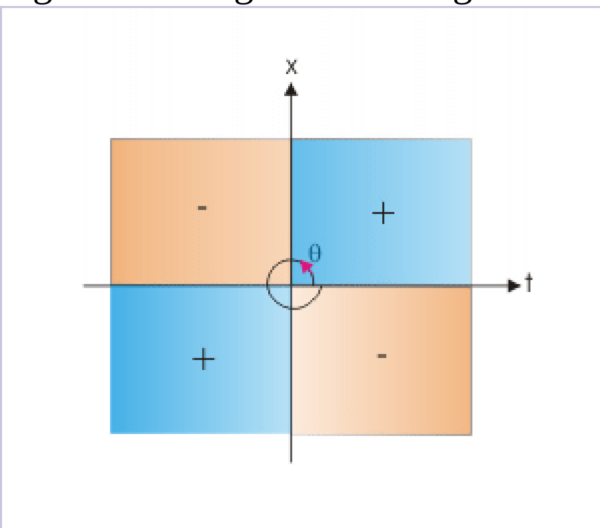
Position - time plot

We use different plots to describe rectilinear motion. Position-time plot is one of them. Position of the point object in motion is drawn against time. Evidently, it is a two dimensional plot. The position is plotted with appropriate sign as described earlier.

Nature of slope

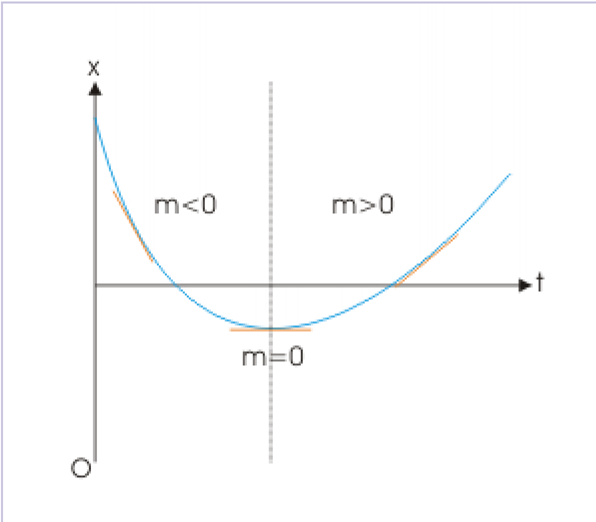
One of the important tool used to understand nature of such plots (as drawn above) is the slope of the tangent drawn on the plot.In particular, we need to qualitatively ascertain whether the slope is positive or negative. In this section, we seek to find out the ways to determine the nature of slope. Mathematically, the slope of a straight line is numerically equal to trigonometric tangent of the angle that the line makes with x – axis. It follows, therefore, that the slope of the straight line may be positive or negative depending on the angle. It is seen as shown in the figure below, the tangent of the angle in first and third quarters is positive, whereas it is negative in the remaining second and fourth quarter. This assessment of the slope of the position - time plot helps us to identify whether velocity is positive or negative?

Sign of the tangent of the angle



We may, however, use yet another simpler and effective technique to judge the nature of the slope. This employs physical interpretation of the plot. We know that the tangent of the angle is equal to the ratio of x (position) and t (time). In order to judge the nature of slope, we progress with the time and determine whether “x” increases or decreases. The increase in “x” corresponds to positive slope and a decrease, on the other hand, corresponds to negative slope. This assessment helps us to quickly identify whether velocity is positive or negative?

Slope of the curve



Direction of motion

The visual representation of the curve might suggest that the tangent to the position – time plot gives the direction of velocity. It is not true. It is contradictory to the assumption of the one dimensional motion. Motion is either in positive or negative x – direction and not in any other direction as would be suggested by the direction of tangent at various points. As a matter of fact, the curve of the position – time plot is not the representation of the path of motion. The path of the motion is simply a straight line. This distinction should always be kept in mind.

In reality, the nature of slope indicates the sense of direction, which can assume either of the two possible directions. A positive slope of the curve denotes motion along the positive direction of the referred axis, whereas negative slope indicates reversal of the direction of motion.

In the position – time plot as shown in the example at the beginning of the module (See Figure) , the slope of the curve from t = 0 s to t = 0.75 s is negative, whereas slope becomes positive for t > 0.75 s. Clearly, an inversion of slope indicates reversal of direction. The particle, in the instant case, changes direction once at t = 0.75 s during the motion.

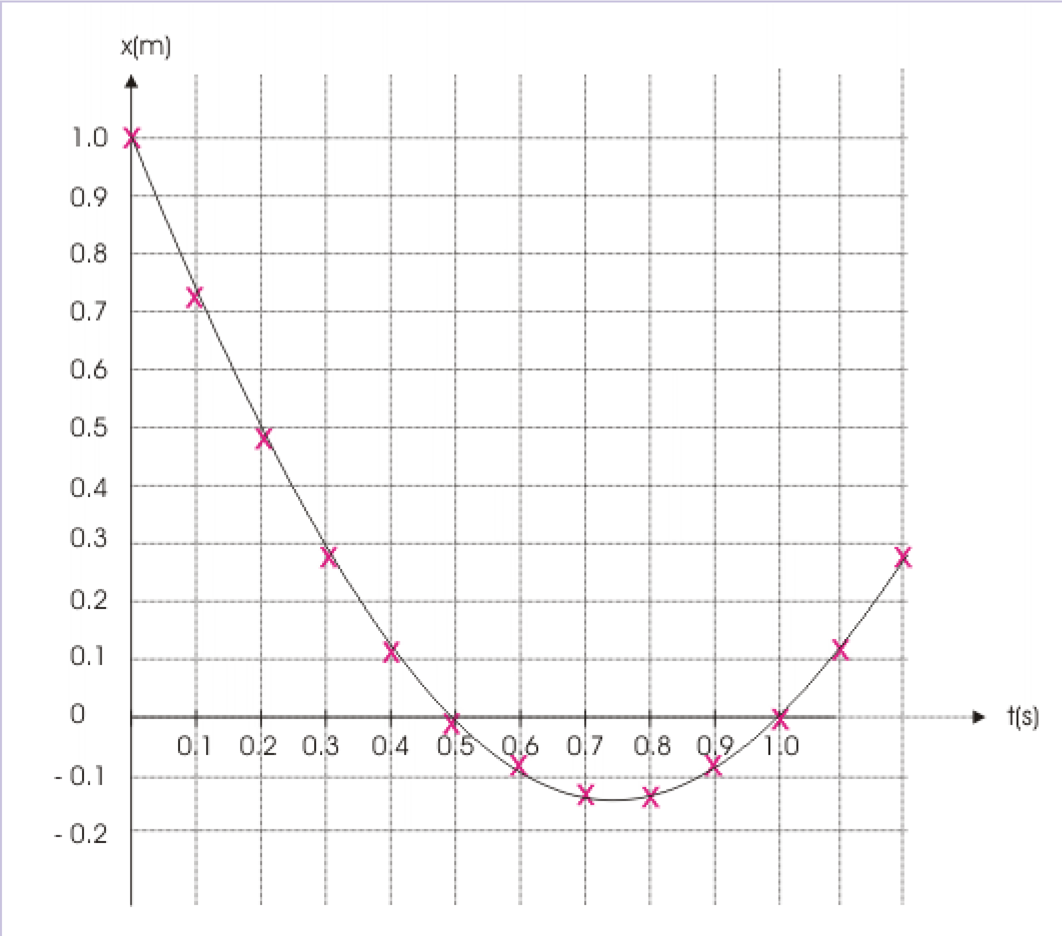
Variation in the velocity

In addtion to the sense of direction, the position - time plot allows us to determine the magnitude of velocity i.e. speed, which is equal to the magnitude of the slope. Here we shall see that the position – time plot is not only helpful in determining magnitude and direction of the velocity, but also in determining whether speed is increasing or decreasing or a constant.

Let us consider the plot generated in the example at the beginning of this module. The data set of the plot is as given here :

```
t(s) x(m) Δx(m) --------------------- 0.0 1.0 0.1
0.72 -0.28 0.2 0.48 -0.24 0.3 0.28 -0.20 0.4 0.12
-0.16 0.5 0.00 -0.12 0.6 -0.08 -0.08 0.7 -0.12
-0.12 0.8 -0.12 0.00 0.9 -0.08 0.04 1.0 0.00 0.08
1.1 0.12 0.12 1.2 0.28 0.16 ---------------------
```

In the beginning of the motion starting from t = 0, we see that particle covers distance in decreasing magnitude in the negative x - direction. The magnitude of difference, Δx, in equal time interval decreases with the progress of time. Accordingly, the curve becomes flatter. This is reflected by the fact that the slope of the tangent becomes gentler till it becomes horizontal at t = 0.75 s. Beyond t = 0.75 s, the velocity is directed in the positive x – direction. We can see that particle covers more and more distances as the time progresses. It means that the velocity of the particle increases with time and the curve gets steeper with the passage of time.
Position – time plot



In general, we can conclude that a gentle slope indicates smaller velocity and a steeper slope indicates a larger velocity.

Velocity - time plot

In general, the velocity is a three dimensional vector quantity. A velocity – time would, therefore, require additional dimension. Hence, it is not possible to draw velocity – time plot on a three dimensional coordinate system. Two dimensional velocity - time plot is possible, but its drawing is complex.

One dimensional motion, having only two directions – along positive or negative direction of axis, allows plotting velocity – time graph. The velocity is treated simply as scalar speed with one qualification that velocity

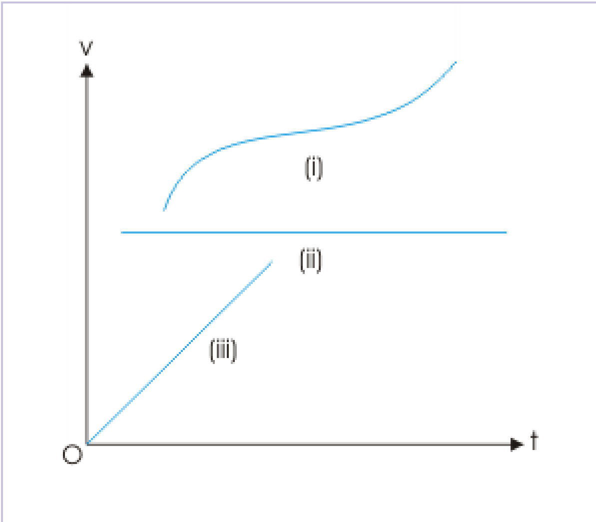
in the direction of chosen reference is considered positive and velocity in the opposite direction to chosen reference is considered negative.

Note: In "speed- time" and "velocity – time" plots use the symbol “v” to represent both speed and velocity. This is likely to create some confusion as these quantities are essentially different. We use the scalar symbol “v” to represent velocity as a special case for rectilinear motion, because scalar value of velocity with appropriate sign gives the direction of motion as well. Therefore, the scalar representation of velocity is consistent with the requirement of representing both magnitude and direction. Though current writings on the subject allows duplication of symbol, we must, however, be aware of the difference between two types of plot. The speed (v) is always positive in speed – time plot and drawn in the first quadrant of the coordinate system. On the other hand, velocity (v) may be positive or negative in velocity – time plot and drawn in first and fourth quadrants of the two dimensional coordinate system.

The nature of velocity – time plot

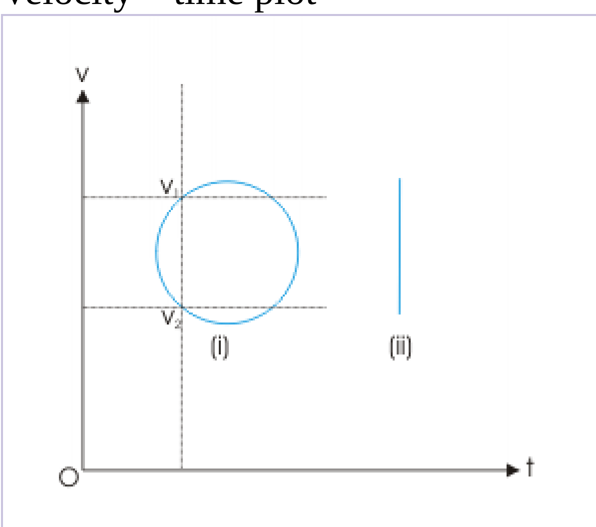
Velocity – time plot for rectilinear motion is a curve (Figure i). The nature of the curve is determined by the nature of motion. If the particle moves with constant velocity, then the plot is a straight line parallel to the time axis (Figure ii). On the other hand, if the velocity changes with respect to time at uniform rate, then the plot is a straight line (Figure iii).

Velocity – time plot



The representation of the variation of velocity with time, however, needs to be consistent with physical interpretation of motion. For example, we can not think of velocity - time plot, which is a vertical line parallel to the axis of velocity (Figure ii). Such plot is inconsistent as this would mean infinite numbers of values, against the reality of one velocity at a given instant. Similarly, the velocity-time plot should not be intersected by a vertical line twice as it would mean that the particle has more than one velocity at a given time (Figure i).

Velocity – time plot



Area under velocity – time plot

The area under the velocity – time plot is equal to displacement. The displacement in the small time period "dt" is given by :

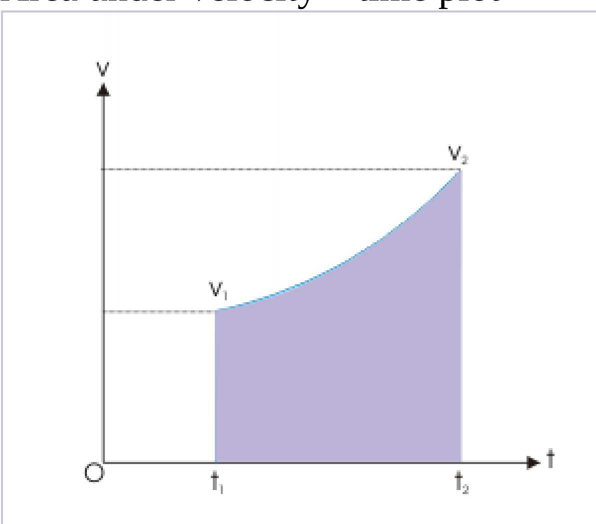
$$đx = v đt$$

Integrating on both sides between time intervals t_1 and t_2,

$$\Delta x = \int_{t_1}^{t_2} v đt$$

The right hand side integral graphically represents an area on a plot drawn between two variables : velocity (v) and time (t). The area is bounded by (i) v-t curve (ii) two time ordinates t_1 and t_2 and (iii) time (t) axis as shown by the shaded region on the plot. Thus, the area under v-t plot bounded by the ordinates give the magnitude of displacement (Δx) in the given time interval.

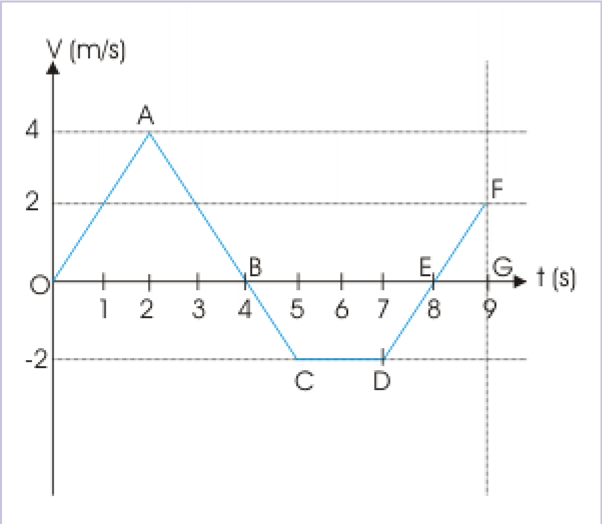
Area under velocity – time plot



When v-t curve consists of negative values of velocity, then the curve extends into fourth quadrant i.e. below time axis. In such cases, it is sometimes easier to evaluate area above and below time axis separately. The area above time axis represents positive displacement, whereas area under time axis represents negative displacement. Finally, areas are added with proper sign to obtain the net displacement during the motion.

To illustrate the working of the process for determining displacement, let us consider the rectilinear motion of a particle represented by the plot shown.

Velocity – time plot



Here,

$$\text{Area of triangle OAB} = \frac{1}{2} \text{ x } 4 \text{ x } 4 = 8 \text{ m}$$
$$\text{Area of trapezium BCDE} = -\frac{1}{2} \text{ x } 2 \text{ x } (2+4) = \text{-6 m}$$
$$\text{Area of triangle EFG} = \frac{1}{2} \text{ x } 1 \text{ x } 2 = 1 \text{ m}$$
$$\Rightarrow \text{Net area} = 8 - 6 + 1 = 3 \text{ m}$$

A switch from positive to negative value of velocity and vice-versa is associated with change of direction of motion. It means that every intersection of the v – t curve with time axis represents a reversal of direction. Note that it is not the change of slope (from postive to negative and and vive versa) like on the position time that indicates a change in the direction of motion; but the intersection of time axis, which indicates change of direction of velocity. In the motion described in figure above, particle undergoes reversal of direction at two occasions at B and E.

It is also clear from the above example that displacement is given by the net area (considering appropriate positive and negative sign), while distance covered during the motion in the time interval is given by the cumulative area without considering the sign. In the above example, distance covered is :

s = Area of triangle OAB + Area of trapezium BCDE + Area of triangle EFG

$$\Rightarrow s = 8 + 6 + 1 = 15 \text{ m}$$

Exercise:

Problem:

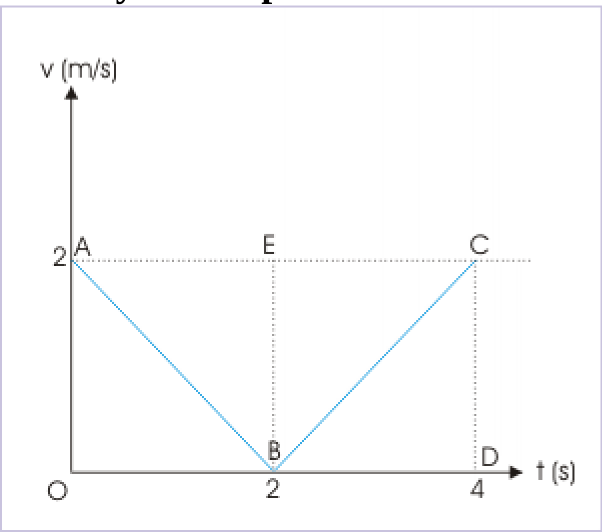
A person walks with a velocity given by |t – 2| along a straight line. Find the distance and displacement for the motion in the first 4 seconds. What is the average velocity in this period?

Solution:

Here, the velocity is equal to the modulus of a function in time. It means that velocity is always positive. An inspection of the function reveals that velocity linearly decreases for the first 2 second from 2 m/s to zero. It, then, increases from zero to 2 m/s in the next 2 seconds. In order to obtain distance and displacement, we draw the Velocity – time plot plot as shown.

The area under the plot gives displacement. In this case, however, there is no negative displacement involved. As such, distance and displacement are equal.

Velocity – time plot



$$\text{Displacement} = \Delta\text{OAB} + \Delta\text{BCD}$$
$$\text{Displacement} = \frac{1}{2} \text{ x OB x OA} + \frac{1}{2} \text{ x BD x CD}$$
$$\text{Displacement} = \frac{1}{2} \text{ x 2 x 2} + \frac{1}{2} \text{ x 2 x 2} = 4 \text{ m}$$

and the average velocity is given by :

$$v_{\text{avg}} = \frac{\text{Displacement}}{\Delta t} = \frac{4}{4} = 1 \text{ m/s}$$

Uniform motion

Uniform motion is a subset of rectilinear motion. It is the most simplified class of motion. In this case, the body under motion moves with constant velocity. It means that the body moves along a straight line without any change of magnitude and direction as velocity is constant. Also, a constant velocity implies that velocity is constant all through out the motion. The velocity at every instant during motion is, therefore, same.

It follows then that instantaneous and averages values of speed and velocity are all equal to a constant value for uniform motion :

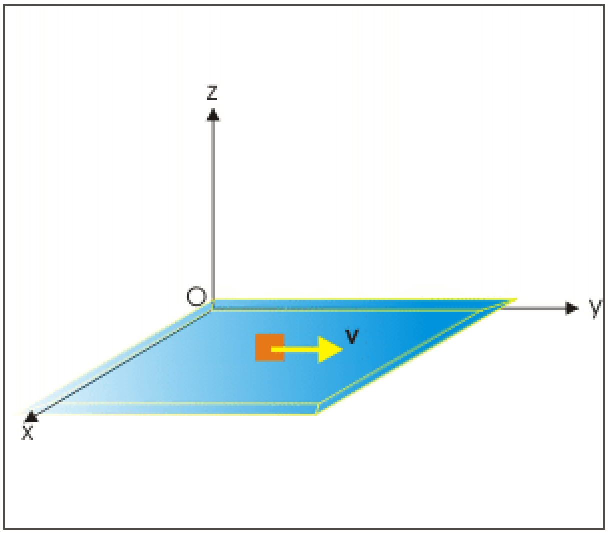
$$v_a = |\mathbf{v_a}| = v = |\mathbf{v}| = \text{Constant}$$

The motion of uniform linear motion has special significance, as this motion exactly echoes the principle enshrined in the first law of motion. The law states that all bodies in the absence of external force maintain their speed and direction. It follows, therefore, that the study of uniform motion is actually the description of motion, when no external force is in play.

The absence of external force is hypothetical in our experience as bodies are always subject to external force(s). The force of gravitation is short of omnipresent force that can not be overlooked - atleast on earth. Nevertheless, the concept of uniform motion has great theoretical significance as it gives us the reference for the accelerated or the non-uniform real motion.

On the earth, a horizontal motion of a block on a smooth plane approximates uniform motion as shown in the figure. The force of gravity acts and normal reaction force at the contact between surfaces act in vertically, but opposite directions. The two forces balances each other. As a result, there is no net force in the vertical direction. As the surface is smooth, we can also neglect horizontal force due to friction, which could have opposed the motion in horizontal direction. This situation is just an approxmiation for we can not think of a flawless smooth surface in the first place; and also there would be intermolecular attraction between the block and surface at the contact. In brief, we can not achieve zero friction - eventhough the surfaces in contact are perfectly smooth. However, the approximation like this is helpful for it provides us a situation, which is equivalent to the motion of an object without any external force(s).

Uniform motion

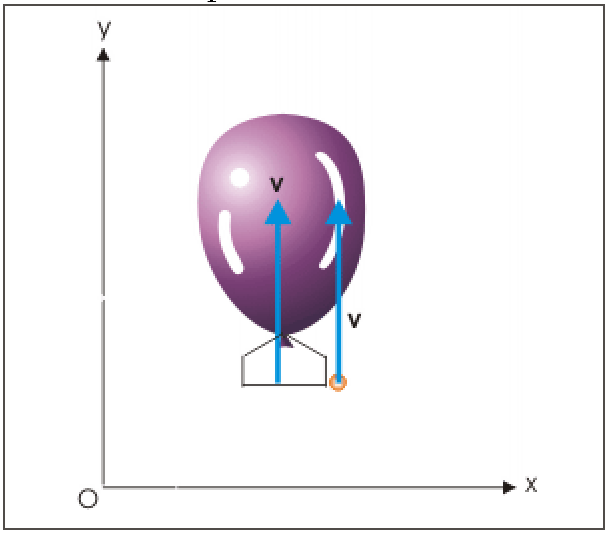


We can also imagine a volumetric space, where massive bodies like planets and stars are not nearby. The motion of an object in that space, therefore, would be free from any external force and the motion would be in accordance with the laws of motion. The study of astronauts walking in the space and doing repairs to the spaceship approximates the situation of the absence of external force. The astronaut in the absence of the force of gravitation and friction (as there is no atmosphere) moves along with the velocity of the spaceship.

Motion of separated bodies

The fact that the astronauts moves with the velocity of spaceship is an important statement about the state of motion of the separated bodies. A separated body acquires the velocity of the containing body. A pebble released from a moving train or dropped from a rising balloon is an example of the motion of separated body. The pebble acquires the velocity of the train or the balloon as the case may be.

Motion of separated bodies



It means that we must assign a velocity to the released body, which is equal in magnitude and direction to that of the body from which the released body has separated. The phenomena of imparting velocity to the separated body is a peculiarity with regard to velocity. We shall learn that acceleration (an attribute of non-uniform motion) does not behave in the same fashion. For example, if the train is accelerating at the time, the pebble is dropped, then the pebble would not acquire the acceleration of the containing body.

The reason that pebble does not acquire the acceleration of the train is very simple. We know that acceleration results from application of external force. In this case, the train is accelerated as the engine of the train pulls the compartment (i.e. applies force on the compartment. The pebble, being part of the compartment), is also accelerated till it is held in the hand of the passenger. However, as the pebble is dropped, the connection of the pebble with the rest of the system or with the engine is broken. No force is applied on the pebble in the horizontal direction. As such, pebble after being dropped has no acceleration in the horizontal direction.

In short, we can conclude that a separated body acquires velocity, but not the acceleration.

Rectilinear motion (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

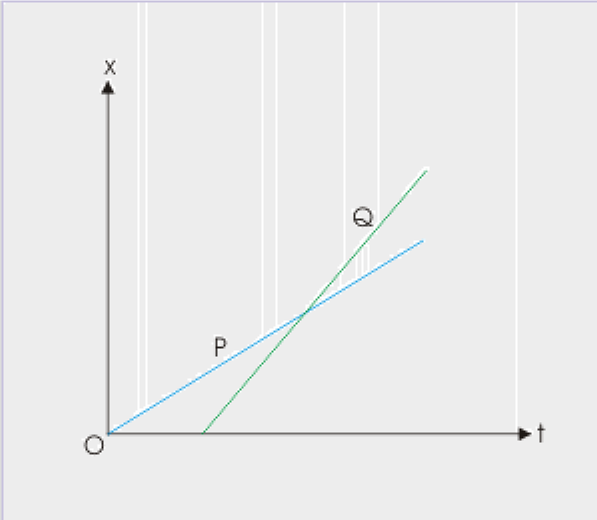
Representative problems and their solutions

We discuss problems, which highlight certain aspects of rectilinear motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Interpretation of position - time plot
- Interpretation of displacement - time plot
- Displacement
- Average velocity

Position vector

Example:
Problem : Two boys (P and Q) walk to their respective school from their homes in the morning on a particular day. Their motions are plotted on a position – time graph for the day as shown. If all schools and homes are situated by the side of a straight road, then answer the followings :
Motion of two boys

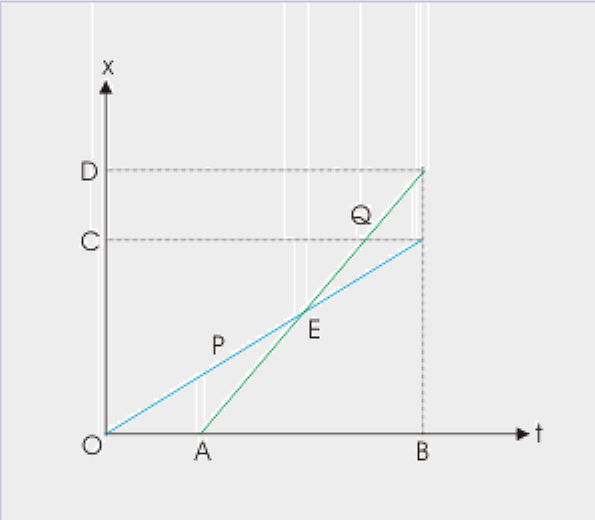


Position – time plot of a rectilinear motion.

1. Which of the two resides closer to the school?
2. Which of the two starts earlier for the school?
3. Which of the two walks faster?
4. Which of the two reaches the school earlier?
5. Which of the two overtakes other during walk?

Solution : In order to answer questions, we complete the drawing with vertical and horizontal lines as shown here. Now,

Motion of two boys



Position – time plot of a rectilinear motion.

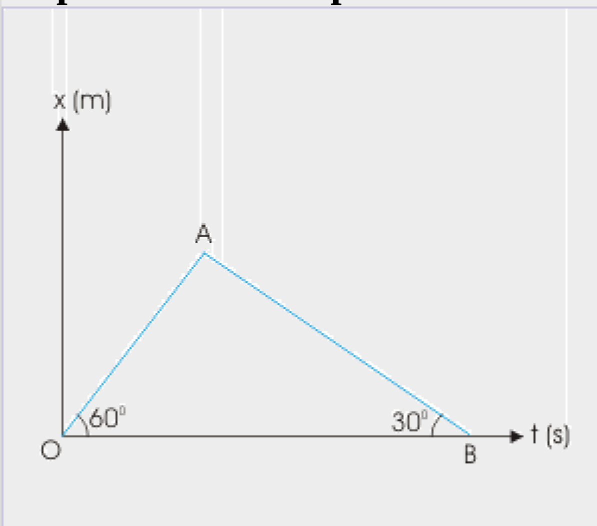
1: Which of the two resides closer to the school?
We can answer this question by knowing the displacements. The total displacements here are OC by P and OD by Q. From figure, OC < OD. Hence, P resides closer to the school.
2: Which of the two starts earlier for the school?
The start times are point O for P and A for Q on the time axis. Hence, P starts earlier for school.
3: Which of the two walks faster?
The speed is given by the slope of the plot. Slope of the motion of P is smaller than that of Q. Hence, Q walks faster.
4: Which of the two reaches the school earlier?
Both boys reach school at the time given by point B on time axis. Hence, they reach school at the same time.
5: Which of the two overtakes other during walk?
The plots intersect at a point E. They are at the same location at this time instant. Since, speed of B is greater, he overtakes A.

Interpretation of displacement - time plot

Example:

Problem : A displacement – time plot in one dimension is as shown. Find the ratio of velocities represented by two straight lines.

Displacement-time plot



There two segments of different velocities.

Solution : The slope of displacement – time plot is equal to velocity. Let v1 and v2 be the velocities in two segments, then magnitudes of velocities in two segments are :

$$|v_1| = \tan 60^\circ = \sqrt{3}$$
$$\left|v_2\right| = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

We note that velocity in the first segment is positive, whereas velocity in the second segment is negative. Hence, the required ratio of two velocities is :

$$\frac{v_1}{v_2} = -\frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} = -3$$

Displacement

Example:

Problem : The displacement “x” of a particle moving in one dimension is related to time “t” as :

$$t = \sqrt{x} + 3$$

where “t” is in seconds and “x” is in meters. Find the displacement of the particle when its velocity is zero.

Solution : We need to find “x” when velocity is zero. In order to find this, we require to have an expression for velocity. This, in turn, requires an expression of displacement in terms of time. The given expression of time, therefore, is required to be re-arranged :

$$\sqrt{x} = t - 3$$

Squaring both sides, we have :

$$\Rightarrow x = (t-3)^2 = t^2 - 6t + 9$$

Now, we obtain the required expression of velocity in one dimension by differentiating the above relation with respect to time,

$$v = \frac{dx}{dt} = 2t - 6$$

According to question,

$$v = 2t - 6 = 0$$
$$t = 3\ s$$

Putting this value of time in the expression of displacement, we have :

$$x = 3^2 - 6 \text{ x } 3 + 9 = 0$$

Average velocity

Example:

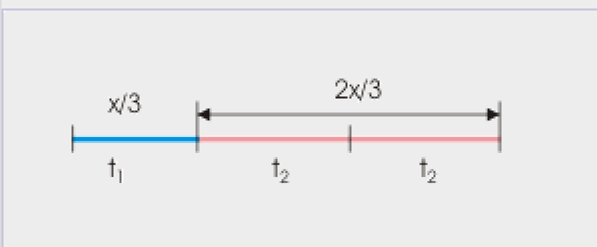
Problem : A particle moving in a straight line covers 1/3rd of the distance with a velocity 4 m/s. The remaining part of the linear distance is covered with velocities 2 m/s and 6 m/s for equal times. What is the average velocity during the total motion?

Solution : The average velocity is ratio of displacement and time. In one dimensional unidirectional motion, distance and displacement are same. As such, average velocity is ratio of total distance and time.

Let the total distance be "x". Now, we need to find total time in order to find the average velocity. Let " t_1 ", " t_2 " and " t_2 " be the time periods of the motion in three parts of the motion as given in the question.

Considering the first part of the motion,

Unidirectional motion



The particle covers segments of the motion with different average velocities.

$$t_1 = \frac{x_1}{v_1} = \frac{\frac{x}{3}}{4} = \frac{x}{12}$$

For the second part of the motion, the particle covers the remaining distance at two different velocities for equal time, " t_2 ",

$$x - \frac{x}{3} = v_2 \text{ x } t_2 + v_3 \text{ x } t_2$$

$$\Rightarrow \frac{2x}{3} = 2t_2 + 6t_2 = 8t_2$$

$$\Rightarrow t_2 = \frac{2x}{24} = \frac{x}{12}$$

Thus, average velocity is :

$$v_a = \frac{x}{t} = \frac{x}{t_1 + 2t_2} = \frac{x}{\frac{x}{12} + \frac{x}{6}} = 4\, m/s$$

Understanding motion

The discussion of different attributes of motion in previous modules has led us to the study of motion from the point of view of a general consideration to a simplified consideration such as uniform or rectilinear motion. The time is now ripe to recapitulate and highlight important results - particularly where distinctions are to be made.

For convenience, we shall refer general motion as the one that involves non-linear, two/ three dimensional motion. The simplified motion, on the other hand, shall refer motion that involves one dimensional, rectilinear and uniform motion.

Consideration of scalar quantities like distance and speed are same for “general” as well as “simplified” cases. We need to score similarities or differences for vector quantities to complete our understanding up to this point. It is relevant here to point out that most of these aspects have already been dealt in detail in previous modules. As such, we shall limit our discussion on main points/ results and shall generally not use figures and details.

Similarities and differences

Similarity / Difference 1 : In general, the magnitude of displacement is not equal to distance.
Equation:

$$|\Delta \mathbf{r}| \le s$$

For rectilinear motion (one dimensional case) also, displacement is not equal to distance as motion may involve reversal of direction along a line.
Equation:

$$|\Delta x| \le s$$

For uniform motion (unidirectional motion),

Equation:

$$|\Delta x| = s$$

Similarity / Difference 2 : The change in the magnitude of position vector is not equal to the magnitude of change in position vector except for uniform motion i.e motion with constant velocity.

For two/three dimensional motion,
Equation:

$$\Delta r \neq |\Delta \mathbf{r}|$$

For one dimensional motion,
Equation:

$$\Delta x \neq |\Delta x|$$

For uniform motion (unidirectional),
Equation:

$$\Delta x = |\Delta x|$$

Similarity / Difference 3 : In all cases, we can draw a distance – time or speed – time plot. The area under speed – time plot equals distance (s).
Equation:

$$s = \int v \, \mathrm{d}t$$

Similarity / Difference 4 : There is an ordered sequence of differentiation with respect to time that gives motional attributes of higher order. For example first differentiation of position vector or displacement yields velocity. We shall come to know subsequently that differentiation of

velocity, in turn, with respect to time yields acceleration. Differentiation, therefore, is a tool to get values for higher order attributes.

These differentiations are defining relations for the attributes of motion and hence applicable in all cases irrespective of the dimensions of motion or nature of velocity (constant or variable).

For two or three dimensional motion,
Equation:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

For one dimensional motion,
Equation:

$$v = \frac{dx}{dt}$$

Similarity / Difference 5 : Just like differentiation, there is an ordered sequence of integration that gives motional attributes of lower attributes. Since these integrations are based on basic/ defining differential equations, the integration is applicable in all cases irrespective of the dimensions of motion or nature of velocity (constant or variable).

For two or three dimensional motion,
Equation:

$$\Delta\mathbf{r} = \int \mathbf{v}d\ \mathrm{t}$$

For one dimensional motion,
Equation:

$$\Delta x = \int v d\ \mathrm{t}$$

Similarity / Difference 6 : We can not draw position – time, displacement – time or velocity – time plots for three dimensional motion. We can draw these plots for two dimensional motion, but the same would be complex and as such we would avoid drawing them.

We can, however, draw the same for one – dimensional motion by treating the vector attributes (position vector, displacement and velocity) as scalar with appropriate sign. Such drawing would be in the first and fourth quarters of two – dimensional plots. We should clearly understand that if we are drawing these plots, then the motion is either one or two dimensional. In general, we draw attribute .vs. time plots mostly for one – dimensional motion. We should also understand that graphical method is an additional tool for analysis in one dimensional motion.

The slope of the curve on these plots enables us to calculate the magnitude of higher attributes. The slope of position – time and displacement – time plot gives the magnitude of velocity; whereas the slope of velocity – time plot gives the magnitude of acceleration (This will be dealt in separate module).

Significantly, the tangent to the slopes on a "time" plot does not represent direction of motion. It is important to understand that the though the nature of slope (positive or negative) gives the direction of motion with respect to reference direction, but the tangent in itself does not indicate direction of motion. We must distinguish these “time” plots with simple position plots. The curve on the simple position plot is actual representation of the path of motion. Hence, tangent to the curve on position plot (plot on a x,y,z coordinate system) gives the direction of motion.

Similarity / Difference 7 : Needless to say that what is valid for one dimensional motion is also valid for the component motion in the case of two or three dimensional motion. This is actually a powerful technique to even treat a complex two or three dimensional motion, using one dimensional techniques. This aspect will be demonstrated on topics such as projectile and circular motion.

Similarity / Difference 8 : The area under velocity – time plot (for one dimensional motion) is equal to displacement.

Equation:

$$x = \int v dt$$

As area represents a vector (displacement), we treat area as scalar with appropriate sign for one dimensional motion. The positive area above the time axis gives the positive displacement, whereas the negative area below time axis gives negative displacement. The algebraic sum with appropriate sign results in net displacement. The algebraic sum without sign results in net distance.

Important thing to realize is that this analysis tool is not available for analysis of three dimensional motion as we can not draw the plot in the first place.

Similarity / Difference 9 : There is a difficulty in giving differentiating symbols to speed and velocity in one dimensional motion as velocity is treated as scalar. Both are represented as simple letter “v”. Recall that a non-bold faced letter “v” represents speed in two/three dimensional case. An equivalent representation of speed, in general, is |v|, but is seldom used in practice. As we do not use vector for one dimensional motion, there is a conflicting representation of the same symbol, “v”. We are left with no other solution as to be elaborate and specific so that we are able to convey the meaning either directly or by context. Some conventions, in this regard, may be helpful :

- If “v” is speed, then it can not be negative.
- If “v” is velocity, then it can be either positive or negative.
- If possible follow the convention as under :

$$v = \text{velocity}$$
$$|v| = \text{speed}$$

Similarity / Difference 10 : In the case of uniform motion (unidirectional motion), there is no distinction between scalar and vector attributes at all. The distance .vs. displacement and speed .vs. velocity differences have no relevance. The paired quantities are treated equal and same. Motion is one

dimensional and unidirectional; there being no question of negative value for attributes with direction.

Similarity / Difference 11 : Since velocity is a vector quantity being the time rate of change of position vector (displacement), there can be change in velocity due to the change in position vector (displacement) in any of the following three ways :

1. change in magnitude
2. change in direction
3. change in both magnitude and direction

This realization brings about important subtle differences in defining terms of velocity and their symbolic representation. In general motion, velocity is read as the "time rate of change of position vector" :
Equation:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

The speed i.e. the magnitude of velocity is read as the "absolute value (magnitude) of the time rate of change of position vector" :
Equation:

$$v = \left|\frac{d\mathbf{r}}{dt}\right|$$

But the important thing to realize is that “time rate of change in the magnitude of position vector” is not same as “magnitude of the time rate of change of position vector”. As such the time rate of change of the magnitude of position vector is not equal to speed. This fact can be stated mathematically in different ways :
Equation:

$$\frac{dr}{dt} = \frac{d|\mathbf{r}|}{dt} \neq v$$
$$\frac{dr}{dt} = \frac{d|\mathbf{r}|}{dt} \neq \left|\frac{d\mathbf{r}}{dt}\right|$$

We shall work out an example of a motion in two dimensions (circular motion) subsequently in this module to illustrate this difference.

However, this difference disappears in the case of one dimensional motion. It is so because we use scalar quantity to represent vector attribute like position vector and velocity. Physically, we can interpret that there is no difference as there is no change of direction in one dimensional motion. It may be argued that there is a change in direction even in one dimensional motion in the form of reversal of motion, but then we should realize that we are interpreting instantaneous terms only – not the average terms which may be affected by reversal of motion. Here, except at the point of reversal of direction, the speed is :

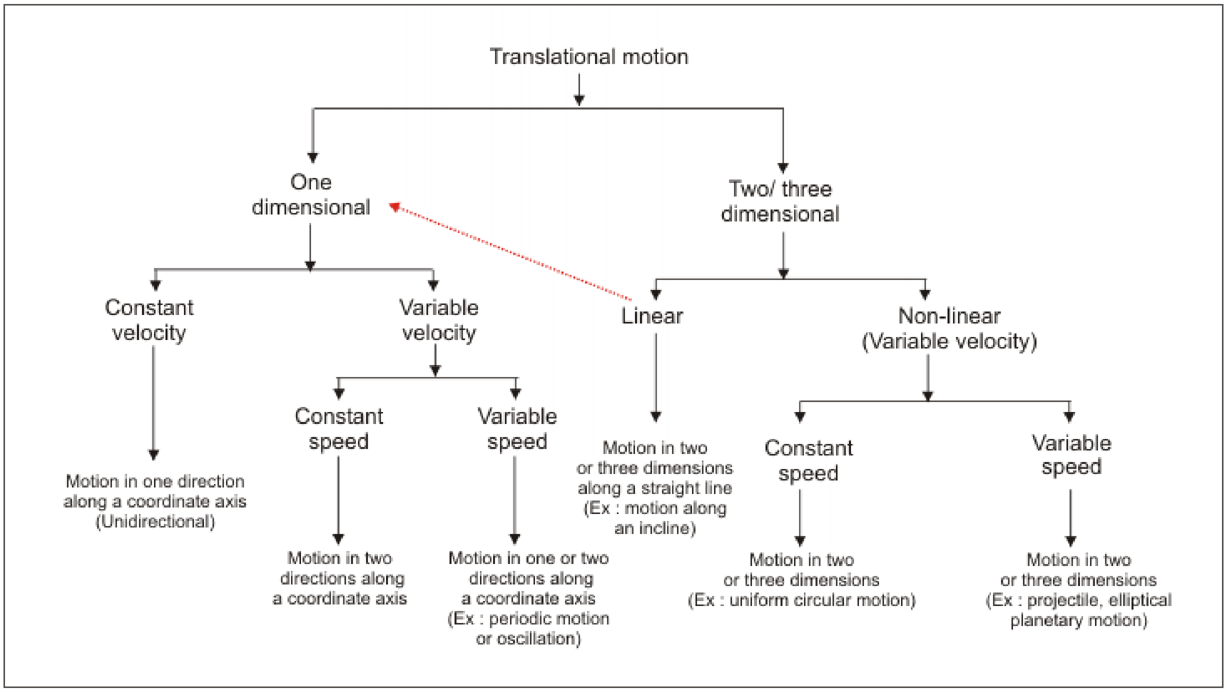
Equation:

$$v = \left|\frac{dx}{dt}\right| = \frac{d|x|}{dt}$$

Similarity / Difference 12 : Understanding of the class of motion is important from the point of view of analysis of motion (solving problem). The classification lets us clearly know which tools are available for analysis and which are not? Basically, our success or failure in understanding motion largely depends on our ability to identify motion according to a certain scheme of classification and then apply appropriate tool (formula/ defining equations etc) to analyze or solve the problem. It is, therefore, always advisable to write down the characteristics of motion for analyzing a situation involving motion in the correct context.

A simple classification of translational motion types, based on the study up to this point is suggested as given in the figure below. This classification is based on two considerations (i) dimensions of motion and (ii) nature of velocity.

Classification of motion



Classification based on (i) dimensions and (ii) velocity.

Example:

Problem : The position vector of a particle in motion is :

$$\mathbf{r} = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$$

where "a" is a constant. Find the time rate of change in the magnitude of position vector.

Solution : We need to know the magnitude of position vector to find its time rate of change. The magnitude of position vector is :

$$r = \left|\mathbf{r}\right| = \surd\left\{(a\cos\omega t)^2 + (a\sin\omega t)^2\right\}$$

$$\Rightarrow r = a\surd\left(\cos^2\omega t + \sin^2\omega t\right)$$

$$\Rightarrow r = a$$

But “a” is a constant. Hence, the time rate of change of the magnitude of position vector is zero :

$$\frac{dr}{dt} = 0$$

This result is an important result. This highlights that time rate of change of the magnitude of position vector is not equal to magnitude of time rate of change of the position vector (speed).
The velocity of the particle is obtained by differentiating the position vector with respect to time as :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\omega \sin \omega t\mathbf{i} + a\omega \cos \omega t\mathbf{j}$$

The speed, which is magnitude of velocity, is :

$$v = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left\{(-a\omega \sin \omega t)^2 + (a\omega \cos \omega t)^2\right\}}$$
$$\Rightarrow v = a\omega\sqrt{\left(\sin^2\omega t + \cos^2\omega t\right)}$$
$$\Rightarrow v = a\omega$$

Clearly, speed of the particle is not zero. This illustrates that even if there is no change in the magnitude of position vector, the particle can have instantaneous velocity owing to the change in the direction.
As a matter of fact, the motion given by the position vector in the question actually represents uniform circular motion, where particle is always at constant distance (position) from the center, but has velocity of constant speed and varying directions. We can verify this by finding the equation of path for the particle in motion, which is nothing but a relation between coordinates. An inspection of the expression for “x” and “y” coordinates suggest that following trigonometric identity would give the desired equation of path,

$$\sin^2\theta + \cos^2\theta = 1$$

For the given case,

$$\mathbf{r} = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$$
$$\Rightarrow x = a\cos\omega t$$
$$\Rightarrow y = a\sin\omega t\mathbf{j}$$

Rearranging, we have :

$$\cos\omega t = \frac{x}{a}$$

and

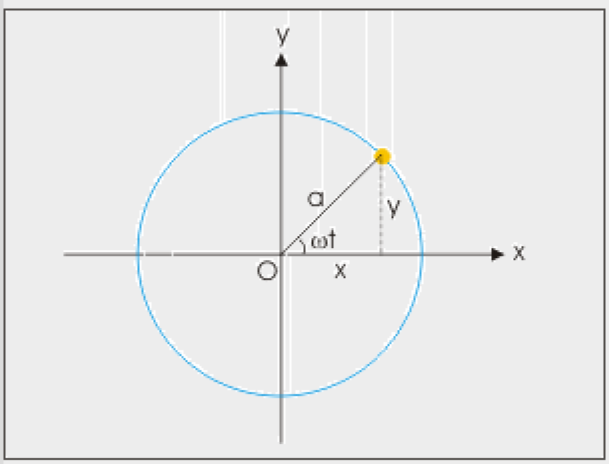
$$\sin\omega t = \frac{y}{a}$$

Now, using trigonometric identity :

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$
$$\Rightarrow x^2 + y^2 = a^2$$

This is an equation of a circle of radius “a”.

Uniform circular motion



The particle moves along a circular path with a constant speed.

In the nutshell, for motion in general (for two/three dimensions),

$$\frac{dr}{dt} = \frac{d|\mathbf{r}|}{dt} \neq v$$

$$\frac{dr}{dt} = \frac{d|\mathbf{r}|}{dt} \neq \left|\frac{d\mathbf{r}}{dt}\right|$$

It is only in one dimensional motion that this distinction disappears as there is no change of direction as far as instantaneous velocity is concerned.

Velocity (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the concept of velocity. For this reason, questions are categorized in terms of the characterizing features of the subject matter :

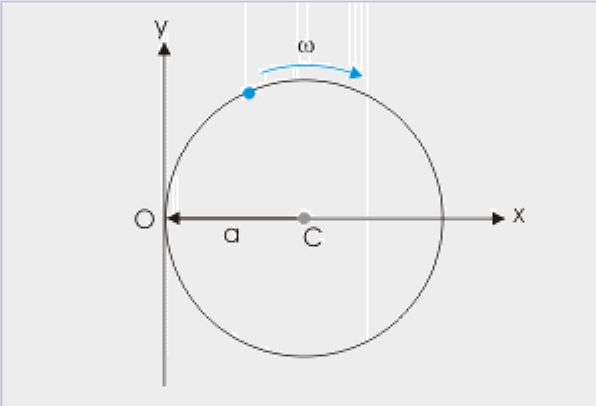
- Position vector
- Displacement
- Constrained motion
- Nature of velocity
- Comparing velocities

Position vector

Example:

Problem : A particle is executing motion along a circle of radius “a” with a constant angular speed “ω” as shown in the figure. If the particle is at “O” at t = 0, then determine the position vector of the particle at an instant in xy - plane with "O" as the origin of the coordinate system.

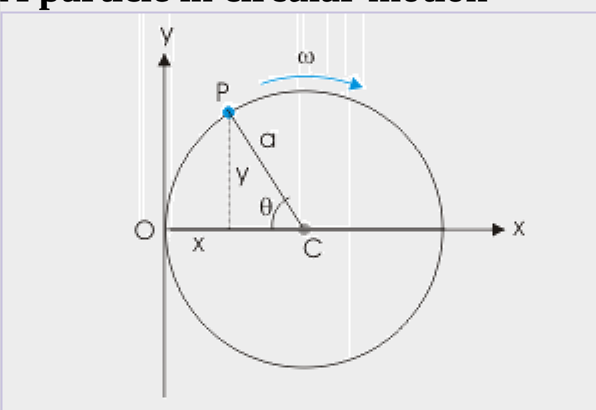
A particle in circular motion



The particle moves with a constant angular velocity.

Solution : Let the particle be at position “P” at a given time “t”. Then the position vector of the particle is :

A particle in circular motion



The particle moves with a constant angular velocity starting from “O” at t = 0.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

Note that "x" and "y" components of position vector is measured from the origin "O". From the figure,

$$y = a\sin\theta = a\sin\omega t$$

and

$$x = a - a\cos\omega t = a(1 - \cos\omega t)$$

Thus, position vector of the particle in circular motion is :

$$\mathbf{r} = a(1 - \cos\omega t)\mathbf{i} + a\sin\omega t\mathbf{j}$$

Example:

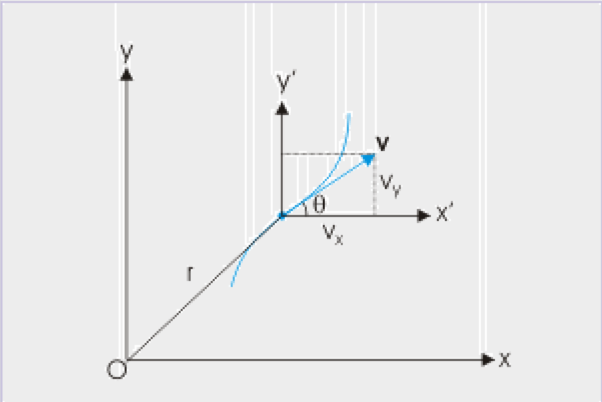
Problem : The position vector of a particle (in meters) is given as a function of time as :

$$\mathbf{r} = 2t\mathbf{i} + 2t^2\mathbf{j}$$

Determine the time rate of change of the angle “θ” made by the velocity vector with positive x-axis at time, t = 2 s.

Solution : It is a two dimensional motion. The figure below shows how velocity vector makes an angle "θ" with x-axis of the coordinate system. In order to find the time rate of change of this angle "θ", we need to express trigonometric ratio of the angle in terms of the components of velocity vector. From the figure :

Velocity of a particle in two dimensions



The velocity has two

components.

$$\tan\theta = \frac{v_y}{v_x}$$

As given by the expression of position vector, its component in coordinate directions are :

$$x = 2t \;\text{ and }\; y = 2t^2$$

We obtain expression of the components of velocity in two directions by differentiating "x" and "y" components of position vector with respect to time :

$$v_x = 2 \;\text{ and }\; v_y = 4t$$

Putting in the trigonometric function, we have :

$$\tan\theta = \frac{v_y}{v_x} = \frac{4t}{2} = 2t$$

Since we are required to know the time rate of the angle, we differentiate the above trigonometric ratio with respect to time as,

$$\sec^2\theta \frac{d\theta}{dt} = 2$$

$$\Rightarrow \left(1 + \tan^2\theta\right)\frac{d\theta}{dt} = 2$$

$$\Rightarrow \left(1 + 4t^2\right)\frac{d\theta}{dt} = 2$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{2}{(1+4t^2)}$$

At t = 2 s,

$$\Rightarrow \frac{d\theta}{dt} = \frac{2}{(1+4 \text{ x } 2^2)} = \frac{2}{17}\text{ rad}/s$$

Displacement

Example:
Problem : The displacement (x) of a particle is given by :

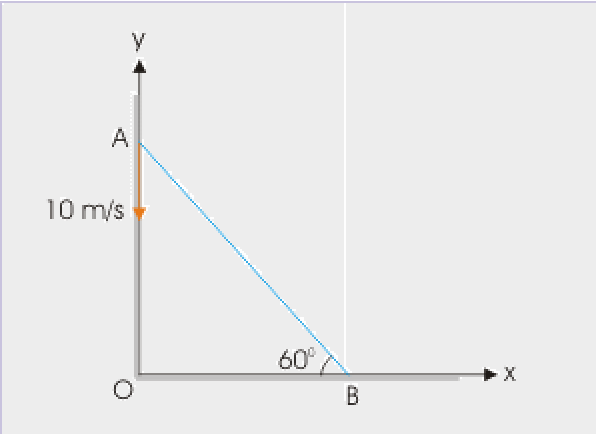
$$x = A\sin(\omega t + \theta)$$

At what time is the displacement maximum?
Solution : The displacement (x) depends on the value of sine function. It will be maximum for maximum value of sin(wt + θ). The maximum value of sine function is 1. Hence,

$$\sin\left(\omega t + \theta\right) = 1 = \sin\left(\frac{\pi}{2}\right)$$
$$\Rightarrow \omega t + \theta = \pi 2$$
$$\Rightarrow t = \frac{\pi}{2\omega} - \frac{\theta}{\omega}$$

Constrained motion

Example:
Problem : Two particles A and B are connected by a rigid rod AB. The rod slides along perpendicular rails as shown here. The velocity of A moving down is 10 m/s. What is the velocity of B when angle θ = 60° ?
Motion of a leaning rod



One end of the ros is moving with a speed 10 m/s in vertically downward direction.

Solution : The velocity of B is not an independent velocity. It is tied to the velocity of the particle "A" as two particles are connected through a rigid rod. The relationship between two velocities is governed by the inter-particles separation, which is equal to the length of rod.
The length of the rod, in turn, is linked to the positions of particles "A" and "B" . From figure,

$$x = \sqrt{(L^2 - y^2)}$$

Differentiatiting, with respect to time :

$$\Rightarrow v_x = \frac{dx}{dt} = -\frac{2y}{2\sqrt{(L^2 - y^2)}} X \frac{dy}{dt} = -\frac{y v_y}{\sqrt{(L^2 - y^2)}} = -v_y \tan\theta$$

Considering magnitude only,

$$\Rightarrow v_x = v_y \tan\theta = 10 \tan 60^0 = 10\sqrt{3} \quad \frac{m}{s}$$

Nature of velocity

Example:

Problem : The position vector of a particle is :

$$\mathbf{r} = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

where “a” is a constant. Show that velocity vector is perpendicular to position vector.

Solution : In order to prove as required, we shall use the fact that scalar (dot) product of two perpendicular vectors is zero. Now, we need to find the expression of velocity to evaluate the dot product as intended. We can obtain the same by differentiating the expression of position vector with respect to time as :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}$$

To check whether velocity is perpendicular to the position vector, we evalaute the scalar product of **r** and **v**, which should be equal to zero.

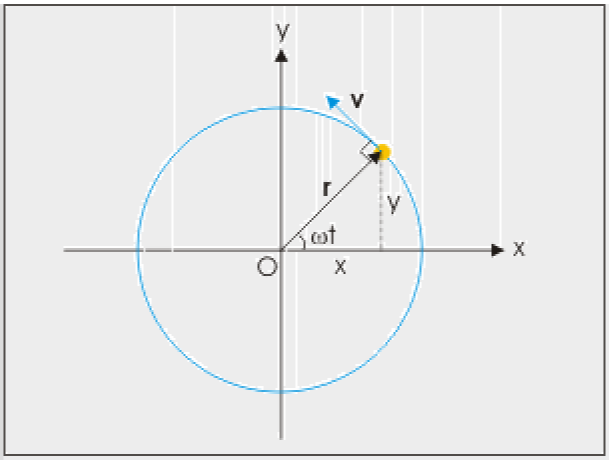
$$\mathbf{r}.\mathbf{v} = 0$$

In this case,

$$\Rightarrow \mathbf{r}.\mathbf{v} = (a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}) \,.\, (-a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j})$$
$$\Rightarrow -a^2\omega \sin \omega t \cos \omega t + a^2\omega \sin \omega t \cos \omega t = 0$$

This means that the angle between position vector and velocity are at right angle to each other. Hence, velocity is perpendicular to position vector. It is pertinent to mention here that this result can also be inferred from the plot of motion. An inspection of position vector reveals that it represents uniform circular motion as shown in the figure here.

Circular motion



The particle describes a circular path.

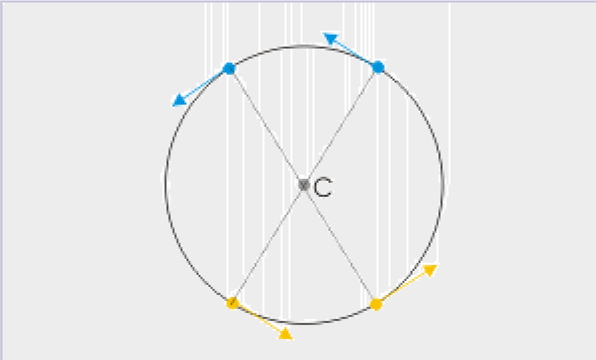
The position vector is always directed radially, whereas velocity vector is always tangential to the circular path. These two vectors are, therefore, perpendicular to each other.

Example:

Problem : Two particles are moving with the same constant speed, but in opposite direction. Under what circumstance will the separation between two remains constant?

Solution : The condition of motion as stated in the question is possible, if particles are at diametrically opposite positions on a circular path. Two particles are always separated by the diameter of the circular path. See the figure below to evaluate the motion and separation between the particles.

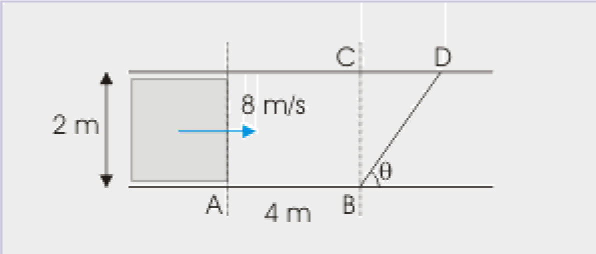
Motion along a circular path



Two particles are always separated by the diameter of the circle transversed by the particles.

Comparing velocities

Example:
Problem : A car of width 2 m is approaching a crossing at a velocity of 8 m/s. A pedestrian at a distance of 4 m wishes to cross the road safely. What should be the minimum speed of pedestrian so that he/she crosses the road safely?
Solution : We draw the figure to illustrate the situation. Here, car travels the linear distance (AB + CD) along the direction in which it moves, by which time the pedestrian travels the linear distance BD. Let pedestrian travels at a speed "v" along BD, which makes an angle "θ" with the direction of car.
Motion of a car and a pedestrian



The pedestrian crosses the road at angle with direction of car.

We must understand here that there may be number of combination of angle and speed for which pedestrian will be able to safely cross before car reaches. However, we are required to find the minimum speed. This speed should, then, correspond to a particular value of θ.
We can also observe that pedestrian should move obliquely. In doing so he/she gains extra time to cross the road.
From triangle BCD,

$$\tan\left(90-\theta\right) = \cot\theta = \frac{\mathrm{CD}}{\mathrm{BC}} = \frac{\mathrm{CD}}{2}$$
$$\Rightarrow \mathrm{CD} = 2\cot\theta$$

Also,

$$\cos\left(90-\theta\right) = \sin\theta = \frac{\mathrm{BC}}{\mathrm{BD}} = \frac{2}{\mathrm{BD}}$$
$$\Rightarrow \mathrm{BD} = \frac{2}{\sin\theta}$$

According to the condition given in the question, the time taken by car and pedestrian should be equal for the situation outlined above :

$$t = \frac{4+2\cot\theta}{8} = \frac{\frac{2}{\sin\theta}}{v}$$

$$v = \frac{8}{2\sin\theta+\cos\theta}$$

For minimum value of speed, $\frac{dv}{d\theta} = 0$,

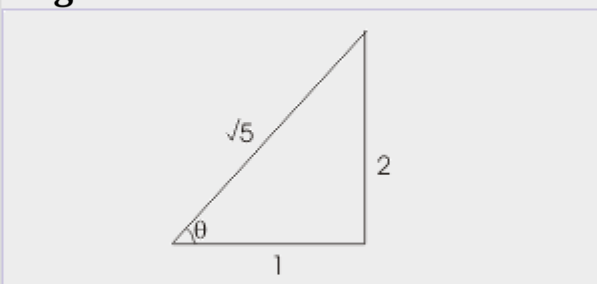
$$\Rightarrow \frac{dv}{d\theta} = \frac{-8 \text{ x } (2\cos\theta - \sin\theta)}{(2\sin\theta + \cos\theta)^2} = 0$$

$$\Rightarrow (2\cos\theta - \sin\theta) = 0$$

$$\Rightarrow \tan\theta = 2$$

In order to evaluate the expression of velocity with trigonometric ratios, we take the help of right angle triangle as shown in the figure, which is consistent with the above result.

Trigonometric ratio



The tangent of angle is equal to 2.

From the triangle, defining angle “θ”, we have :

$$\sin\theta = 2\sqrt{5}$$

and

$$\cos\theta = \frac{1}{\sqrt{5}}$$

The minimum velocity is :

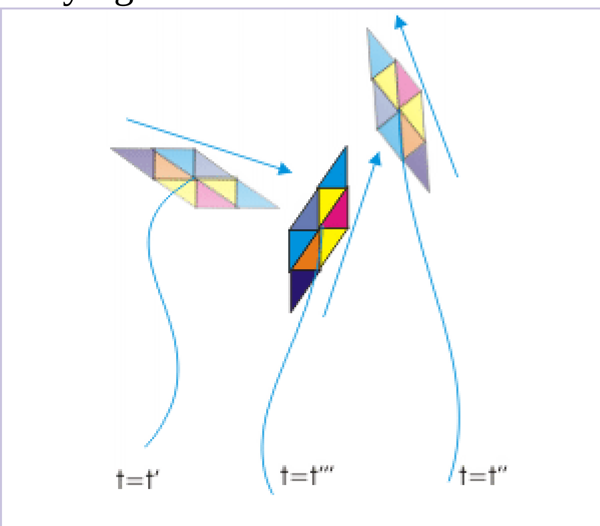
$$v = \frac{8}{2 \text{ x } 2\sqrt{5 + \frac{1}{\sqrt{5}}}} = \frac{8}{\sqrt{5}} = 3.57\, m/s$$

Acceleration

All bodies have intrinsic property to maintain its velocity. This is a fundamental nature of matter. However, a change in the velocity results when a net external force is applied. In that situation, velocity is not constant and is a function of time.

In our daily life, we are often subjected to the change in velocity. The incidence of the change in velocity is so common that we subconsciously treat constant velocity more as a theoretical consideration. We drive car with varying velocity, while negotiating traffic and curves. We take a ride on the train, which starts from rest and comes to rest. We use lift to ascend and descend floors at varying speeds. All these daily life routines involve change in the velocity. The spontaneous natural phenomena are also largely subjected to force and change in velocity. A flying kite changes its velocity in response to wind force as shown in the figure below.

A flying kite



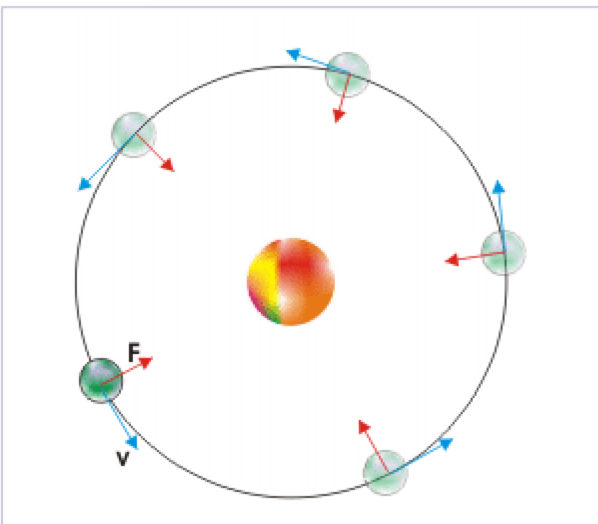
The kite changes its velocity in response to wind force.

The presence of external force is a common feature of our life and not an exception. Our existence on earth, as a matter of fact, is under the moderation of force due to gravity and friction. It is worth while here to

point out that the interaction of external force with bodies is not limited to the earth, but extends to all bodies like stars, planets and other mass aggregation.

For example, we may consider the motion of Earth around Sun that takes one year. For illustration purpose, let us approximate the path of motion of the earth as circle. Now, the natural tendency of the earth is to move linearly along the straight line in accordance with the Newton's first law of motion. But, the earth is made to change its direction continuously by the force of gravitation (shown with red arrow in the figure) that operates between the Earth and the Sun. The change in velocity in this simplified illustration is limited to the change in the direction of the velocity (shown in with blue arrow).

Motion of earth around sun



Gravitational force changes the direction of the motion

This example points to an interesting aspect of the change in velocity under the action of an external force. The change in velocity need not be a change in the magnitude of velocity alone, but may involve change of magnitude or direction or both. Also, the change in velocity (effect) essentially indicates the presence of a net external force (cause).

Acceleration

Acceleration

Acceleration is the rate of change of velocity with respect to time.

It is evident from the definition that acceleration is a vector quantity having both magnitude and direction, being a ratio involving velocity vector and scalar time. Mathematically,

$$\mathbf{a} = \frac{\Delta\mathbf{v}}{\Delta t}$$

The ratio denotes average acceleration, when the measurement involves finite time interval; whereas the ratio denotes instantaneous acceleration for infinitesimally small time interval, $\Delta t \rightarrow 0$.

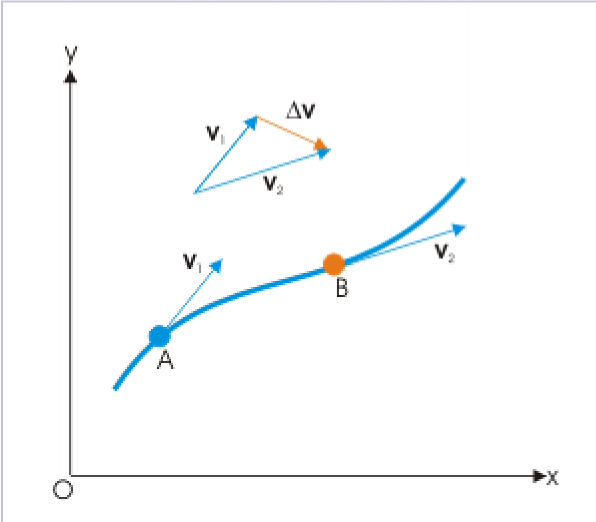
We know that velocity itself has the dimension of length divided by time; the dimension of the acceleration, which is equal to the change in velocity divided by time, involves division of length by squared time and hence its dimensional formula is LT^{-2} . The SI unit of acceleration is $\text{meter}/\,\text{second}^2$ i.e. m/s^2 .

Average acceleration

Average acceleration gives the overall acceleration over a finite interval of time. The magnitude of the average acceleration tells us the rapidity with which the velocity of the object changes in a given time interval.

$$\mathbf{a}_{\text{avg}} = \frac{\Delta\mathbf{v}}{\Delta t}$$

Average acceleration



The direction of acceleration is along the vector $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ and not required to be in the direction of either of the velocities. If the initial velocity is zero, then $\Delta \mathbf{v} = \mathbf{v}_2 = \mathbf{v}$ and average acceleration is in the direction of final velocity.

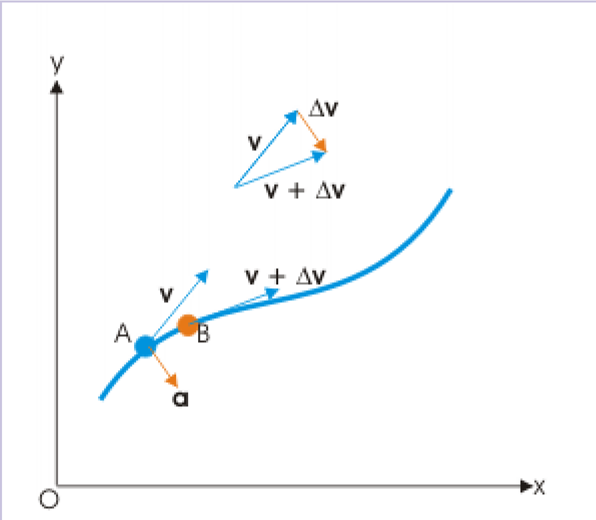
Note: Difference of two vectors $\mathbf{v}_2 - \mathbf{v}_1$ can be drawn conveniently following certain convention (We can take a mental note of the procedure for future use). We draw a straight line, starting from the arrow tip (i.e. head) of the vector being subtracted $\mathbf{v}_1$ to the arrow tip (i.e. head) of the vector from which subtraction is to be made $\mathbf{v}_2$. Then, from the triangle law of vector addition,

$$\mathbf{v}_1 + \Delta \mathbf{v} = \mathbf{v}_2$$
$$\Rightarrow \Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$$

Instantaneous acceleration

Instantaneous acceleration, as the name suggests, is the acceleration at a given instant, which is obtained by evaluating the limit of the average acceleration as $\Delta t \rightarrow 0$.

Instantaneous acceleration



$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\mathbf{\Delta v}}{\Delta \mathrm{t}} = \frac{đ\mathbf{v}}{đt}$$

As the point B approaches towards A, the limit of the ratio evaluates to a finite value. Note that the ratio evaluates not along the tangent to the curve as in the case of velocity, but along the direction shown by the red arrow. This is a significant result as it tells us that direction of acceleration is independent of the direction of velocity.

Instantaneous acceleration

> Instantaneous acceleration is equal to the first derivative of velocity with respect to time.

It is evident that a body might undergo different phases of acceleration during the motion, depending on the external forces acting on the body. It means that accelerations in a given time interval may vary. As such, the average and the instantaneous accelerations need not be equal.

A general reference to the term acceleration (**a**) refers to the instantaneous acceleration – not average acceleration. The absolute value of acceleration gives the magnitude of acceleration :

$$|\mathbf{a}| = a$$

Acceleration in terms of position vector

Velocity is defined as derivative of position vector :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

Combining this expression of velocity into the expression for acceleration, we obtain,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \quad = \frac{d^2\mathbf{r}}{dt^2}$$

Acceleration
: Acceleration of a point body is equal to the second derivative of position vector with respect to time.

Now, position vector is represented in terms of components as :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Substituting in the expression of acceleration, we have :

$$\mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$
$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Example:
Acceleration
Problem : The position of a particle, in meters, moving in space is described by following functions in time.

$$x = 2t^2 - 2t + 3; y = -4t \text{ and } z = 5$$

Find accelerations of the particle at t = 1 and 4 seconds from the start of motion.
Solution : Here scalar components of accelerations in x,y and z directions are given as :

$$a_x = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}\left(2t^2 - 2t + 3\right) = 4$$

$$a_y = \frac{d^2y}{dt^2} = \frac{d^2}{dt^2}\left(-4t\right) = 0$$

$$a_z = \frac{d^2z}{dt^2} = \frac{d^2}{dt^2}\left(5\right) = 0$$

Thus, acceleration of the particle is :

$$\mathbf{a} = 4\mathbf{i}$$

The acceleration of the particle is constant and is along x-direction. As acceleration is not a function of time, the accelerations at t = 1 and 4 seconds are same being equal to $4m/s^2$.

Understanding acceleration

The relationship among velocity, acceleration and force is central to the study of mechanics. Text book treatment normally divides the scope of study between kinematics (study of velocity and acceleration) and dynamics (study of acceleration and force). This approach is perfectly fine till we do not encounter some inherent problem as to the understanding of the subject matter.

One important short coming of this approach is that we tend to assume that acceleration depends on the state of motion i.e. velocity. Such inference is not all too uncommon as acceleration is defined in terms of velocities. Now the question is "Does acceleration actually depend on the velocity?"

There are many other such inferences that need to be checked. Following paragraphs investigate these fundamental issues in detail and enrich our understanding of acceleration.

Velocity, acceleration and force

Acceleration is related to external force. This relationship is given by Newton's second law of motion. For a constant mass system,

$$\mathbf{F} = m\mathbf{a}$$

In words, Newton's second law states that acceleration (effect) is the result of the application of net external force (cause). Thus, the relationship between the two quantities is that of cause and effect. Further, force is equal to the product of a scalar quantity, m, and a vector quantity, **a**, implying that the direction of the acceleration is same as that of the net force. This means that acceleration is though the measurement of the change of velocity, but is strictly determined by the external force and the mass of the body; and expressed in terms of "change of velocity" per unit time.

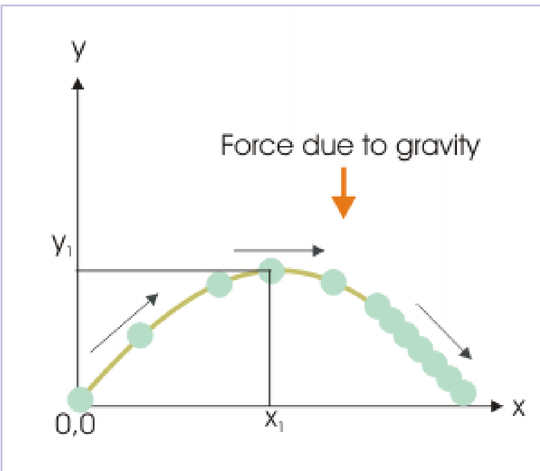
If we look around ourselves, we find that force modifies the state of motion of the objects. The immediate effect of a net force on a body is that the state of motion of the body changes. In other words, the velocity of the body changes in response to the application of net force. Here, the word "net" is important. The motion of the body responds to the net or resultant force. In this sense, acceleration is mere statement of the effect of the force as measurement of the rate of change of the velocity with time.

Now, there is complete freedom as to the magnitude and direction of force being applied. From our real time experience, we may substantiate this assertion. For example, we can deflect a foot ball, applying force as we wish (in both magnitude and direction). The motion of the ball has no bearing on how we apply force. Simply put : the magnitude and direction of the force (and that of acceleration) is not dependent on the magnitude and direction of the velocity of the body.

In the nutshell, **we conclude that force and hence acceleration is independent of the velocity of the body**. The magnitude and direction of the acceleration is determined by the magnitude and direction of the force and mass of the body. This is an important clarification.

To elucidate the assertion further, let us consider parabolic motion of a ball as shown in the figure. The important aspect of the parabolic motion is that the acceleration associated with motion is simply ‘g’ as there is no other force present except the force of gravity. The resultant force and mass of the ball together determine acceleration of the ball.

Parabolic motion

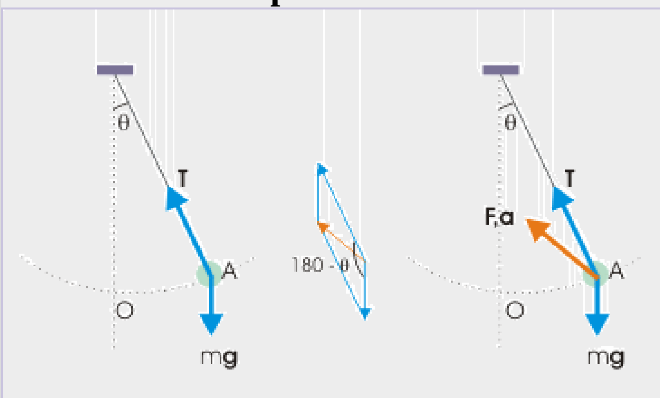


The resultant force and mass of the ball together determine acceleration of the ball

Example:
Acceleration

Problem : If the tension in the string is T, when the string makes an angle θ with the vertical. Find the acceleration of a pendulum bob, having a mass "m".
Solution : As pointed out in our discussion, we need not study or consider velocities of bob to get the answer. Instead we should strive to know the resultant force to find out acceleration, using Newton's second law of motion.
The forces, acting on the bob, are (i) force of gravity, mg, acting in the downward direction and (ii) Tension, T, acting along the string. Hence, the acceleration of the bob is determined by the resultant force, arising from the two forces.

Acceleration of a pendulum bob



Using parallelogram theorem for vector addition, the resultant force is :

$$F = \sqrt{\{m^2g^2 + T^2 + 2mgT\cos(180^\circ - \theta)\}} = \sqrt{(m^2g^2 + T^2 + 2mgT\cos\theta)}$$

The acceleration is in the direction of force as shown in the figure, whereas the magnitude of the acceleration, a, is given by :

$$a = \frac{\sqrt{(m^2g^2+T^2+2mgT\cos\theta)}}{m}$$

External force and possible scenarios

The change in velocity, resulting from the application of external force, may occur in magnitude or direction or both.

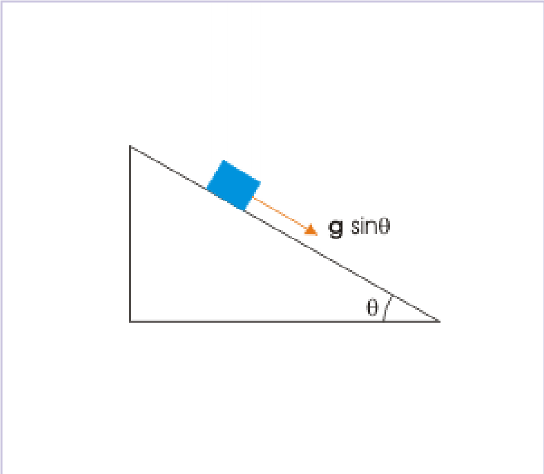
1: When force is applied in a given direction and the object is stationary.

Under this situation, the magnitude of velocity increases with time, while the body follows a linear path in the direction of force (or acceleration).

2: When force is applied in the direction of the motion, then it increases the magnitude of the velocity without any change in the direction.

Let us consider a block sliding on a smooth incline surface as shown in the figure. The component of the force due to gravity applies in the direction of motion. Under this situation, the magnitude of velocity increases with time, while the body follows a linear path. There is no change in the direction of motion.

A block sliding along smooth incline

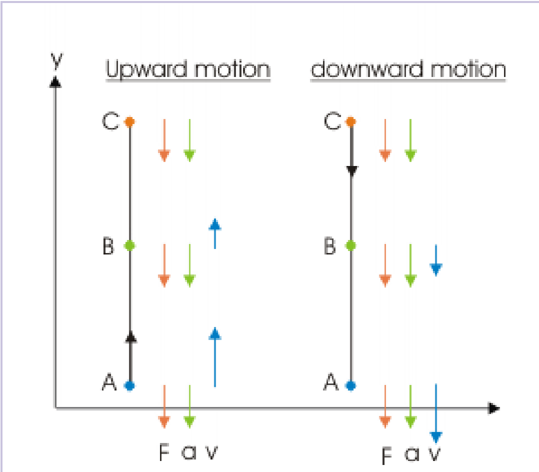


The block moves faster as it moves down the slope

3: When force is applied in the opposite direction to the motion, then it decreases the magnitude of the velocity without any change in the direction.

Take the example of a ball thrown vertically in the upward direction with certain velocity. Here, force due to gravity is acting downwards. The ball linearly rises to the maximum height till the velocity of ball reduces to zero.

A ball thrown in vertical direction



The ball reverses its direction during motion.

During upward motion, we see that the force acts in the opposite direction to that of the velocity. Under this situation, the magnitude of velocity decreases with time, while the body follows a linear path. There is no change in the direction of motion.

The velocity of the object at the point, where motion changes direction, is zero and force is acting downwards. This situation is same as the case 1. The object is at rest. Hence, object moves in the direction of force i.e. in the downward direction.

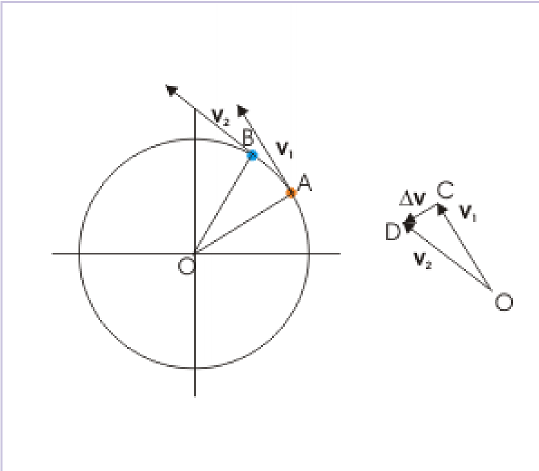
In the figure above, the vectors drawn at various points represent the direction and magnitude of force (**F**), acceleration (**a**) and velocity (**v**) during the motion. Note that both force and acceleration act in the downward direction during the motion.

4: When force is applied perpendicular to the direction of motion, then it causes change in the direction of the velocity.

A simple change of direction also constitutes change in velocity and, therefore, acceleration. Consider the case of a uniform circular motion in which a particle moves along a circular path at constant speed "v". Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities of the particle at two time instants, then

$$|\mathbf{v}_1| = |\mathbf{v}_2| = \text{ a constant}$$

Uniform circular motion



A central force perpendicular to motion causes change in direction.

In the adjoining figure, the vector segments OC and OD represent $\mathbf{v}_1$ and $\mathbf{v}_2$. Knowing that vector difference Δv is directed from initial to final position, it is represented by vector CD. Using the adjoining vector triangle,

$$\mathbf{OC} + \mathbf{CD} = \mathbf{OD}$$
$$\mathbf{v} = \mathbf{CD} = \mathbf{v}_2 - \mathbf{v}_1$$

The important thing to realize here is that direction of Δ **v** is along CD, which is directed towards the origin. This result is in complete agreement of what we know about uniform circular motion (The topic of uniform circular motion is covered in separate module). We need to apply a force (causing acceleration to the moving particle) across (i.e. perpendicular) to the motion to change direction. If the force (hence acceleration) is perpendicular to velocity, then magnitude of velocity i.e. speed remains same, whereas the direction of motion keeps changing.

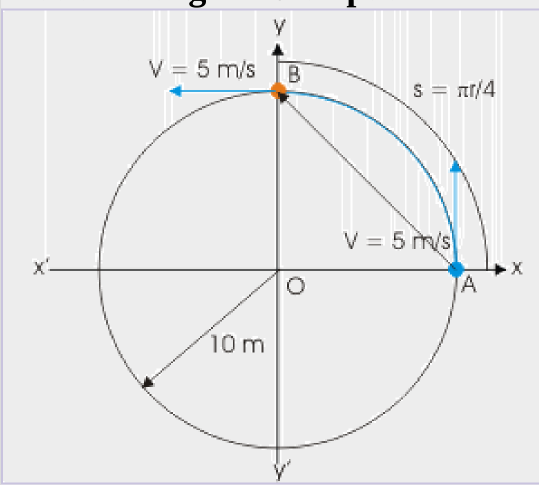
Example:
Acceleration
Problem : An object moves along a quadrant AB of a circle of radius 10 m with a constant speed 5 m/s. Find the average velocity and average acceleration in this interval.

Solution : Here, we can determine time interval form the first statement of the question. The particle covers a distance of 2πr/4 with a speed v. Hence, time interval,

$$t = \frac{\text{distance}}{\text{speed}} = \frac{2\pi r}{4v} = \frac{3.14x10}{2x5} = 3.14s$$

Magnitude of average velocity is given by the ratio of the magnitude of displacement and time :

Motion along circular path

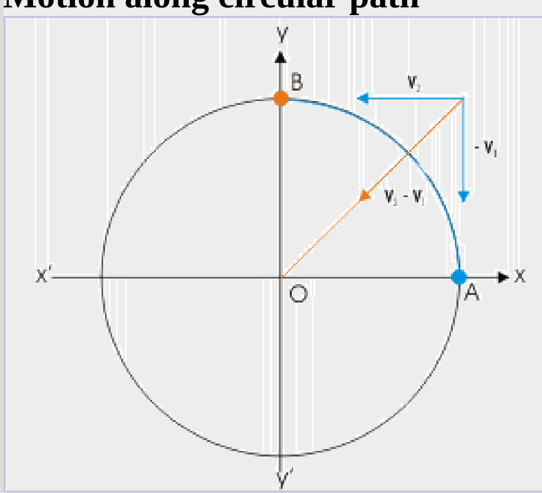


$$\left|\mathbf{v}_{\text{avg}}\right| == \frac{|\mathbf{AB}|}{\text{Time}} = \frac{\sqrt{(10^2+10^2)}}{3.14}\ \frac{14.14}{3.14} = 4.5\ m/s$$

Average velocity is directed along vector AB i.e. in the direction of displacement vector.

Magnitude of average acceleration is given by the ratio :

Motion along circular path



Direction of acceleration

$$a_{\text{avg}} == \frac{|\Delta\mathbf{v}|}{\Delta t}$$

Here,

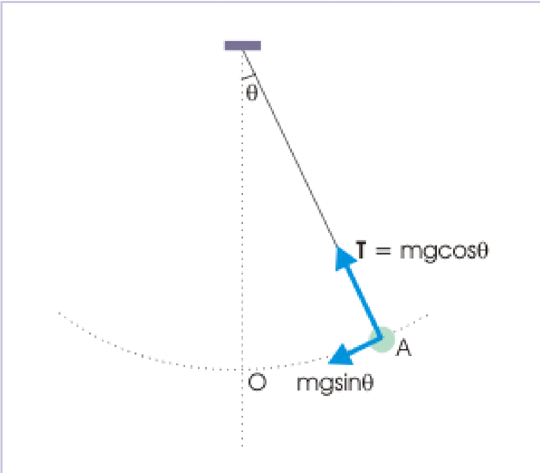
$$|\Delta\mathbf{v}| = |\mathbf{v}_2 - \mathbf{v}_1|$$
$$\left|\Delta\mathbf{v}\right| = \sqrt{(5^2 + 5^2)} = 5\sqrt{2}\ m/s$$
$$\Rightarrow a_{\text{avg}} = \frac{5\sqrt{2}}{3.14} = 3.1\ m/s^2$$

Average acceleration is directed along the direction of vector $\mathbf{v}_2 - \mathbf{v}_1$i.e. directed towards the center of the circle (as shown in the figure).

5: When force is applied at a certain angle with the motion, then it causes change in both magnitude and direction of the velocity. The component of force in the direction of motion changes its magnitude (an increase or decrease), while the component of force perpendicular to the direction of motion changes its direction.

The motion of a small spherical mass “m”, tied to a fixed point with the help of a string illustrates the changes taking place in both the direction and magnitude of the velocity. Saving the details for discussion at a later point in the course, here tension in the string and tangential force provide for the change in the direction and magnitude of velocity respectively.

When the string makes at angle, θ, with the vertical, then :
Motion of a pendulum



Forces acting on the pendulum bob

(a) Tension in the string, T is :

$$T = \mathrm{mg}\cos\theta$$

This force acts normal to the path of motion at all points and causes the body to continuously change its direction along the circular path. Note that this force acts perpendicular to the direction of velocity, which is along the tangent at any given point on the arc.

(b) Tangential force, F, is :

$$F = \mathrm{mg}\sin\theta$$

This force acts tangentially to the path and is in the direction of velocity of the spherical mass, which is also along the tangent at that point. As the force and velocity are in the same direction, this force changes the magnitude of velocity. The magnitude of velocity increases when force acts in the direction of velocity and magnitude of velocity decreases when force acts in the opposite direction to that of velocity.

Acceleration and deceleration
Negative vector is a relative term - not an independent concept. We completely loose the significance of a negative vector when we consider it in isolation. A negative vector assumes meaning only in relation with another vector or some reference direction.

Acceleration and deceleration are terms which are generally considered to have opposite meaning. However, there is difference between literal and scientific meanings of these terms. In literal sense, acceleration is considered to describe an increase or positive change of speed or velocity. On the other hand, deceleration is considered to describe a decrease or negative change of speed or velocity. Both these descriptions are incorrect in physics. We need to form accurate and exact meaning of these two terms. In this module, we shall explore these terms in the context of general properties of vector and scalar quantities.

A total of six (6) attributes viz time, distance, displacement, speed, velocity and acceleration are used to describe motion. Three of these namely time, distance and speed are scalar quantities, whereas the remaining three attributes namely displacement, velocity and acceleration are vectors. Interpretations of these two groups are different with respect to (ii) negative and positive sign and (i) sense of increase and decrease. Further these interpretations are also affected by whether we consider these terms in one or two/three dimensional motion.

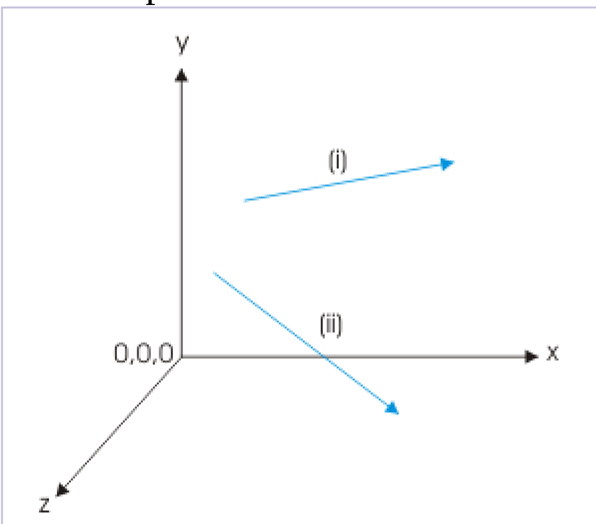
The meaning of scalar quantities is more and less clear. The scalar attributes have only magnitude and no sense of direction. The attributes “distance” and “speed” are positive quantities. There is no possibility of negative values for these two quantities. In general, time is also positive. However, it can be assigned negative value to represent a time instant that occurs before the start of observation. For this reason, it is entirely possible that we may get negative time as solution of kinematics consideration.

Negative vector quantities

A vector like acceleration may be directed in any direction in three dimensional space, which is defined by the reference coordinate system.

Now, why should we call vector (as shown in the figure below) represented by line (i) positive and that by line (ii) negative? What is the qualification for a vector being positive or negative? There is none. Hence, in pure mathematical sense, a negative vector can not be identified by itself.

Vector representation in three dimensional reference



Vectors in isolation have no sense of being negative.

In terms of component vectors, let us represent two accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ as :

$$\mathbf{a}_1 = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$$
$$\mathbf{a}_2 = -2\mathbf{i} +\text{-}\ 3\mathbf{j} + 4\mathbf{k}$$

Why should we call one of the above vectors as positive and other as negative acceleration? Which sign should be considered to identify a positive or negative vector? Further, the negative of $\mathbf{a}_1$ expressed in component form is another vector given by $-\mathbf{a}_1$:

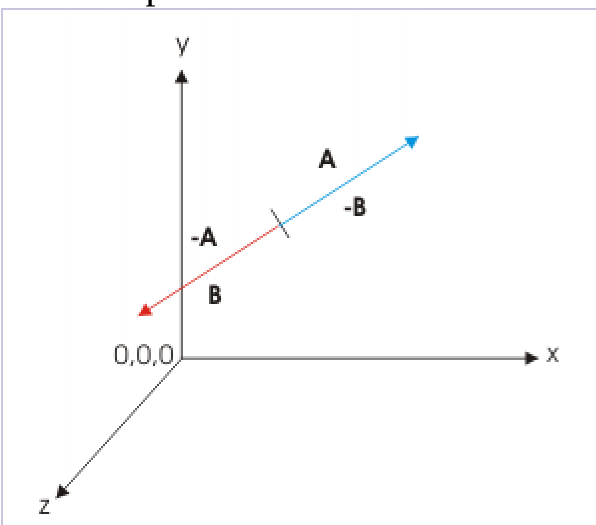
$$-\mathbf{a}_1 = -(-2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \quad = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

So, what is the actual position? The concept of negative vector is essentially a relative concept. If we represent a vector $\mathbf{A}$ as shown in the figure below,

then negative to this vector –**A** is just another vector, which is directed in the opposite direction to that of vector **A** and has equal magnitude as that of **A**.

In addition, if we denote "**-A**" as "**B**", then "**A**" is "**-B**". We, thus, completely loose the significance of a negative vector when we consider it in isolation. We can call the same vector either "**A**" or "**-A**". We conclude, therefore, that a negative vector assumes meaning only in relation with another vector.

Vector representation in three dimensional reference



Vectors in isolation have no sense of being negative.

In one dimensional motion, however, it is possible to assign distinct negative values. In this case, there are only two directions; one of which is in the direction of reference axis (positive) and the other is in the opposite direction (negative). The significance of negative vector in one dimensional motion is limited to relative orientation with respect to reference direction. In the nutshell, sign of vector quantity in one dimensional motion represents the directional property of vector. It has only this meaning. We can not attach any other meaning for negating a vector quantity.

It is important to note that the sequence in “-5**i**" is misleading in the sense that a vector quantity can not have negative magnitude. The negative sign, as a matter of fact, is meant for unit vector “**i**”. The correct reading sequence would be “5 x -“**i**”, meaning that it has a magnitude of “5” and is directed in “-“**i**” direction i.e. opposite to reference direction. Also, since we are free to choose our reference, the previously assigned “-5 **i**” can always be changed to “5 **i**” and vice-versa.

We summarize the discussion so far as :

- There is no independent meaning of a negative vector attribute.
- In general, a negative vector attribute is defined with respect to another vector attribute having equal magnitude, but opposite direction.
- In the case of one dimensional motion, the sign represents direction with respect to reference direction.

“Increase” and “decrease” of vectors quantities

The vector consist of both magnitude and direction. There can be infinite directions of a vector. On the other hand, increase and decrease are bi-directional and opposite concepts. Can we attach meaning to a phrase “increase in direction” or “decrease in direction”? There is no sense in saying that direction of the moving particle has increased or decreased. In the nutshell, we can associate the concepts of increase and decrease with quantities which are scalar – not quantities which are vector. Clearly, we can attach the sense of increase or decrease with the magnitudes of velocity or acceleration, but not with velocity and acceleration.

For this reason, we may recall that velocity is defined as the time rate of “change” – not “increase or decrease” in displacement. Similarly, acceleration is defined as time rate of change of velocity – not “increase or decrease” in velocity. It is so because the term “change” conveys the meaning of “change” in direction as well that of “change” as increase or decrease in the magnitude of a vector.

However, we see that phrases like “increase or decrease in velocity” or “increase or decrease in acceleration” are used frequently. We should be

aware that these references are correct only in very specific context of motion. If motion is unidirectional, then the vector quantities associated with motion is treated as either positive or negative scalar according as it is measured in the reference direction or opposite to it. Even in this situation, we can not associate concepts of increase and decrease to vector quantities. For example, how would we interpret two particles moving in negative x-direction with negative accelerations $-10m/s^2$ and $-20m/s^2$ respectively ? Which of the two accelerations is greater acceleration ? Algebraically, “-10” is greater than “-20”. But, we know that second particle is moving with higher rate of change in velocity. The second particle is accelerating at a higher rate than first particle. Negative sign only indicates that particle is moving in a direction opposite to a reference direction.

Clearly, the phrases like “increase or decrease in velocity” or “increase or decrease in acceleration” are correct only when motion is "unidirectional" and in "positive" reference direction. Only in this restricted context, we can say that acceleration and velocity are increasing or decreasing. In order to be consistent with algebraic meaning, however, we may prefer to associate relative measure (increase or decrease) with magnitude of the quantity and not with the vector quantity itself.

Deceleration

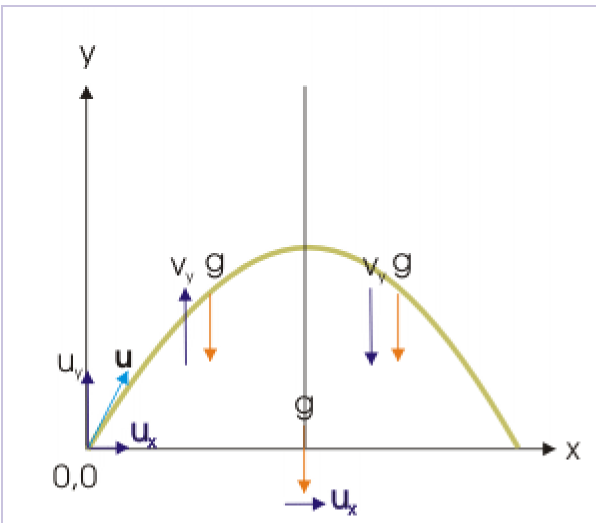
Acceleration is defined strictly as the time rate of change of velocity vector. Deceleration, on the other hand, is acceleration that causes reduction in "speed". Deceleration is not opposite of acceleration. It is certainly not negative time rate of change of velocity. It is a very restricted term as explained below.

We have seen that speed of a particle in motion decreases when component of acceleration is opposite to the direction of velocity. In this situation, we can say that particle is being decelerated. Even in this situation, we can not say that deceleration is opposite to acceleration. Here, only a component of acceleration is opposite to velocity – not the entire acceleration. However, if acceleration itself (not a component of it) is opposite to velocity, then deceleration is indeed opposite to acceleration.

If we consider motion in one dimension, then the deceleration occurs when signs of velocity and acceleration are opposite. A negative velocity and a positive acceleration mean deceleration; a positive velocity and a negative acceleration mean deceleration; a positive velocity and a positive acceleration mean acceleration; a negative velocity and a negative acceleration mean acceleration.

Take the case of projectile motion of a ball. We study this motion as two equivalent linear motions; one along x-direction and another along y-direction.

Parabolic motion



The projected ball undergoes deceleration in y-direction.

For the upward flight, velocity is positive and acceleration is negative. As such the projectile is decelerated and the speed of the ball in + y direction decreases (deceleration). For downward flight from the maximum height, velocity and acceleration both are negative. As such the projectile is accelerated and the speed of the ball in - y direction increases (acceleration).

In the nutshell, we summarize the discussion as :

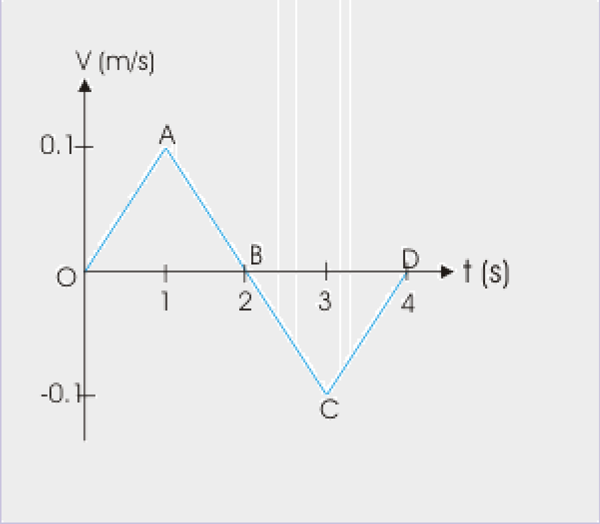
- Deceleration results in decrease in speed i.e magnitude of velocity.
- In one dimensional motion, the “deceleration” is defined as the acceleration which is opposite to the velocity.

Example:

Acceleration and deceleration

Problem : The velocity of a particle along a straight line is plotted with respect to time as shown in the figure. Find acceleration of the particle between OA and CD. What is acceleration at t = 0.5 second and 1.5 second. What is the nature of accelerations in different segments of motion? Also investigate acceleration at A.

Velocity – time plot



Solution : Average acceleration between O and A is given by the slope of straight line OA :

$$a_{\mathrm{OA}} = \frac{v_2 - v_1}{t} = \frac{0.1 - 0}{1} = 0.1 m/s^2$$

Average acceleration between C and D is given by the slope of straight line CD :

$$a_{\mathrm{CD}} = \frac{v_2 - v_1}{t} = \frac{0 - (-0.1)}{1} = 0.1 m/s^2$$

Accelerations at t = 0.5 second and 1.5 second are obtained by determining slopes of the curve at these time instants. In the example, the slopes at

these times are equal to the slope of the lines OA and AB.
Instantaneous acceleration at t = 0.5 s :

$$a_{0.5} = a_{\text{OA}} = 0.1 m/s^2$$

Instantaneous acceleration at t = 1.5 s :

$$a_{1.5} = a_{\text{AB}} = \frac{v_2 - v_1}{t} = \frac{0 - 0.1}{1} = -0.1 m/s^2$$

We check the direction of velocity and acceleration in different segments of the motion in order to determine deceleration. To enable comparision, we determine directions with respect to the assumed positive direction of velocity. In OA segment, both acceleration and velocity are positive (hence particle is accelerated). In AB segment, acceleration is negative, but velocity is positive (hence particle is decelerated). In BC segment, both acceleration and velocity are negative (hence particle is accelerated). In CD segment, acceleration is positive but velocity is negative (hence particle is decelerated).
Alternatively, the speed increases in segment OA and BC (hence acceleration); decreases in segments AB and CD (hence deceleration).
We note that it is not possible to draw an unique tangent at point A. We may draw infinite numbers of tangent at this point. In other words, limit of average acceleration can not be evaluated at A. Acceleration at A, therefore, is indeterminate.

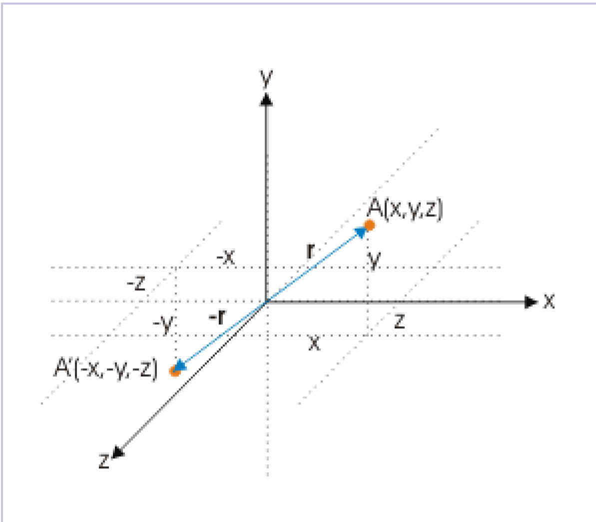
Graphical interpretation of negative vector quantities describing motion

Position vector

We may recall that position vector is drawn from the origin of reference to the position occupied by the body on a scale taken for drawing coordinate axes. This implies that the position vector is rooted to the origin of reference system and the position of the particle. Thus, we find that position vector is tied at both ends of its graphical representation.

Also if position vector ‘**r**’ denotes a particular position (A), then “**-r**” denotes another position (A’), which is lying on the opposite side of the reference point (origin).

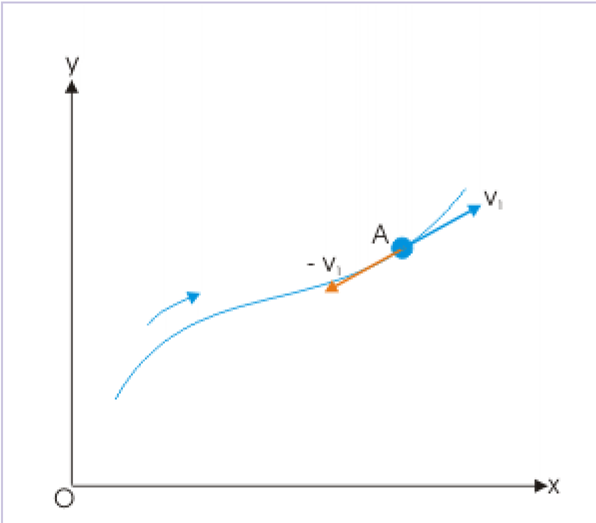
Position vector



Velocity vector

The velocity vector, on the other hand, is drawn on a scale from a particular position of the object with its tail and takes the direction of the tangent to the position curve at that point. Also, if velocity vector ‘**v**’ denotes the velocity of a particle at a particular position, then “**-v**” denotes another velocity vector, which is reversed in direction with respect to the velocity vector, **v**.

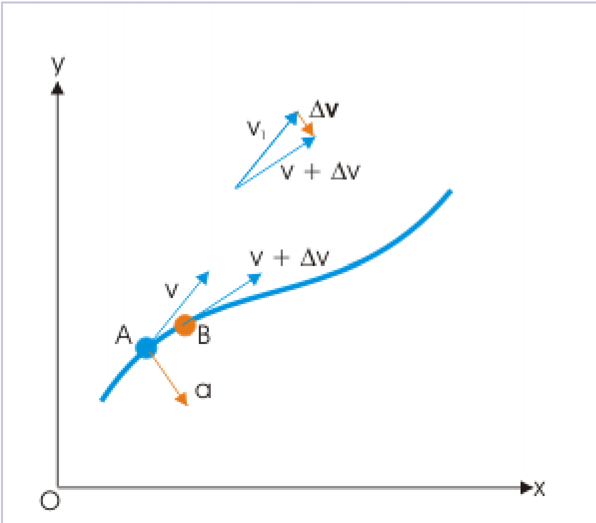
Velocity vector



In either case (positive or negative), the velocity vectors originate from the position of the particle and are drawn along the tangent to the motion curve at that point. It must be noted that velocity vector, **v**, is not rooted to the origin of the coordinate system like position vector.

Acceleration vector

Acceleration vector is drawn from the position of the object with its tail. It is independent of the origin (unlike position vector) and the direction of the tangent to the curve (unlike velocity vector). Its direction is along the direction of the force or alternatively along the direction of the vector representing change in velocities.
Acceleration vector



Further, if acceleration vector ‘**a**’ denotes the acceleration of a particle at a particular position, then “-**a**” denotes another acceleration vector, which is reversed in direction with respect to the acceleration vector, **a**.

Summarizing the discussion held so far :

1: Position vector is rooted to a pair of points i.e. the origin of the coordinate system and the position of the particle.

2: Velocity vector originates at the position of the particle and acts along the tangent to the curve, showing path of the motion.

3: Acceleration vector originates at the position of the particle and acts along the direction of force or equivalently along the direction of the change in velocity.

4: In all cases, negative vector is another vector of the same magnitude but reversed in direction with respect to another vector. The negative vector essentially indicates a change in direction and not the change in magnitude and hence, there may not be any sense of relative measurement (smaller or bigger) as in the case of scalar quantities associated with negative quantities. For example, -4° C is a smaller temperature than +4° C. Such is not the case with vector quantities. A 4 Newton force is as big as -4 Newton. Negative sign simply indicates the direction.

We must also emphasize here that we can shift these vectors laterally without changing direction and magnitude for vector operations like vector addition and multiplication. This independence is characteristic of vector operation and is not influenced by the fact that they are actually tied to certain positions in the coordinate system or not. Once vector operation is completed, then we can shift the resulting vector to the appropriate positions like the position of the particle (for velocity and acceleration vectors) or the origin of the coordinate system (for position vector).

Accelerated motion in one dimension

The nature of acceleration is determined by the net external force for constant mass system. Depending on the nature of force, there exists wide range of possibilities like zero, constant or varying accelerations in one dimensional motion.

Motion in one dimension is the basic component of all motion. A general three dimensional motion is equivalent to a system of three linear motion along three axes of a rectangular coordinate system. Thus, study of one dimensional accelerated motion forms the building block for studying accelerated motion in general. The basic defining diferential equations for velocity and acceleration retain the form in terms of displacement and position, except that they consist of displacement or position component along a particular direction. In terms of represntation, position vector **r** is replaced with x**i** or y**j** or z**k** in accordance with the direction of motion considered.

The defining equations of velocity and acceleration, in terms of position, for one dimensional motion are (say in x-direction) :

$$\mathbf{v} = \frac{đx}{đt}\mathbf{i}$$

and

$$\mathbf{a} = \frac{đ^2x}{đt^2}\mathbf{i}$$

Only possible change of direction in one dimensional motion is reversal of motion. Hence, we can define velocity and acceleration in a particular direction, say x - direction, with equivalent scalar system, in which positive and negative values of scalar quantities defining motion represent the two possible direction.

The corresponding scalar form of the defining equations of velocity and acceleration for one dimensional motion are :

$$v = \frac{đx}{đt}$$

and

$$a = \frac{đ^2 x}{đt^2}$$

It must be clearly understood that the scalar forms are completely equivalent to vector forms. In the scalar form, the sign of various quantities describing motion serves to represent direction.

Example:

Problem : The displacement of a particle along x – axis is given by :

$$x = t^3 - 3t^2 + 4t - 12$$

Find the velocity when acceleration is zero.

Solution : Here, displacement is :

$$x = t^3 - 3t^2 + 4t - 12$$

We obtain expression for velocity by differentiating the expression of displacement with respect to time,

$$\Rightarrow v = \frac{đx}{đt} = 3t^2 - 6t + 4$$

Similarly, we obtain expression for acceleration by differentiating the expression of velocity with respect to time,

$$\Rightarrow a = \frac{đv}{đt} = 6t - 6$$

Note that acceleration is a function of time "t" and is not constant. For acceleration, a = 0,

$$\Rightarrow 6t - 6 = 0$$
$$\Rightarrow t = 1$$

Putting this value of time in the expression of velocity, we have :

$$\Rightarrow v = 3\ \mathrm{t}^2 - 6t + 4$$

$$\Rightarrow v = 3\ \mathrm{x}\ 1^2 - 6\ \mathrm{x}\ 1 + 4 = 1\ \ \mathrm{m} / \mathrm{s}$$

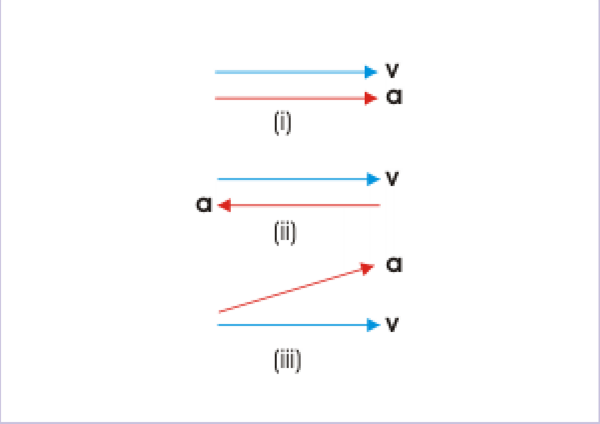
Nature of acceleration in one dimensional motion

One dimensional motion results from the action of net external force that applies along the direction of motion. It is a requirement for motion to be in one dimension. In case, force and velocity are at certain angle to each other, then there is sideway deflection of the object and the resulting motion is no more in one dimension.

If velocity and force are in the same direction, then magnitude of velocity increases; If velocity and force are in the opposite direction, then magnitude of velocity decreases.

The valid combination (i and ii)and invalid combination (iii) of velocity and acceleration for one dimensional motion are shown in the figure.

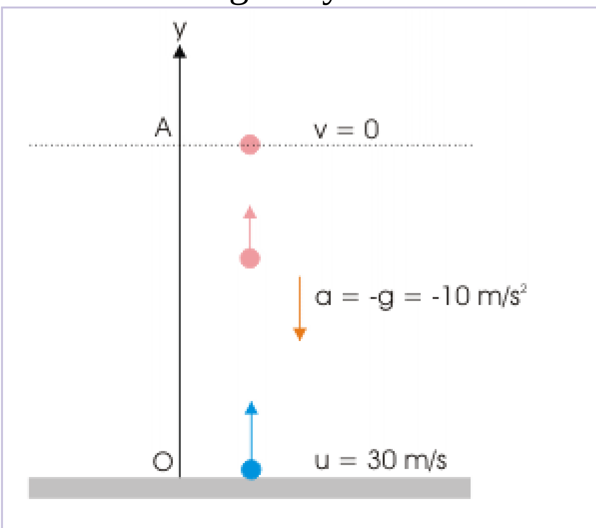
Acceleration – time plot



The requirement of one dimensional motion characterizes the nature of acceleration involved. The acceleration may vary in magnitude only. No sideway directional change in acceleration of the motion is possible for a given external force. We must emphasize that there may be reversal of motion i.e. velocity even without any directional change in acceleration. A projectile, thrown up in vertical direction, for example, returns to ground

with motion reversed at the maximum height, but acceleration at all time during the motion is directed downwards and there is no change in the direction of acceleration.

Motion under gravity



In mathematical parlance, if **v** = A**i**, then **a** = B**i**, where A and B are positive or negative numbers. For one dimensional motion, no other combination of unit vectors is possible. For example, acceleration can not be **a** = B**j** or **a** = B(**i** + **j**).

We summarize the discussion as :

- The velocity and force (hence acceleration) are directed along a straight line.
- For a given external force, the direction of acceleration remains unchanged in one dimension.

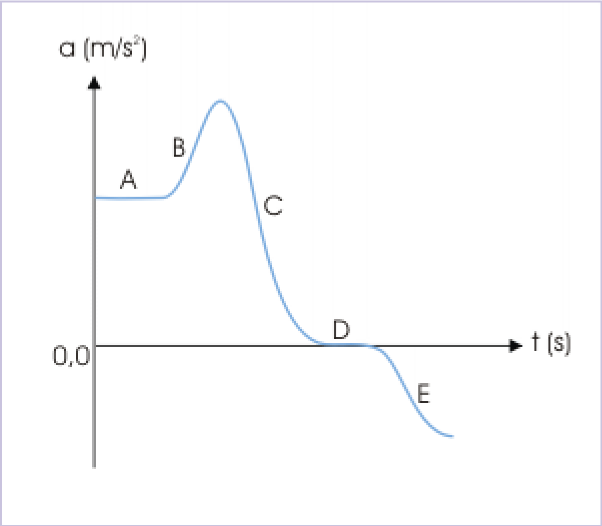
Graphs of one dimensional motion

Graphs are signature of motion. Here, we shall discuss broad categories of motion types in terms of acceleration and velocity.

Acceleration – time plot

Acceleration can be zero, constant or varying, depending upon the net external force and mass of the body. It is imperative that a single motion such as the motion of a car on the road may involve all kinds of variations in acceleration. A representation of an arbitrary real time acceleration - time variation may look like :

Acceleration – time plot



We can interpret this plot if we know the direction of velocity. We consider that positive direction of velocity is same as that of acceleration. Section, A, in the figure, represents constant acceleration. Section B represents an increasing magnitude of acceleration, whereas section C represents a decreasing magnitude of acceleration. Nonetheless, all of these accelerations result in the increase of speed with time as both velocity and acceleration are positive (hence in the same direction). The section E, however, represents deceleration as velocity (positive) and acceleration (negative) are in opposite direction. There is no acceleration during motion corresponding to section D. This section represents uniform motion.

$$\text{For section A : Constant acceleration :} a = \frac{đv}{đt} = \text{Constant}$$

$$\text{For section B and C : Positive acceleration :} a = \frac{đv}{đt} > 0$$

$$\text{For section D : Zero acceleration :} a = \frac{đv}{đt} = 0$$

$$\text{For section E : Negative acceleration :} a = \frac{đv}{đt} < 0$$

Area under acceleration - time plot

We know that :

$$a = \frac{đv}{đt}$$
$$\Rightarrow đv = ađt$$

Integrating both sides, we have :

$$\Delta v = a\Delta t$$

Thus, areas under acceleration – time plot gives the change in velocity in a given interval.

Velocity – time plot

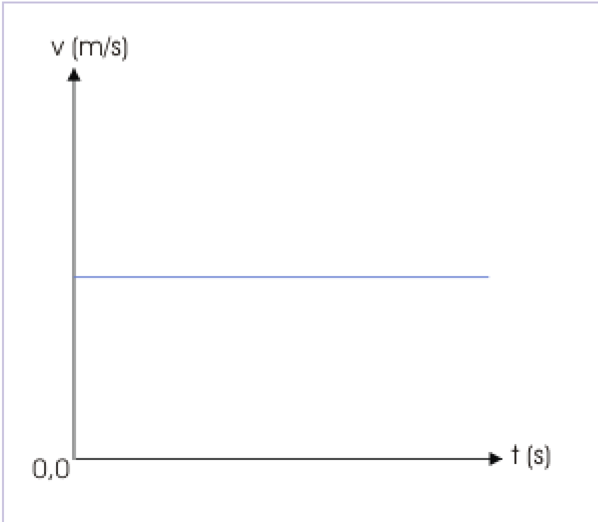
Here, we discuss velocity – time plot for various scenarios of motion in one dimension :

1: Acceleration is zero

If acceleration is zero, then velocity remains constant. As such the velocity time plot is a straight line parallel to time axis.

$$a = \frac{đv}{đt} = 0$$
$$\Rightarrow đv = 0$$
$$\Rightarrow v = \text{Constant}$$

Velocity – time plot



2: Acceleration is constant

Constant acceleration means that instantaneous acceleration at all points during motion is same. It, therefore, implies that instantaneous acceleration at any time instant and average acceleration in any time interval during motion are equal.

$$a_{\text{avg}} = a$$

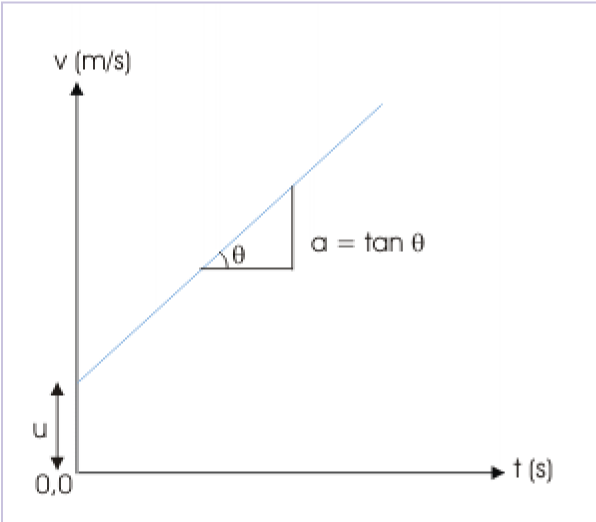
Velocity of the object under motion changes by an equal value in equal time interval. It implies that the velocity – time plot for constant acceleration should be a straight line. Here,

$$\frac{đv}{đt} = \text{constant} = a$$
$$\Rightarrow đv = ađt$$

Integrating both sides,

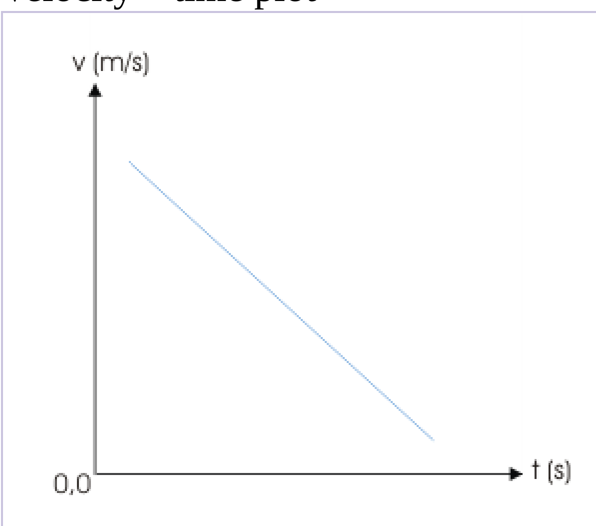
$$\Rightarrow v = at + u$$

This is a linear equation in time “t” representing a straight line, where “a” is acceleration and is equal to the slope of the straight line and "u" is the intercept on velocity axis, representing velocity at t = 0. The plot of velocity with respect to time, therefore, is a straight line as shown in the figure here.
Velocity – time plot



For constant deceleration, the velocity – time plot has negative slope. Here, speed decreases with the passage of time :

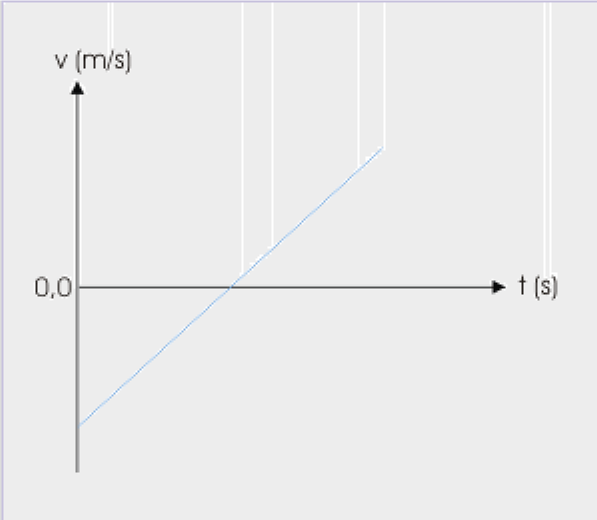
Velocity – time plot



Example:

Problem : A velocity – time plot describing motion of a particle in one dimension is shown in the figure.

Velocity – time plot



Determine (i) Whether the direction of velocity is reversed during motion? (ii) Does the particle stop? (iii) Does the motion involve deceleration and (iv) Is the acceleration constant?

Solution : (i) We see that velocity in the beginning is negative and changes to positive after some time. Hence, there is reversal of the direction of motion.

(ii) Yes. Velocity of the particle becomes zero at a particular time, when plot crosses time axis. This observation indicates an interesting aspect of reversal of direction of motion (i.e. velocity): A reversal of direction of a motion requires that the particle is stopped before the direction is reversed.

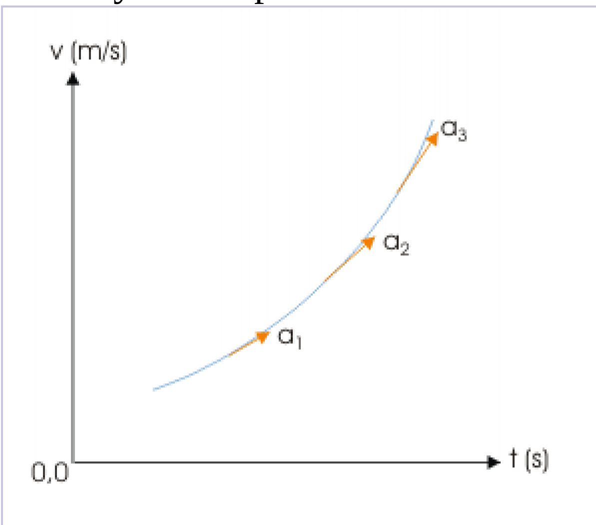
(iii) Yes. Occurance of decelration is determined by comparing directions of velocity and acceleration. In the period before the particle comes to rest, velocity is negative, wheras acceleration is always positive with respect to the positive direction of velocity (slope of the line is positive on velocity - time plot). Thus, velocity and acceleration are in opposite direction for this part of motion and is decelerated. Further, it is also seen that speed of the particle is decreasing in this period.

(iv) The plot is straight line with a constant slope. Thus, motion is under constant acceleration.

3: The magnitude of acceleration is increasing

Since slope of velocity – time curve is equal to the acceleration at that instant, it is expected that velocity – time plot should be a curve, whose

slope increases with time as shown in the figure.
Velocity – time plot

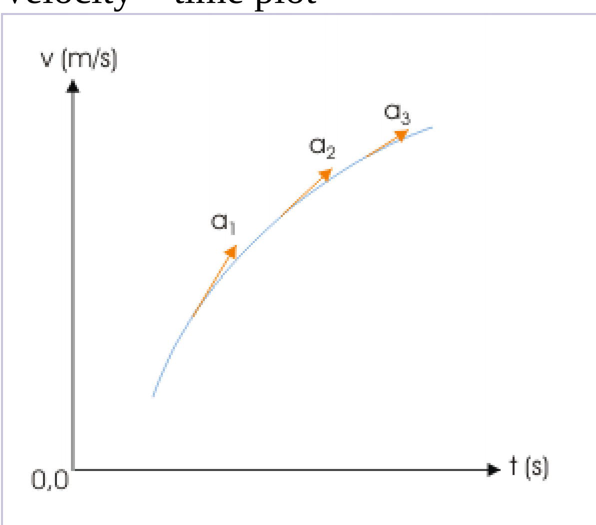


Here,

$$a_1 < a_2 < a_3$$

4: The magnitude of acceleration is decreasing

The velocity – time plot should be a curve, whose slope decreases with time as shown in the figure here :
Velocity – time plot



Here,

$$a_1 > a_2 > a_3$$

Example:

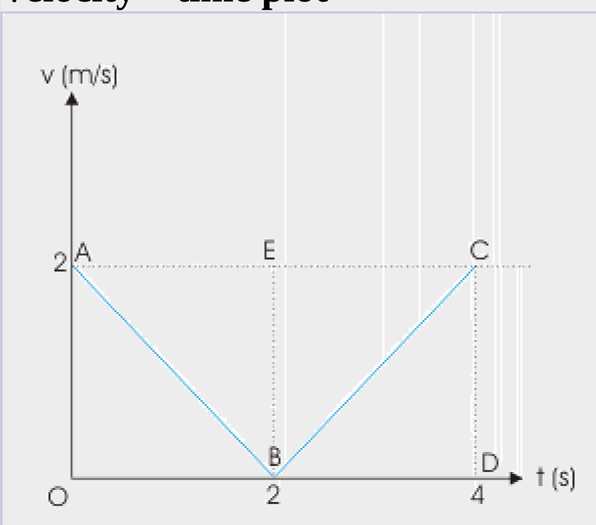
Problem : A person walks with a velocity given by |t – 2| along a straight line. Find the distance and displacement for the motion in the first 4 seconds. What are the average velocity and acceleration in this period? Discuss the nature of acceleration.

Solution : Here, velocity is equal to the modulus of a function in time. It means that velocity is always positive. The representative values at the end of every second in the interval of 4 seconds are tabulated as below :

`---------------------------- Time (s) Velocity (m/s) ---------------------------- 0 2 1 1 2 0 3 1 4 2 ----------------------------`

An inspection of the values in the table reveals that velocity linearly decreases for the first 2 second from 2 m/s to zero. It, then, increases from zero to 2 m/s. In order to obtain distance and displacement, we draw the plot between displacement and time as shown.

Velocity – time plot



Now, area under the plot gives displacement.

$$\text{Displacement} = \Delta\text{OAB} + \Delta\,\text{BCD}$$

$$\Rightarrow \text{Displacement} = \frac{1}{2}\text{ x OB x OA} + \frac{1}{2}\text{ x BD x CD}$$

$$\Rightarrow \text{Displacement} = \frac{1}{2}\text{ x 2 x 2} + \frac{1}{2}\text{ x 2 x 2} = 4\,m$$

We see that velocity does not change its sign. Thus, motion is unidirectional apart from being rectilinear. As such, distance is equal to displacement and is also 4 m.
Now, average velocity is given by :

$$v_{\text{avg}} = \frac{\text{Displacement}}{\text{Time}} = \frac{4}{4} = 1\, m/s$$

Similarly, average acceleration during the motion is :

$$v_{\text{avg}} = \frac{\Delta v}{\Delta t} = \frac{0}{4} = 0$$

Though average acceleration is zero, the instantaneous acceleration of the motion during whole period is not constant as the velocity – time plot is not a single straight line. An inspection of the plot reveals that velocity – time has two line segments AB and BC, each of which separately represents constant acceleration.
Magnitude of constant acceleration (instantaneous) in the first part of the motion is equal to the slope of the displacement – time plot AB :

$$a_{\text{AB}} = \frac{\text{AO}}{\text{OB}} = \frac{-2}{2} = -1\, m/s^2$$

Similarly, acceleration in the second part of the motion is :

$$a_{\text{AB}} = \frac{\text{DC}}{\text{BD}} = \frac{2}{2} = 1\, m/s^2$$

Thus, the acceleration of the motion is negative for the first part of motion and positive for the second of motion with respect to velocity on velocity-time plot. We can, therefore, conclude that acceleration is not constant.

Constant acceleration

A constant acceleration is a special case of the accelerated motion.

The constant acceleration is a special case of accelerated motion. There are vast instances of motions, which can be approximated to be under constant force and hence constant acceleration. Two of the most important forces controlling motions in our daily life are force due to gravity and friction force. Incidentally, these two forces are constant in the range of motions of bodies in which we are interested. For example, force due to gravity is given by :

$$F = \frac{\mathrm{GMm}}{r^2} = \mathrm{mg}$$

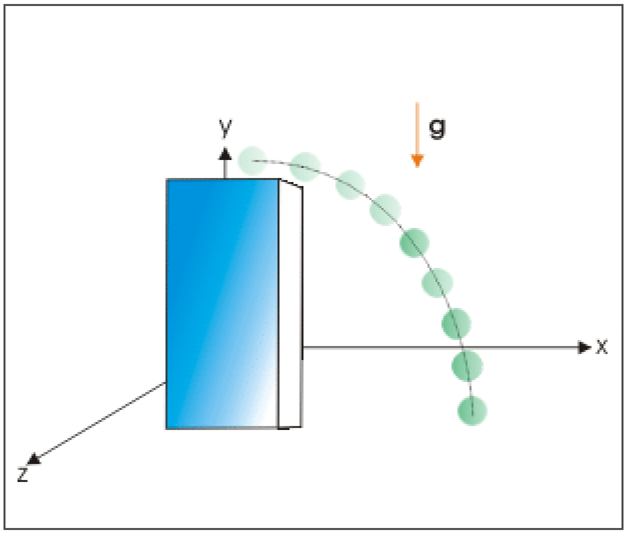
where "G" is the universal constant, "M" is the mass of earth, "m" is the mass of the body and "r" is the distance between center of earth and the body.

The resulting acceleration due to gravity, g, is a constant in the immediate neighborhood of the earth surface and is given by :

$$g = \frac{\mathrm{GM}}{r^2}$$

The only variable for a given mass is "r", which changes with the position of the body. The distance "r", however, is equal to Earth's radius for all practical purposes as any difference arising from the position of body on earth can be ignored. Therefore, acceleration due to gravity can safely be considered to be constant for motions close to the surface of the earth. Significantly acceleration due to gravity is a constant irrespective of the mass "m" of the body.

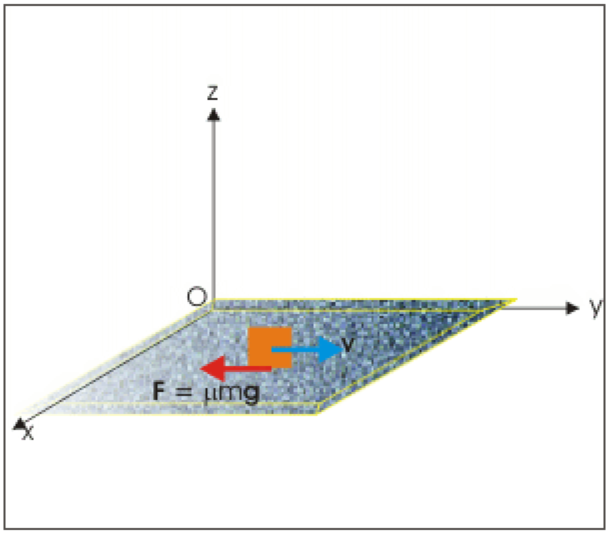
A ball kicked from a height



The ball follows a parabolic path
as it approaches ground

The figure above shows the motion of a ball kicked from the top of a tower. The ball moves under constant acceleration of gravity during its flight to the ground. The constant acceleration, therefore, assumes significance in relation to the motion that takes place under the influence of gravity. In the same manner, motion on a rough plane is acted upon by the force of friction in the direction opposite to the motion. The force of friction is a constant force for the moving body and characteristic of the surfaces in contact. As a result, the object slows down at a constant rate.

Motion influenced by friction force



The block decelerates due to friction.

Understanding constant acceleration

The constant acceleration means that the acceleration is independent of time and is equal to a constant value. The implication of a constant acceleration is discussed here as under :

1: As acceleration is same at all instants during the motion, it follows that average acceleration is equal to instantaneous acceleration during the motion. Mathematically,

$$\mathbf{a_{avg}} = \mathbf{a}$$
$$\Rightarrow \frac{\Delta \mathbf{v}}{\Delta t} = \frac{đ\mathbf{v}}{đt}$$

2: When Δt = 1 second, then

$$\mathbf{a_{avg}} = \mathbf{a} = \Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$$

This means that initial velocity , on an average, is changed by the acceleration vector after every second.

Example:
Constant acceleration
Problem : The position of a particle, in meters, moving in the coordinate space is described by the following functions in time.

$$x = 2t^2 - 4t + 3;\ y = -2t;\ \text{and}\ z = 5$$

Find the velocity and acceleration at t = 2 seconds from the start of motion. Also, calculate average acceleration in the first four seconds.
Solution : The component velocities in three directions are :

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(2t^2 - 4t + 3\right) = 4t - 4$$
$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-2t\right) = -2$$
$$v_z = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(5\right) = 0$$

and the velocity is given by :

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$
$$\Rightarrow \mathbf{v} = (4t - 4)\mathbf{i} - 2\mathbf{j}$$

Thus, velocity at t = 2 seconds,

$$\mathbf{v}_2 = 4\mathbf{i} - 2\mathbf{j}$$

Acceleration of the particle along three axes are given as :

$$a_x = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(2t^2 - 4t + 3\right) = 4$$
$$a_y = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(4t\right) = 0$$
$$a_z = \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(5\right) = 0$$

The resultant acceleration is given by :

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\Rightarrow \mathbf{a} = 4\mathbf{i}\, m/s^2$$

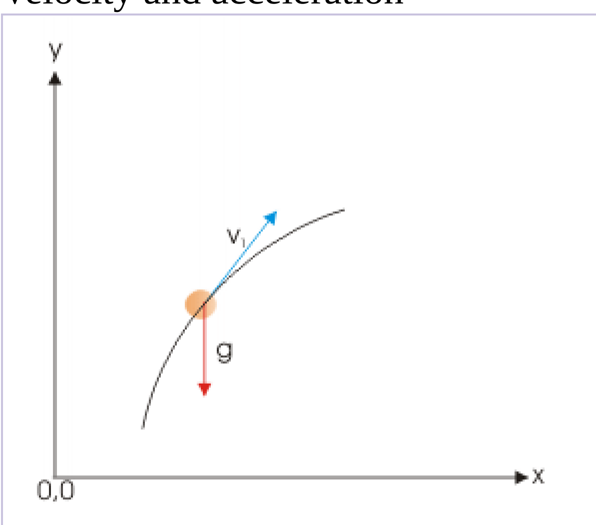
which is a constant and is independent of time. The accelerations at all time instants are, therefore, same. We know that the average and instantaneous accelerations are equal when acceleration is constant. Hence, $a_{\text{avg}} = 4\mathbf{i}$. We can check the result calculating average acceleration for the first four seconds as :

$$\mathbf{a}_{\text{avg}} = \frac{\mathbf{v}_4 - \mathbf{v}_0}{4} = \frac{12\mathbf{i} - 2\mathbf{j} + 4\mathbf{i} + 2\mathbf{j}}{4} = 4\mathbf{i}\, m/s^2$$

The important fall out of a constant acceleration is that its magnitude has a constant value and its direction is fixed. A change in either of the two attributes, constituting acceleration, shall render acceleration variable. This means that acceleration is along a straight line. But does this linear nature of acceleration mean that the associated motion is also linear? Answer is no.

Reason is again the “disconnect” between acceleration and velocity. We know that magnitude and direction of acceleration are solely determined by the mass of the object and net external force applied on it. Thus, a constant acceleration only indicates that the force i.e the cause that induces change in motion is linear. It does not impose any restriction on velocity to be linear.

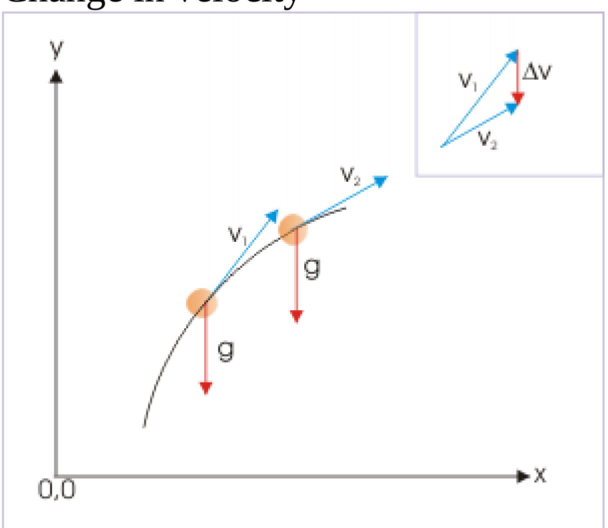
Velocity and acceleration



Force and velocity are in two different directions

It is imperative that if the initial velocity of the object is not aligned with linear constant acceleration like in the figure above, then the immediate effect of the applied force, causing acceleration, is to change the velocity. Since acceleration is defined as the time rate of change in velocity, the resulting velocity would be so directed and its magnitude so moderated that the change in velocity (not the resulting velocity itself) is aligned in the direction of force.

Change in velocity

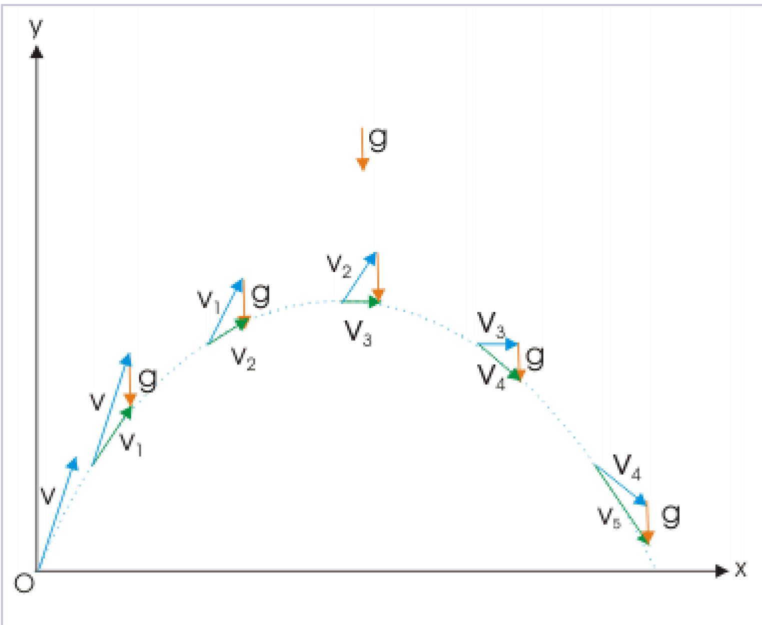


The change in velocity has the same direction as that of acceleration.

As the resulting velocity may not be aligned with the direction of force (acceleration), the resulting motion may not be linear either. For motion being linear, it is essential that the initial velocity and the force applied (and the resulting acceleration) are aligned along a straight line.

Examples of motions in more than one dimension with constant acceleration abound in nature. We have already seen that motion of a projectile in vertical plane has constant acceleration due to gravity, having constant magnitude, g, and fixed downward direction. If we neglect air resistance, we can assume that all non- propelled projectile motions above ground are accelerated with constant acceleration. In the nutshell, we can say that constant acceleration is unidirectional and linear, but the resulting velocity may not be linear. Let us apply this understanding to the motion of a projectile, which is essentially a motion under constant acceleration due to gravity.

Parabolic motion

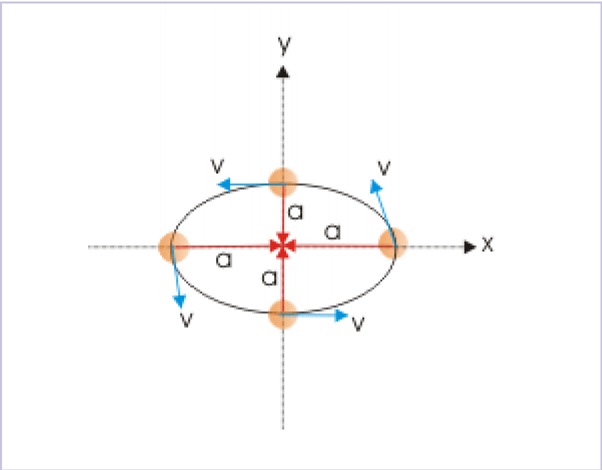


The velocity of the ball is moderated by acceleration.

In the figure, see qualitatively, how the initial velocity vector, v, is modified by the constant acceleration vector, g, at the end of successive seconds. Note that combined change in both magnitude and direction of the velocity is taking place at a constant rate and is in vertically downward direction.

In the context of constant acceleration, we must also emphasize that both magnitude and direction are constant. A constant acceleration in magnitude only is not sufficient. For constant acceleration, the direction of acceleration should also be same (i.e constant). We can have a look at a uniform circular motion in horizontal plane, which follows a horizontal circular path with a constant speed.

Uniform circular motion



The object moves along a circular path constant speed.

Notwithstanding the constant magnitude, acceleration of uniform circular motion is a variable acceleration in the horizontal plane, because direction of radial centripetal acceleration (shown with red arrow) keeps changing with time. Therefore, the acceleration of the motion keeps changing and is not independent of time as required for acceleration to be constant.

Equation of motion

The motion under constant acceleration allows us to describe accelerated motion, using simple mathematical construct. Here, we set out to arrive at these relationships in the form of equations. In these equations, “**u**” stands for initial velocity, “**v**” for final velocity, “**a**” for constant acceleration and “t” for time interval of motion under consideration. This is short of

convention followed by many text books and hence the convention has been retained here.

First equation

$$1: \quad \mathbf{v} = \mathbf{u} + \mathbf{a}t$$

This relation can be established in many ways. One of the fundamental ways is to think that velocity is changed by constant acceleration (vector) at the end of successive seconds. Following this logic, the velocity at the end of t successive seconds are as under :

$$\begin{array}{lcl} \text{At the end of } 1^{\text{st}} \text{ second} & : & \mathbf{u} + \mathbf{a} \\ \text{At the end of } 2^{\text{nd}} \text{ second} & : & \mathbf{u} + 2\mathbf{a} \\ \text{At the end of } 3^{\text{rd}} \text{ second} & : & \mathbf{u} + 3\mathbf{a} \\ \text{At the end of } 4^{\text{th}} \text{ second} & : & \mathbf{u} + 4\mathbf{a} \\ \text{------------------------------} & & \\ \text{------------------------------} & & \\ \text{At the end of } \text{t}^{\text{th}} \text{ second} & : & \mathbf{u} + t\mathbf{a} = \mathbf{u} + \mathbf{a}t \end{array}$$

We can also derive this equation, using the defining concept of constant acceleration. We know that :

$$\begin{array}{l} \mathbf{a} = \mathbf{a}_{\text{avg}} \\ \Rightarrow \mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v} - \mathbf{u}}{t} \\ \Rightarrow \mathbf{a}t = \mathbf{v} - \mathbf{u} \\ \Rightarrow \mathbf{v} = \mathbf{u} + \mathbf{a}t \end{array}$$

Alternatively (using calculus), we know that :

$$\begin{array}{l} \mathbf{a} = \frac{đ\mathbf{v}}{đt} \\ \Rightarrow đ\mathbf{v} = \mathbf{a}đt \end{array}$$

Integrating on both sides, we have :

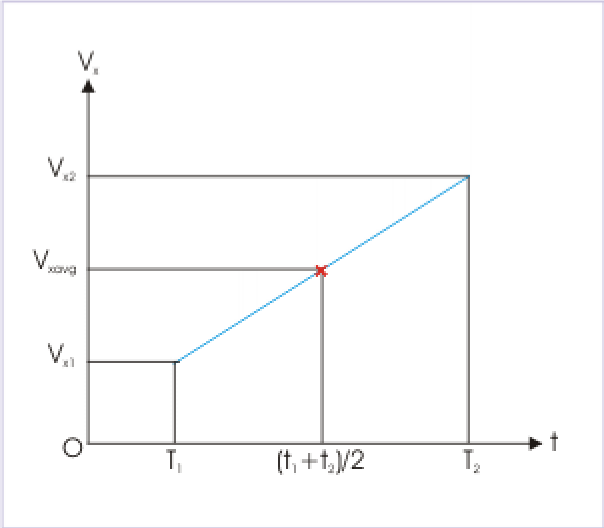
$$\begin{aligned}
&\Delta \mathbf{v} = \mathbf{a}\Delta t \\
&\Rightarrow \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{a}t \\
&\Rightarrow \mathbf{v} - \mathbf{u} = \mathbf{a}t \\
&\Rightarrow \mathbf{v} = \mathbf{u} + \mathbf{a}t
\end{aligned}$$

Second equation

2: $\mathbf{v}_{\text{avg}} = \frac{\mathbf{u}+\mathbf{v}}{2}$

The equation $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ is a linear relationship. Hence, average velocity is arithmatic mean of initial and final velocities :

Average velocity



Average velocity is equal to mean velocity.

$$\Rightarrow \mathbf{v}_{\text{avg}} = \frac{\mathbf{u}+\mathbf{v}}{2}$$

Third equation

$$3: \quad \mathbf{s} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2$$

This equation is derived by combining the two expressions available for the average velocity.

$$\mathbf{v}_{\text{avg}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t}$$

and

$$\mathbf{v}_{\text{avg}} = \frac{(\mathbf{v}_1 + \mathbf{v}_2)}{2} = \frac{(\mathbf{v} + \mathbf{u})}{2}$$

Combining two expressions of average acceleration and rearranging, we have :

$$\Rightarrow \left(\mathbf{r}_2 - \mathbf{r}_1\right) = \frac{(\mathbf{v} + \mathbf{u})t}{2}$$

Using the relation, $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ and substituting for v, we have :

$$\Rightarrow \left(\mathbf{r}_2 - \mathbf{r}_1\right) = \frac{(\mathbf{u} + \mathbf{a}t + \mathbf{u})t}{2}$$
$$\Rightarrow \mathbf{s} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 == \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

Alternatively, we can derive this equation using calculus. Here,

$$\mathbf{v} = \frac{đ\mathbf{r}}{đt}$$
$$\Rightarrow đ\mathbf{r} = \mathbf{v}đt$$

Integrating between the limits on both sides,

$$\int đ\mathbf{r} = \int \mathbf{v}đt$$

Now substituting **v**,

$$\Rightarrow \int đ\mathbf{r} = \int \left(\mathbf{u} + \mathbf{a}t\right) đt$$
$$\Rightarrow \Delta\mathbf{r} = \int \mathbf{u}t + \int \mathbf{a}tđt$$
$$\Rightarrow \mathbf{s} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

The expression for the displacement has two terms : one varies linearly (**u**t) with the time and the other ($\frac{1}{2}\mathbf{a}t^2$) varies with the square of time. The first term is equal to the displacement due to non-accelerated motion i.e the displacement when the particle moves with uniform velocity, **u**. The second term represents the contribution of the acceleration (change in velocity) towards displacement.

This equation is used for determining either displacement($\Delta\mathbf{r}$) or position($\mathbf{r}_1$). A common simplification, used widely, is to consider beginning of motion as the origin of coordinate system so that

$$\mathbf{r}_1 = 0$$
$$\Rightarrow \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}\ (\text{say})$$

In this case, both final position vector and displacement are equal. This simplification, therefore, allows us to represent both displacement and position with a single vector variable **r**.

Graphical interpretation of equations of motion

The three basic equations of motion with constant acceleration, as derived above, are :

$$1\text{: } \mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$2\text{: } \mathbf{v}_{\text{avg}} = \frac{(\mathbf{u}+\mathbf{v})}{2}$$
$$3\text{: } \mathbf{s} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

These three equations completely describe motion of a point like mass, moving with constant acceleration. We need exactly five parameters to describe the motion under constant acceleration : $\mathbf{u}, \mathbf{v}, \mathbf{r}_1, \mathbf{r}_2$ and t .

It can be emphasized here that we can not use these equations if the acceleration is not constant. We should use basic differentiation or integration techniques for motion having variable acceleration (non-uniform acceleration). These equations serve to be a ready to use equations that avoids differentiation and integration. Further, it is evident that equations of motion are vector equations, involving vector addition. We can evaluate a motion under constant acceleration, using either graphical or algebraic method based on components.

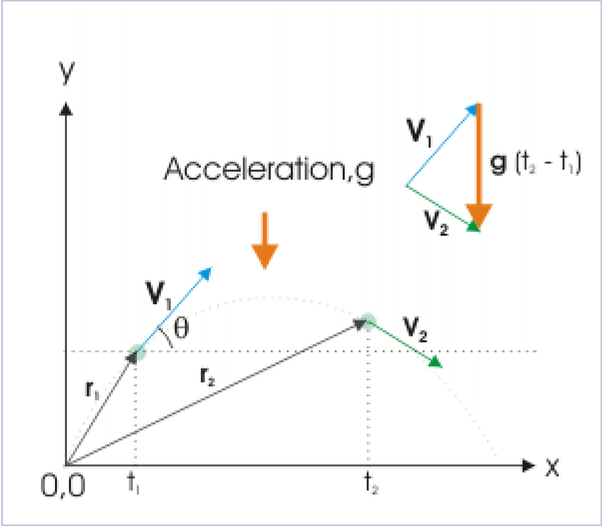
Here, we interpret these vector equations, using graphical technique. For illustration purpose, we apply these equations to a motion of an object, which is thrown at an angle θ from the horizontal. The magnitude of acceleration is "g", which is directed vertically downward. Let acceleration vector be represented by corresponding bold faced symbol $\mathbf{g}$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities at time instants t_1 and t_2 respectively and corresponding position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$.

The final velocity at time instant t_2 , is given by :

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$\Rightarrow \mathbf{v}_2 = \mathbf{v}_1 + \mathbf{g}(t_2 - t_1)$$

Graphically, the final velocity is obtained by modifying initial vector $\mathbf{v}_1$ by the vector $\mathbf{g}(t_2 - t_1)$.

Graphical representation of first equation

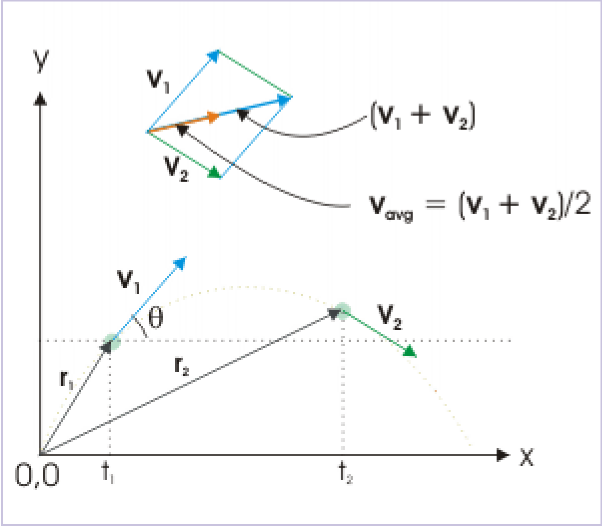


Now, we discuss graphical representation of second equation of motion. The average velocity between two time instants or two positions is given by :

$$\mathbf{v}_{\mathrm{avg}} = \frac{(\mathbf{u}+\mathbf{v})}{2}$$
$$\Rightarrow \mathbf{v}_{\mathrm{avg}} = \frac{(\mathbf{v}_1+\mathbf{v}_2)}{2}$$

The vector addition involved in the equation is graphically represented as shown in the figure. Note that average velocity is equal to half of the vector sum $\mathbf{v}_1 + \mathbf{v}_2$.

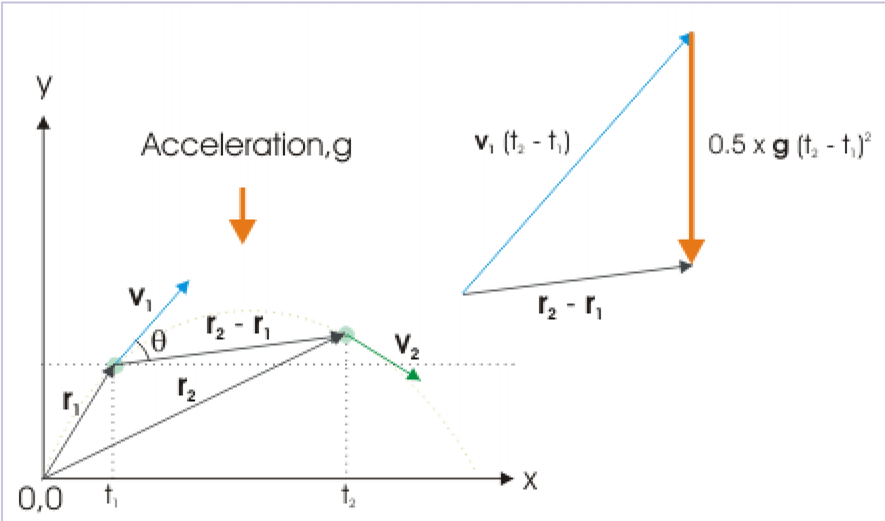
Graphical representation of second equation



Third equation of motion provides for displacement in terms of two vector quantities - initial velocity and acceleration. The displacement, **s**, is equal to addition of two vector terms :

$$\mathbf{s} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$\Rightarrow \mathbf{s} = \Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{v}_1\left(t_2 - t_1\right) + \frac{1}{2}\mathbf{g}(t_2 - t_1)^2$$

Graphical representation of third equation



Equations of motion in component form

The application of equations of motion graphically is tedious. In general, we use component representation that allows us to apply equations algebraically. We use equations of motion, using component forms of vector quantities involved in the equations of motion. The component form of the various vector quantities are :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$\Delta\mathbf{r} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}$$
$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$
$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$
$$\mathbf{v} = v_{\text{avgx}}\mathbf{i} + v_{\text{avgy}}\mathbf{j} + v_{\text{avgz}}\mathbf{k}$$
$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Using above relations, equations of motion are :

$$1: \quad v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) + (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\, t$$
$$2: \quad v_{\text{avgx}}\mathbf{i} + v_{\text{avgy}}\mathbf{j} + v_{\text{avgz}}\mathbf{k} = \frac{(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})+(v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k})}{2}$$
$$3: \quad \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k} = \left(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}\right) t + \frac{1}{2}\left(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}\right) t^2$$

Example:
Acceleration in component form
Problem : A particle is moving with an initial velocity $(8\mathbf{i} + 2\mathbf{j})\,m/s$, having an acceleration $\left(0.4\mathbf{i} + 0.3\mathbf{j}\right)\,m/s^2$. Calculate its speed after 10 seconds.
Solution : The particle has acceleration of $\left(0.4\mathbf{i} + 0.3\mathbf{j}\right)\,m/s^2$, which is a constant acceleration. Its magnitude is $\sqrt{\left(0.4^2 + 0.3^2\right)} = 0.5\,m/s^2$ making an angle with x-direction. $\theta = \tan^{-1}\left(\frac{3}{4}\right)$. Time interval is 10 seconds. Thus, applying equation of motion, we have :

$$v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = \left(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}\right) + \left(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}\right)t$$
$$\Rightarrow \mathbf{v} = (8\mathbf{i} + 2\mathbf{j}) + (0.4\mathbf{i} + 0.3\mathbf{j}) \text{ x } 10 = 12\mathbf{i} + 5\mathbf{j}$$

Speed i.e. magnitude of velocity is :

$$\Rightarrow v = \sqrt{\left(12^2 + 5^2\right)} = 13\,m/s$$

Note: This example illustrates the basic nature of the equations of motion. If we treat them as scalar equations, we may be led to wrong answers. For example, magnitude of initial velocity i.e. speed is $\sqrt{(8^2 + 2^2)} = 8.25\,m/s$, Whereas magnitude of acceleration is $\sqrt{\left(0.4^2 + 0.3^2\right)} = 0.5\,m/s^2$. Now, using equation of motion as scalar equation, we have :

$$v = u + at = 8.25 + 0.5 \text{ x } 10 = 13.25\,m/s$$

Equivalent scalar system of equations of motion

We have discussed earlier that a vector quantity in one dimension can be conveniently expressed in terms of an equivalent system of scalar representation. The advantage of linear motion is that we can completely do away with vector notation with an appropriate scheme of assigning plus or minus signs to the quantities involved. The equivalent scalar representation takes advantage of the fact that vectors involved in linear motion has only two possible directions. The one in the direction of chosen axis is considered positive and the other against the direction of the chosen axis is considered negative.

At the same time, the concept of component allows us to treat a motion into an equivalent system of the three rectilinear motions in the mutually perpendicular directions along the axes. The two concepts, when combined together, renders it possible to treat equations of motion in scalar terms in mutually three perpendicular directions.

Once we follow the rules of equivalent scalar representation, we can treat equations of motion as scalar equations in the direction of an axis, say x - axis, as :

$$\begin{array}{ll} \text{1x:} & v_x = u_x + a_x t \\ \text{2x:} & v_{\text{avgx}} = \frac{(u_x + v_x)}{2} = \frac{(x_2 - x_1)}{t} \\ \text{3x:} & \Delta x = x_2 - x_1 = u_x t + \frac{1}{2} a_x t^2 \end{array}$$

We have similar set of equations in the remaining two directions. We can obtain the composite interpretation of the motion by combing the individual result in each direction. In order to grasp the method, we rework the earlier example.

Example:

Acceleration in scalar form

Problem : A particle is moving with an initial velocity $(8\mathbf{i} + 2\mathbf{j})\ m/s$, having an acceleration $(0.4\mathbf{i} + 0.3\mathbf{j})\ m/s^2$. Calculate its speed after 10 seconds.

Solution : The motion in x – direction :

$$u_x = 8\,m/s\,; a_x = 0.4\,m/s^2; 10\,s \text{ and,}$$
$$v_x = u_x + a_x t$$
$$\Rightarrow v_x = 8 + 0.4 \text{ x } 10 = 12\,m/s$$

The motion in y – direction :

$$u_y = 2\,m/s\,; a_y = 0.3\,m/s^2; 10\,s \text{ and,}$$
$$v_y = u_y + a_y t$$
$$\Rightarrow v_y = 2 + 0.3 \text{ x } 10 = 5\,m/s$$

Therefore, the velocity is :

$$\Rightarrow \mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$$
$$\Rightarrow v = \sqrt{\left(12^2 + 5^2\right)} = 13 m/s^2$$

Constant acceleration (application)
Solving problems is an essential part of the understanding

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1. Though acceleration is constant and hence one – dimensional, but the resulting motion can be one, two or three dimensional – depending on the directional relation between velocity and acceleration.
2. Identify : what is given and what is required. Establish relative order between given and required attribute.
3. Use differentiation method to get a higher order attribute in the following order : displacement (position vector) → velocity → acceleration.
4. Use integration method to get a lower order attribute in the following order : acceleration → velocity → displacement (position vector).

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the motion with constant acceleration. The questions are categorized in terms of the characterizing features of the subject matter :

- Average velocity
- Differentiation and Integration method
- Components of constant acceleration
- Rectilinear motion with constant acceleration
- Equations of motion

Average velocity

Example:
Problem : A particle moves with an initial velocity “**u**” and a constant acceleration “**a**”. What is average velocity in the first “t” seconds?
Solution : The particle is moving with constant acceleration. Since directional relation between velocity and acceleration is not known, the motion can have any dimension. For this reason, we shall be using vector form of equation of motion. Now, the average velocity is given by :

$$\mathbf{v}_a = \frac{\Delta \mathbf{r}}{\Delta t}$$

The displacement for motion with constant acceleration is given as :

$$\Delta \mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

Thus, average velocity is :

$$\mathbf{v}_a = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{u}t + \frac{1}{2}\mathbf{a}t^2}{t} = \mathbf{u} + \frac{1}{2}\mathbf{a}t$$

Differentiation and Integration methods

Example:
Problem : A particle is moving with a velocity $2\mathbf{i} + 2t\mathbf{j}$ in m/s. Find (i) acceleration and (ii) displacement at t = 1 s.
Solution : Since velocity is given as a function in “t”, we can find acceleration by differentiating the function with respect to time.

$$\mathbf{a} = \frac{đ}{đt}\left(2\mathbf{i} + 2t\mathbf{j}\right) = 2\mathbf{j}$$

Thus, acceleration is constant and is directed in y-direction. However, as velocity and acceleration vectors are not along the same direction, the motion is in two dimensions. Since acceleration is constant, we can employ equation of motion for constant acceleration in vector form,

$$\Delta \mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$\Delta \mathbf{r} = \left(2\mathbf{i} + 2t\mathbf{j}\right)t + \frac{1}{2}\ \text{x}\ 2\mathbf{j}\ \text{x}\ t^2$$

For t = 1 s

$$\Delta \mathbf{r} = \left(2\mathbf{i} + 2\ \text{x}\ 1\mathbf{j}\right)\ \text{x}\ 1 + \frac{1}{2}\ \text{x}\ 2\mathbf{j}\ \text{x}\ 1^2$$
$$\Delta \mathbf{r} = 2\mathbf{i} + 3\mathbf{j}$$

Note 1 : We should remind ourselves that we obtained displacement using equation of motion for constant acceleration. Had the acceleration been variable, then we would have used integration method to find displacement.
Note 2 : A constant acceleration means that neither its magnitude or direction is changing. Therefore, we may be tempted to think that a constant acceleration is associated with one dimensional motion. As we see in the example, this is not the case. A constant acceleration can be associated with two or three dimensional motion as well. It is the relative directions of acceleration with velocity that determines the dimension of motion – not the dimension of acceleration itself.

Example:
Problem : The coordinates of a particle (m/s) in a plane at a given time "t" is $2t,\ \ t^2$. Find (i) path of motion (ii) velocity at time "t" and (iii) acceleration at time "t".
Solution : Clearly, the position of the particle is a function of time and the particle moves in a two dimensional xy - plane. Here,

$$x = 2t$$
$$y = t^2$$

In order to find the relation between "x" and "y", we substitute "t" from the first equation in to second as :

$$y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$$
$$\Rightarrow x^2 = 4y$$

Hence, path of motion is parabolic. Now, the position vector is :

$$\mathbf{r} = 2t\mathbf{i} + t^2\mathbf{j}$$

Differentiating with respect to time, the velocity of the particle is :

$$\mathbf{v} = \frac{đ\mathbf{r}}{đt} = 2\mathbf{i} + 2t\mathbf{j}\ m/s$$

Further differentiating with respect to time, the acceleration of the particle is :

$$\mathbf{a} = \frac{đ\mathbf{v}}{đt} = 2\mathbf{j}\, m/s^2$$

Components of acceleration

Example:
Problem : At a certain instant, the components of velocity and acceleration are given as :

$$v_x = 4\, m/s;\ \ v_y = 3\, m/s;\ \ a_x = 2\, m/s^2;\ \ a_y = 1\, m/s^2.$$

What is the rate of change of speed?
Solution : Here, phrasing of question is important. We are required to find the rate of change of speed – not the rate of change of velocity or magnitude of rate of change of velocity. Let us have a look at the subtle differences in the meaning here :
1: The rate of change of velocity
The rate of change of velocity is equal to acceleration. For the given two dimensional motion,

$$\mathbf{a} = \frac{đ\mathbf{v}}{đt} = \frac{đ}{đt}\left(a_x\mathbf{i} + a_y\mathbf{j}\right)$$

$$\mathbf{a} = \frac{đ\mathbf{v}}{đt} = 2\mathbf{i} + 1\mathbf{j}$$

2: The magnitude of rate of change of velocity
The magnitude of rate of change of velocity is equal to magnitude of acceleration. For the given two dimensional motion,

$$\left|\mathbf{a}\right| = \left|\frac{đ\mathbf{v}}{đt}\right| = \sqrt{\left(2^2 + 1^2\right)} = \sqrt{5}\, m/s^2$$

3: The rate of change of speed
The rate of change of speed (dv/dt) is not equal to the magnitude of acceleration, which is equal to the absolute value of the rate of change of velocity. It is so because speed is devoid of direction, whereas acceleration consists of both magnitude and direction.
Let “v” be the instantaneous speed, which is given in terms of its component as :

$$v^2 = {v_x}^2 + {v_y}^2$$

We need to find rate of change of speed i.e dv/dt, using the values given in the question. Therefore, we need to differentiate speed with respect to time,

$$\Rightarrow 2v\frac{đv}{đt} = 2v_x\left(\frac{đv_x}{đt}\right) + 2v_y\left(\frac{đv_y}{đt}\right)$$

If we ponder a bit, then we would realize that when we deal with component speed or magnitude of component velocity then we are essentially dealing with unidirectional motion. No change in direction is possible as components are aligned to a fixed axis. As such, equating rate of change in speed with the magnitude of acceleration in component direction is valid. Now, proceeding ahead,

$$\Rightarrow \frac{đv}{đt} = \frac{v_x a_x + v_y a_y}{v}$$

$$\Rightarrow \frac{đv}{đt} = \frac{v_x a_x + v_y a_y}{\sqrt{({v_x}^2 + {v_y}^2)}}$$

Putting values, we have :

$$\Rightarrow \frac{đv}{đt} = \frac{4 \text{ x } 2 + 3 \text{ x } 1}{\sqrt{(4^2+3^2)}}$$

$$\Rightarrow \frac{đv}{đt} = \frac{11}{5} = 2.2\, m/s$$

Note: This is an important question as it brings out differences in interpretation of familiar terms. In order to emphasize the difference, we summarize the discussion as hereunder :

i: In general (i.e two or three dimensions),

$$\Rightarrow \frac{đv}{đt} \neq \left|\frac{đ\mathbf{v}}{đt}\right|$$

$$\Rightarrow \frac{đv}{đt} \neq \left|\mathbf{a}\right|$$

$$\Rightarrow \frac{đv}{đt} \neq a$$

ii: In the case of one dimensional motion, the inequality as above disappears.

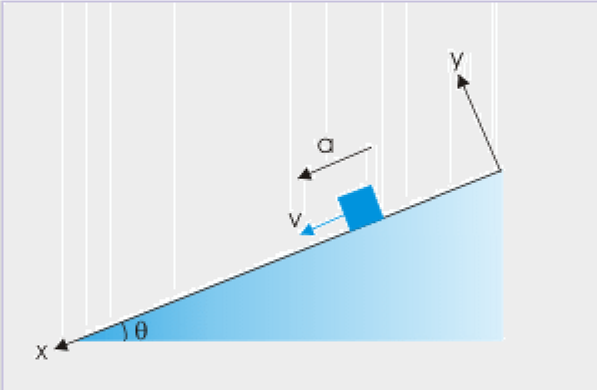
$$\Rightarrow \frac{đv}{đt} = a$$

Rectilinear motion with constant acceleration

Example:

Problem : A block is released from rest on a smooth inclined plane. If S_n denotes the distance traveled by it from t = n - 1 second to t = n seconds, then find the ratio :

Motion along an incline



The block moves with a constant acceleration.

$$A = \frac{S_n}{S_{n+1}}$$

Solution : It must be noted that the description of linear motion is governed by the equations of motion whether particle moves on a horizontal surface (one dimensional description) or on an inclined surface (two dimensional description). Let us orient our coordinates so that the motion can be treated as one dimensional unidirectional motion. This allows us to use equations of motion in scalar form,

$$\Rightarrow S_n = u + \frac{a}{2}\left(2n - 1\right)$$

Here, u = 0, thus

$$\Rightarrow S_n = \frac{a}{2}\left(2n - 1\right)$$

Following the description of term Sn as given by the question, we can define Sn+1 as the linear distance from t = n second to t = n + 1 seconds. Thus, substituting "n" by "n+1" in the formulae, we have :

$$\Rightarrow S_{n+1} = \frac{a}{2}\left(2n + 2 - 1\right)$$
$$\Rightarrow S_{n+1} = \frac{a}{2}\left(2n + 1\right)$$

The required ratio is :

$$\Rightarrow A = \frac{S_n}{S_{n+1}} = \frac{(2n-1)}{(2n+1)}$$

Equations of motion

Example:
Problem : A force of 2 N is applied on a particle of mass 1 kg, which is moving with a velocity 4 m/s in a perpendicular direction. If the force is applied all through the motion, then find displacement and velocity after 2 seconds.
Solution : It is a two dimensional motion, but having a constant acceleration. Notably, velocity and accelerations are not in the same direction. In order to find the displacement at the end of 2 seconds, we shall use algebraic method. Let the direction of initial velocity and acceleration be along “x” and “y” coordinates (they are perpendicular to each other). Also, let “A” be the initial position and “B” be the final position of the particle. The displacement between A (position at time t =0) and B (position at time t = 2 s) is given as :

$$\mathbf{AB} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

For time t = 2 s,

$$\mathbf{AB}_{\mathrm{t=2}} = 2\mathbf{u} + \frac{1}{2}\mathbf{a}\ \ \mathrm{x}\ \ 2^2 = 2\left(\mathbf{u} + \mathbf{a}\right)$$

This is a vector equation involving sum of two vectors at right angles. According to question,

$$u = 4\,m/s; a = \frac{F}{m} = \frac{2}{1} = 2\,m/s^2$$

Since **u** and **a** perpendicular to each other, the magnitude of the vector sum (**u** + **a**) is :

$$|\mathbf{u}+\mathbf{a}| = \surd\left(u^2 + a^2\right) = \surd\left(4^2 + 2^2\right) = 2\surd 5$$

Hence, magnitude of displacement is :

$$\mathrm{AB}_{\mathrm{t}=2} = 2|\mathbf{u}+\mathbf{a}| == 2 \ \text{ x } \ 2\surd 5 = 4\surd 5\, m$$

Let the displacement vector makes an angle "θ" with the direction of initial velocity.

$$\tan\theta = \frac{a}{u} = \frac{2}{4} = \frac{1}{2}$$

Let the direction of initial velocity and acceleration be along "x" and "y" coordinates (they are perpendicular to each other). Then,

$$u = 4\mathbf{i}$$
$$a = 2\mathbf{j}$$

Using equation of motion for constant acceleration, the final velocity is :

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$\Rightarrow \mathbf{v} = 4\mathbf{i} + 2\mathbf{j} \ \text{ x } \ 1 = 4(\mathbf{i}+\mathbf{j})$$

The magnitude of velocity is :

$$v = |\mathbf{v}| = 4\surd\left(1^2 + 1^2\right) = 4\surd 2\, m$$

Let the final velocity vector makes an angle "θ" with the direction of initial velocity.

$$\tan\theta = \frac{1}{1} = 1$$
$$\theta = 45^\circ$$

One dimensional motion with constant acceleration

Free falling bodies under gravity represents typical case of motion in one dimension with constant acceleration. A body projected vertically upwards is also a case of constant acceleration in one dimension, but with the difference that body undergoes reversal of direction as well after reaching the maximum height. Yet another set of examples of constant accelerations may include object sliding on an incline plane, motion of an abject impeded by rough surfaces and many other motions under the influence of gravitational and frictional forces.

The defining differential equations of velocity and acceleration involve only one position variable (say x). In the case of motion under constant acceleration, the differential equation defining acceleration must evaluate to a constant value.

$$\mathbf{v} = \frac{đx}{đt}\mathbf{i}$$

and

$$\mathbf{a} = \frac{đ^2x}{đt^2}\mathbf{i} = k\mathbf{i}$$

where k is a positive or negative constant.

The corresponding scalar form of the defining equations of velocity and acceleration for one dimensional motion with constant acceleration are :

$$v = \frac{đx}{đt}$$

and

$$a = \frac{đ^2x}{đt^2} = k$$

Example:
Constant acceleration

Problem : The position "x" in meter of a particle moving in one dimension is described by the equation :

$$t = \sqrt{x} + 1$$

where "t" is in second.

1. Find the time when velocity is zero.
2. Does the velocity changes its direction?
3. Locate position of the particle in the successive seconds for first 3 seconds.
4. Find the displacement of the particle in first three seconds.
5. Find the distance of the particle in first three seconds.
6. Find the displacement of the particle when the velocity becomes zero.
7. Determine, whether the particle is under constant or variable force.

Solution : Velocity is equal to the first differential of the position with respect to time, while acceleration is equal to the second differential of the position with respect to time. The given equation, however, expresses time, t, in terms of position, x. Hence, we need to obtain expression of position as a function in time.

$$t = \sqrt{x} + 1$$
$$\Rightarrow \sqrt{x} = t - 1$$

Squaring both sides, we have :

$$\Rightarrow x = t^2 - 2t + 1$$

This is the desired expression to work upon. Now, taking first differential w.r.t time, we have :

$$v = \frac{đx}{đt} = \frac{đ}{đt}\left(t^2 - 2t + 1\right) = 2t - 2$$

1: When v = 0, we have v = 2t – 2 = 0

$$\Rightarrow t = 1s$$

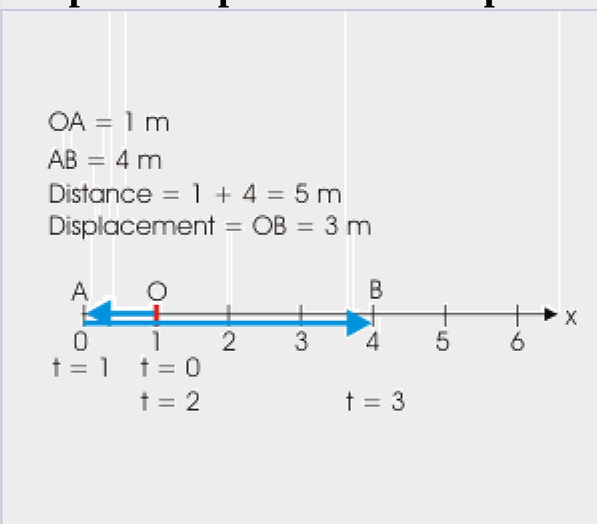
2. Velocity is expressed in terms of time as :

$$v = 2t - 2$$

It is clear from the expression that velocity is negative for t < 1 second, while positive for t > 1. As such velocity changes its direction during motion.
3: Positions of the particle at successive seconds for first three seconds are :

$$t = 0; x = t^2 - 2t + 1 = 0 - 0 + 1 = 1\,m$$
$$t = 1; x = t^2 - 2t + 1 = 1 - 2 + 1 = 0\,m$$
$$t = 2; x = t^2 - 2t + 1 = 4 - 4 + 1 = 1\,m$$
$$t = 3; x = t^2 - 2t + 1 = 9 - 6 + 1 = 4\,m$$

Graphical representation of position



4: Positions of the particle at t = 0 and t = 3 s are 1 m and 4 m from the origin.
Hence, displacement in first three seconds is 4 – 1 = 3 m
5: The particle moves from the start position, x = 1 m, in the negative direction for 1 second. At t = 1, the particle comes to rest. For the time interval from 1 to 3 seconds, the particle moves in the positive direction. Distance in the interval t = 0 to 1 s is :

$$s_1 = 1 - 0 = 1\, m$$
$$s_2 = 4 - 0 = 4\, m$$

Total distance is 1 + 4 = 5 m.
6: Velocity is zero, when t = 1 s. In this period, displacement is 1 m.
7: In order to determine the nature of force on the particle, we first determine the acceleration as :

$$a = \frac{đv}{đt} = \frac{đ}{đt}\left(2t - 2\right) = 2\, m/s^2$$

Acceleration of the motion is constant, independent of time. Hence, force on the particle is also constant during the motion.

Equation of motion for one dimensional motion with constant acceleration

The equation of motions in one dimension for constant acceleration is obtained from the equations of motion established for the general case i.e. for the three dimensional motion. In one dimension, the equation of motion is simplified (**r** is replaced by x or y or z with corresponding unit vector). The three basic equations of motions are (say in x - direction) :

$$1\text{: } \mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$2\text{: } \mathbf{v}_{\text{avg}} = \frac{(\mathbf{u}+\mathbf{v})}{2}$$
$$3\text{: } \Delta x\mathbf{i} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

Significantly, we can treat vector equivalently as scalars with appropriate sign. Following is the construct used for this purpose :
Sign convention

1. Assign an axis along the motion. Treat direction of axis as positive.
2. Assign the origin with the start of motion or start of observation. It is, however, a matter of convenience and is not a requirement of the construct.

3. Use all quantities describing motion in the direction of axis as positive.
4. Use all quantities describing motion in the opposite direction of axis as negative.

Once, we follow the rules as above, we can treat equations of motion as scalar equations as :

$$1: v = u + at$$
$$2: v_{\text{avg}} = \frac{(u+v)}{2}$$
$$3: \Delta x = ut + \frac{1}{2}at^2$$

Example:
Constant acceleration
Problem : A car moving with constant acceleration covers two successive kilometers in 20 s and 30 s respectively. Find the acceleration of the car.
Solution : Let "u" and "a" be the initial velocity and acceleration of the car. Applying third equation of motion for first kilometer, we have :

$$1000 = u \text{ x } 20 + \frac{1}{2}a20^2 = 20u + 200a$$
$$\Rightarrow 100 = 2u + 20a$$

At the end of second kilometer, total displacement is 2 kilometer (=2000 m) and total time is 20 + 30 = 50 s. Again applying third equation of motion,

$$2000 = u \text{ x } 50 + \frac{1}{2}a50^2 = 50u + 2500a$$
$$\Rightarrow 200 = 5u + 125a$$

Solving two equations,

$$a = -2.86\, m/s^2$$

Note that acceleration is negative to the positive direction (direction of velocity) and as such it is termed “deceleration”. This interpretation is

valid as we observe that the car covers second kilometer in longer time that for the first kilometer, which means that the car has slowed down.
It is important to emphasize here that mere negative value of acceleration does not mean it to be deceleration. The deciding criterion for deceleration is that acceleration should be opposite to the direction of velocity.

Motion under gravity

We have observed that when a feather and an iron ball are released from a height, they reach earth surface with different velocity and at different times. These objects are under the action of different forces like gravity, friction, wind and buoyancy force. In case forces other than gravity are absent like in vacuum, the bodies are only acted by the gravitational pull towards earth. In such situation, acceleration due to gravity, denoted by g, is the only acceleration.

The acceleration due to gravity near the earth surface is nearly constant and equal to 9.8 m/s^2 . Value of 'g' is taken as 10 m/s^2 as an approximation to facilitate ease of calculation.

When only acceleration due to gravity is considered, neglecting other forces, each of the bodies (feather and iron ball) starting from rest is accelerated at the same rate. Velocity of each bodies increases by 9.8 m/s at the end of every second. As such, the feather and the iron ball reach the surface at the same time and at the same velocity.

Additional equations of motion

A close scrutiny of three equations of motion derived so far reveals that they relate specific quantities, which define the motion. There is possibility that we may encounter problems where inputs are not provided in the manner required by equation of motion.

For example, a problem involvoing calaculation of displacement may identify initial velocity, final velocity and acceleration as input. Now, the

equation of motion for displacement is expressed in terms of initial velocity, time and acceleration. Evidently, there is a mis-match between what is given and what is required. We can, no doubt, find out time from the set of given inputs, using equation for velocity and then we can solve equation for the displacement. But what if we develop a relation-ship which relates the quantities as given in the input set! This would certainly help.

From first equation :

$$v = u + at$$
$$\Rightarrow v - u = at$$
$$\Rightarrow t = \frac{(v-u)}{a} = at$$

From second equation :

$$\frac{(u+v)}{2} = \frac{(x_2-x_1)}{t}$$
$$\Rightarrow \left(u + v\right) = \frac{2(x_2-x_1)}{t}$$

Eliminating ‘t’,

$$\Rightarrow \left(u + v\right) = \frac{2a(x_2-x_1)}{(v-u)}$$
$$\Rightarrow v^2 - u^2 = 2a\left(x_2 - x_1\right)$$

Equation:

$$\Rightarrow v^2 - u^2 = 2a\left(x_2 - x_1\right)$$

This equation relates initial velocity, final velocity, acceleration and displacement.

Also, we observe that equation for displacement calculates displacement when initial velocity, acceleration and time are given. If final velocity - instead of initial velocity - is given, then displacement can be obtained with slight modification.

$$\Delta x = x_2 - x_1 = ut = \frac{1}{2}at^2$$

Using v = u + at,

$$\Rightarrow \Delta x = x_2 - x_1 = \left(v - at\right)t + \frac{1}{2}at^2$$

Equation:

$$\Rightarrow \Delta x = x_2 - x_1 = vt - \frac{1}{2}at^2$$

Example:

Constant acceleration in one dimensional motion

Problem : A train traveling on a straight track moves with a speed of 20 m/s. Brake is applied uniformly such that its speed is reduced to 10 m/s, while covering a distance of 200 m. With the same rate of deceleration, how far will the train go before coming to rest.

Solution : To know the distance before train stops, we need to know the deceleration. We can find out deceleration from the first set of data. Here, u = 20 m/s ; v = 10 m/s ; x = 200 m ; a = ? An inspection of above data reveals that none of the first three equations of motion fits the requirement in hand, while the additional form of equation $v^2 - u^2 = 2ax$ serves the purpose :

$$a = \frac{v^2 - u^2}{2x} = \frac{10^2 - 20^2}{2\text{x}200} = -\frac{3}{4}\,m/s^2$$

For the train to stops, we have

$$u = 20\,m/s;\; v = 0\,m/s;\; a = -\frac{3}{4}\,m/s^2;\;\; x = ?$$

and,

$$x = \frac{v^2 - u^2}{2a} = \frac{0^2 - 20^2}{2\text{x}\left(-\frac{3}{4}\right)} = 266.67\,m$$

Displacement in a particular second

The displacement in a second is obtained by subtracting two displacements in successive seconds. We calculate displacements in n^{th} second and (n-1)$^{\text{th}}$ seconds. The difference of two displacements is the displacement in the n^{th} second. Now, the displacements at the end of n and (n-1) seconds as measured from origin are given by :

$$x_n = un + \frac{1}{2}an^2$$
$$x_{n-1} = u\left(n-1\right) + \frac{1}{2}a(n-1)^2$$

The displacement in the n^{th} second, therefore, is :

$$s_n = x_2 - x_1 = un + \frac{1}{2}an^2 - un + u - \frac{1}{2}an^2 - \frac{1}{2}a + an$$
$$\Rightarrow s_n = x_2 - x_1 = u - \frac{1}{2}a + an$$

Equation:

$$\Rightarrow s_n = x_2 - x_1 = u + \frac{a}{2}\left(2n-1\right)$$

Example:
Problem : The equation of motion for displacement in n^{th} second is given by :

$$x_n = u + \frac{a}{2}\left(2n-1\right)$$

This equation is dimensionally incompatible, yet correct. Explain.
Solution : Since "n" is a number, the dimension of the terms of the equation is indicated as :

$$\left[x_n\right] = \left[\text{Displacment}\right] = \left[L\right]$$
$$\left[u\right] = \left[\text{velocity}\right] = \left[LT^{-1}\right]$$
$$\left[\frac{a}{2}\left(2n-1\right)\right] = \left[\text{acceleration}\right] = \left[LT^{-2}\right]$$

Clearly, dimensions of the terms are not same and hence equation is apparently incompatible in terms of dimensions.
In order to resolve this apparent incompatibility, we need to show that each term of right hand side of the equation has dimension that of displacement i.e. length. Now, we know that the relation is derived for a displacement for time equal to “1” second. As multiplication of “1” or " 1^2" with any term is not visible, the apparent discrepancy has appeared in otherwise correct equation. We can, therefore, re-write the equation with explicit time as :

$$x_n = u \ \text{x} \ 1 + \frac{a}{2}\left(2n-1\right) \ \text{x} \ 1^2$$
$$\Rightarrow x_n = u \ \text{x} \ t + \frac{a}{2}\left(2n-1\right) \ \text{x} \ t^2$$

Now, the dimensions of the first and second terms on the right side are :

$$\left[u \ \text{x} \ t\right] = \left[LT^{-1}T\right] = \left[L\right]$$
$$\left[\frac{a}{2}\left(2n-1\right) \ \text{x} \ t^2\right] = \left[LT^{-2}T^2\right] = \left[L\right]$$

Thus, we see that the equation is implicitly correct in terms of dimensions.

Exercise:

Problem:

If a particle, moving in straight line, covers distances “x”, “y” and “z” in p^{th} , q^{th} and r^{th} seconds respectively, then prove that :

$$(p-q)z + (r-p)y + (q-r)x = 0$$

Solution:

This is a motion with constant acceleration in one dimension. Let the particle moves with a constant acceleration, “a”. Now, the distances in particular seconds as given in the question are :

$$x = u + \frac{a}{2}\left(2p - 1\right)$$
$$y = u + \frac{a}{2}\left(2q - 1\right)$$
$$z = u + \frac{a}{2}\left(2r - 1\right)$$

Subtracting, second from first equation,

$$x - y = \frac{a}{2}\left(2p - 1 - 2q + 1\right) = a\left(p - q\right)$$
$$\Rightarrow \left(p - q\right) = \frac{(x-y)}{a}$$
$$\Rightarrow \left(p - q\right)z = \frac{(x-y)z}{a}$$

Similarly,

$$\Rightarrow \left(r - p\right)y = \frac{(z-x)y}{a}$$
$$\Rightarrow \left(q - r\right)x = \frac{(y-z)x}{a}$$

Adding three equations,

$$\left(p - q\right)z + \left(r - p\right)y + \left(q - r\right)x = \frac{1}{a}\left(xy - yz + yz - xy + xy - xz\right)$$
$$\Rightarrow (p - q)z + (r - p)y + (q - r)x = 0$$

Average acceleration

Average acceleration is ratio of change in velocity and time. This is an useful concept where acceleration is not constant throughout the motion. There may be motion, which has constant but different values of acceleration in different segments of motion. Our job is to find an equivalent constant acceleration, which may be used to determine attributes

for the whole of motion. Clearly, the single value equivalent acceleration should be such that it yields same value of displacement and velocity for the entire motion. This is the underlying principle for determining equivalent or average acceleration for the motion.

Example:

Problem : A particle starting with velocity “u” covers two equal distances in a straight line with accelerations a_1 and a_2 . What is the equivalent acceleration for the complete motion?

Solution : The equivalent acceleration for the complete motion should yield same value for the attributes of the motion at the end of the journey. For example, the final velocity at the end of the journey with the equivalent acceleration should be same as the one calculated with individual accelerations.

Here initial velocity is "u". The velocity at the end of half of the distance (say “x”) is :

$$v_1{}^2 = u^2 + 2a_1x$$

Clearly, v_1 is the initial velocity for the second leg of motion,

$$v_2{}^2 = v_1{}^2 + 2a_2x$$

Adding above two equations, we have :

$$v_2{}^2 = u^2 + 2\left(a_1 + a_2\right)x$$

This is the square of velocity at the end of journey. Now, let “a” be the equivalent acceleration, then applying equation of motion for the whole distance (2x),

$$v_2{}^2 = u^2 + 2a\left(2x\right)$$

Comparing equations,

$$a = \frac{a_1 + a_2}{2}$$

Interpretation of equations of motion

One dimensional motion felicitates simplified paradigm for interpreting equations of motion. Description of motion in one dimension involves mostly the issue of “magnitude” and only one aspect of direction. The only possible issue of direction here is that the body undergoing motion in one dimension may reverse its direction during the course of motion. This means that the body may either keep moving in the direction of initial velocity or may start moving in the opposite direction of the initial velocity at certain point of time during the motion. This depends on the relative direction of initial velocity and acceleration. Thus, there are two paradigms :

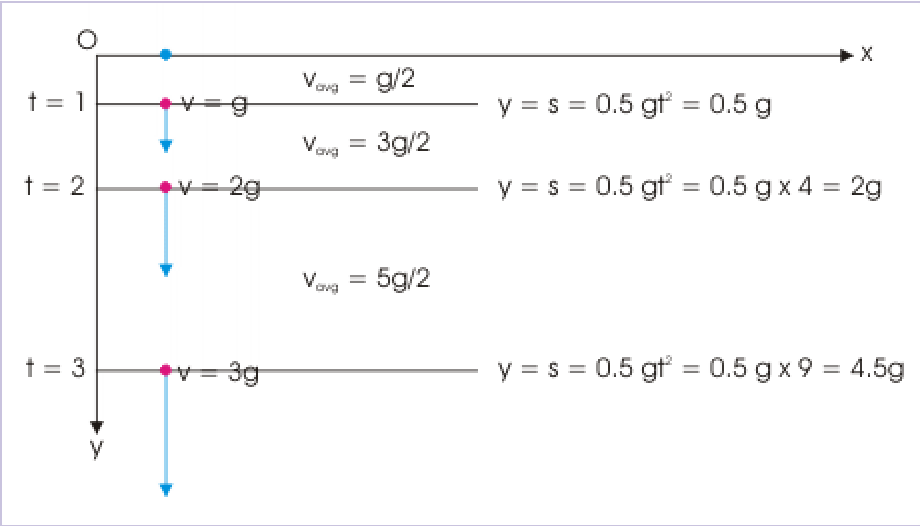
1. Constant force is applied in the direction of initial velocity.
2. Constant force is applied in the opposite direction of initial velocity.

Irrespective of the above possibilities, one fundamental attribute of motion in one dimension is that all parameters defining motion i.e initial velocity, final velocity and acceleration act along a straight line.

Constant acceleration (force) is applied in the direction of velocity

The magnitude of velocity increases by the magnitude of acceleration at the end of every second (unit time interval). In this case, final velocity at any time instant is greater than velocity at an earlier instant. The motion is not only in one dimension i.e. linear , but also unidirectional. Take the example of a ball released (initial velocity is zero) at a certain height ‘h’ from the surface. The velocity of the ball increases by the magnitude of ‘g’ at the end of every second. If the body has traveled for 3 seconds, then the velocity after 3 seconds is 3g ($v= 0 + 3 \times g = 3g$ m/s).

Attributes of motion



Attributes of motion as the ball falls under gravity

In this case, all parameters defining motion i.e initial velocity, final velocity and acceleration not only act along a straight line, but also in the same direction. As a consequence, displacement is always increasing during the motion like distance. This fact results in one of the interesting aspect of the motion that magnitude of displacement is equal to distance. For this reason, average speed is also equal to the magnitude of average velocity.

$$s = |x|$$

and

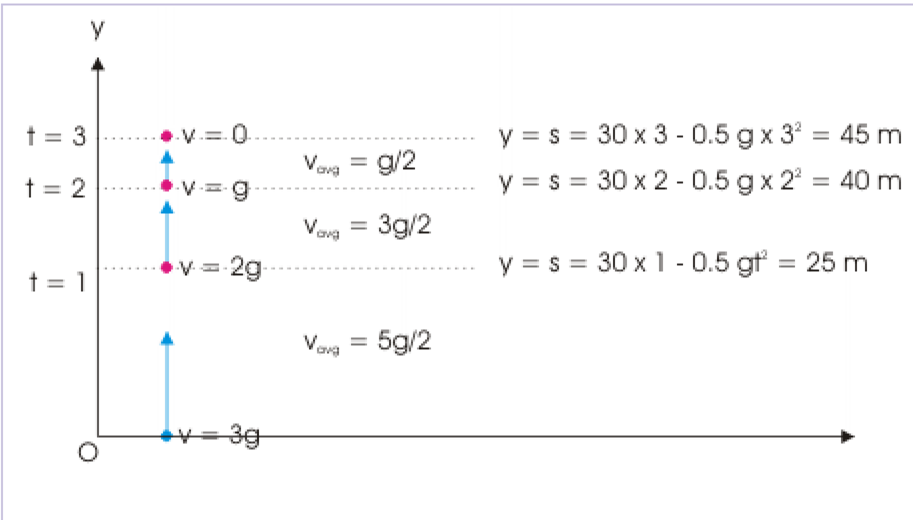
$$\frac{\Delta s}{\Delta \mathrm{t}} = \left|\frac{\Delta x}{\Delta \mathrm{t}}\right|$$

Constant acceleration (force) is applied in the opposite direction of velocity

The magnitude of velocity decreases by the magnitude of acceleration at the end of every second (unit time interval). In this case, final velocity at any time instant is either less than velocity at an earlier instant or has reversed

its direction. The motion is in one dimension i.e. linear, but may be unidirectional or bidirectional. Take the example of a ball thrown (initial velocity is ,say, 30 m/s) vertically from the surface. The velocity of the ball decreases by the magnitude of ‘g’ at the end of every second. If the body has traveled for 3 seconds, then the velocity after 3 seconds is 30 - 3g = 0 (assume g = 10 m/s^2).

Attributes of motion



Attributes of motion as the ball moves up against gravity

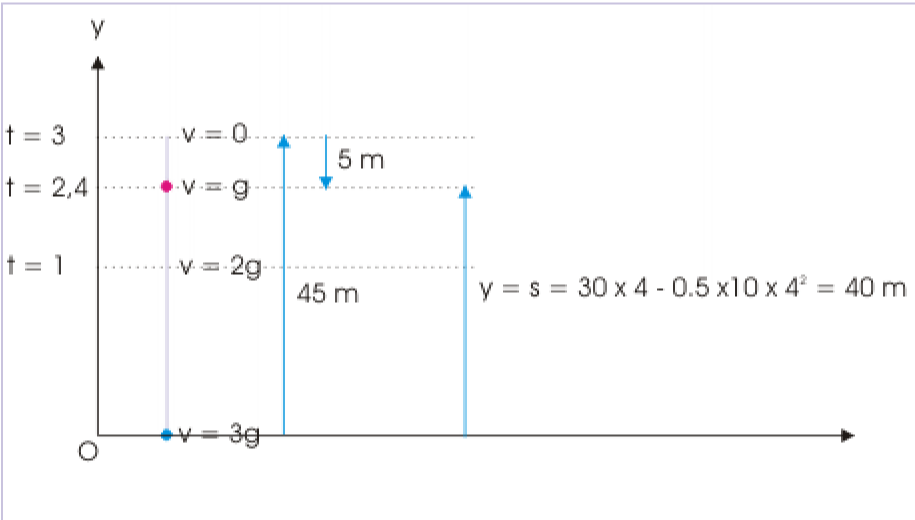
During upward motion, velocity and acceleration due to gravity are in opposite direction. As a result, velocity decreases till it achieves the terminal velocity of zero at the end of 3rd second. Note that displacement during the motion is increasing till the ball reaches the maximum height.

At the maximum height, the velocity of the ball is zero and is under the action of force due to gravity as always during the motion. As such, the ball begins moving in downward direction with the acceleration due to gravity. The directions of velocity and acceleration, in this part of motion, are same. Note that displacement with respect to point of projection is decreasing.

In the overall analysis of motion when initial velocity is against acceleration, parameters defining motion i.e initial velocity, final velocity

and acceleration act along a straight line, but in different directions. As a consequence, displacement may either be increasing or decreasing during the motion. This means that magnitude of displacement may not be equal to distance. For example, consider the motion of ball from the point of projection, A, to maximum height, B, to point, C, at the end of 4 seconds. The displacement is 40 m, while distance is 45 + 5 = 50 m as shown in the figure below.

Attributes of motion



For this reason, average speed is not always equal to the magnitude of average velocity.

$$s \neq |x|$$

and

$$\frac{\Delta s}{\Delta \mathrm{t}} \neq \left|\frac{\Delta x}{\Delta \mathrm{t}}\right|$$

Exercises

Exercise:

Problem:

Two cyclists start off a race with initial velocities 2 m/s and 4 m/s respectively. Their linear accelerations are 2 and 1 m/s^2 respectively. If they reach the finish line simultaneously, then what is the length of the track?

Solution:

This is a case of one dimensional motion with constant acceleration. Since both cyclists cross the finish line simultaneously, they cover same displacement in equal times. Hence,

$$x_1 = x_2$$

$$u_1 t + \frac{1}{2} a_1 t^2 = u_2 t + \frac{1}{2} a_2 t^2$$

Putting values as given in the question, we have :

$$2t + \frac{1}{2} \text{ x } 2 \text{ x } t^2 = 4t + \frac{1}{2} \text{ x } 1 \text{ x } t^2$$
$$\Rightarrow t^2 - 4t = 0$$
$$\Rightarrow t = 0\,s, 4\,s$$

Neglecting zero value,

$$\Rightarrow t = 4\,s$$

The linear distance covered by the cyclist is obtained by evaluating the equation of displacement of any of the cyclists as :

$$x = 2t + \frac{1}{2} \text{ x } 2 \text{ x } t^2 = 2 \text{ x } 4 + \frac{1}{2} \text{ x } 2 \text{ x } 4^2 = 24\,t$$

Exercise:

Problem:

Two cars are flagged off from the starting point. They move with accelerations a_1 and a_2 respectively. The car "A" takes time "t" less than car "B" to reach the end point and passes the end point with a difference of speed, "v", with respect to car "B". Find the ratio v/t.

Solution:

Both cars start from rest. They move with different accelerations and hence take different times to reach equal distance, say t_1 and t_2 for cars A and B respectively. Their final speeds at the end points are also different, say v_1 and v_2 for cars A and B respectively. According to the question, the difference of time is "t", whereas difference of speeds is "v".

As car A is faster, it takes lesser time. Here, $t_2 > t_1$. The difference of time, "t", is :

$$t = t_2 - t_1$$

From equation of motion,

$$x = \frac{1}{2} a_1 {t_1}^2$$
$$\Rightarrow t_1 = \sqrt{\left(\frac{2x}{a_1}\right)}$$

Similarly,

$$\Rightarrow t_2 = \sqrt{\left(\frac{2x}{a_2}\right)}$$

Hence,

$$\Rightarrow t = t_2 - t_1 = \sqrt{\left(\frac{2x}{a_2}\right)} - \sqrt{\left(\frac{2x}{a_1}\right)}$$

Car A is faster. Hence, $v_1 > v_2$. The difference of time, "v", is :

$$\Rightarrow v = v_1 - v_2$$

From equation of motion,

$$\Rightarrow {v_1}^2 = 2a_1 x$$
$$\Rightarrow v_1 = \surd(2a_1 x)$$

Similarly,

$$\Rightarrow v_2 = \surd(2a_2 x)$$

Hence,

$$v = v_1 - v_2 = \surd(2a_1 x) - \surd(2a_2 x)$$

The required ratio, therefore, is :

$$\frac{v}{t} = \frac{\surd(2a_1 x) - \surd(2a_2 x)}{\sqrt{\left(\frac{2x}{a_2}\right)} - \sqrt{\left(\frac{2x}{a_1}\right)}} = \frac{\{\surd(2a_1) - \surd(2a_2)\}\surd a_1 \surd a_2}{\{\surd(2a_1) - \surd(2a_2)\}}$$

$$\frac{v}{t} = \sqrt{\left(a_1 a_2\right)}$$

Exercise:

Problem:

Two particles start to move from same position. One moves with constant linear velocity, “v”; whereas the other, starting from rest, moves with constant acceleration, “a”. Before the second catches up with the first, what is maximum separation between two?.

Solution:

One of the particles begins with a constant velocity, “v” and continues to move with that velocity. The second particle starts with zero velocity and continues to move with a constant acceleration, “a”. At a given instant, “t”, the first covers a linear distance,

$$x_1 = vt$$

In this period, the second particle travels a linear distance given by :

$$x_2 = \frac{1}{2}at^2$$

First particle starts with certain velocity as against second one, which is at rest. It means that the first particle will be ahead of second particle in the beginning. The separation between two particles is :

$$\Delta x = x_1 - x_2$$
$$\Delta x = vt - \frac{1}{2}at^2$$

For the separation to be maximum, its first time derivative should be equal to zero and second time derivative should be negative. Now, first and second time derivatives are :

$$\frac{đ(\Delta x)}{đt} = v - at$$
$$\frac{đ^2(\Delta x)}{đt^2} = -a < 0$$

For maximum separation,

$$\Rightarrow \frac{đ(\Delta x)}{đt} = v - at = 0$$

$$\Rightarrow t = \frac{v}{a}$$

The separation at this time instant,

$$\Delta x = vt - \frac{1}{2}at^2 = v \text{ x } \frac{v}{a} - \frac{1}{2}a\left(\frac{v}{a}\right)^2$$
$$\Delta x = \frac{v^2}{2a}$$

Graphs of motion with constant acceleration

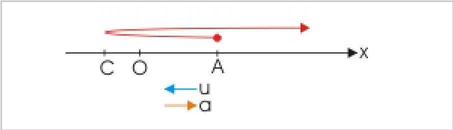
Graphical analysis of motion with constant acceleration in one dimension is based on the equation defining position and displacement, which is a quadratic function. Nature of motion under constant acceleration and hence its graph is characterized by the relative orientation of initial velocity and acceleration. The relative orientation of these parameters controls the nature of motion under constant acceleration. If initial velocity and acceleration are in the same direction, then particle is accelerated such that speed of the particle keeps increasing with time. However, if velocity and acceleration are directed opposite to each other, then particle comes to rest momentarily. As a result, the motion is divided in two segments - one with deceleration in which particle moves with decreasing speed and second with acceleration in which particle moves with increasing speed.

We study various scenarios of motion of constant acceleration by analyzing equation of position, which is quadratic expression in time. The position of particle is given by :

$$x = x_0 + ut + \frac{1}{2}at^2$$

$$\Rightarrow x = \frac{1}{2}at^2 + ut + x_0$$

Clearly, the equation for position of the particle is a quadratic expression in time, “t”. A diagram showing a generalized depiction of motion represented by quadratic expression is shown here :
Motion under constant acceleration



Motion under constant acceleration

In case initial position of the particle coincides with origin of reference, then initial position $x_0 = 0$ and in that case x denotes position as well as displacement :

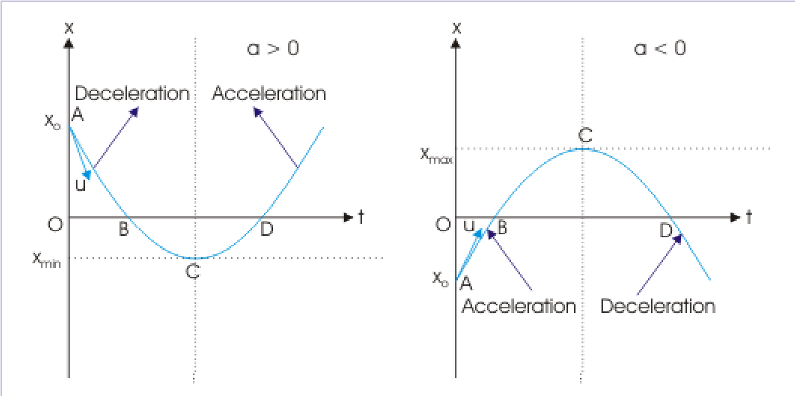
$$\Rightarrow x = ut + \frac{1}{2}at^2$$

Nature of graphs

The nature of quadratic polynomial is determined by two controlling factors (i) nature of coefficient of squared term, t^2 and (ii) nature of discriminant of the quadratic equation, which is formed by equating quadratic expression to zero.

Nature of coefficient of squared term

The coefficient of squared term is “a/2”. Thus, its nature is completely described by the nature of “a” i.e. acceleration. If “a” is positive, then graph is a parabola opening up. On the other hand, if “a” is negative, then graph is a parabola opening down. These two possibilities are shown in the picture.
Motion under constant acceleration



Motion under constant acceleration

It may be interesting to know that sign of acceleration is actually a matter of choice. A positive acceleration, for example, is negative acceleration if we reverse the reference direction. Does it mean that mere selection of reference direction will change the nature of motion? As expected, it is not so. The parabola comprises two symmetric sections about a line passing through the minimum or maximum point (C as shown in the figure). One section represents a motion in which speed of the particle is decreasing as velocity and acceleration are in opposite directions and second section represents a motion in which speed of the particle is increasing as velocity and acceleration are in the same direction. In other words, one section represents deceleration, whereas other section represents acceleration. The two sections are simply exchanged in two graphs. As such, a particular motion of acceleration is described by different sections of two graphs when we change the sign of acceleration. That is all. The nature of motion remains same. Only the section describing motion is exchanged.

Nature of discriminant

Comparing with general quadratic equation $Ax^2 + Bx + C = 0$, we see that determinant of the corresponding quadratic equation is given by :

$$D = B^2 - 4AC = u^2 - 4X\frac{a}{2}Xx_0$$

$$\Rightarrow D = u^2 - 2ax_0$$

The important aspect of discriminant is that it comprises of three variable parameters. However, simplifying aspect of the discriminant is that all parameters are rendered constant by the “initial” setting of motion. Initial position, initial velocity and acceleration are all set up by the initial conditions of motion.

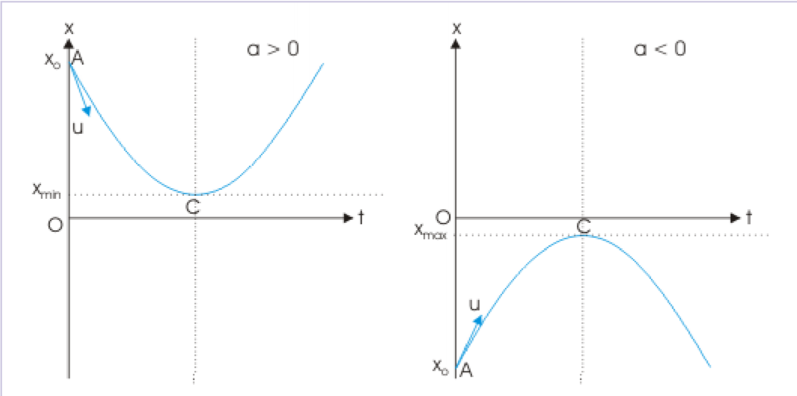
The points on the graph intersecting t-axis gives the time instants when particle is at the origin i.e. x=0. The curve of the graph intersects t-axis when corresponding quadratic equation (quadratic expression equated to zero) has real roots. For this, discriminant of the corresponding quadratic equation is non-negative (either zero or positive). It means :

$$D = u^2 - 2ax_0 \geq 0$$

$$\Rightarrow 2ax_0 \leq u^2$$

Note that squared term u^2 is always positive irrespective of sign of initial velocity. Thus, this condition is always fulfilled if the signs of acceleration and initial position are opposite. However, if two parameters have same sign, then above inequality should be satisfied for the curve to intersect t-axis. In earlier graphs, we have seen that parabola intersects t-axis at two points corresponding two real roots of corresponding quadratic equation. However, if discriminant is negative, then parabola does not intersect t-axis. Such possibilities are shown in the figure :

Motion under constant acceleration



Motion under constant acceleration

Clearly, motion of particle is limited by the minimum or maximum positions. It is given by :

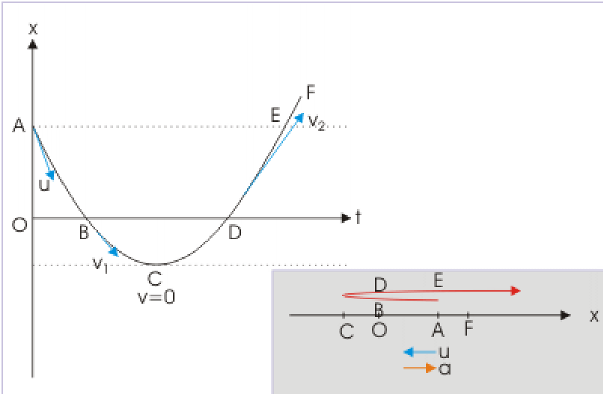
$$x = -\frac{D}{4A} = -\frac{u^2 - 2ax_0}{4 \text{ X } \frac{a}{2}} = -\frac{u^2 - 2ax_0}{2a}$$

It may be noted that motion of particle may be restricted to some other reference points as well. Depending on combination of initial velocity and acceleration, a particle may not reach a particular point.

Reading of graph

Our consideration, here, considers only positive values of time. It means that we are discussing motion since the start of observation at t=0 and subsequently as the time passes by. Mathematically, the time parameter, "t" is a non-negative number. It can be either zero or positive, but not negative. In the following paragraphs, we describe critical segments or points of a typical position – time graph as shown in the figure above :

Motion under constant acceleration



Motion under constant acceleration

Point A : This is initial position at t = 0. The position corresponding to this time is denoted as x_0 . This point is the beginning of graph, which lies on x-axis. The tangent to the curve at this point is the direction of initial velocity, “u”. Note that initial position and origin of reference may be different. Further, this point is revisited by the particle as it reaches E. Thus, A and E denotes the same start position - though represented separately on the graph.

Curve AC : The tangent to the curve part AB has negative slope. Velocity is directed in negative x-direction. The slope to the curve keeps decreasing in magnitude as we move from A to C. It means that speed of the particle keeps decreasing in this segment. In other words, particle is decelerated during motion in this segment.

Points B, D : The curve intersects t-axis. The particle is at the origin of the reference, O, chosen for the motion. The time corresponding to these points (B and D) are real roots of the quadratic equation, which is obtained by equating quadratic expression to zero.

Point C : The slope of tangent to the curve at this point is zero. It means that the speed of the particle has reduced to zero. The particle at this point is at rest. The slopes of the tangent to curve about this point changes sign. It means that velocity is oppositely directed about this point. The point B, therefore, is a point, where reversal of direction of motion occurs. Note that particle can reverse its direction only once during its motion under constant acceleration.

Point E : The particle reaches start point E (i.e. A) again in its motion after reversal of motion at C.

Point F : This is the end point of motion.

The graph of a motion under constant acceleration is bounded by set of parameters defining a motion. In a particular case, we may be considering only a segment of the curve starting from point A and ending at point F – not necessarily covering the whole of graph as shown in the figure. The point F may lie anywhere on the graph. Further, nature of curve will be determined by values and signs of various parameters like initial position, initial velocity and acceleration. Here, we have divided our study in two categories based on the sign of acceleration : (i) acceleration is positive and (ii) acceleration is negative.

Acceleration is positive (in the reference direction)

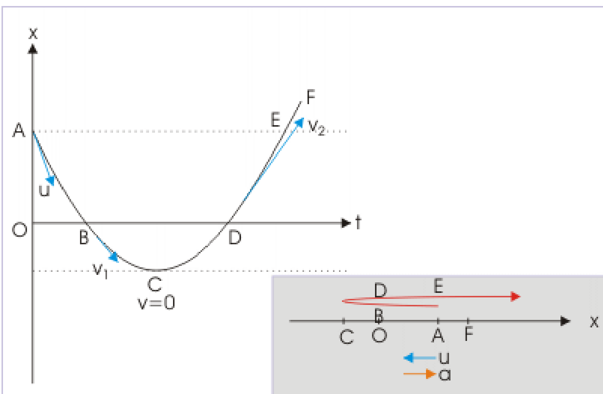
The graph of quadratic equation is a parabola opening upwards as coefficient of squared term t^2 is positive i.e. a > 0. The minimum value of expression i.e. x is :

$$x_{\min} = -\frac{D}{4A} = -\frac{u^2 - 2ax_0}{4 \text{ X } \frac{a}{2}} = -\frac{u^2 - 2ax_0}{2a}$$

The various possibilities of initial positions of the particle with time are discussed here :

Case 1: Initial position and origin of reference are different. Initial position is positive. Initial velocity is negative i.e. it is directed in negative reference direction. The velocity and acceleration are oppositely directed.

Motion under constant acceleration



Motion under constant acceleration

Initially particle is decelerated as the speed of the particle keeps on decreasing till it becomes zero at point C. This is indicated by the diminishing slope (magnitude) of the tangents to the curve. Subsequently, particle is accelerated so long force causing acceleration is applied on the particle.

The segment DF with origin at D is typical graph of free fall of particle under gravity, considering DF as the height of fall and downward direction as positive direction. Only differing aspect is that particle has initial velocity. Nevertheless, the nature of curve of free fall is similar. Note that speed of the particle keeps increasing till it hits the ground.

Example:

Problem : Given $x_0 = 10 \quad m; \quad u = -15 \quad m/s; \quad a = 10 \quad m/s^2$

Find the position when particle reverses its motion.

Solution : Acceleration is negative. The position-time graph is a parabola opening up. Therefore, the particle changes its direction of motion at its minimum position, which is given as :

$$x_{\min} = -\frac{u^2 - 2ax_0}{2a}$$

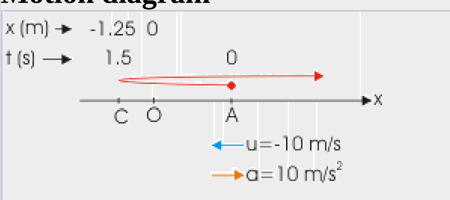
$$\Rightarrow x_{\min} = -\frac{(-15)^2 - 2X10X10}{2X10} = -\frac{25}{20} = -1.25 \quad m$$

Alternatively,
At the point of reversal, velocity is zero. Using v=u+at, :

$$\Rightarrow 0 = -15 + 10t$$

$$\Rightarrow t = \frac{3}{2} = 1.5 \quad s$$

Position of particle in 1.5 s is :
Motion diagram



Motion diagram

$$x = x_0 + ut + \frac{1}{2}at^2$$

$$\Rightarrow x = 10 + (-15)X_1.5 + \frac{1}{2}X10X2.25 = 10 - 22.5 + 5X2.25 = -12.5 + 11.25 = -1.25 \quad m$$

Exercise:

Problem: Given : $x_0 = 10 \quad m; \quad u = -15 \quad m/s; \quad a = 10 \quad m/s^2$

Find the time instants when the particle is at origin of reference and initial position. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = x_0 + ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put x=0,

$$0 = 10 - 15t + \frac{1}{2} X 10t^2$$

$$\Rightarrow t^2 - 3t + 2 = 0$$

$$\Rightarrow t^2 - t - 2t + 2 = 0$$

$$\Rightarrow t(t-1) - 2(t-1) = 0$$

$$\Rightarrow t = 1 \quad s \quad \text{and} \quad 2 \quad s$$

In order to determine time instant when particle is at initial position, we put x=10,

$$\Rightarrow 10 = 10 - 15t + \frac{1}{2} X 10t^2$$

$$\Rightarrow 5t_2 - 15t = 0$$

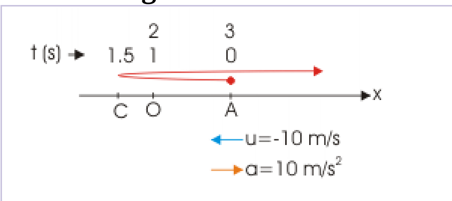
$$\Rightarrow t = 0 \quad \text{and} \quad 3 \quad s$$

The particle is at the initial position twice, including start of motion. Now, at the point of reveral of direction, speed of the particle is zero. Putting v = 0 and using v=u+at,

$$\Rightarrow 0 = -15 + 10t$$

$$\Rightarrow t = 1.5 \quad s$$

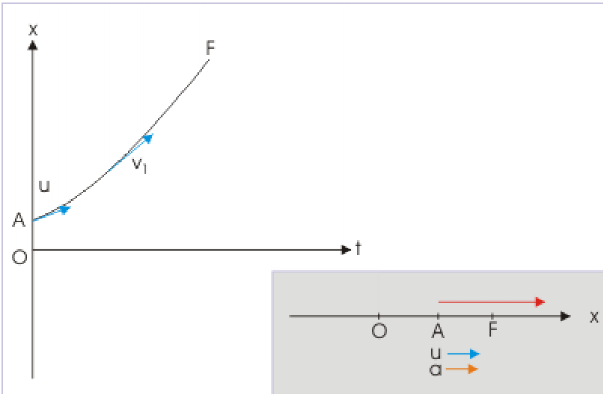
Motion diagram



Motion diagram

Case 2: Initial position and origin of reference are different. Initial position is positive. Initial velocity is positive i.e. it is directed in reference direction. The velocity and acceleration are in same direction.

Motion under constant acceleration



Motion under constant acceleration

The particle is accelerated so long force causing acceleration is applied on the particle. The segement AF with origin at D is typical graph of free fall of particle under gravity, considering AF as the height of fall and downward direction as positive direction. Only differing aspect is that particle has initial velocity. Neverthless, the nature of curve of free fall is similar. Note that speed of the particle keeps increasing till it hits the ground.

Exercise:

Problem: Given $x_0 = 10 \quad m; \quad u = 15 \quad m/s; \quad a = 10 \quad m/s^2$

Find the time instants when the particle is at origin of reference and initial position. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = x_0 + ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put x=0,

$$\Rightarrow 0 = 10 + 15t + \frac{1}{2}X10t^2$$

$$\Rightarrow t^2 + 3t + 2 = 0$$

$$\Rightarrow t^2 + t + 2t + 2 = 0$$

$$\Rightarrow t = -1 \quad s \quad \text{and} \quad -2 \quad s$$

We neglect both these negative values and deduce that the particle never reaches point of origin. In order to determine time instant when particle is at initial position, we put x=10,

$$\Rightarrow 10 = 10 + 15t + \frac{1}{2}X10t^2$$

$$\Rightarrow 5t^2 + 15t = 0$$

$$\Rightarrow t = 0 \quad \text{and} \quad -3 \quad s$$

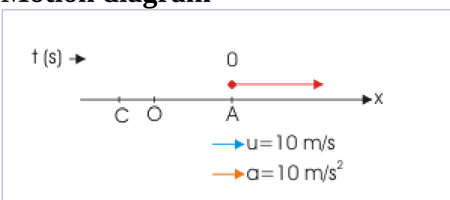
We neglect negative value. The particle is at the initial position only at the start of motion. Now, at the point of reveral of direction, speed of the particle is zero. Putting v = 0 and using v=u+at,

$$\Rightarrow 0 = 15 + 10t$$

$$\Rightarrow t = -1.5 \quad s$$

The particle never changes its direction.

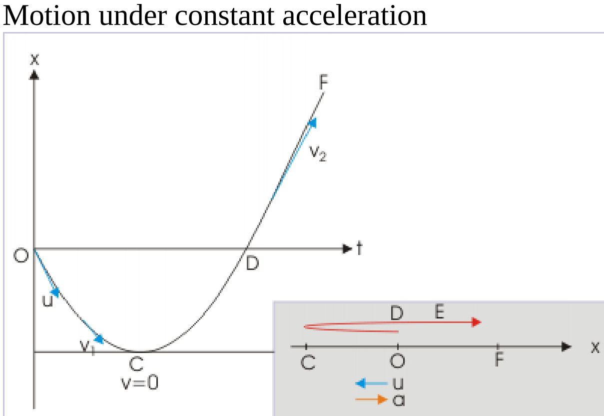
Motion diagram



Motion diagram

Case 3: Initial position and origin of reference are same. Initial velocity is negative i.e. it is directed in negative reference direction. The velocity and acceleration are oppositely directed.

Motion under constant acceleration



Motion under constant acceleration

Initially particle is decelerated as the speed of the particle keeps on decreasing till it becomes zero at point C. This is indicated by the diminishing slope (magnitude) of the tangents to the curve. Subsequently, particle is accelerated so long force causing accleration is applied on the particle.

Exercise:

Problem: Given $x_0 = 0 \quad m; \quad u = -15 \quad m/s; \quad a = 10 \quad m/s^2$

Find the time instants when the particle is at origin of reference. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put x=0,

$$\Rightarrow 0 = -15t + \frac{1}{2}X10t^2$$

$$\Rightarrow t^2 - 3t = 0$$

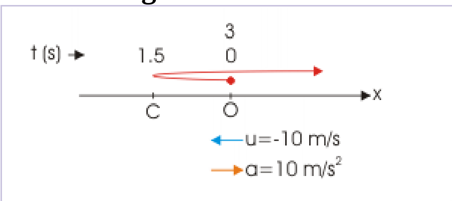
$$\Rightarrow t = 0 \quad \text{and} \quad 3 \quad s$$

The particle is at the origin of reference twice, including start of motion. Now, at the point of reversal of direction, speed of the particle is zero. Putting v = 0 and using v=u+at,

$$\Rightarrow 0 = -15 + 10t$$

$$\Rightarrow t = 1.5 \quad s$$

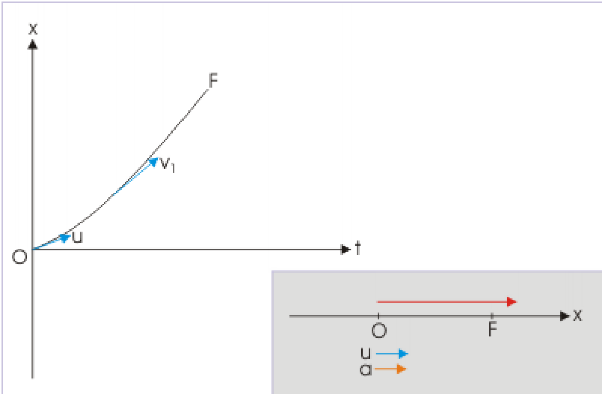
Motion diagram



Motion diagram

Case 4: Initial position and origin of reference are same. Initial velocity is positive i.e. it is directed in reference direction. The velocity and acceleration are in same direction.

Motion under constant acceleration



Motion under constant acceleration

The particle is accelerated so long force causing acceleration is applied on the particle.

The segement OF is typical graph of free fall of particle under gravity, considering OF as the height of fall and downward direction as positive direction. Only differing aspect is that particle has initial velocity. Neverthless, the nature of curve of free fall is similar. Note that speed of the particle keeps increasing till it hits the ground.

Exercise:

Problem: Given $x_0 = 0 \quad m; \quad u = 15 \quad m/s; \quad a = 10 \quad m/s^2$

Find the time instants when the particle is at origin of reference. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put x=0,

$$0 = 15t + \frac{1}{2}X10t^2$$

$$\Rightarrow t^2 + 3t = 0$$

$$\Rightarrow t = 0 \quad \text{and} \quad -3 \quad s$$

Neglecting negative value of time,

$$\Rightarrow t = 0$$

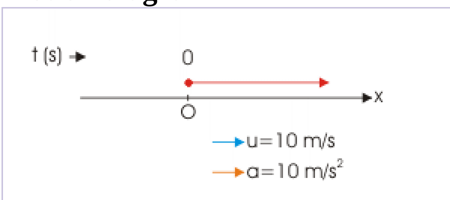
The particle is at the origin of reference only at the start of motion. Now, at the point of reversal of direction, speed of the particle is zero. Putting v = 0 and using v=u+at,

$$\Rightarrow 0 = 15 + 10t$$

$$\Rightarrow t = -1.5 \quad s$$

We neglect negative value and deduce that particle never ceases to move and as such there is no reversal of motion.

Motion diagram



Motion diagram

Acceleration is negative (opposite to the reference direction)

In this case, the graph of quadratic equation is a parabola opening downwards as coefficient of squared term t^2 is negative i.e. a > 0. The maximum value of expression i.e. x is :

$$x_{\max} = -\frac{D}{4A} = -\frac{u^2 - 2ax_0}{4 \text{ X } \frac{a}{2}} = -\frac{u^2 - 2ax_0}{2a}$$

The analysis for this case is similar to the first case. We shall, therefore, not describe this case here.

Exercise:

Problem: Given $x_0 = 10 \quad m; \quad u = 15 \quad m/s; \quad a = -10 \quad m/s^2$

Find the time instants when the particle is at origin of reference and initial position. Also find the time when velocity is zero.

Solution:

The position of the particle with respect to origin of reference is given by :

$$x = x_0 + ut + \frac{1}{2}at^2$$

In order to determine time instants when the particle is at origin of reference, we put x=0,

$$\Rightarrow 0 = 10 + 15t + \frac{1}{2}X - 10Xt^2$$

$$\Rightarrow -t^2 + 3t + 2 = 0$$

$$\Rightarrow t = \frac{-3 \pm \sqrt{\{9 - (4X - 1X2)\}}}{2X - 1}$$

$$\Rightarrow t = \frac{-3 \pm 4.12}{-2} = -0.56 \quad s \quad \text{or} \quad 3.56 \quad s$$

We neglect negative value of time. The particle reaches origin of reference only once at t = 2 s. In order to determine time instant when particle is at initial position, we put x=10,

$$\Rightarrow 10 = 10 + 15t + \frac{1}{2}X - 10t^2$$

$$\Rightarrow -t^2 + 3t = 0$$

$$\Rightarrow t = 0, \quad 3 \quad s$$

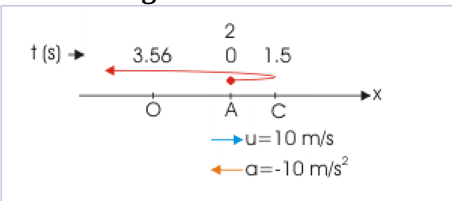
The particle is at the initial position twice, including start point. Now, at the point of reversal of direction, speed of the particle is zero. Putting v = 0 and using v=u+at,

$$\Rightarrow 0 = 15 - 10t$$

$$\Rightarrow t = 1.5 \quad s$$

We neglect negative value and deduce that particle never ceases to move and as such there is no reversal of motion.

Motion diagram



Motion diagram

Example

Example:

Problem : A particle's velocity in "m/s" is given by a function in time "t" as :

$$v = 40 - 10t$$

If the particle is at x = 0 at t = 0, find the time (s) when the particle is 60 m away from the initial position.

Solution : The velocity is given as a function in time "t". Thus, we can know its acceleration by differentiating with respect to time :

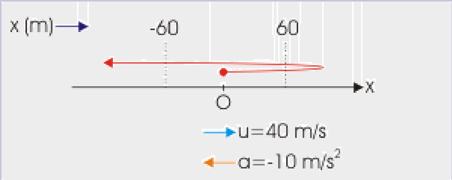
$$a = \frac{dv}{dt} = -10\, m/s^2$$

Alternatively, we can see that expression of velocity has the form v = u + at. Evidently, the motion has a constant acceleration of -10 m/s^2 . Also, note that origin of reference and initial position are same. Applying equation of motion for position as :

$$x = ut + \frac{1}{2}at^2 = 40t + \frac{1}{2} \text{ x } -10 \text{ x } t^2 = 40t - 5t^2$$

According to question, we have to find the time when particle is 60 m away from the initial position. Since it is one dimensional motion, the particle can be 60 m away either in the positive direction of the reference or opposite to it. Considering that it is 60 m away in the positive reference direction, we have :

Motion diagram



Motion diagram

$$60 = 40t - 5t^2$$

Re-arranging,

$$t^2 - 8t + 12 = 0$$
$$t = 2\ s \ \text{ or } \ 6\ s$$

We observe here that the particle is ultimately moving in the direction opposite to reference direction. As such it will again be 60 away from the initial position in the negative reference direction. For considering that the particle is 60 m away in the negative reference direction,

$$-60 = 40t - 5t^2$$

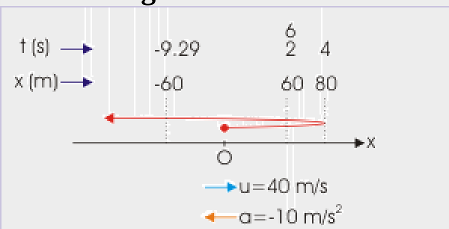
$$\Rightarrow t^2 - 8t - 12 = 0$$
$$t = \frac{-\left(-8\right) \pm \sqrt{\left\{(-8)^2 - 4 \text{ x } 1 \text{ x } \left(-12\right)\right\}}}{2} = -1.29\ s, 9.29\ s$$

Neglecting negative value of time,

$$\Rightarrow t = 9.29\ s$$

Note : We can also solve the quadratic equation for zero displacement to find the time for the particle to return to initial position. This time is found to be 8 seconds. We note here that particle takes 2 seconds to reach the linear distance of 60 m in the positive direction for the first time, whereas it takes only 9.29 – 8 = 1.29 second to reach 60 m from the initial position in the negative direction.

The particle is decelerated while moving in positive direction as velocity is positive, but acceleration is negative. On the other hand, both velocity and acceleration are negative while going away from the initial position in the negative direction and as such particle is accelerated. Therefore, the particle

takes lesser time to travel same linear distance in the negative direction than in the positive direction from initial position..
Hence, the particle is 60 m away from the initial position at t = 2 s, 6 s and 9.29 s.
`t (s) 0 2 4 6 8 9.29 x (m) 0 60 80 60 0 -60 v (m/s) 40 20 0 -20 -40 -52.9`
Motion diagram


Motion diagram

Example:
Problem : A particle moves along x-axis with a velocity 9 m/s and acceleration -2 m/s^2 . Find the distance covered in 5^{th} second.
Solution : The displacement in 5^{th} second is :

$$x_n = u + \frac{a}{2}\left(2n - 1\right)$$
$$x_n = 9 + \frac{-2}{2}\left(2 \text{ x } 5 - 1\right)$$
$$x_n = 0$$

The displacement in the 5^{th} second is zero. A zero displacement, however, does not mean that distance covered is zero. We can see here that particle is decelerated at the rate of -2 m/s^2 and as such there is reversal of direction when v = 0.

$$v = u + at$$
$$2 = 9 - 2t$$
$$t = 4.5 \ s$$

This means that particle reverses its motion at t = 4.5 s i.e in the period when we are required to find distance. The particle, here, travels in the positive direction from t = 4 s to 4.5 s and then travels in the negative direction from t = 4.5 s to 5 s. In order to find the distance in 5^{th} second, we need to find displacement in each of these time intervals and then sum their magnitude to find the required distance.
Motion along a straight line
[missing_resource: odm1a.gif]

The partilcle reverses its direction in 5th second .

The displacement for the period t = 0 s to 4 s is :

$$x = ut + \frac{1}{2}at^2 = 9 \text{ x } 4 + \frac{1}{2} \text{ x } 2 \text{ x } 4^2 = 36 - 16 = 20\, m$$

The displacement for the period t = 0 s to 4.5 s is :

$$x = ut + \frac{1}{2}at^2 = 9 \text{ x } 4.5 + \frac{1}{2} \text{ x } 2 \text{ x } 4.5^2 = 40.5 - 20.25 = 0.25\, m$$

The displacement for the period t = 4 s to 4.5 s is :

$$\Delta x_1 = 20.25 - 20 = 0.25\, m$$

Since particle is moving with constant acceleration in one dimension, it travels same distance on its return for the same period. It means that it travels 0.25 m in the period from t = 4.5 s to t = 5 s.

$$\Delta x_2 = 0.25\, m$$

Thus, total distance covered between t = 4 s to t = 5s i.e. in the 5^{th} second is :

$$\Delta x = \Delta x_2 + \Delta x_2 = 0.25 + 0.25 = 0.5\, m$$

Vertical motion under gravity

Vertical motion under gravity is a specific case of one dimensional motion with constant acceleration. Here, acceleration is always directed in vertically downward direction and its magnitude is "g".

As the force due to gravity may be opposite to the direction of motion, there exists the possibility that the body under force of gravity reverses its direction. It is, therefore, important to understand that the quantities involved in the equations of motion may evaluate to positive or negative values with the exception of time (t). We must appropriately assign sign to various inputs that goes into the equation and correctly interpret the result with reference to the assumed positive direction. Further, some of them evaluate to two values one for one direction and another of reversed direction.

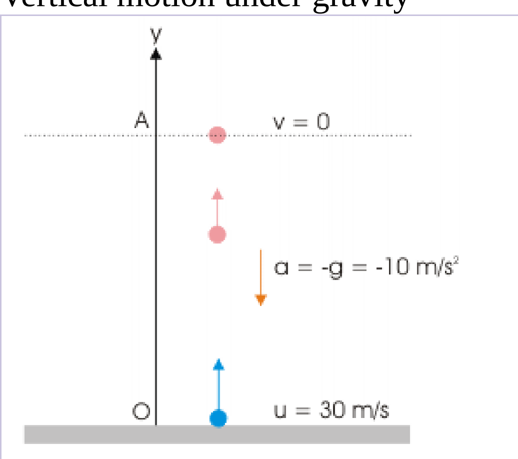
As pointed out earlier in the course, we must also realize that a change in reference direction may actually change the sign of the attributes, but their physical interpretation remains same. What it means that an attribute such as velocity, for example, can be either 5 m/s or -5 m/s, conveying the same velocity. The interpretation must be done with respect to the assigned positive reference direction.

Velocity

Let us analyze the equation "v = u + at" for the vertical motion under gravity with the help of an example. We consider a ball thrown upwards from ground with an initial speed of 30 m/s. In the frame of reference with upward direction as positive,

$$u = 30\, m/s \text{ and } a = -g = -10\, m/s^2$$

Vertical motion under gravity



The ball reaches maximum
height when its velocity

height when its velocity
becomes zero

Putting this value in the equation, we have :

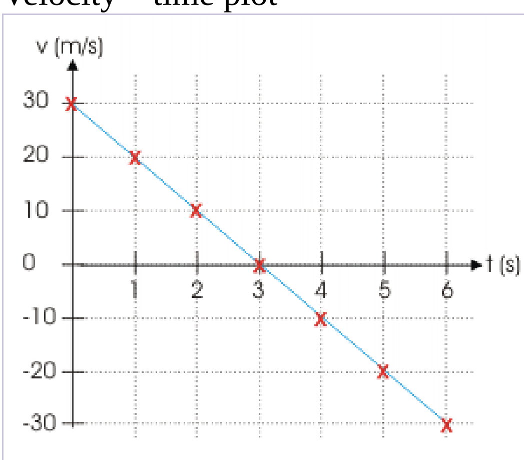
v = 30 – 10 t

The important aspect of this equation is that velocity evaluates to both positive and negative values; positive for upward motion and negative for downward motion. The final velocity (v) is positive for t < 3 seconds, zero for t = 3 seconds and negative for t > 3 seconds. The total time taken for the complete up and down journey is 3 (for upward motion) + 3 (for downward motion) = 6 seconds.

The velocities of the ball at successive seconds are :

```
---------------------------------- Time (t) Final velocity (v) in seconds in m/s ---------------------------------- 0.0 30 1.0 20 2.0 10 3.0 0 4.0 -10 5.0 -20 6.0 -30 ----------------------------------
```

The corresponding velocity – time plot looks like as shown in the figure.
Velocity – time plot



We notice following important characteristics of the motion :

1: The velocity at the maximum height is zero (v=0).

2: The time taken by the ball to reach maximum height is obtained as :

$$\begin{aligned}
&\text{For } v = 0, \\
&v = u + at = u - gt = 0 \\
&\Rightarrow u = gt \\
&\Rightarrow t = \frac{u}{g}
\end{aligned}$$

3: The ball completely regains its speed when it returns to ground, but the motion is directed in the opposite direction i.e.

$$v = -u$$

4: The time taken for the complete round trip is :

$$\begin{aligned}
&\text{For } v = -u, \\
&v = u + at = u - gt \\
&\Rightarrow -u = u - gt \\
&\Rightarrow t = \frac{2u}{g}
\end{aligned}$$

The time taken for the complete journey is twice the time taken to reach the maximum height. It means that the ball takes equal time in upward and downward journey. Thus, the total motion can be considered to be divided in two parts of equal duration.

5: The velocity of the ball is positive in the first half of motion; Zero at the maximum height; negative in the second of the motion.

6: The velocity is decreasing all through the motion from a positive value to less positive value in the first half and from a less negative value to more negative value in the second half of the motion. This renders acceleration to be always negative (directed in -y direction), which is actually the case.

7: The velocity (positive) and acceleration (negative) in the first part are opposite in direction and the resulting speed is decreasing. On the other hand, the velocity (negative) and acceleration (negative) in the second part are in the same direction and the resulting speed is increasing.

Displacement and distance

Let us analyze the equation $\varDelta x = x_2 - x_1 = ut + \frac{1}{2}at^2$ for the vertical motion under gravity with the help of earlier example. If we choose initial position as the origin, then $x_1 = 0,\ \Delta x = x_2 = x\ (\text{say})$ and $x = ut + \frac{1}{2}at^2$, where x denotes position and displacement as well. In the frame of reference with upward direction as positive,

$$u = 30\, m/s \text{ and } a = -g = -10\, m/s^2$$

Putting these values in the equation, we have :

$$\Rightarrow x = 30t - 5t^2$$

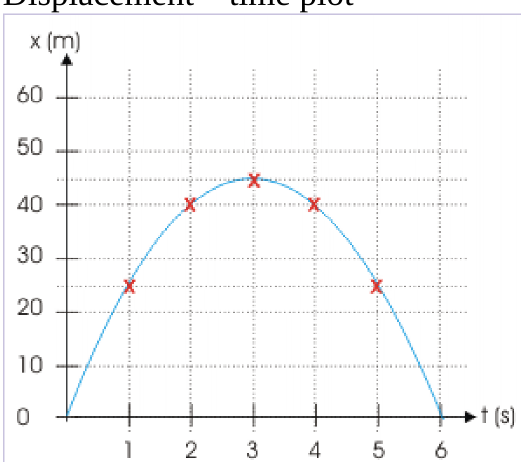
The important aspect of this equation is that it is a quadratic equation in time “t”. This equation yields two values of time “t” for every position and displacement. This outcome is in complete agreement with the actual motion as the ball reaches a given position twice (during upward and downward motion). Only exception is point at the maximum height, which is reached only once. We have seen earlier that ball reaches maximum height at t = 3 s. Therefore, maximum height,H, is given as :

$$\Rightarrow H = 30 \text{ X } 3 - 5 \text{ X } 9 = 45\, m$$

The displacement values for the motion at successive seconds are :

```
-----------------------------------------------------------
--- Time (t) ut 5txt Displacement in seconds or position
(x) in meters ---------------------------------------------
----------------- 0.0 0 0 0 1.0 30 5 25 2.0 60 20 40 3.0
90 45 45 4.0 120 80 40 5.0 150 125 25 6.0 180 180 0 ------
--------------------------------------------------------
```

The corresponding displacement – time plot looks like as shown in the figure.
Displacement – time plot



We notice following important characteristics of the motion :

1: The ball retraces every position during motion except the point at maximum height.

2: The net displacement when ball return to initial position is zero. Thus, the total time of journey (T) is obtained using displacement, x = 0,

$$\begin{aligned}&\text{For } x = 0,\\ &x = ut + \tfrac{1}{2}at^2 = uT - \tfrac{1}{2}gt^2 = 0\\ &\Rightarrow 2uT - gT^2 = 0\\ &\Rightarrow T = \tfrac{2\mathrm{u}}{g}\end{aligned}$$

Here, we neglect T = 0, which corresponds to initial position.

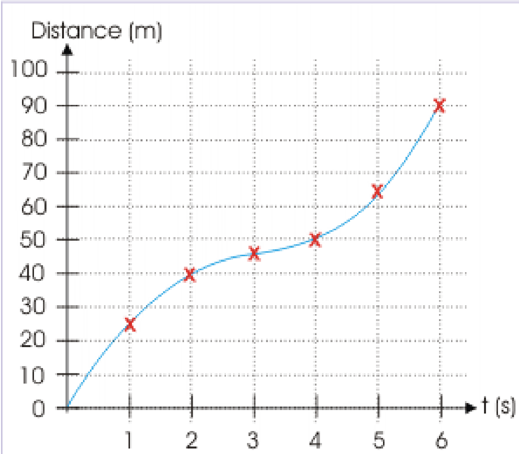
3: The “x” in equation $x = ut + \frac{1}{2}at^2$ denotes displacement and not distance. Hence, it is not possible to use this equation directly to obtain distance, when motion is not unidirectional.

Let us answer the question with respect to the motion of the ball under consideration : what is the distance traveled in first 4 seconds? Obviously, the ball travels 30 m in the upward direction to reach maximum height in 3 seconds and then it travels 5 m in the 4$^{\text{th}}$ second in downward direction. Hence, the total distance traveled is 45 + 5 = 50 m in 4 s. This means that we need to apply the equation of motion in two parts : one for the upward motion and the second for the downward motion. Thus, we find displacement for each segment of the motion and then we can add their magnitude to obtain distance.

The distance values for the motion at successive seconds are :

```
----------------------------------------------------------
-- Time (t) ut 5t*t Displacement Distance in seconds or
position in meters (x) in meters ------------------------
---------------------------------- 0.0 0 0 0 0 0 1.0 30 5
25 25 2.0 60 20 40 40 3.0 90 45 45 45 4.0 120 80 40 50 5.0
150 125 25 65 6.0 180 180 0 90 --------------------------
---------------------------------
```

Distance – time plot



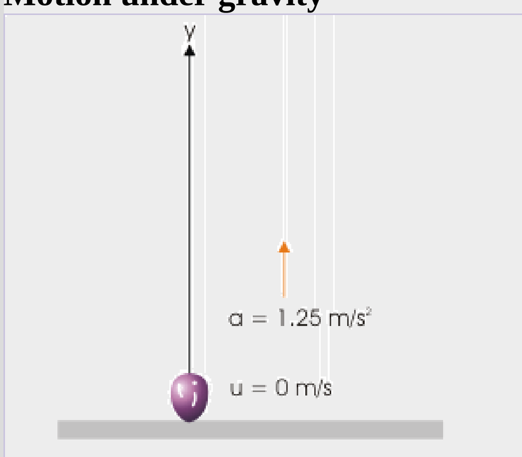
Example:

Constant acceleration

Problem : A balloon starts rising from the ground with an acceleration of 1.25 m/s. After 8 second, a stone is released from the balloon. Starting from the release of stone, find the displacement and distance traveled by the stone on reaching the ground. Also, find the time taken to reach the ground (take g = 10 m/s^2).

Solution : This question raises few important issues. First the rise of balloon is at a constant acceleration of 1.25 m/s^2 . This acceleration is the "measured" acceleration, which is net of the downward acceleration due to gravity. This means that the balloon rises with this net vertical acceleration of 1.25 m/s^2 in the upward direction.

Motion under gravity



Here, u = 0; a = 1.25 m/s^2 ; t = 8 s. Let the balloon rises to a height "h" during this time, then (considering origin on ground and upward direction as positive) the displacement of the balloon after 8 seconds is :

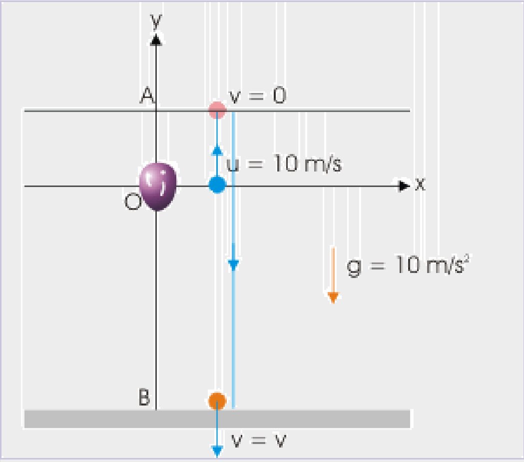
$$y = u t + \frac{1}{2} a t^2 = 0 \text{ x } 8 + 0.5 \text{ x } 1.25 \text{ x } 8^2 = 40 \quad m$$

Now, we know that, the body released from moving body acquires the velocity but not the acceleration of the container body under motion. The velocity of the balloon at the instant of separation is equal to the velocity of the balloon at that instant.

$$v = u + a t = 0 + 1.25 \text{ x } 8 = 10 \quad m / s$$

Thus, this is the initial velocity of the stone and is directed upward as that of the velocity of balloon. Once released, the stone is acted upon by the force of gravity alone. The role of the acceleration of the balloon is over. Now, the acceleration for the motion of stone is equal to the acceleration due to gravity, g.
The path of motion of the stone is depicted in the figure. Stone rises due to its initial upward velocity to a certain height above 40 m where it was released till its velocity is zero. From this highest vertical point, the stone falls freely under gravity and hits the ground.
1: In order to describe motion of the stone once it is released, we realize that it would be easier for us if we shift the origin to the point where stone is released. Considering origin at the point of release and upward direction as positive as shown in the figure, the displacement during the motion of stone is :
y = OB = - 40 m

Motion under gravity



2: Distance, on the other hand, is equal to :
s = OA + AO + OB = 2 OA + OB
In order to obtain, OA, we consider this part of rectilinear motion (origin at the point of release and upward direction as positive as shown in the figure).
Here, u = 10 m/s; a = -10 m/s^2 and v = 0. Applying equation of motion, we have :

$$v^2 = u^2 + 2ay$$
$$y = \frac{v^2 - u^2}{2a} = \frac{0^2 - 10^2}{-2\text{x}10} = 5\, m$$

Hence, OA = 5 m, and distance is :
s = 2 OA + OB = 2 x 5 + 40 = 50 m

3: The time of the journey of stone after its release from the balloon is obtained using equation of motion (origin at the point of release and upward direction as positive as shown in the figure).
Here, u = 12 m/s; a = -10 m/s^2 and y = - 40 m.

$$y = ut + \frac{1}{2}at^2$$
$$\Rightarrow -40 = 10t - 0.5 \text{ x } 10t^2$$
$$\Rightarrow 5t^2 - 10t - 40 = 0$$

This is a quadratic equation in “t”. Its solution is :

$$\Rightarrow 5t^2 + 10t - 20t - 40 = 0$$
$$\Rightarrow 5t(t+2) - 20(t+2) = 0$$
$$\Rightarrow (5t - 20)(t+2) = 0$$
$$\Rightarrow 4, -2\ s$$

As negative value of time is not acceptable, time to reach the ground is 4s.

Note: It is important to realize that we are at liberty to switch origin or direction of reference after making suitable change in the sign of attributes.

Position

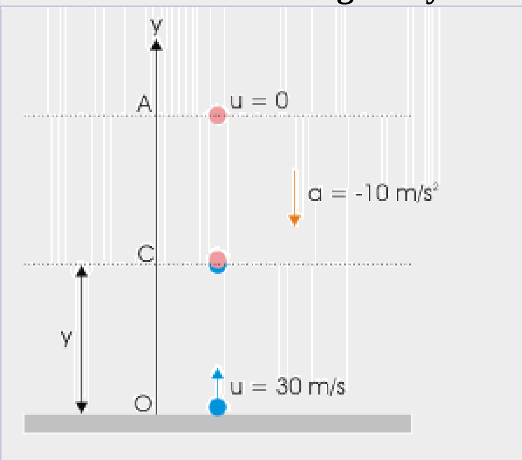
We use the equation $\mathit{\Delta} x = x_2 - x_1 = ut + \frac{1}{2}at^2$ normally in the context of displacement, even though the equation is also designed to determine initial (x_1) or final position (x_2). In certain situations, however, using this equation to determine position rather than displacement provides more elegant adaptability to the situation.

Let us consider a typical problem highlighting this aspect of the equation of motion.

Example:
Problem : A ball is thrown vertically from the ground at a velocity 30 m/s, when another ball is dropped along the same line, simultaneously from the top of tower 120

m in height. Find the time (i) when the two balls meet and (ii) where do they meet.
Solution : This question puts the position as the central concept. In addition to equal time of travel for each of the balls, rhe coordinate positions of the two balls are also same at the time they meet. Let this position be "y". Considering upward direction as the positive reference direction, we have :

Vertical motion under gravity



The balls have same coordinate value when they meet.

For ball thrown from the ground :

$$u = 30\, m/s\ ,\ a = -10\, m/s^2\ ,\ y_1 = 0\ ,\ y_1 = y$$

$$y_2 - y_1 = ut + \frac{1}{2}at^2$$
$$\Rightarrow y - 0 = 30t - \frac{1}{2}10 \text{ x } t^2$$

Equation:

$$\Rightarrow y = 30t - 5 \text{ x } t^2$$

For ball dropped from the top of the tower :

$$u = 0\, m/s\ ,\ a = -10\, m/s^2\ ,\ y_1 = 120\ ,\ y_2 = y$$

$$y_2 - y_1 = ut + \frac{1}{2}at^2$$

Equation:

$$\Rightarrow y - 120 = -5xt^2$$

Now, deducting equation (2) from (1), we have :

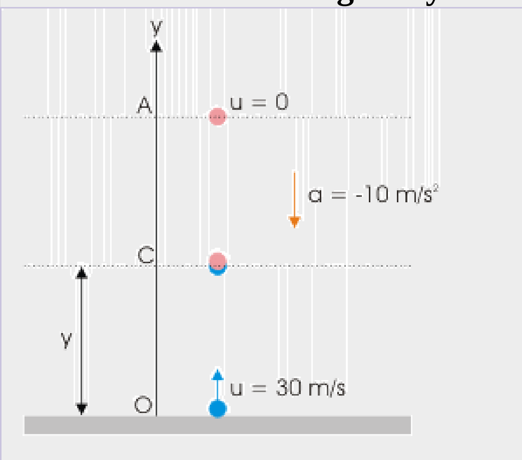
$$30t = 120$$
$$\Rightarrow t = 4\,s$$

Putting this value in equation – 1, we have :

$$\Rightarrow y = 30x4 - 5x4^2 = 120 - 80 = 40\,m$$

One interesting aspect of this simultaneous motion of two balls is that the ball dropped from the tower meets the ball thrown from the ground, when the ball thrown from the ground is actually returning from after attaining the maximum height in 3 seconds. For maximum height of the ball thrown from the ground,

Vertical motion under gravity



When returning from the maximum height, the ball thrown up from the ground is hit by the ball dropped from towers.

$$u = 30\,m/s\,,\, a = -10\,m/s^2 \text{ and } v = 0$$

$$v = u + at$$
$$= 30 - 10t$$
$$t = 3\,s$$

This means that this ball has actually traveled for 1 second (4 – 3 = 1 s) in the downward direction, when it is hit by the ball dropped from the tower!

Exercises

Exercise:

Problem:

A ball is thrown up in vertical direction with an initial speed of 40 m/s. Find acceleration of the ball at the highest point.

Solution:

The velocity of the ball at the highest point is zero. The only force on the ball is due to gravity. The accleration of ball all through out its motion is acceleration due to gravity "g", which is directed downwards. The acceleration of the ball is constant and is not dependent on the state of motion - whether it is moving or is stationary.

Exercise:

Problem:

A ball is released from a height of 45 m. Find the magnitude of average velocity during its motion till it reaches the ground.

Solution:

The average velocity is ratio of displacement and time. Here, displacement is given. We need to find the time of travel. For the motion of ball, we consider the point of release as origin and upward direction as positive.

$$y = ut + \frac{1}{2}at^2$$

$$\Rightarrow -45 = 0Xt + \frac{1}{2}X - 10Xt^2$$

$$\Rightarrow t^2 = \frac{45}{5} = 9$$

$$\Rightarrow t = \pm 3 \quad s$$

Neglecting negative time, t = 3 s. Magnitude of average velocity is :

$$v_{\text{avg}} = \frac{45}{3} = 15 \quad m/s$$

Exercise:

Problem:

A ball is released from an elevator moving upward with an acceleration $3 \quad m/s^2$. What is the acceleration of the ball after it is released from the elevator ?

Solution:

On separation, ball acquires the velocity of elevator – not its acceleration. Once it is released, the only force acting on it is that due to gravity. Hence, acceleration of the ball is same as that due to gravity.

$$a = 10 \quad \frac{m}{s^2} \quad \text{downward}$$

Exercise:

Problem:

A ball is released from an elevator moving upward with an acceleration $5 \quad m/s^2$. What is the acceleration of the ball with respect to elevator after it is released from the elevator ?

Solution:

On separation, ball acquires the velocity of elevator – not its acceleration. Once it is released, the only force acting on it is that due to gravity. Hence, acceleration of the ball is same as that due to gravity. The relative acceleration of the ball (considering downward direction as positive) :

$$a_{\text{rel}} = a_{\text{ball}} - a_{\text{elevator}}$$

$$\Rightarrow a_{\text{rel}} = 10 - (-5) = 15 \quad \frac{m}{s^2}$$

Exercise:

Problem:

A balloon ascends vertically with a constant speed for 5 second, when a pebble falls from it reaching the ground in 5 s. Find the speed of balloon.

Solution:

The velocity of balloon is constant and is a measured value. Let the ball moves up with a velocity u. At the time of release, the pebble acquires velocity of balloon. For the motion of pebble, we consider the point of release as origin and upward direction as positive. Here,

$$y = vt = -uX5 = -5u; \quad u = u; \quad a = -g; \quad t = 5 \quad s$$

Using equation for displacement :

$$y = ut + \frac{1}{2}at^2$$

$$\Rightarrow -5u = uX5 + \frac{1}{2}X - gX5^2$$

$$\Rightarrow 10u = 5X25u = 5 \quad \frac{m}{s}$$

Exercise:

Problem:

A balloon ascends vertically with a constant speed of 10 m/s. At a certain height, a pebble falls from it reaching the ground in 5 s. Find the height of ballon when pebble is released from the balloon.

Solution:

The velocity of balloon is constant. Let the ball moves up with a velocity u. At the time of release, the pebble acquires velocity of balloon. For the motion of pebble, we consider the point of release as origin and upward direction as positive. Here,

$$y = ?; \quad u = 10 \quad m/s; \quad a = -g; \quad t = 5 \quad s$$

Using equation for displacement :

$$y = ut + \frac{1}{2}at^2$$

$$\Rightarrow y = 10X5 + \frac{1}{2}X - gX5^2$$

$$\Rightarrow y = 50 - 5X25$$

$$\Rightarrow y = -75 \quad m$$

Height of ballon when pebble is released from it,

$$\Rightarrow H = 75 \quad m$$

Exercise:

Problem:

A ball is released from a top. Another ball is dropped from a point 15 m below the top, when the first ball reaches a point 5 m below the top. Both balls reach the ground simultaneously. Determine the height of the top.

Solution:

We compare motion of two balls under gravity, when second ball is dropped. At that moment, two balls are 10 m apart. The first ball moves with certain velocity, whereas first ball starts with zero velocity.

Let us consider downward direction as positive. The velocity of the first ball when it reaches 10 m below the top is :

$$v^2 = u^2 + 2ax$$

$$\Rightarrow v^2 = 0 + 2X10X5$$

$$\Rightarrow v = 10 \quad m/s$$

Let the balls take time "t" to reach the gorund. First ball travels 10 m more than second ball. Let 1 and 2 denote first and second ball, then,

$$\Rightarrow 10 + 10t + \frac{1}{2}X10t^2 = \frac{1}{2}X10t^2$$

$$10t = 10$$

$$t = 1 \quad s$$

In this time, second ball travels a distance given by :

$$\Rightarrow y = \frac{1}{2}X10t^2 = 5X12 = 5 \quad m$$

But, second ball is 15 m below the top. Hence, height of the top is 15 + 5 = 20 m.

Exercise:

Problem:

One ball is dropped from the top at a height 60 m, when another ball is projected up in the same line of motion. Two balls hit each other 20 m below the top. Compare the speeds of the ball when they strike.

Solution:

For the motion of first ball dropped from the top, let downward direction be positive :

$$v_1 = u + at$$

$$\Rightarrow v_1 = at = 10t$$

For the ball dropped from the top,

$$x = ut + \frac{1}{2}at^2$$

$$\Rightarrow -20 = 0Xt + \frac{1}{2}X - 10Xt^2$$

$$\Rightarrow 20 = 5t^2$$

$$\Rightarrow t = \pm 2 \quad s$$

Neglecting negative value, t = 2s. Hence, velocity of the ball dropped from the top is :

$$v_1 = 10t = 10X2 = 30 \quad m/s$$

For the motion of second ball projected from the bottom, let upward direction be positive :

$$v_2 = u + \text{at}$$

$$\Rightarrow v_2 = u - 10 \text{ X } 2 = u - 20$$

Clearly, we need to know u. For upward motion,

$$x = ut + \frac{1}{2}at^2$$

$$\Rightarrow 40 = uX2 + \frac{1}{2}X - 10X2^2$$

$$\Rightarrow 40 = 2u - 20 = 30 m/s$$

$$\Rightarrow v_2 = 30 - 20 = 10 \quad m/s$$

Thus,

$$\Rightarrow \frac{v_1}{v_2} = \frac{20}{10} = \frac{2}{1}$$

Note: See module titled " Vertical motion under gravity (application) for more questions.

Vertical motion under gravity (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the accelerated motion under gravity. The questions are categorized in terms of the characterizing features of the subject matter :

- Motion plots
- Equal displacement
- Equal time
- Displacement in a particular second
- Twice in a position
- Collision in air

Motion plots

Example:
Problem : A ball is dropped from a height of 80 m. If the ball looses half its speed after each strike with the horizontal floor, draw (i) speed – time and (ii) velocity – time plots for two strikes with the floor. Consider vertical downward direction as positive and g = 10 m/s^2 .
Solution : In order to draw the plot, we need to know the values of speed and velocity against time. However, the ball moves under gravity with a constant acceleration 10 m/s^2 . As such, the speed and velocity between strikes are uniformly increasing or decreasing at constant rate. It means that we need to know end values, when the ball strikes the floor or when it reaches the maximum height.

In the beginning when the ball is released, the initial speed and velocity both are equal to zero. Its velocity, at the time first strike, is obtained from the equation of motion as :

$$v^2 = 0 + 2gh$$
$$\Rightarrow v = \surd(2gh) = \surd(2 \text{ x } 10 \text{ x } 80) = 40\, m/s$$

Corresponding speed is :

$$|v| = 40\, m/s$$

The time to reach the floor is :

$$t = \frac{v-u}{a} = \frac{40-0}{10} = 4\, s$$

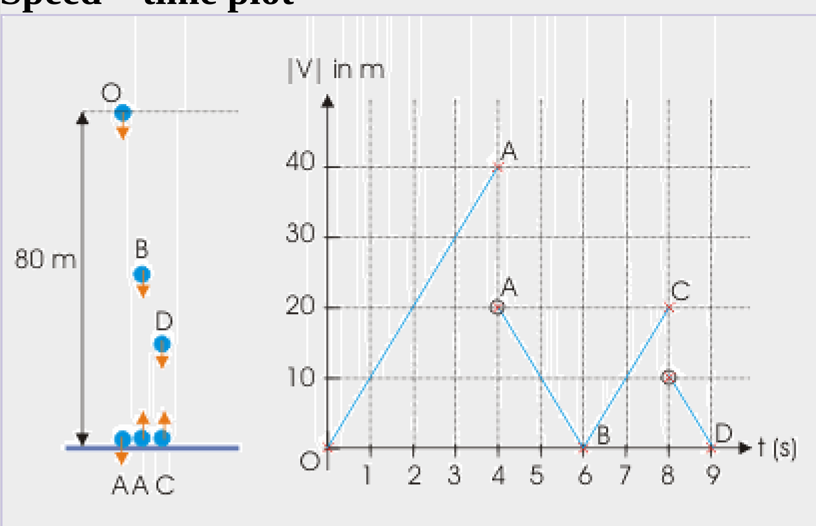
According to question, the ball moves up with half the speed. Hence, its speed after first strike is :

$$|v| = 20\, m/s$$

The corresponding upward velocity after first strike is :

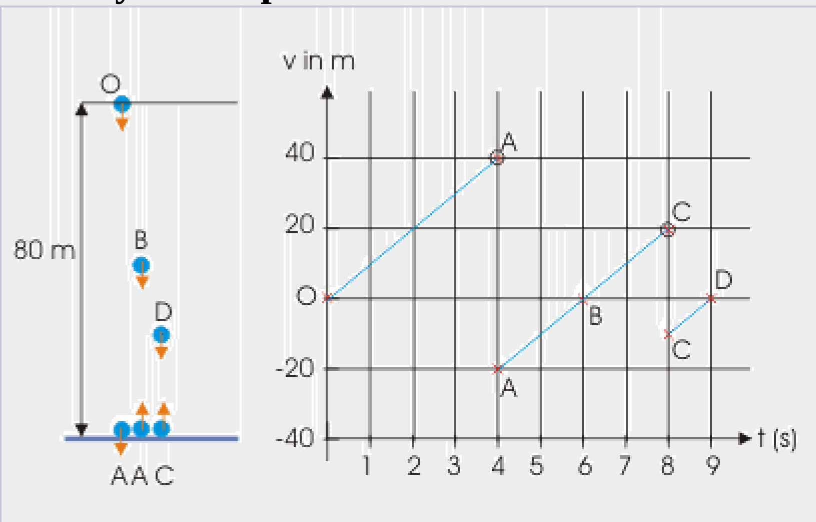
$$v = -20\, m/s$$

Speed – time plot



The ball dropped from a height looses half its height on each strike with the horizontal surface.

Velocity – time plot



The ball dropped from a height looses half its height on each strike with the horizontal surface.

It reaches a maximum height, when its speed and velocity both are equal to zero. The time to reach maximum height is :

$$t = \frac{v-u}{a} = \frac{0+20}{10} = 2\ s$$

After reaching the maximum height, the ball returns towards floor and hits it with the same speed with which it was projected up,

$$v = 20\ m/s$$

Corresponding speed is :

$$|v| = 20\ m/s$$

The time to reach the floor is :

$$t = 2\,s$$

Again, the ball moves up with half the speed with which ball strikes the floor. Hence, its speed after second strike is :

$$|v| = 10\,m/s$$

The corresponding upward velocity after first strike is :

$$v = -10\,m/s$$

It reaches a maximum height, when its speed and velocity both are equal to zero. The time to reach maximum height is :

$$t = \frac{v-u}{a} = \frac{0+10}{10} = 1\,s$$

Example:
Problem : A ball is dropped vertically from a height "h" above the ground. It hits the ground and bounces up vertically to a height "h/3". Neglecting subsequent motion and air resistance, plot its velocity "v" with the height qualitatively.
Solution : We can proceed to plot first by fixing the origin of coordinate system. Let this be the ground. Let us also assume that vertical upward direction is the positive y-direction of the coordinate system. The ball remains above ground during the motion. Hence, height (y) is always positive. Since we are required to plot velocity .vs. height (displacement), we can use the equation of motion that relates these two quantities :

$$v^2 = u^2 + 2ay$$

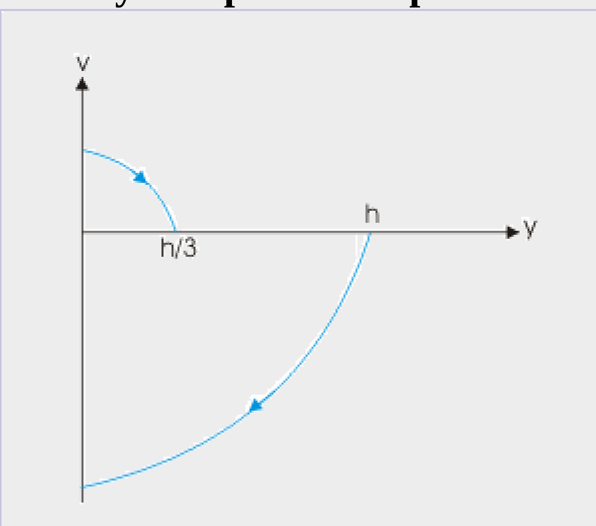
During the downward motion, u = 0, a = -g and displacement (y) is positive. Hence,

$$v^2 = -2gy$$

This is a quadratic equation. As such the plot is a parabola as motion progresses i.e. as y decreases till it becomes equal to zero (see plot below y-axis).
Similarly, during the upward motion, the ball has certain velocity so that it reaches 1/3 rd of the height. It means that ball has a velocity less than that with which it strikes the ground. However, this velocity of rebound is in the upwards direction i.e positive direction of the coordinate system. In the nutshell, we should start drawing upward motion with a smaller positive velocity. The equation of motion, now, is :

$$v^2 = 2gy$$

Again, the nature of the plot is a parabola the relation being a quadratic equation. The plot progresses till displacement becomes equal to y/3 (see plot above y-axis). The two plots should look like as given here :
Velocity – displacement plot



The nature of plot is parabola.

Equal displacement

Example:

Problem : A particle is released from a height of “3h”. Find the ratios of time taken to fall through equal heights “h”.

Solution : Let t_1 , t_2 and t_3 be the time taken to fall through successive heights “h”. Then, the vertical linear distances traveled are :

$$h = \frac{1}{2} g {t_1}^2$$
$$2\text{h} = \frac{1}{2} g (t_1 + t_2)^2$$
$$3\text{h} = \frac{1}{2} g (t_1 + t_2 + t_3)^2$$

$$\Rightarrow {t_1}^2 : (t_1 + t_2)^2 : (t_1 + t_2 + t_3)^2 :: 1 : 2 : 3$$

$$\Rightarrow t_1 : t_1 + t_2 : t_1 + t_2 + t_3 :: 1 : \sqrt{2} : \sqrt{3}$$

In order to reduce this proportion into the one as required in the question, we carry out a general reduction of the similar type. In the end, we shall apply the result to the above proportion. Now, a general reduction of the proportion of this type is carried out as below. Let :

$$t_1 : t_1 + t_2 : t_1 + t_2 + t_3 :: a : b : c$$

$$\Rightarrow \frac{t_1}{a} = \frac{t_1 + t_2}{b} = \frac{t_1 + t_2 + t_3}{c}$$

From the first two ratios,

$$\Rightarrow \frac{t_1}{a} = \frac{t_1 + t_2 - t_1}{b - a} = \frac{t_2}{b - a}$$

Similarly, from the last two ratios,

$$\Rightarrow \frac{t_1 + t_2}{b} = \frac{t_1 + t_2 + t_3 - t_1 - t_2}{c - b} = \frac{t_3}{c - b}$$

Now, combining the reduced ratios, we have :

$$\Rightarrow \frac{t_1}{a} = \frac{t_2}{b - a} = \frac{t_3}{c - b}$$

$$\Rightarrow t_1 : t_2 : t_3 :: a : b - a : c - b$$

In the nutshell, we conclude that if

$$t_1 : t_1 + t_2 : t_1 + t_2 + t_3 :: a : b : c$$

Then,

$$t_1 : t_2 : t_3 :: a : b - a : c - b$$

Applying this result as obtained to the question in hand,

$$t_1 : t_1 + t_2 : t_1 + t_2 + t_3 :: 1 : \sqrt{2} : \sqrt{3}$$

We have :

$$\Rightarrow t_1 : t_2 : t_3 :: 1 : \sqrt{2} - 1 : \sqrt{3} - \sqrt{2}$$

Equal time

Example:
Problem : Balls are successively dropped one after another at an equal interval from a tower. At the instant, 9^{th} ball is released, the first ball hits the ground. Which of the ball in series is at ¾ of the height of the tower?
Solution : Let "t" be the equal time interval. The first ball hits the ground when 9^{th} ball is dropped. It means that the first ball has fallen for a total time (9-1)t = 8t. Let n^{th} ball is at 3/4th of the height of tower. Then,
For the first ball,

$$\Rightarrow h = \frac{1}{2} g \text{ x } 8t^2$$

For the nth ball,

$$\Rightarrow h - \frac{3h}{4} = \frac{h}{4} = \frac{1}{2} g \text{ x } (9 - n)^2 t^2$$

Combining two equations, we have :

$$\Rightarrow h = 2g \text{ x } (9-n)^2 t^2 = \frac{1}{2} g \text{ x } 8t^2 = 32gt^2$$
$$\Rightarrow (9-t)^2 = 16$$
$$\Rightarrow 9-n = 4$$
$$\Rightarrow n = 5$$

Displacement in a particular second

Example:
Problem : A ball is dropped vertically from a tower. If the vertical distance covered in the last second is equal to the distance covered in first 3 seconds, then find the height of the tower. Consider g = 10 m/s^2 .
Solution : Let us consider that the ball covers the height y_n in n^{th} second. Then, the distance covered in the n^{th} second is given as :

$$y_n = u + \frac{g}{2} \text{ x } \left(2n-1\right) = 0 + \frac{10}{2} \text{ x } \left(2n-1\right) = 10n - 5$$

On the other hand, the distance covered in first 3 seconds is :

$$y_3 = ut + \frac{1}{2} g t^2 = 0 + \frac{1}{2} \text{ x } 10 \text{ x } 3^2 = 45\, m$$

According to question,

$$y_n = y_3$$
$$\Rightarrow 10n - 5 = 45$$
$$\Rightarrow n = 5\, s$$

Therefre, the height of the tower is :

$$\Rightarrow y = \frac{1}{2} \text{ x } 10 \text{ x } 5^2 = 125\, m$$

Twice in a position

Example:

Problem : A ball, thrown vertically upward from the ground, crosses a point "A" in time " t_1 ". If the ball continues to move up and then return to the ground in additional time " t_2 ", then find the height of "A" from the ground. .

Solution : Let the upward direction be the positive reference direction. The displacement equation for vertical projection from the ground to the point "A", is :

$$h = ut_1 + \frac{1}{2} g{t_1}^2$$

In this equation, the only variable that we do not know is initial velocity. In order to determine initial velocity, we consider the complete upward motion till the ball reaches the maximum height. We know that the ball takes half of the total time to reach maximum height. It means that time for upward motion till maximum height is :

$$t_{\frac{1}{2}} = \frac{t_1 + t_2}{2}$$

As we know the time of flight, final velocity and acceleration, we can know initial velocity :

$$0 = u - gt_{\frac{1}{2}}$$
$$\Rightarrow u = gt_{\frac{1}{2}} = \frac{(t_1 + t_2)g}{2}$$

Substituting this value, the height of point "A" is :

$$h = ut_1 + \frac{1}{2} g{t_1}^2$$
$$\Rightarrow h = \frac{(t_1 + t_2)g}{2} \text{ x } t_1 - \frac{1}{2} g{t_1}^2$$
$$\Rightarrow h = \frac{g{t_1}^2 + gt_1 t_2 - g{t_1}^2}{2}$$
$$\Rightarrow h = \frac{gt_1 t_2}{2}$$

Example:

Problem : A ball, thrown vertically upward from the ground, crosses a point "A" at a height 80 meters from the ground. If the ball returns to the same position after 6 second, then find the velocity of projection. (Consider g = 10 m/s^2).

Solution : Since two time instants for the same position is given, it is indicative that we may use the displacement equation as it may turn out to be quadratic equation in time "t". The displacement, "y", is (considering upward direction as positive y-direction) :

$$y = ut + \frac{1}{2}gt^2$$

$$80 = ut + \frac{1}{2} \text{ x } -10t^2 = ut - 5t^2$$

$$5t^2 - ut + 80 = 0$$

We can express time "t", using quadratic formulae,

$$\Rightarrow t_2 = \frac{-\left(-u\right)+\sqrt{\left\{(-u)^2 - 4 \text{ x } 5 \text{ x } 80\right\}}}{2 \text{ x } 5} = \frac{u+\sqrt{\left(u^2 - 1600\right)}}{10}$$

Similarly,

$$\Rightarrow t_1 = \frac{u-\sqrt{\left(u^2 - 1600\right)}}{10}$$

According to question,

$$\Rightarrow t_2 - t_1 = \frac{\sqrt{\left(u^2 - 1600\right)}}{5}$$

$$\Rightarrow 6 = \frac{\sqrt{\left(u^2 - 1600\right)}}{5}$$

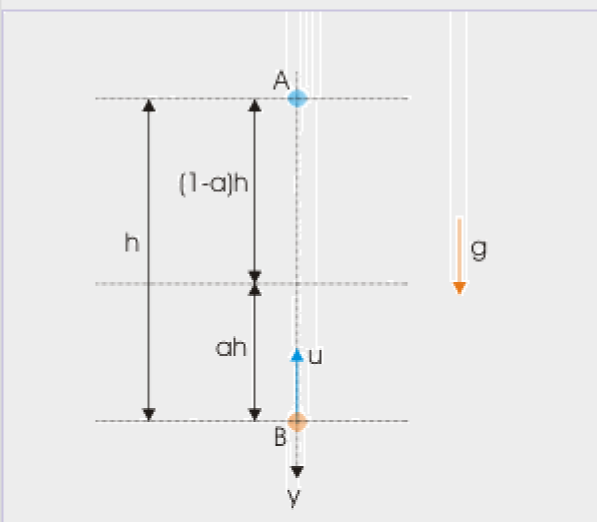
$$\Rightarrow \sqrt{\left(u^2 - 1600\right)} = 30$$

Squaring both sides we have,

$$\Rightarrow u^2 - 1600 = 900$$
$$\Rightarrow u = 50\, m/s$$

Collision in air

Example:
Problem : A ball “A” is dropped from a height, “h”, when another ball, “B” is thrown upward in the same vertical line from the ground. At the instant the balls collide in the air, the speed of “A” is twice that of “B”. Find the height at which the balls collide.
Solution : Let the velocity of projection of “B” be “u” and let the collision occurs at a fraction “a” of the height “h” from the ground. We should pause and make mental note of this new approach to express an intermediate height in terms of fraction of total height (take the help of figure).
Collision in the mid air



The balls collide as they move in same vertical line.

Adhering to the conventional approach, we could have denoted the height at which collision takes place. For example, we could have considered collision at a vertical displacement y from the ground. Then displacements of two balls would have been :
“y” and “h – y”
Obviously, the second expression of displacement is polynomial of two terms because of minus sign involved. On the other hand, the displacements of two balls in terms of fraction, are :
“ha” and “(1-a)h”
The advantage of the second approach (using fraction) is that displacements are stated in terms of the product of two quantities. The variable “h” appearing in each of the terms cancel out if they appear on ether side of the equation and we ultimately get equation in one variable i.e. “a”. In the nutshell, the approach using fraction reduces variables in the resulting equation. This point will be highlighted at the appropriate point in the solution to appreciate why we should use fraction?
Now proceeding with the question, the vertical displacement of “B” is ah and that of “A” is (1-a)h. We observe here that each of the balls takes the same time to cover respective displacements. Using equation of constant acceleration in one dimension (considering downward vertical direction positive),
For ball “A”,

$$\left(1-a\right)h=\frac{1}{2}gt^2$$

For ball “B”,

$$-ah=-ut+\frac{1}{2}gt^2$$

The value of “t” obtained from the equation of ball “A” is :

$$\Rightarrow t=\sqrt{\left\{\frac{2(1-a)h}{g}\right\}}$$

Substituting in the equation of ball “B”, we have :

$$\Rightarrow -ah = -u\sqrt{\left\{\frac{2(1-a)h}{g}\right\}} + \left(1-a\right)h$$

$$\Rightarrow u\sqrt{\left\{\frac{2(1-a)h}{g}\right\}} = h$$

$$\Rightarrow u = \sqrt{\left\{\frac{gh}{2(1-a)}\right\}}$$

Now, according to the condition given in the question,

$$v_A = 2v_B$$

$$\Rightarrow {v_A}^2 = 4{v_B}^2$$

Using equations of motion for constant acceleration, we have :

$$\Rightarrow 2g\left(1-a\right)h = 4\left(u^2 - 2gah\right)$$

$$\Rightarrow 2g\left(1-a\right)h = 4\left\{\frac{gh}{2(1-a)} - 2gah\right\}$$

$$\Rightarrow \left(1-a\right) = \left\{\frac{1}{(1-a)} - 4a\right\}$$

Recall our discussion on the use of fraction. We can write down the corresponding steps for conventional method to express displacements side by side and find that the conventional approach will lead us to an equation as :

$$\left(h-y\right) = \left\{\frac{h}{(h-y)} - 4y\right\}$$

Evidently, this is an equation in two variables and hence can not be solved. It is this precise limitation of conventional approach that we switched to fraction approach to get an equation in one variable,

$$\Rightarrow \left(1-a\right) = \left\{\frac{1}{(1-a)} - 4a\right\}$$

$$\Rightarrow (1-a)^2 = 1 - 4a\left(1-a\right)$$
$$\Rightarrow 1 + a^2 - 2a = 1 - 4a + a^2$$
$$\Rightarrow 3a^2 - 2a = 0$$
$$\Rightarrow a(3a-2) = 0$$
$$\Rightarrow a = 0,\ {}^2/_3$$

Neglecting, a =0,

$$\Rightarrow a = {}^2/_3$$

The height from the ground at which collision takes place is :

$$\Rightarrow y = ah = \frac{2h}{3}$$

Non-uniform acceleration

Non-uniform acceleration constitutes the most general description of motion. It refers to variation in the rate of change in velocity. Simply put, it means that acceleration changes during motion. This variation can be expressed either in terms of position (x) or time (t). We understand that if we can describe non-uniform acceleration in one dimension, we can easily extend the analysis to two or three dimensions using composition of motions in component directions. For this reason, we shall confine ourselves to the consideration of non-uniform i.e. variable acceleration in one dimension.

In this module, we shall describe non-uniform acceleration using expressions of velocity or acceleration in terms of either of time, “t”, or position, “x”. We shall also consider description of non-uniform acceleration by expressing acceleration in terms of velocity. As a matter of fact, there can be various possibilities. Besides, non-uniform acceleration may involve interpretation acceleration - time or velocity - time graphs.

Accordingly, analysis of non-uniform acceleration motion is carried out in two ways :

- Using calculus
- Using graphs

Analysis using calculus is generic and accurate, but is limited to the availability of expression of velocity and acceleration. It is not always possible to obtain an expression of motional attributes in terms of “x” or “t”. On the other hand, graphical method lacks accuracy, but this method can be used with precision if the graphs are composed of regular shapes.

Using calculus involves differentiation and integration. The integration allows us to evaluate expression of acceleration for velocity and evaluate expression of velocity for displacement. Similarly, differentiation allows us to evaluate expression of position for velocity and evaluate expression of velocity for acceleration. We have already worked with expression of position in time. We shall work here with other expressions. Clearly, we need to know a bit about differentiation and integration before we proceed to analyze non-uniform motion.

Important calculus results

Integration is anti-differentiation i.e. an inverse process. We can compare differentiation and integration of basic algebraic, trigonometric, exponential and logarithmic functions to understand the inverse relation between processes. In the next section, we list few important differentiation and integration results for reference.

Differentiation

$$\frac{đ}{đx}x^n = nx^{n-1}; \qquad \frac{đ}{đx}(ax+b)^n = na(ax+b)^{n-1}$$

$$\frac{đ}{đx}\sin ax = a\cos ax; \qquad \frac{đ}{đx}\cos ax = -a\sin ax$$

$$\frac{đ}{đx}e^x = e^x; \qquad \frac{đ}{đx}\log_e x = \frac{1}{x}$$

Integration

$$\int x^n đx = \frac{x^{n+1}}{n+1}; \qquad \int (ax+b)^n đx = \frac{(ax+b)^{n+1}}{a(n+1)}$$

$$\int \sin ax đx = -\frac{\cos ax}{a}; \qquad \int \cos ax đx = \frac{\sin ax}{a}$$

$$\int e^x đx = e^x; \qquad \int \frac{đx}{x} = \log{}_e x$$

Velocity and acceleration is expressed in terms of time “t

Let the expression of acceleration in x is given as function a(t). Now, acceleration is related to velocity as :

$$a(t) = \frac{\text{đ}v}{\text{đ}t}$$

We obtain expression for velocity by rearranging and integrating :

$$\Rightarrow \text{đ}v = a(t)\text{đ}t$$

$$\Rightarrow \Delta v = \int a(t)\text{đ}t$$

This relation yields an expression of velocity in "t" after using initial conditions of motion. We obtain expression for position/ displacement by using defining equation, rearranging and integrating :

$$v(t) = \frac{dx}{dt}$$

$$\Rightarrow \text{đ}x = v(t)\text{đ}t$$

$$\Rightarrow \Delta x = \int v(t)\text{đ}t$$

This relation yields an expression of position in "t" after using initial conditions of motion.

Example:

Problem : The acceleration of a particle along x-axis varies with time as :

$$a = t - 1$$

Velocity and position both are zero at t= 0. Find displacement of the particle in the interval from t=1 to t = 3 s. Consider all measurements in SI unit.

Solution : Using integration result obtained earlier for expression of acceleration :

$$\Delta v = v_2 - v_1 = \int a(t)\text{đ}t$$

Here, $v_1 = 0$. Let $v_2 = v$, then :

$$v = \int_0^t a(t)dt = \int_0^t (t-1)dt$$

Note that we have integrated RHS between time 0 ans t seconds in order to get an expression of velocity in “t”.

$$\Rightarrow v = \frac{t^2}{2} - t$$

Using integration result obtained earlier for expression of velocity :

$$\Delta x = \int v(t)dt = \int \left(\frac{t^2}{2} - t\right) dt$$

Integrating between t=1 and t=3, we have :

$$\Delta x = \int_1^3 \left(\frac{t^2}{2} - t\right) dt = \left[\frac{t^3}{6} - \frac{t^2}{2}\right]_1^3 = \frac{27}{6} - \frac{9}{2} - \frac{1}{6} + \frac{1}{2} = 5 - 4 = 1m$$

Example:

Problem : If the velocity of particle moving along a straight line is proportional to $\frac{3}{4}^{\text{th}}$ power of time, then how do its displacement and acceleration vary with time?

Solution : Here, we need to find a higher order attribute (acceleration) and a lower order attribute displacement. We can find high order attribute by differentiation, whereas we can find lower order attribute by integration.

$$v \propto t^{\frac{3}{4}}$$

Let,

$$v = kt^{\frac{3}{4}}$$

where k is a constant. The acceleration of the particle is :

$$a = \frac{đv(t)}{đt} = \frac{3kt^{\frac{3}{4}-1}}{4} = \frac{3kt^{-\frac{1}{4}}}{4}$$

Hence,

$$v \propto t^{-\frac{1}{4}}$$

For lower order attribute “displacement”, we integrate the function :

$$đx = vđt$$
$$\Rightarrow \Delta x = \int vđt = \int kt^{\frac{3}{4}}đt = \frac{4kt^{\frac{7}{4}}}{7}$$
$$\Delta x \propto t^{\frac{7}{4}}$$

Velocity and acceleration is expressed in terms of time “x”

Let the expression of acceleration in x given as function a(x). Now, the defining equation of instantaneous acceleration is:

$$a(x) = \frac{đv}{đt}$$

In order to incorporate differentiation with position, “x”, we rearrange the equation as :

$$a(x) = \frac{đv}{đx} X \frac{đx}{đt} = \frac{vđv}{đx}$$

We obtain expression for velocity by rearranging and integrating :

$$\int vdv = \int a(x)dx$$

This relation yields an expression of velocity in x. We obtain expression for position/ displacement by using defining equation, rearranging and integrating

:

$$v(x) = \frac{đx}{đt}$$

$$\Rightarrow đt = \frac{đx}{v(x)}$$

$$\Rightarrow \Delta t = \int \frac{đx}{v(x)}$$

This relation yields an expression of position in t.

Example:
Problem : The acceleration of a particle along x-axis varies with position as :

$$a = 9x$$

Velocity is zero at t = 0 and x=2m. Find speed at x=4. Consider all measurements in SI unit.
Solution : Using integration,

$$\int vđv = \int a(x)đx$$

$$\int_0^v vđv = \int_2^x 9xđx$$

$$\Rightarrow \left[\frac{v^2}{2}\right]_0^v = 9\left[\frac{x^2}{2}\right]_2^x$$

$$\Rightarrow \frac{v^2}{2} = \frac{9x^2}{2} - \frac{36}{2}$$

$$\Rightarrow v^2 = 9x^2 - 36 = 9\left(x^2 - 4\right)$$

$$\Rightarrow v = \pm 3\sqrt{(x^2 - 4)}$$

We neglect negative sign as particle is moving in x-direction with positive acceleration. At x =4 m,

$$\Rightarrow v = 3\sqrt{(4^2 - 4)} = 3\sqrt{12} = 6\sqrt{3} \quad m/s$$

Acceleration in terms of velocity

Let acceleration is expressed as function of velocity as :

$$a = a(v)$$

Obtaining expression of velocity in time, "t"

The acceleration is defined as :

Arranging terms with same variable on one side of the equation, we have :

$$đt = \frac{đv}{a(v)}$$

Integrating, we have :

$$\Rightarrow \Delta t = \int \frac{đv}{a(v)}$$

Evaluation of this relation results an expression of velocity in t. Clearly, we can proceed as before to obtain expression of position in t.

Obtaining expression of velocity in time, "x"

In order to incorporate differentiation with position, “x”, we rearrange the defining equation of acceleration as :

$$a(v) = \frac{đv}{đx} X \frac{đx}{đt} = \frac{vđv}{đx}$$

Arranging terms with same variable on one side of the equation, we have :

$$đx = \frac{vđv}{a(v)}$$

Integrating, we have :

$$\Rightarrow \Delta x = \int \frac{vđv}{a(v)}$$

Evaluation of this relation results an expression of velocity in x. Clearly, we can proceed as before to obtain expression of position in t.

Example:

Problem : The motion of a body in one dimension is given by the equation dv(t)/dt = 6.0 -3 v (t), where v(t) is the velocity in m/s and “t” in seconds. If the body was at rest at t = 0, then find the expression of velocity.

Solution : Here, acceleration is given as a function in velocity as independent variable. We are required to find expression of velocity in time. Using integration result obtained earlier for expression of time :

$$\Rightarrow \Delta t = \int \frac{đv}{a(v)}$$

Substituting expression of acceleration and integrating between t=0 and t=t, we have :

$$\Rightarrow \Delta t = t = \int \frac{đv(t)}{6 - 3v(t)}$$

$$\Rightarrow t = \frac{\ln\left[\frac{\{6 - 3v(t)}{6}\right]}{-3}$$

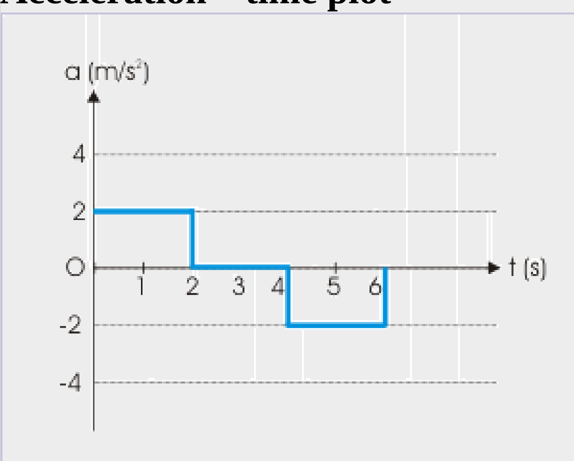
$$\Rightarrow 6 - 3v(t) = 6e^{-3t}$$
$$\Rightarrow 3v(t) = 6(1 - e^{-3t})$$
$$\Rightarrow v(t) = 2(1 - e^{-3t})$$

Graphical method

We analyze graphs of motion in parts keeping in mind regular geometric shapes involved in the graphical representation. Here, we shall work with two examples. One depicts variation of velocity with respect to time. Other depicts variation of acceleration with respect to time.

Velocity .vs. time

Example:
Problem : A particle moving in a straight line is subjected to accelerations as given in the figure below :
Acceleration – time plot



Acceleration – time plot

If v = 0 and t = 0, then draw velocity – time plot for the same time interval.
Solution : In the time interval between t = 0 and t = 2 s, acceleration is constant and is equal to 2 m/s^2 . Hence, applying equation of motion for final velocity, we have :

$$v = u + at$$

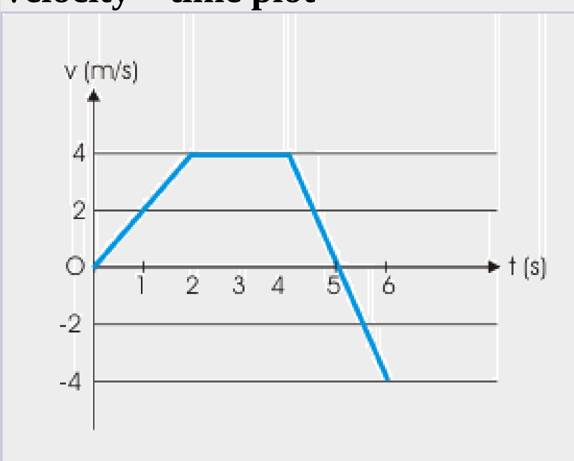
But, u = 0 and a = 2 m/s^2 .

$$\Rightarrow v = 2t$$

The velocity at the end of 2 seconds is 4 m/s. Since acceleration is constant, the velocity – time plot for the interval is a straight line. On the other hand, the acceleration is zero in the time interval between t = 2 and t = 4 s. Hence, velocity remains constant in this time interval. For time interval t = 4 and 6 s, acceleration is - 4 m/s^2 .
From the plot it is clear that the velocity at t = 4 s is equal to the velocity at t = 2 s, which is given by :

$$\Rightarrow u = 2 \text{ x } 2 = 4\, m/s$$

Velocity – time plot



Corresponding velocity – time plot.

Putting this value as initial velocity in the equation of velocity, we have :

$$\Rightarrow v = 4 - 4t$$

We should, however, be careful in drawing velocity plot for t = 4 s to t = 6 s. We should relaize that the equation above is valid in the time interval considered. The time t = 4 s from the beginning corresponds to t = 0 s and t = 6 s corresponds to t = 2 s for this equation.
Velocity at t = 5 s is obtained by putting t = 1 s in the equation,

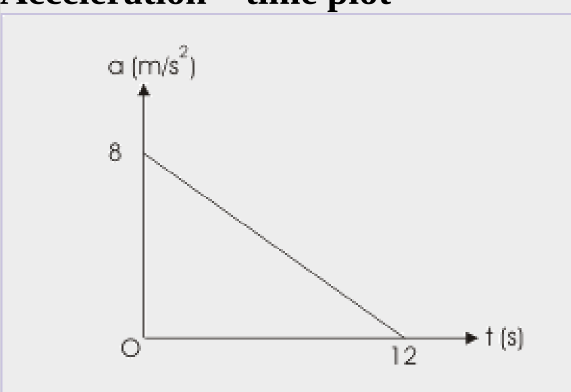
$$\Rightarrow v = 4 - 4 \text{ x } 1 = 0$$

Velocity at t = 6 s is obtained by putting t = 2 s in the equation,

$$\Rightarrow v = 4 - 4 \text{ x } 2 = -4\, m/s$$

Acceleration .vs. time

Example:
Problem : A particle starting from rest and undergoes a rectilinear motion with acceleration "a". The variation of "a" with time "t" is shown in the figure. Find the maximum velocity attained by the particle during the motion.
Acceleration – time plot



The area under plot gives change in velocity.

Solution : We see here that the particle begins from rest and is continuously accelerated in one direction (acceleration is always positive through out the motion) at a diminishing rate. Note that though acceleration is decreasing, but remains positive for the motion. It means that the particle attains maximum velocity at the end of motion i.e. at t = 12 s.
The area under the plot on acceleration – time graph gives change in velocity. Since the plot start at t = 0 i.e. the beginning of the motion, the areas under the plot gives the velocity at the end of motion,

$$\Delta v = v_2 - v_1 = v - 0 = v = \frac{1}{2} \text{ x } 8 \text{ x } 12 = 48\, m/s$$

$$v_{\text{max}} = 48\, m/s$$

Exercises

- First identify : what is given and what is required. Establish relative order between given and required attribute.
- Use differentiation method to get a higher order attribute in the following order : displacement (position vector) → velocity → acceleration.
- Use integration method to get a lower order attribute in the following order : acceleration → velocity → displacement (position vector).
- Since we are considering accelerated motion in one dimension, graphical representation of motion is valid. The interpretation of plot in terms of slope of the curve gives higher order attribute, whereas interpretation of plot in terms of area under the plot gives lower order attribute.

Exercise:

Problem:

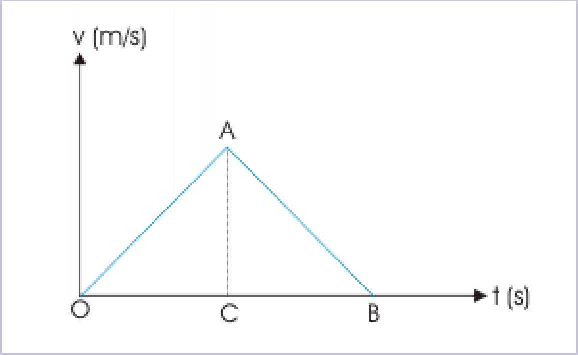
A particle of mass m moves on the x-axis. It starts from rest t = 0 from the point x = 0, and comes to rest at t = 1 at the point x = 1. No other information is available about its motion at intermediate time (0 < t < 1). Prove that magnitude of instantaneous acceleration during the motion can not be less than 4 m/s^2 .

Solution:

In between the starting and end point, the particle may undergo any combination of acceleration and deceleration. According to question, we are required to know the minimum value of acceleration for any combination possible.

Let us check the magnitude of acceleration for the simplest combination and then we evaluate other complex possible scenarios. Now, the simplest scenario would be that the particle first accelerates and then decelerates for equal time and at the same rate to complete the motion.

Velocity – time plot



Velocity – time plot

In order to assess the acceleration for a linear motion , we would make use of the fact that area under v-t plot gives the displacement (= 1 m). Hence,

$$\Delta \text{OAB} = \frac{1}{2}\ \text{x}\ vt = \frac{1}{2}\ \text{x}\ v\ \text{x}\ 1 = 1\,m$$
$$\Rightarrow v_{\text{max}} = 2\,m/s$$

Thus, the maximum velocity during motion under this condition is 2 m/s. On the other hand, the acceleration for this motion is equal to the slope of the line,

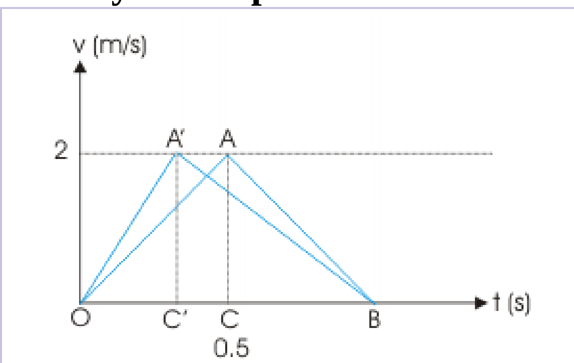
$$a = \tan\theta = \frac{2}{0.5} = 4\,m/s^2$$

Now let us complicate the situation, in which particle accelerates for shorter period and decelerates gently for a longer period (see figure

below). This situation would result velocity being equal to 2 m/s and acceleration greater than 4 m/s^2 .

It is so because time period is fixed (1 s) and hence base of the triangle is fixed. Also as displacement in given time is 1 m. It means that area under the triangle is also fixed (1 m). As area is half of the product of base and height, it follows that the height of the triangle should remain same i.e 2 m/s. For this reason the graphical representations of possible variation of acceleration and deceleration may look like as shown in the figure here.

Velocity – time plot



Area under velocity – time plot gives displacement , whereas slope of the plot gives the acceleration.

Clearly, the slope of OA' is greater than 4 m/s^2 .

We may argue that why to have a single combination of acceleration and deceleration? What if the particle undergoes two cycles of acceleration and deceleration? It is obvious that such consideration will again lead to similar analysis for symmetric and non-symmetric acceleration, which would be bounded by the value of time and area of the triangle.

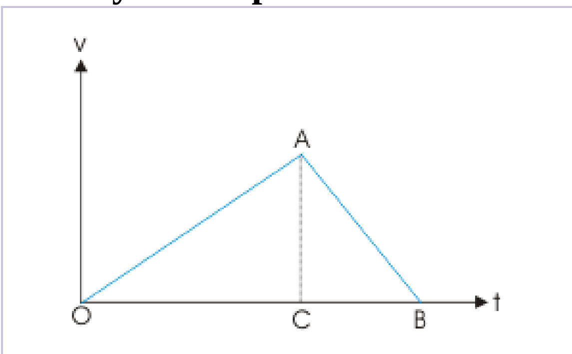
Thus we conclude that the minimum value of acceleration is 4 m/s^2 .

Exercise:

Problem:

A particle accelerates at constant rate “a” from rest at and then decelerates at constant rate “b” to come to rest. If the total time elapsed is “t” during this motion, then find (i) maximum velocity achieved and (ii) total displacement during the motion.

Velocity – time plot



Area under velocity – time plot gives displacement , whereas slope of the plot gives the acceleration.

Solution:

The particle initially accelerates at a constant rate. It means that its velocity increases with time. The particle, therefore, gains maximum velocity say, v_{max} , before it begins to decelerate. Let t_1 and t_2 be the time intervals, for which particle is in acceleration and deceleration respectively. Then,

$$t = t_1 + t_2$$

From the triangle OAC, the magnitude of acceleration is equal to the slope of straight line OA. Hence,

$$a = \frac{v_{\text{max}}}{t_1}$$

Similarly from the triangle CAB, the magnitude of acceleration is equal to the slope of straight line AB. Hence,

$$b = \frac{v_{\max}}{t_2}$$

Substituting values of time, we have :

$$t = \frac{v_{\max}}{a} + \frac{v_{\max}}{b} = v_{\max}\left(\frac{a+b}{ab}\right)$$

$$v_{\max} = \frac{abt}{(a+b)}$$

In order to find the displacement, we shall use the fact that area under velocity – time plot is equal to displacement. Hence,

$$x = \frac{1}{2} \text{ x } v_{\max}t$$

Putting the value of maximum velocity, we have :

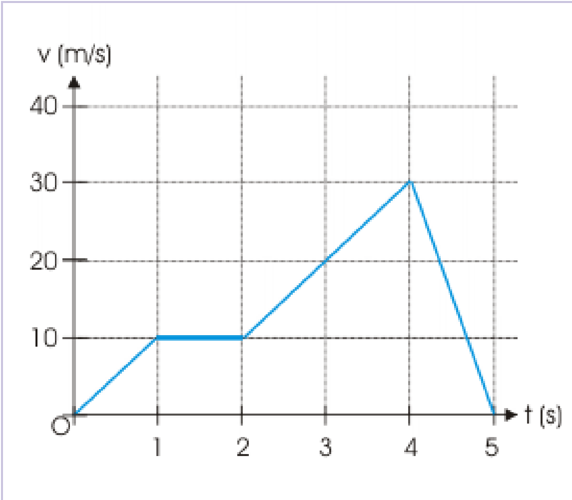
$$\Rightarrow x = \frac{abt^2}{2(a+b)}$$

Exercise:

Problem:

The velocity – time plot of the motion of a particle along x - axis is as shown in the figure. If x = 0 at t= 0, then (i) analyze acceleration of particle during the motion (ii) draw corresponding displacement – time plot (showing nature of the plot) and (iii) find total displacement.

Velocity – time plot



Slope of the plot gives acceleration; area under plot gives displacement.

Solution:

Graphically, slope of the plot yields acceleration and area under the plot yields displacement.

Since the plot involves regular geometric shapes, we shall find areas of these geometric shapes individually and then add them up to get various points for displacement – time plot.

The displacement between times 0 and 1 s is :

$$\Delta x = \frac{1}{2} \text{ x } 1 \text{ x } 10 = 5m$$

The velocity between 0 and 1 s is increasing at constant rate. The acceleration is constant and its magnitude is equal to the slope, which is ${}^{10}/{}_{1} = 10\, m/s^2$. Since acceleration is in the direction of motion, the particle is accelerated. The resulting displacement – time curve is a parabola with an increasing slope.

The displacement between times 1 and 2 s is :

$$\Delta x = 1 \ \text{x} \ 10 = 10m$$

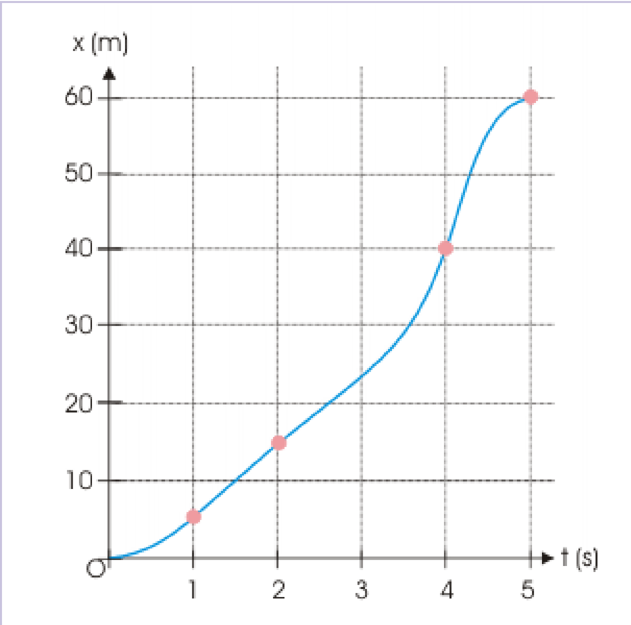
The velocity between 1 and 2 s is constant and as such acceleration is zero. The resulting displacement – time curve for this time interval is a straight line having slope equal to that of the magnitude of velocity.

The displacement at the end of 2 s is 5 + 10 = 15 m

Now, the displacement between times 2 and 4 s is :

$$\Delta x = \frac{1}{2} \ \text{x} \ \left(20 + 30\right) \ \text{x} \ 1 = 25m$$

Displacement – time plot



Corresponding displacement – time plot of the given velocity- - time plot.

The velocity between 2 and 4 s is increasing at constant rate. The acceleration, therefore, is constant and its magnitude is equal to the slope of the plot, $^{20}/_{2} = 10 \, m/s^2$. Since acceleration is in the direction of motion, the particle is accelerated and the resulting displacement – time curve is a parabola with increasing slope.

The displacement at the end of 4 s is 5 + 10 + 25 = 40 m

Now, the displacement between times 4 and 5 s is :

$$\Delta x = \frac{1}{2} \text{ x } 30 \text{ x } 1 = 15m$$

The velocity between 2 and 4 s is decreasing at constant rate, but always remains positive. As such, the acceleration is constant, but negative and its magnitude is equal to the slope, which is ${}^{30}/_{1} = 30\, m/s^2$. Since acceleration is in the opposite direction of motion, the particle is decelerated. The resulting displacement – time curve is an inverted parabola with decreasing slope.

The final displacement at the end of 5 s is 5 + 10 + 25 + 15 = 55 m

Characteristics of motion : One dimensional, Unidirectional, variable acceleration in one dimension (magnitude)

Exercise:

Problem:

The motion of a body in one dimension is given by the equation dv(t)/dt = 6.0 -3 v (t), where v(t) is the velocity in m/s and “t” in seconds. If the body was at rest at t = 0, then find the terminal speed.

Solution:

In order to answer this question, we need to know the meaning of terminal speed. The terminal speed is the speed when the body begins to move with constant speed. Now, the first time derivate of velocity gives the acceleration of the body :

$$\frac{dv(t)}{dt} = a = 6 - 3v\left(t\right)$$

For terminal motion, a = 0. Hence,

$$a = 6 - 3v(t) = 0$$

$$v\left(t\right) = \frac{6}{3} = 2\, m/s$$

Exercise:

Problem:

The motion of a body in one dimension is given by the equation dv(t)/dt = 6.0 -3 v (t), where v(t) is the velocity in m/s and “t” in seconds. If the body was at rest at t = 0, then find the magnitude of the initial acceleration and also find the velocity, when the acceleration is half the initial value.

Solution:

We have to find the acceleration at t = 0 from the equation given as :

$$\frac{đv(t)}{đt} = a = 6 - 3v\left(t\right)$$

Generally, we would have been required to know the velocity function so that we can evaluate the function for time t = 0. In this instant case, though the function of velocity is not known, but we know the value of velocity at time t = 0. Thus, we can know acceleration at t =0 by just substituting the value for velocity function at that time instant as :

$$a = 6 - 3v\left(t\right) = 6 - 3 \text{ x } 0 = 6\, m/s^2$$

Now when the acceleration is half the initial acceleration,

$$\Rightarrow \text{a'} = \frac{a}{2} = \frac{6}{2} = 3\, m/s^2$$

Hence,

$$\Rightarrow 6 - 3v(t) = 3$$
$$\Rightarrow v\left(t\right) = \frac{3}{3} = 1\, m/s$$

Relative velocity in one dimension

The measurements, describing motion, are subject to the state of motion of the frame of reference with respect to which measurements are made. Our day to day perception of motion is generally earth's view – a view common to all bodies at rest with respect to earth. However, we encounter occasions when there is perceptible change to our earth's view. One such occasion is traveling on the city trains. We find that it takes lot longer to overtake another train on a parallel track. Also, we see two people talking while driving separate cars in the parallel lane, as if they were stationary to each other!. In terms of kinematics, as a matter of fact, they are actually stationary to each other - even though each of them are in motion with respect to ground.

In this module, we set ourselves to study motion from a perspective other than that of earth. Only condition we subject ourselves is that two references or two observers making the measurements of motion of an object, are moving at constant velocity (We shall learn afterward that two such reference systems moving with constant velocity is known as inertial frames, where Newton's laws of motion are valid.).

The observers themselves are not accelerated. There is, however, no restriction on the motion of the object itself, which the observers are going to observe from different reference systems. The motion of the object can very well be accelerated. Further, we shall study relative motion for two categories of motion : (i) one dimension (in this module) and (ii) two dimensions (in another module). We shall skip three dimensional motion – though two dimensional study can easily be extended to three dimensional motion as well.

Relative motion in one dimension

We start here with relative motion in one dimension. It means that the individual motions of the object and observers are along a straight line with only two possible directions of motion.

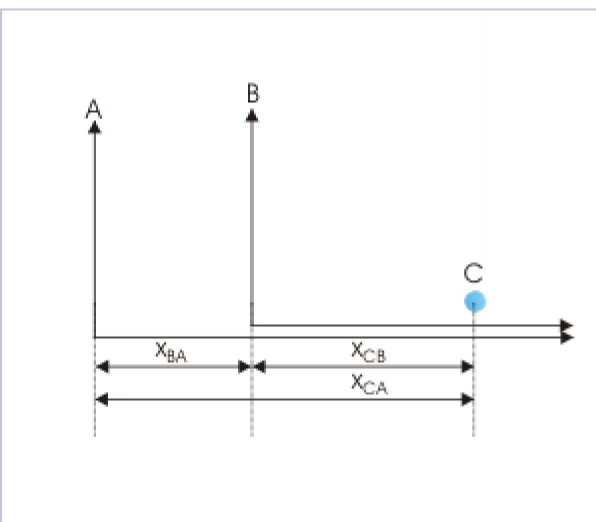
Position of the point object

We consider two observers "A" and "B". The observer "A" is at rest with earth, whereas observer "B" moves with a velocity v_{BA} with respect to the observer "A". The two observers watch the motion of the point like object "C". The motions of "B" and "C" are along the same straight line.

Note: It helps to have a convention about writing subscripted symbol such as v_{BA} . The first subscript indicates the entity possessing the attribute (here velocity) and second subscript indicates the entity with respect to which measurement is made. A velocity like v_{BA} shall, therefore, mean velocity of "B" with respect to "A".

The position of the object "C" as measured by the two observers "A" and "B" are x_{CA} and x_{CB} as shown in the figure. The observers are represented by their respective frame of reference in the figure.

Position



Here,

$$x_{CA} = x_{BA} + x_{CB}$$

Velocity of the point object

We can obtain velocity of the object by differentiating its position with respect to time. As the measurements of position in two references are different, it is expected that velocities in two references are different, because one observer is at rest, whereas other observer is moving with constant velocity.

$$v_{CA} = \frac{dx_{CA}}{dt}$$

and

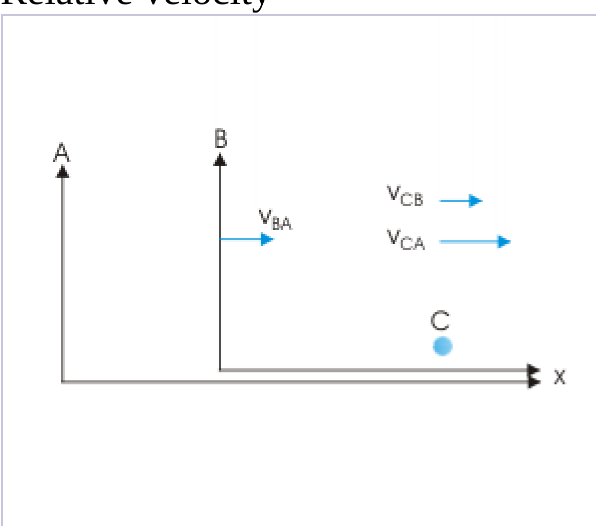
$$v_{CB} = \frac{dx_{CB}}{dt}$$

Now, we can obtain relation between these two velocities, using the relation $x_{CA} = x_{BA} + x_{CB}$ and differentiating the terms of the equation with respect to time :

$$\frac{dx_{CA}}{dt} = \frac{dx_{BA}}{dt} + \frac{dx_{CB}}{dt}$$

$$\Rightarrow v_{CA} = v_{BA} + v_{CB}$$

Relative velocity



The meaning of the subscripted velocities are :

- v_{CA} : velocity of object "C" with respect to "A"
- v_{CB} : velocity of object "C" with respect to "B"
- v_{BA} : velocity of object "B" with respect to "A"

Example:

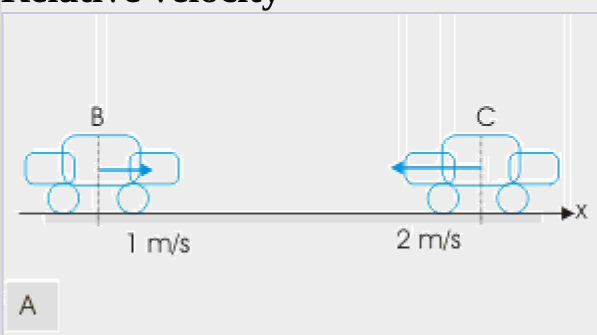
Problem : Two cars, standing a distance apart, start moving towards each other with speeds 1 m/s and 2 m/s along a straight road. What is the speed with which they approach each other ?

Solution : Let us consider that "A" denotes Earth, "B" denotes first car and "C" denotes second car. The equation of relative velocity for this case is :

$$\Rightarrow v_{CA} = v_{BA} + v_{CB}$$

Here, we need to fix a reference direction to assign sign to the velocities as they are moving opposite to each other and should have opposite signs. Let us consider that the direction of the velocity of B is in the reference direction, then

Relative velocity



$$v_{BA} = 1\, m/s \text{ and } v_{CA} = -2\, m/s.$$

Now :

$$v_{CA} = v_{BA} + v_{CB}$$

$$\Rightarrow -2 = 1 + v_{CB}$$
$$\Rightarrow v_{CB} = -2 - 1 = -3\, m/s$$

This means that the car "C" is approaching "B" with a speed of -3 m/s along the straight road. Equivalently, it means that the car "B" is approaching "C" with a speed of 3 m/s along the straight road. We, therefore, say that the two cars approach each other with a relative speed of 3 m/s.

Acceleration of the point object

If the object being observed is accelerated, then its acceleration is obtained by the time derivative of velocity. Differentiating equation of relative velocity, we have :

$$v_{CA} = v_{BA} + v_{CB}$$

$$\Rightarrow \frac{đv_{CA}}{đt} = \frac{đv_{BA}}{đt} + \frac{đv_{CB}}{đt}$$

$$\Rightarrow a_{CA} = a_{BA} + a_{CB}$$

The meaning of the subscripted accelerations are :

- a_{CA} : acceleration of object "C" with respect to "A"
- a_{CB} : acceleration of object "C" with respect to "B"
- a_{BA} : acceleration of object "B" with respect to "A"

But we have restricted ourselves to reference systems which are moving at constant velocity. This means that relative velocity of "B" with respect to "A" is a constant. In other words, the acceleration of "B" with respect to "A" is zero i.e. $a_{BA} = 0$. Hence,

$$\Rightarrow a_{CA} = a_{CB}$$

The observers moving at constant velocity, therefore, measure same acceleration of the object. As a matter of fact, this result is characteristics of inertial frame of reference. The reference frames, which measure same acceleration of an object, are inertial frames of reference.

Interpretation of the equation of relative velocity

The important aspect of relative velocity in one dimension is that velocity has only two possible directions. We need not use vector notation to write or evaluate equation of relative velocities in one dimension. The velocity, therefore, can be treated as signed scalar variable; plus sign indicating velocity in the reference direction and minus sign indicating velocity in opposite to the reference direction.

Equation with reference to earth

The equation of relative velocities refers velocities in relation to different reference system.

$$v_{CA} = v_{BA} + v_{CB}$$

We note that two of the velocities are referred to A. In case, "A" denotes Earth's reference, then we can conveniently drop the reference. A velocity without reference to any frame shall then mean Earth's frame of reference.

$$\Rightarrow v_C = v_B + v_{CB}$$
$$\Rightarrow v_{CB} = v_C - v_B$$

This is an important relation. This is the working relation for relative motion in one dimension. We shall be using this form of equation most of the time, while working with problems in relative motion. This equation can be used effectively to determine relative velocity of two moving objects with uniform velocities (C and B), when their velocities in Earth's reference are known. Let us work out an exercise, using new notation and see the ease of working.

Example:
Problem : Two cars, initially 100 m distant apart, start moving towards each other with speeds 1 m/s and 2 m/s along a straight road. When would

they meet ?

Solution : The relative velocity of two cars (say 1 and 2) is :

$$v_{21} = v_2 - v_1$$

Let us consider that the direction v_1 is the positive reference direction. Here, v_1 = 1 m/s and v_2 = -2 m/s. Thus, relative velocity of two cars (of 2 w.r.t 1) is :

$$\Rightarrow v_{21} = -2 - 1 = -3\, m/s$$

This means that car "2" is approaching car "1" with a speed of -3 m/s along the straight road. Similarly, car "1" is approaching car "2" with a speed of 3 m/s along the straight road. Therefore, we can say that two cars are approaching at a speed of 3 m/s. Now, let the two cars meet after time “t” :

$$t = \frac{\text{Displacement}}{\text{Relative velocity}} = \frac{100}{3} = 33.3\, s$$

Order of subscript

There is slight possibility of misunderstanding or confusion as a result of the order of subscript in the equation. However, if we observe the subscript in the equation, it is easy to formulate a rule as far as writing subscript in the equation for relative motion is concerned. For any two subscripts say “A” and “B”, the relative velocity of “A” (first subscript) with respect to “B” (second subscript) is equal to velocity of “A” (first subscript) subtracted by the velocity of “B” (second subscript) :

$$v_{AB} = v_A - v_B$$

and the relative velocity of B (first subscript) with respect to A (second subscript) is equal to velocity of B (first subscript) subtracted by the velocity of A (second subscript):

$$v_{BA} = v_B - v_A$$

Evaluating relative velocity by making reference object stationary

An inspection of the equation of relative velocity points to an interesting feature of the equation. We need to emphasize that the equation of relative velocity is essentially a vector equation. In one dimensional motion, we have taken the liberty to write them as scalar equation :

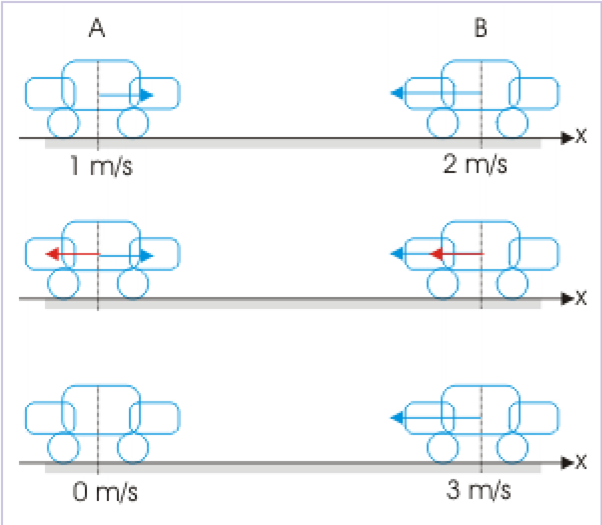
$$v_{BA} = v_B - v_A$$

Now, the equation comprises of two vector quantities (v_B and $-v_A$) on the right hand side of the equation. The vector " $-v_A$ " is actually the negative vector i.e. a vector equal in magnitude, but opposite in direction to " v_A ". Thus, we can evaluate relative velocity as following :

1. Apply velocity of the reference object (say object "A") to both objects and render the reference object at rest.
2. The resultant velocity of the other object ("B") is equal to relative velocity of "B" with respect to "A".

This concept of rendering the reference object stationary is explained in the figure below. In order to determine relative velocity of car "B" with reference to car "A", we apply velocity vector of car "A" to both cars. The relative velocity of car "B" with respect to car "A" is equal to the resultant velocity of car "B".

Relative velocity



This technique is a very useful tool for consideration of relative motion in two dimensions.

Direction of relative velocities

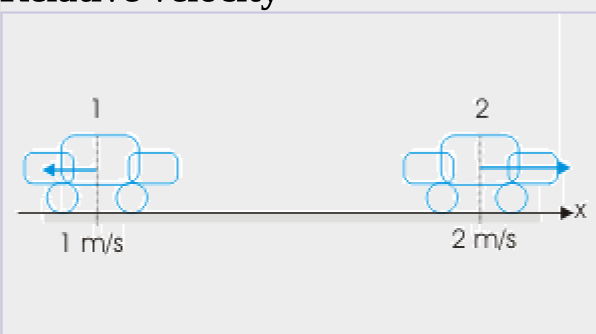
For a pair of two moving objects moving uniformly, there are two values of relative velocity corresponding to two reference frames. The values differ only in sign – not in magnitude. This is clear from the example here.

Example:

Problem : Two cars start moving away from each other with speeds 1 m/s and 2 m/s along a straight road. What are relative velocities ? Discuss the significance of their sign.

Solution : Let the cars be denoted by subscripts "1" and "2". Let us also consider that the direction v_2 is the positive reference direction, then relative velocities are :

Relative velocity



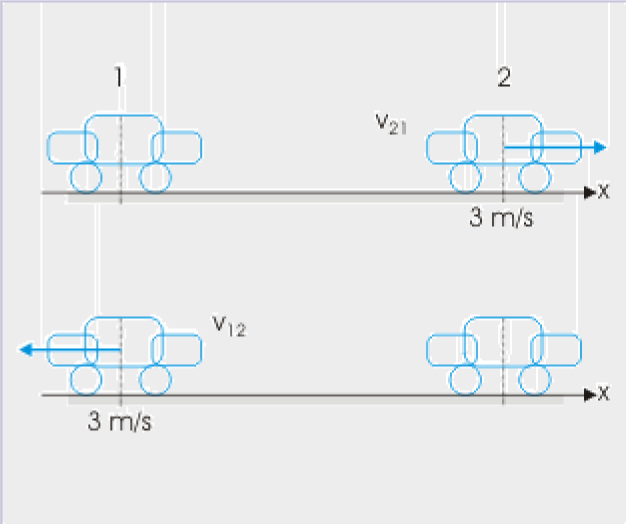
$$\Rightarrow v_{12} = v_1 - v_2 = -1 - 2 = -3\, m/s$$
$$\Rightarrow v_{21} = v_2 - v_1 = 2 - (-1) = 3\, m/s$$

The sign attached to relative velocity indicates the direction of relative velocity with respect to reference direction. The directions of relative velocity are different, depending on the reference object.

However, two relative velocities with different directions mean same physical situation. Let us read the negative value first. It means that car 1

moves away from car 2 at a speed of 3 m/s in the direction opposite to that of car 2. This is exactly the physical situation. Now for positive value of relative velocity, the value reads as car 2 moves from car 1 in the direction of its own velocity. This also is exactly the physical situation. There is no contradiction as far as physical interpretation is concerned. Importantly, the magnitude of approach – whatever be the sign of relative velocity – is same.

Relative velocity



Relative velocity .vs. difference in velocities

It is very important to understand that relative velocity refers to two moving bodies – not a single body. Also that relative velocity is a different concept than the concept of "difference of two velocities", which may pertain to the same or different objects. The difference in velocities represents difference of “final” velocity and “initial” velocity and is independent of any order of subscript. In the case of relative velocity, the order of subscripts are important. The expression for two concepts viz relative velocity and difference in velocities may look similar, but they are different concepts.

Relative acceleration

We had restricted out discussion up to this point for objects, which moved with constant velocity. The question, now, is whether we can extend the concept of relative velocity to acceleration as well. The answer is yes. We can attach similar meaning to most of the quantities - scalar and vector both. It all depends on attaching physical meaning to the relative concept with respect to a particular quantity. For example, we measure potential energy (a scalar quantity) with respect to an assumed datum.

Extending concept of relative velocity to acceleration is done with the restriction that measurements of individual accelerations are made from the same reference.

If two objects are moving with different accelerations in one dimension, then the relative acceleration is equal to the net acceleration following the same working relation as that for relative velocity. For example, let us consider than an object designated as "1" moves with acceleration "a_1 " and the other object designated as "2" moves with acceleration " a_2" along a straight line. Then, relative acceleration of "1" with respect to "2" is given by :

$$\Rightarrow a_{12} = a_1 - a_2$$

Similarly,relative acceleration of "2" with respect to "1" is given by :

$$\Rightarrow a_{21} = a_2 - a_1$$

Worked out problems

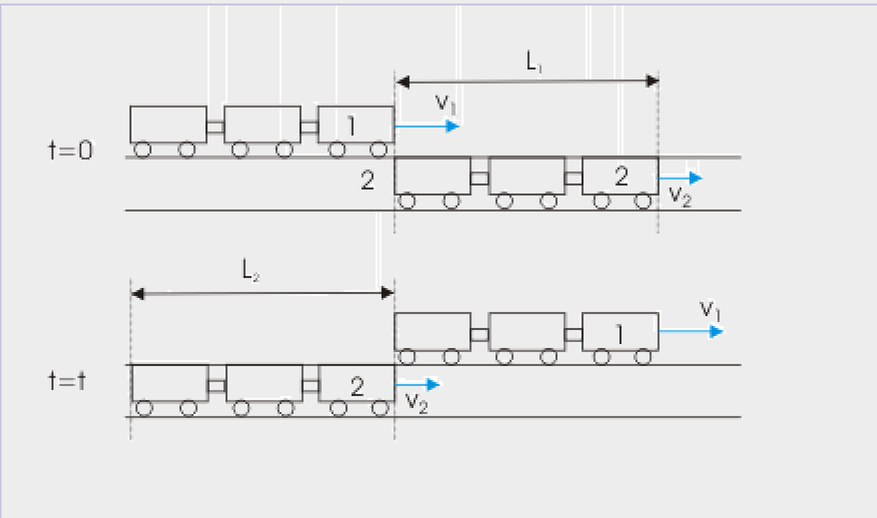
Example:
Relative motion
Problem : Two trains are running on parallel straight tracks in the same direction. The train, moving with the speed of 30 m/s overtakes the train ahead, which is moving with the speed of 20 m/s. If the train lengths are 200 m each, then find the time elapsed and the ground distance covered by the trains during overtake.

Solution : First train, moving with the speed of 30 m/s overtakes the second train, moving with the speed of 20 m/s. The relative speed with which first train overtakes the second train,

$$v_{12} = v_1 - v_2 = 30 - 20 = 10 \text{ m/s}.$$

The figure here shows the initial situation, when faster train begins to overtake and the final situation, when faster train goes past the slower train. The total distance to be covered is equal to the sum of each length of the trains (L1 + L2) i.e. 200 + 200 = 400 m. Thus, time taken to overtake is :

The total relative distance



The total relative distance to cover during overtake is equal to the sum of lengths of each train.

$$t = \frac{400}{10} = 40 \text{ s}.$$

In this time interval, the two trains cover the ground distance given by:

$$s = 30 \text{ x } 40 + 20 \text{ x } 40 = 1200 + 800 = 2000 \text{ m}.$$

Exercise:

Problem:

In the question given in the example, if the trains travel in the opposite direction, then find the time elapsed and the ground distance covered by the trains during the period in which they cross each other.

Solution:

$$v_{12} = v_1 - v_2 = 30 - (-20) = 50 \text{ m/s}.$$

The total distance to be covered is equal to the sum of each length of the trains i.e. 200 + 200 = 400 m. Thus, time taken to overtake is :

$$t = \frac{400}{50} = 8 \text{ s}.$$

Now, in this time interval, the two trains cover the ground distance given by:

$$s = 30 \text{ x } 8 + 20 \text{ x } 8 = 240 + 160 = 400 \text{ m}.$$

In this case, we find that the sum of the lengths of the trains is equal to the ground distance covered by the trains, while crossing each other.

Check your understanding

Check the module titled Relative velocity in one dimension (Check your understanding) to test your understanding of the topics covered in this module.

Relative velocity in one dimension(application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1. Foremost thing in solving problems of relative motion is about visualizing measurement. If we say a body "A" has relative velocity "v" with respect to another moving body "B", then we simply mean that we are making measurement from the moving frame (reference) of "B".
2. The concept of relative velocity applies to two objects. It is always intuitive to designate one of the objects as moving and other as reference object.
3. It is helpful in solving problem to make reference object stationary by applying negative of its velocity to both objects. The resultant velocity of the moving object is equal to the relative velocity of the moving object with respect to reference object. If we interpret relative velocity in this manner, it gives easy visualization as we are accustomed to observing motion from stationary state.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the relative velocity in one dimension.

Example:
Problem : A jet cruising at a speed of 1000 km/hr ejects hot air in the opposite direction. If the speed of hot air with respect to Jet is 800 km/hr, then find its speed with respect to ground.

Solution : Let the direction of Jet be x – direction. Also, let us denote jet with “A” and hot air with “B”. Here,

$$v_A = 1000 \text{ km / hr}$$
$$v_B = \text{ ?}$$
$$v_{\text{BA}} = -800 \text{ km / hr}$$

Now,

$$v_{BA} = v_B - v_A$$
$$v_B = v_A + v_{AB}$$
$$= 1000 + (-800) = 200 \text{ km / hr}$$

The speed of the hot air with respect to ground is 200 km/hr.

Example:
Problem : If two bodies, at constant speeds, move towards each other, then the linear distance between them decreases at 6 km/hr. If the bodies move in the same direction with same speeds, then the linear distance between them increases at 2 km/hr. Find the speeds of two bodies (in km/hr).
Solution : Let the speeds of the bodies are “u” and “v” respectively. When they move towards each other, the relative velocity between them is :

$$v_{AB} = v_A - v_B = u - (-v) = u + v = 6 \text{ km / hr}$$

When they are moving in the same direction, the relative velocity between them is :

$$v_{AB} = v_A - v_B = u - v = 2 \text{ km / hr}$$

Solving two linear equations, we have :
u = 4 km/hr
v = 2 km/hr

Example:

Problem : Two cars A and B move along parallel paths from a common point in a given direction. If “u” and “v” be their speeds (u > v), then find the separation between them after time “t”.

Solution : The relative velocity of the cars is :

$$v_{12} = v_1 - v_2 = u - v$$

The separation between the cars is :

$$x = v_{12}t = (u - v)t$$

Example:

Problem : Two trains of length 100 m each, running on parallel track, take 20 seconds to overtake and 10 seconds to cross each other. Find their speeds (in m/s).

Solution : The distance traveled in the two cases is 100 + 100 = 200m. Let their speeds be “u” and “v”. Now, relative velocity for the overtake is :

$$v_{12} = v_1 - v_2 = u - v$$

$$200 = (u - v) \text{ x } 20$$
$$\Rightarrow u - v = 10$$

The relative velocity to cross each other is :

$$v_{12}‘ = v_1 - v_2 = u - (-v) = u + v$$

$$200 = (u + v) \text{ x } 10$$
$$\Rightarrow u + v = 20$$

Solving two linear equations, we have :
u = 15 m/s and v = 5 m/s.

Example:
Problem : Two cars are moving in the same direction at the same speed,”u”. The cars maintain a linear distance "x" between them. An another car (third car) coming from opposite direction meets the two cars at an interval of “t”, then find the speed of the third car.
Solution : Let “u” be the speed of either of the two cars and “v” be the speed of the third car. The relative velocity of third car with respect to either of the two cars is :

$$v_{\text{rel}} = u + v$$

The distance between the two cars remains same as they are moving at equal speeds in the same direction. The events of meeting two cars separated by a distance “x” is described in the context of relative velocity. We say equivalently that the third car is moving with relative velocity "u+v", whereas the first two cars are stationary. Thus, distance covered by the third car with relative velocity is given by :

$$x = (u + v)t$$

$$v = \frac{(x - ut)}{t} = \frac{x}{t} - u$$

Example:
Problem : Two cars, initially at a separation of 12 m, start simultaneously. First car “A”, starting from rest, moves with an acceleration 2 m/s^2 , whereas the car “B”, which is ahead, moves with a constant velocity 1 m/s along the same direction. Find the time when car “A” overtakes car “B”.
Solution : We shall attempt this question, using concept of relative velocity. First we need to understand what does the event “overtaking” mean? Simply said, it is the event when both cars are at the same position at a particular time instant. Subsequently, the car coming from behind moves ahead.
Now relative velocity has an appropriate interpretation suiting to this situation. The relative velocity of “A” with respect to “B” means that the

car “B” is stationary, while car “A” moves with relative velocity. Thus, problem reduces to a simple motion of a single body moving with a certain velocity (relative velocity) and covering a certain linear distance (here sepeartion of 12 m) with certain acceleration. Note that car “B” has no acceleration.
Relative velocity of “A” with respect to “B”, i.e. v_{AB} , at the start of motion is :

$$v_{\mathrm{AB}} = v_A - v_B = 0 - 1 = 1\, m/s$$

Relative acceleration of “A” with respect to “B”, i.e. aAB is :

$$a_{\mathrm{AB}} = a_A - a_B = 2 - 0 = 2\, m/s^2$$

Applying equation of motion for a single body (remember the second body is rendered stationary),

$$\begin{aligned}
& x = ut + \frac{1}{2}at^2 \\
& \Rightarrow 12 = -1 \text{ x } t + \frac{1}{2} \text{ x } 2 \text{ x } t^2 \\
& \Rightarrow t^2 - t - 12 = 0 \\
& \Rightarrow t^2 + 3t - 4t - 12 = 0 \\
& \Rightarrow (t+3)(t-4) = 0
\end{aligned}$$

Neglecting negative value,

$$\Rightarrow t = 4\, s$$

Note : We can solve this problem without using the concept of relative velocity as well. The car “A” moves 12 m more than the distance covered by “B” at the time of overtake. If car “B” moves by “x” meters, then car “A” travels “x+12” meters. It means that :

$$\text{Displacement of “A”} = \text{Displacement of “B”} + 12$$

$$\frac{1}{2}at^2 = vt + 12$$

Putting values,

$$\Rightarrow \frac{1}{2} \text{ x } 2 \text{ x } t^2 = 1 \text{ x } t + 12$$
$$\Rightarrow t^2 - t - 12 = 0$$

This is the same equation as obtained earlier. Hence,

$$\Rightarrow t = 4\,s$$

Evidently, it appears to be much easier when we do not use the concept of relative velocity. This is the case with most situations in one dimensional motion of two bodies. It is easier not to use concept of relative velocity in one dimensional motion to the extent possible. We shall learn subsequently, however, that the concept of relative velocity has an edge in analyzing two dimensional motion, involving two bodies.

Relative velocity in two dimensions

The concept of relative motion in two or three dimensions is exactly same as discussed for the case of one dimension. The motion of an object is observed in two reference systems as before – the earth and a reference system, which moves with constant velocity with respect to earth. The only difference here is that the motion of the reference system and the object ,being observed, can take place in two dimensions. The condition that observations be carried out in inertial frames is still a requirement to the scope of our study of relative motion in two dimensions.

As a matter of fact, theoretical development of the equation of relative velocity is so much alike with one dimensional case that the treatment in this module may appear repetition of the text of earlier module. However, application of relative velocity concept in two dimensions is different in content and details, requiring a separate module to study the topic.

Relative motion in two dimensions

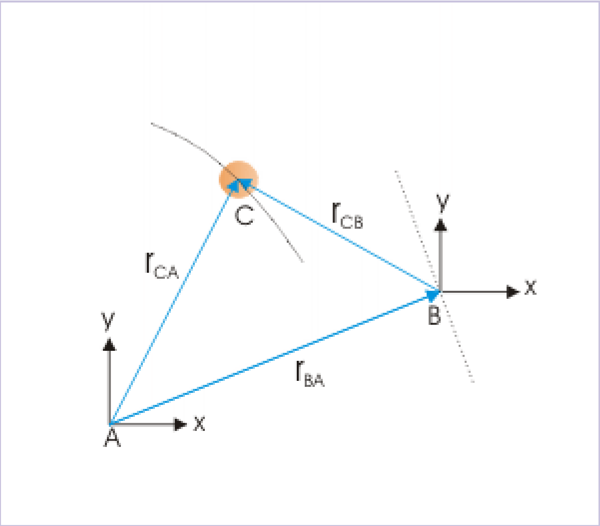
The important aspect of relative motion in two dimensions is that we can not denote vector attributes of motion like position, velocity and acceleration as signed scalars as in the case of one dimension. These attributes can now have any direction in two dimensional plane (say “xy” plane) and as such they should be denoted with either vector notations or component scalars with unit vectors.

Position of the point object

We consider two observers A and B. The observer “A” is at rest with respect to earth, whereas observer “B” moves with a constant velocity with respect to the observer on earth i.e. “A”. The two observers watch the motion of the point like object “C”. The motions of “B” and “C” are as shown along dotted curves in the figure below. Note that the path of observer "B" is a straight line as it is moving with constant velocity. However, there is no such restriction on the motion of object C, which can be accelerated as well.

The position of the object "C" as measured by the two observers "A" and "B" are $\mathbf{r}_{CA}$ and $\mathbf{r}_{CB}$. The observers are represented by their respective frame of reference in the figure.

Positions



The observers are represented by their respective frame of reference.

Here,

$$\mathbf{r}_{CA} = \mathbf{r}_{BA} + \mathbf{r}_{CB}$$

Velocity of the point object

We can obtain velocity of the object by differentiating its position with respect to time. As the measurements of position in two references are different, it is expected that velocities in two references are different,

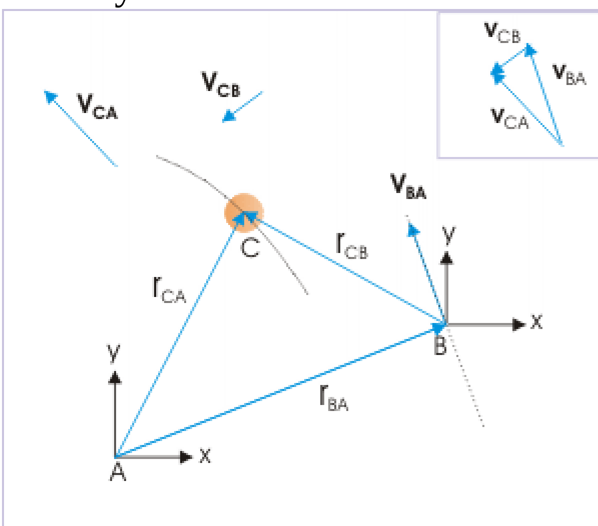
$$\mathbf{v}_{CA} = \frac{đ\mathbf{r}_{CA}}{đt}$$

and

$$\mathbf{v}_{CB} = \frac{đ\mathbf{r}_{CB}}{đt}$$

The velocities of the moving object “C” ($\mathbf{v}_{CA}$ and $\mathbf{v}_{CB}$) as measured in two reference systems are shown in the figure. Since the figure is drawn from the perspective of “A” i.e. the observer on the ground, the velocity $\mathbf{v}_{CA}$ of the object "C" with respect to "A" is tangent to the curved path.

Velocity



The observers measure different velocities.

Now, we can obtain relation between these two velocities, using the relation $\mathbf{r}_{CA} = \mathbf{r}_{BA} + \mathbf{r}_{CB}$ and differentiating the terms of the equation with respect to time as :

$$\frac{đ\mathbf{r}_{CA}}{đt} = \frac{đ\mathbf{r}_{BA}}{đt} + \frac{đ\mathbf{r}_{CB}}{đt}$$

$$\Rightarrow \mathbf{v}_{CA} = \mathbf{v}_{BA} + \mathbf{v}_{CB}$$

The meaning of the subscripted velocities are :

- $\mathbf{v}_{CA}$: velocity of object "C" with respect to "A"
- $\mathbf{v}_{CB}$: velocity of object "C" with respect to "B"

- $\mathbf{v}_{BA}$: velocity of object "B" with respect to "A"

Example:
Problem : A boy is riding a cycle with a speed of 5√3 m/s towards east along a straight line. It starts raining with a speed of 15 m/s in the vertical direction. What is the direction of rainfall as seen by the boy.
Solution : Let us denote Earth, boy and rain with symbols A, B and C respectively. The question here provides the velocity of B and C with respect to A (Earth).

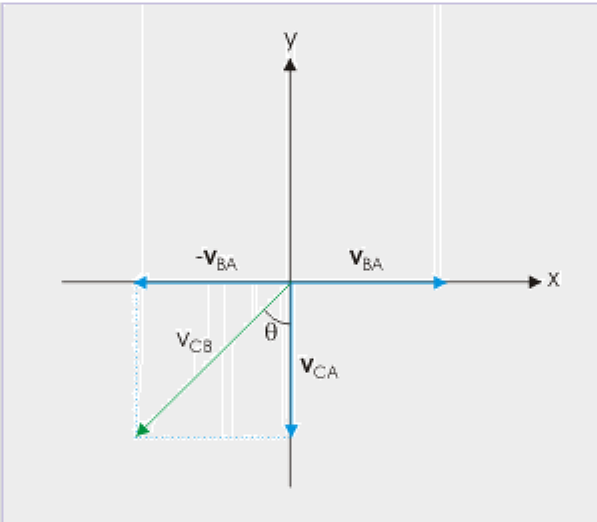
$$v_{BA} = 5\sqrt{3}\, m/s$$
$$v_{CA} = 15\, m/s$$

We need to determine the direction of rain (C) with respect to boy (B) i.e. $\mathbf{v}_{CB}$.

$$\mathbf{v}_{CA} = \mathbf{v}_{BA} + \mathbf{v}_{CB}$$

$$\Rightarrow \mathbf{v}_{CB} = \mathbf{v}_{CA} - \mathbf{v}_{BA}$$

We now draw the vector diagram to evaluate the terms on the right side of the equation. Here, we need to evaluate " $\mathbf{v}_{CA} - \mathbf{v}_{BA}$ ", which is equivalent to " $\mathbf{v}_{CA} + (-\mathbf{v}_{BA})$ ". We apply parallelogram theorem to obtain vctor sum as reprsesented in the figure.
Relative velocity



For the boy (B), the rain appears to fall, making an angle "θ" with the vertical (-y direction).

$$\Rightarrow \tan\theta = \frac{v_{BA}}{v_{CA}} = \frac{5\sqrt{3}}{15} = \frac{1}{\sqrt{3}} = \tan 30^0$$

$$\Rightarrow \theta = 30^0$$

Acceleration of the point object

If the object being observed is accelerated, then its acceleration is obtained by the time derivative of velocity. Differentiating equation of relative velocity, we have :

$$\Rightarrow \frac{đ\mathbf{v}_{CA}}{đt} = \frac{đ\mathbf{v}_{BA}}{đt} + \frac{đ\mathbf{v}_{CB}}{đt}$$

$$\Rightarrow \mathbf{a}_{CA} = \mathbf{a}_{BA} + \mathbf{a}_{CB}$$

The meaning of the subscripted accelerations are :

- $\mathbf{a}_{CA}$: acceleration of object "C" with respect to "A"
- $\mathbf{a}_{CB}$: acceleration of object "C" with respect to "B"
- $\mathbf{a}_{BA}$: acceleration of object "B" with respect to "A"

But we have restricted ourselves to reference systems which are moving at constant velocity. This means that relative velocity of "B" with respect to "A" is a constant. In other words, the acceleration of "B" with respect to "A" is zero i.e. $\mathbf{a}_{BA} = 0$. Hence,

$$\Rightarrow \mathbf{a}_{CA} = \mathbf{a}_{CB}$$

The observers moving at constant velocities, therefore, measure same acceleration of the object (C).

Interpretation of the equation of relative velocity

The interpretation of the equation of relative motion in two dimensional motion is slightly tricky. The trick is entirely about the ability to analyze vector quantities as against scalar quantities. There are few alternatives at our disposal about the way we handle the vector equation.

In broad terms, we can either use graphical techniques or vector algebraic techniques. In graphical method, we can analyze the equation with graphical representation along with analytical tools like Pythagoras or Parallelogram theorem of vector addition as the case may be. Alternatively, we can use vector algebra based on components of vectors.

Our approach shall largely be determined by the nature of inputs available for interpreting the equation.

Equation with reference to earth

The equation of relative velocities refers velocities in relation to different reference system.

$$\mathbf{v}_{CA} = \mathbf{v}_{BA} + \mathbf{v}_{CB}$$

We note that two of the velocities are referred to A. In case, “A” denotes Earth’s reference, then we can conveniently drop the reference. A velocity without reference to any frame shall then mean Earth’s frame of reference.

$$\Rightarrow \mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$$
$$\Rightarrow \mathbf{v}_{CB} = \mathbf{v}_C - \mathbf{v}_B$$

This is an important relation. As a matter of fact, we shall require this form of equation most of the time, while working with problems in relative motion. This equation can be used effectively to determine relative velocity of two moving objects with uniform velocity (C and B), when their velocities in Earth's reference are known.

Note : As in the case of one dimensional case, we can have a working methodology to find the relative velocity in two dimensions. In brief, we drop the reference to ground all together. We simply draw two velocities as given $\mathbf{v}_A$, $\mathbf{v}_B$. Then, we reverse the direction of reference velocity $\mathbf{v}_B$ and find the resultant relative velocity, $\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$, applying parallelogram theorem or using algebraic method involving unit vectors.

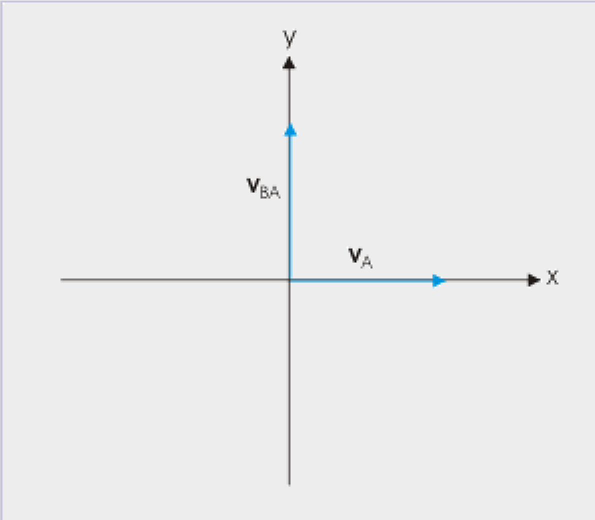
In general, for any two objects "A" and "B", moving with constant velocities,

$$\Rightarrow \mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

Example:
Problem : A person is driving a car towards east at a speed of 80 km/hr. A train appears to move towards north with a velocity of 80√3 km/hr to the person driving the car. Find the speed of the train as measured with respect to earth.
Solution : Let us first identify the car and train as "A" and "B". Here, we are provided with the speed of car ("A") with respect to Earth i.e. " v_A " and speed of train ("B") with respect to "A" i.e v_{BA} .

$$v_A = 80 \text{ km / hr}$$
$$v_{BA} = 80\sqrt{3} \text{ km / hr}$$

Relative velocity

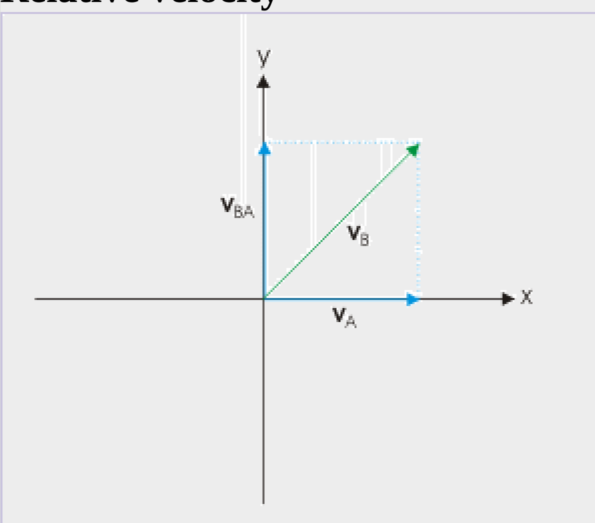


We are required to find the speed of train ("B") with respect to Earth i.e. $\mathbf{v}_B$, . From equation of relative motion, we have :

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$$
$$\Rightarrow \mathbf{v}_B = \mathbf{v}_{BA} + \mathbf{v}_A$$

To evaluate the right hand side of the equation, we draw vectors " $\mathbf{v}_{BA}$ " and " $\mathbf{v}_A$ " and use parallelogram law to find the actual speed of the train.

Relative velocity



$$\Rightarrow v_B = \sqrt{\left(v_{BA}{}^2 + v_A{}^2\right)} = \sqrt{\left\{(80\sqrt{3})^2 + 80^2\right\}} = 160 \text{ km / hr}$$

Evaluation of equation using analytical technique

We have already used analytical method to evaluate vector equation of relative velocity. Analytical method makes use of Pythagoras or Parallelogram theorem to determine velocities.

Analytical method, however, is not limited to making use of Pythagoras or Parallelogram theorem. Depending on situation, we may use simple trigonometric relation as well to evaluate equation of relative motion in two dimensions. Let us work out an exercise to emphasize application of such geometric (trigonometric) analytical technique.

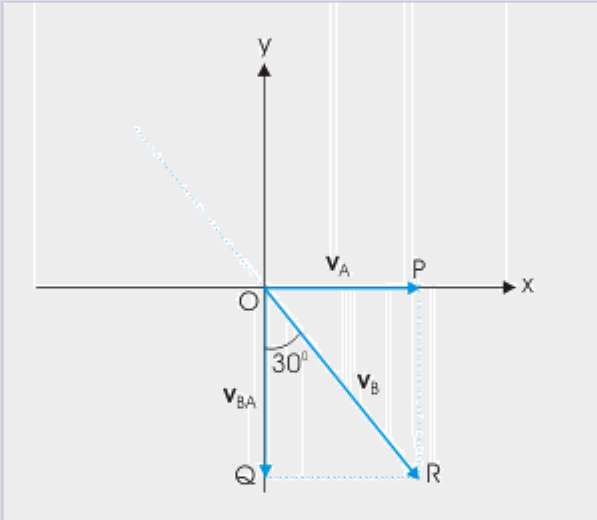
Example:

Problem : A person, standing on the road, holds his umbrella to his back at an angle 30° with the vertical to protect himself from rain. He starts running at a speed of 10 m/s along a straight line. He finds that he now has to hold his umbrella vertically to protect himself from the rain. Find the speed of raindrops as measured with respect to (i) ground and (ii) the moving person.

Solution : Let us first examine the inputs available in this problem. To do this let us first identify different entities with symbols. Let A and B denote the person and the rain respectively. The initial condition of the person gives the information about the direction of rain with respect to ground - notably not the speed with which rain falls. It means that we know the direction of velocity $\mathbf{v}_B$. The subsequent condition, when person starts moving, tells us the velocity of the person "A" with respect to ground i.e $\mathbf{v}_A$. Also, it is given that the direction of relative velocity of rain "B" with respect to the moving person "A" is vertical i.e. we know the direction of relative velocity $\mathbf{v}_{BA}$.

We draw three vectors involved in the problem as shown in the figure. OP represents $\mathbf{v}_A$; OQ represents $\mathbf{v}_B$; OR represents $\mathbf{v}_{BA}$.

Relative velocity



In ΔOCB,

$$v_B = \mathrm{OR} = \frac{\mathrm{QR}}{\sin 30^0}$$
$$\Rightarrow v_B = \frac{10}{\frac{1}{2}} = 20\, m/s$$

and

$$v_{BA} = \mathrm{OQ} = \frac{\mathrm{QR}}{\tan 30^0}$$
$$\Rightarrow v_{BA} = \frac{10}{\frac{1}{\sqrt{3}}} = 10\sqrt{3}\, m/s$$

Equation in component form

So far we have used analytical method to evaluate vector equation of relative velocity. It is evident that vector equation also renders to component form – particularly when inputs are given in component form along with unit vectors.

Here, we shall highlight one very important aspect of component analysis, which helps us to analyze complex problems. The underlying concept is that consideration of motion in mutually perpendicular direction is independent of each other. This aspect of independence is emphasized in

analyzing projectile motion, where motions in vertical and horizontal directions are found to be independent of each other (it is an experimental fact).

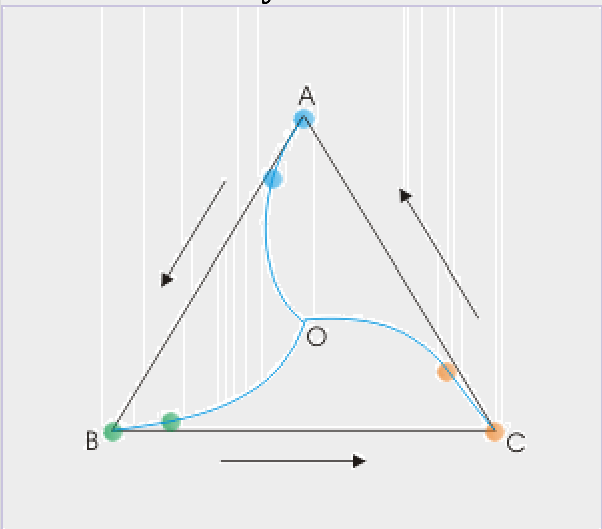
We work out the exercise to illustrate the application of the technique, involving component analysis.

Example:

Problem : Three particles A,B and C situated at the vertices of an equilateral triangle starts moving simultaneously at a constant speed “v” in the direction of adjacent particle, which falls ahead in the anti-clockwise direction. If “a” be the side of the triangle, then find the time when they meet.

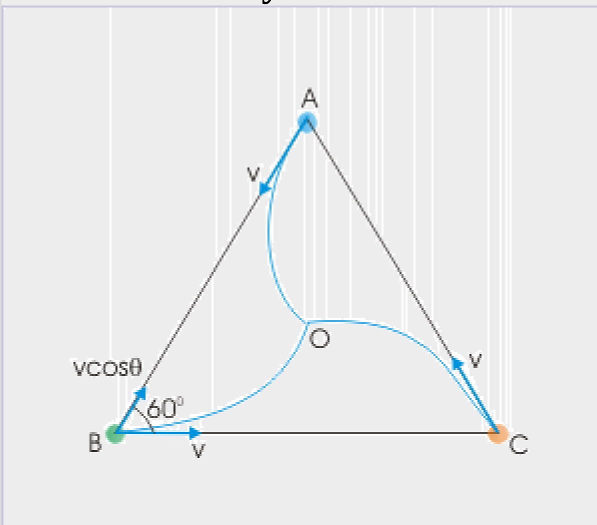
Solution : Here, particle “A” follows “B”, “B” follows “C” and “C” follows “A”. The direction of motion of each particle keeps changing as motion of each particle is always directed towards other particle. The situation after a time “t” is shown in the figure with a possible outline of path followed by the particles before they meet.

Relative velocity



This problem appears to be complex as the path of motion is difficult to be defined. But, it has a simple solution in component analysis. Let us consider the pair “A” and “B”. The initial component of velocities in the direction of line joining the initial position of the two particles is “v” and “vcosθ” as shown in the figure here :

Relative velocity



The component velocities are directed towards eachother. Now, considering the linear (one dimensional) motion in the direction of AB, the relative velocity of "A" with respect to "B" is :

$$v_{AB} = v_A - v_B$$
$$v_{AB} = v - (-v\cos\theta) = v + v\cos\theta$$

In equilateral triangle, θ = 60°,

$$v_{AB} = v + v\cos 60^0 = v + \frac{v}{2} = \frac{3v}{2}$$

The time taken to cover the displacement "a" i..e. the side of the triangle,

$$t = \frac{2a}{3v}$$

Check your understanding

Check the module titled Relative velocity in two dimensions (application) to test your understanding of the topics covered in this module.

Relative velocity in two dimensions (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1. Solution of problems involving relative motion in two dimensions involves evaluation of vector equation. The evaluation or analysis of vector equation is not limited to the use of pythogoras theorem, but significantly makes use of goemtric consideration like evaluating trigonometirc ratios.
2. Generally, we attempt graphical solution. This is so because graphical solution is intuitive and indicative of actual physical phenomenon. However, most of the problem can equally be handled with the help of algebraic vector analysis, involving unit vectors.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the relative velocity in two dimensions. The questions are categorized in terms of the characterizing features of the subject matter :

- Velocity of an individual object
- Relative velocity
- Closest approach

Velocity of an individual object

Example:
Problem : A man, moving at 3 km/hr along a straight line, finds that the rain drops are falling at 4 km/hr in vertical direction. Find the angle with which rain drop hits the ground.

Solution : Let the man be moving in x-direction. Let us also denote man with "A" and rain drop with "B". Here, we need to know the direction of rain drop with respect to ground i.e. the direction of $\mathbf{v}_B$.
Here,

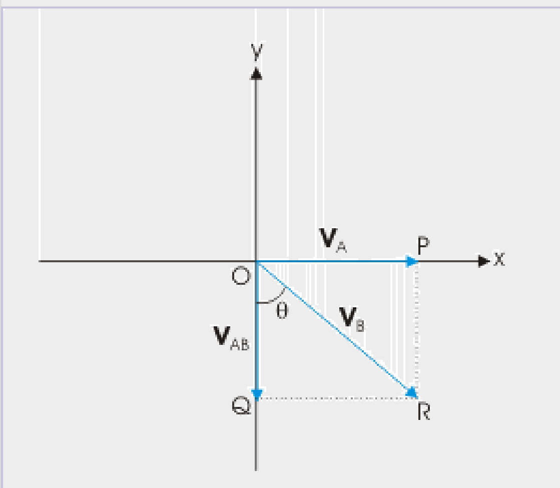
$$v_A = 3 \text{ km / hr}$$
$$v_B = ?$$
$$v_{\text{BA}} = 4 \text{ km /hr : } \text{ in the vertical direction}$$

Using equation, $\mathbf{v}_{\text{BA}} = \mathbf{v}_B - \mathbf{v}_A$,

$$\Rightarrow \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{\text{BA}}$$

In order to evaluate the right hand side of the equation, we construct the vector diagram as shown in the figure.

Relative motion in two dimensions



From inspection of given data and using appropriate trigonometric function in ΔOBR, we have :

$$\tan\theta = \frac{3}{4} = \tan 37^0$$
$$\Rightarrow \theta = 37^0$$

Example:

Problem : A person, moving at a speed of 1 m/s, finds rain drops falling (from back) at 2 m/s at an angle 30° with the vertical. Find the speed of raindrop (m/s)

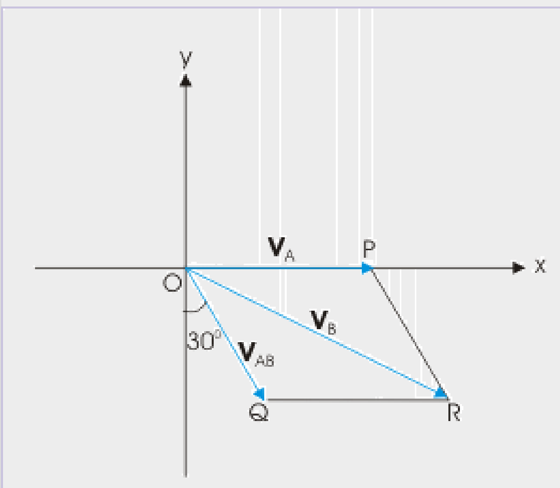
with which it hits the ground.
Solution : Let the person be moving in x-direction. Let us also denote man with "A" and rain drop with "B". Here we need to know the speed of the rain drops with respect to ground i.e. $\mathbf{v}_B$.
Here,

$$v_A = 1\, m/s$$
$$v_B = \text{ ?}$$
$$v_{\text{BA}} = 2\, m/s$$

Using equation, $\mathbf{v}_{\text{BA}} = \mathbf{v}_B - \mathbf{v}_A$,

$$\Rightarrow \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{\text{BA}}$$

In order to evaluate the right hand side of the equation, we construct the vector diagram as shown in the figure.
Relative motion in two dimensions



From parallelogram theorem,

$$v_B = \sqrt{\left({v_A}^2 + {v_{\text{BA}}}^2 + 2v_A v_{\text{BA}} \cos 60^0\right)}$$
$$\Rightarrow v_B = \sqrt{\left(1^2 + 2^2 + 2 \text{ x } 1 \text{ x } 2 \text{ x } \tfrac{1}{2}\right)}$$
$$\Rightarrow v_B = \sqrt{(1 + 4 + 2)} = \sqrt{7}\, m/s$$

Example:

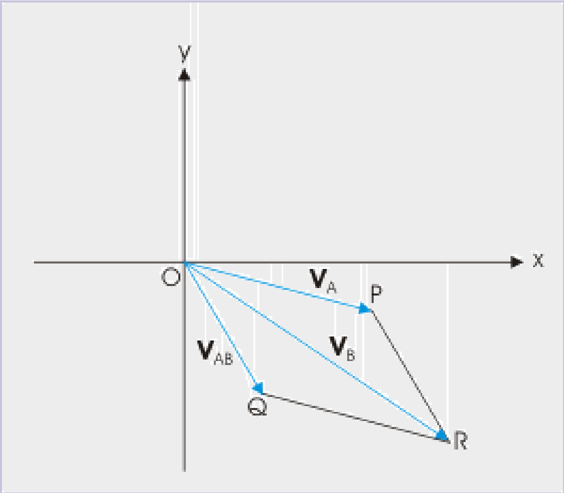
Problem : A boy moves with a velocity 0.5 **i** – **j** in m/s. He receives rains at a velocity 0.5 **i** – 2**j** in m/s. Find the speed at which rain drops meet the ground.
Solution : Let the person be moving along OA. Let us also denote man with "A" and rain drop with "B". Here we need to know the speed at which rain drops fall on the ground (v_B).
Here,

$$\mathbf{v}_A = 0.5\mathbf{i} - \mathbf{j}$$
$$v_B = ?$$
$$\mathbf{v}_{\mathrm{BA}} = 0.5\mathbf{i} - 2\mathbf{j}$$

Using equation, $\mathbf{v}_{\mathrm{BA}} = \mathbf{v}_B - \mathbf{v}_A$,

$$\Rightarrow \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{\mathrm{BA}}$$

Relative motion in two dimensions



$$\Rightarrow \mathbf{v}_B = 0.5\mathbf{i} - \mathbf{j} + 0.5\mathbf{i} - 2\mathbf{j} = \mathbf{i} - 3\mathbf{j}$$
$$\Rightarrow \mathbf{v}_B = \sqrt{}(1+9) = \sqrt{}(10)\ m/s$$

Relative velocity

Example:

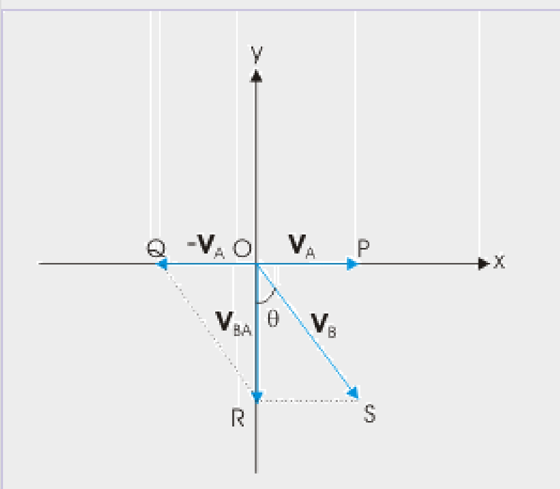
Problem : Rain drop appears to fall in vertical direction to a person, who is walking at a velocity 3 m/s in a given direction. When the person doubles his velocity in the same direction, the rain drop appears to come to make an angle 45° from the vertical. Find the speed of the rain drop.

Solution : This is a slightly tricky question. Readers may like to visualize the problem and solve on their own before going through the solution given here.

Let us draw the situations under two cases. Here, only the directions of relative velocities in two conditions are given. The figure on left represents initial situation. Here, the vector OP represents velocity of the person ($\mathbf{v}_A$); OR represents relative velocity of rain drop with respect to person ($\mathbf{v}_{\mathrm{BA}}$); OS represents velocity of rain drop.

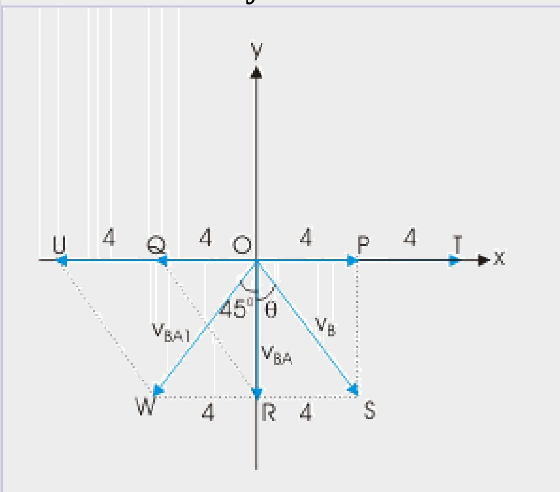
The figure on right represents situation when person starts moving with double velocity. Here, the vector OT represents velocity of the person ($\mathbf{v}_{\mathrm{A1}}$); OW represents relative velocity of rain drop with respect to person ($\mathbf{v}_{\mathrm{BA1}}$). We should note that velocity of rain ($\mathbf{v}_B$) drop remains same and as such, it is represented by OS represents as before.

Relative velocity

Raindrop appears to fall vertically.	Raindrop appears to fall at angle of 45°.
	[missing_resource: rmt6.gif]

According to question, we are required to know the speed of raindrop. It means that we need to know the angle “θ” and the side OS, which is the magnitude of velocity of raindrop. It is intuitive from the situation that it would help if consider the vector diagram and carry out geometric analysis to find these quantities. For this, we substitute the vector notations with known magnitudes as shown here.

Relative velocity



Values of different segments are shown.

We note here that

$$WR = UQ = 4 \quad m\ /\ s$$

Clearly, triangles ORS and ORW are congruent triangles as two sides and one enclosed angle are equal.

$$WR = RS = 4m/s$$

$$OR = OR$$

$$\angle ORW = \angle ORS = 90^0$$

Hence,

$$\angle WOR = \angle SOR = 45^0$$

In triangle ORS,

$$\sin 45^0 = \frac{RS}{OS}$$

$$\Rightarrow OS = \frac{RS}{\sin 45^0} = 4\sqrt{2} \quad m/s$$

Example:

Problem : Rain drop appears to fall in vertical direction to a person, who is wlaking at a velocity 3 m/s in a given direction. When the person doubles his velocity in the same direction, the rain drop appears to come to make an angle 45° from the vertical. Find the speed of the rain drop, using unit vectors.

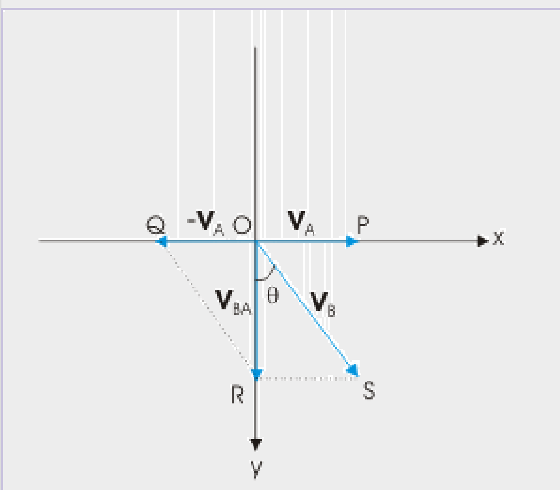
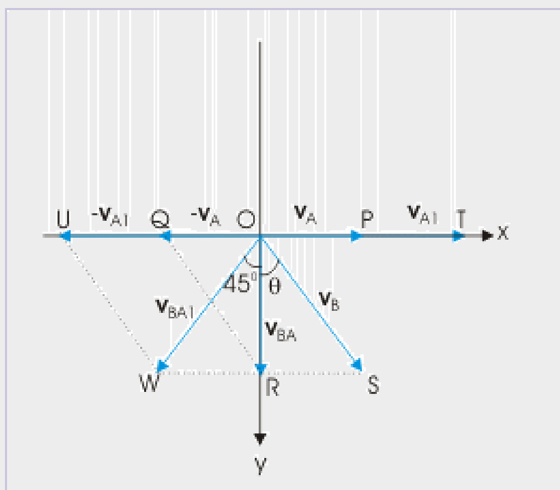
Solution : It is the same question as earlier one, but is required to be solved using unit vectors. The solution of the problem in terms of unit vectors gives us an insight into the working of algebraic analysis and also let us appreciate the power and elegance of using unit vectors to find solution of the problem, which otherwise appears to be difficult.

Let the velocity of raindrop be :

$$\mathbf{v}_B = a\mathbf{i} + b\mathbf{j}$$

where "a" and "b" are constants. Note here that we have considered vertically downward direction as positive.

Relative velocity

Raindrop appears to fall vertically.	Raindrop appears to fall at angle of 45°.
	

According to question,

$$\mathbf{v}_A = 4\mathbf{i}$$

The relative velocity of raindrop with respect to person is :

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A = a\mathbf{i} + b\mathbf{j} - 4\mathbf{i} = (a-4)\mathbf{i} + b\mathbf{j}$$

However, it is given that the raindrop appears to fall in vertical direction. It means that relative velocity has no component in x-direction. Hence,

$$a - 4 = 0$$

$$a = 4 \quad m/s$$

and

$$\mathbf{v}_{BA} = b\mathbf{j}$$

In the second case,

$$\mathbf{v}_{A_1} = 8\mathbf{i}$$

$$\mathbf{v}_{BA_1} = \mathbf{v}_B - \mathbf{v}_A = a\mathbf{i} + b\mathbf{j} - 8\mathbf{i} = (a-8)\mathbf{i} + b\mathbf{j} = (4-8)\mathbf{i} + b\mathbf{j} = -4\mathbf{i} + b\mathbf{j}$$

It is given that the raindrop appears to fall in the direction, making an angle 45° with the vertical.

$$\tan 135^0 = -1 = -\frac{4}{b}$$

Note that tangent of the angle is measured from positive x – direction (90° + 45° = 135°) of the coordinate system.

$$b = 4$$

Thus, velocity of the raindrop is :

$$\mathbf{v}_B = a\mathbf{i} + b\mathbf{j} = -4\mathbf{i} + b\mathbf{j}$$

The speed of raindrop is :

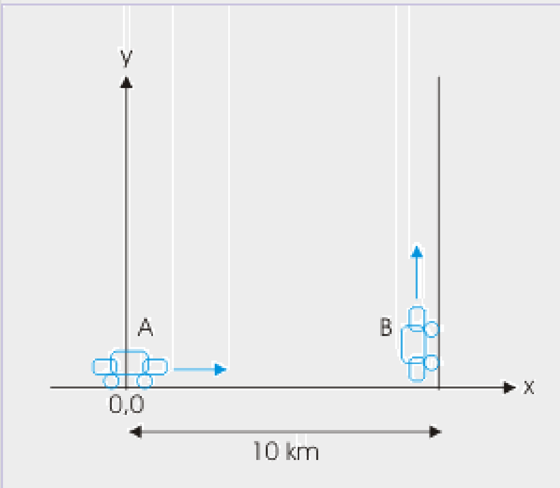
$$\Rightarrow \mathbf{v}_B = \sqrt{(4^2 + 4^2)} = 4\sqrt{2} \quad m/s$$

Closest approach

Example:

Problem : The car "B" is ahead of car "A" by 10 km on a straight road at a given time. At this instant, car "B" turns left on a road, which makes right angle to the original direction. If both cars are moving at a speed of 40 km/hr, then find the closest approach between the cars and the time taken to reach closest approach.

Motion of two cars



Motion of two cars in perpendicular directions

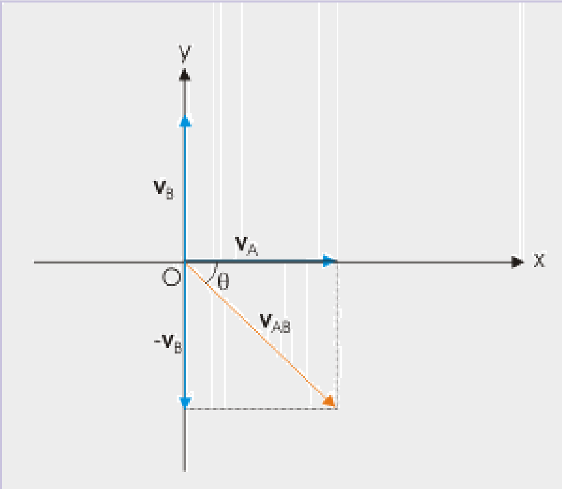
Solution : Closest approach means that the linear distance between the cars is mimimum. There are two ways to handle this question. We can proceed in the conventional manner, considering individual velocity as observed from ground reference. We use calculus to find the closest approach. Alternatively, we can use the concept of relative velocity and determine the closest approach. Here, we shall first use the relative velocity technique. By solving in two ways, we shall reinforce the conceptual meaning of relative velocity and see that its interpretation in terms of a stationary reference is a valid conception.

The car "A" moves with certain relative velocity with respect to "B", which is given by :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

Since they are moving with same speed and at right angle to each other, relative velocity is as shown in the figure. Using pythagoras theorem, the magnitude of relative velocity is :

Relative velocity



Relative velocity is obtained as sum of vectors.

$$\mathbf{v}_{AB} = \sqrt{(v^2 + v^2)} = \sqrt{\left(40^2 + 40^2\right)} = 40\sqrt{2} \quad m/s$$

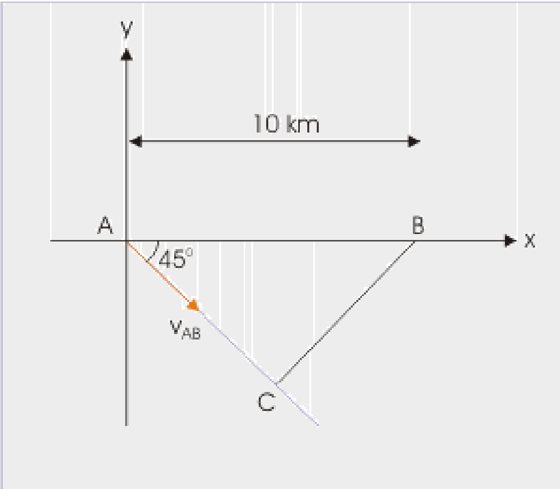
Its direction with respect to original direction is :

$$\tan\theta = \frac{40}{40} = 1$$

$$\theta = 45^0$$

As we have discussed, relative velocity can be conceptually interpreted as if car "B" is stationary and car "A" is moving with a velocity $40\sqrt{2}$ making an angle 45°. Since perpendicular drawn from the position of "B" (which is stationary) to the path of motion of "A" is the shortest linear distance and hence the closest approach. Therefore, closets approach by trigonometry of Δ ABC,

Relative velocity



Closest approach

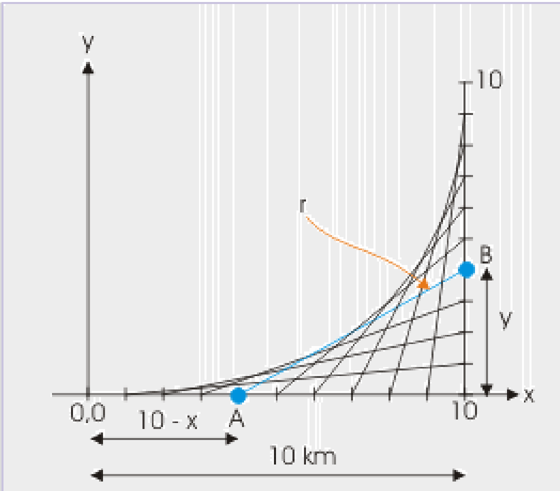
$$r_{\min} = AB \sin 45^0 = 10x\frac{1}{\sqrt{2}} = 5\sqrt{2} \quad km$$

We should understand here that car “A” is moving in the direction as shown from the perspective of car “B”.
The time taken by car “A” to travel $5\sqrt{2}$ km with a velocity of $40\sqrt{2}$ m/s along the direction is :

$$t = \frac{5\sqrt{2}}{40\sqrt{2}} = \frac{1}{8} \quad hr = 7.5 \quad \text{min}$$

Now, let us now attempt to analyze motion from the ground’s perpective. The figure here shows the positions of the car and linear distance between them at a displacement of 1 km. The linear distance first decreases and then increases. At a given time, the linear distance between the cars is :

Motion of two cars



Closest approach

$$r = \sqrt{\left\{(10-x)^2 + y^2\right\}}$$

For minimum distance, first time derivative is equal to zero. Hence,

$$\Rightarrow \frac{dr}{dt} = \frac{-2(10-x)\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{\left\{(10-x)^2 + y^2\right\}}} = 0$$

$$\Rightarrow -2(10-x)\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow -10 + x + y = 0$$

$$\Rightarrow x + y = 10$$

Since cars are moving with same speed, x = y. Hence,

$$x = y = 5 \quad km$$

The closest approach, therefore, is :

$$r_{\min} = \sqrt{\left\{(10 - x)^2 + y^2\right\}} = \sqrt{\left\{(10 - 5)^2 + 5^2\right\}} = 5\sqrt{2}$$

Resultant motion

Motion of an object in a medium like air, water or other moving body is affected by the motion of the medium itself. It is all so evident on a flight that the aircraft covers the same distance faster or slower, depending on the wind force working on it. The motion of aircraft is influenced by the velocity (read force exerted by the wind) of the wind - in both its magnitude and direction.

Similarly, the motion of boat, steamer or sea liner is influenced by the velocity of water stream. We can analyze these problems, using concept of relative motion (velocity), but with certain specific consideration.

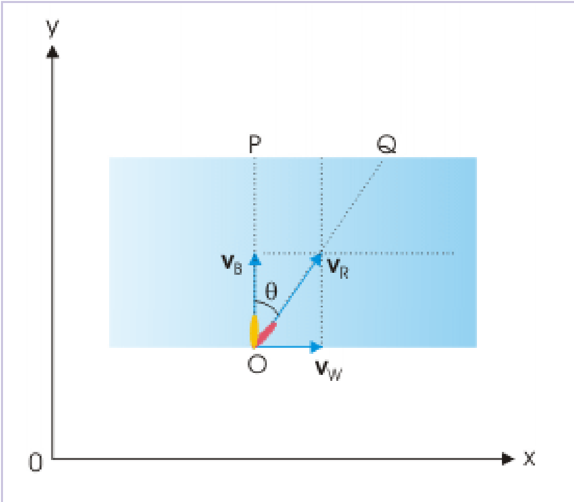
Understanding motion in a medium

First thing, we note that two bodies under consideration in the study of relative motion are essentially separated bodies. This is not so here. The body moves right within the body of the medium. They are in contact with each other. The body, in question, acquires a net velocity which comprises of its own velocity and that of the medium. Importantly, the two mass systems are in contact during motion unlike consideration in relative motion, where bodies are moving separately.

Resultant motion

The velocity of a boat in a stream, for example, is the resultant of velocities of the velocity of boat in still water and the velocity of the stream. The boat, therefore, moves having resultant velocity with respect to ground reference. This is the velocity with which boat ultimately moves in the stream and covers distance along a path.

Resultant velocity



The boat moves with the resultant velocity as seen by an observer on the ground.

The important point here to understand is that all velocities are measured in ground reference. The velocity of boat in still water is an indirect reference to ground. Velocity of stream, ofcourse, is measured with respect to ground. The resultant velocity of the boat is what an observer observes on the ground.

$$\mathbf{v}_R = \mathbf{v}_B + \mathbf{v}_W$$

where " $\mathbf{v}_R$ " is the resultant velocity of the object; " $\mathbf{v}_B$ " is the velocity of the boat in still water. and " $\mathbf{v}_W$ " is the velocity of the water stream.

The question now is that if velocities of entities are all measured with respect to a common reference, then where is the question of relative motion? We can simply treat the velocity of the body as seen from the ground equal to the resultant velocity, comprising of velocity of the object in a standstill medium and velocity of the medium itself.

Resultant velocity and relative velocity

This interpretation or understanding of resultant motion is perfectly valid except when a problem situation specifically involves terms such as “relative speed of boat with respect to stream” or “relative velocity of an aircraft with respect to air”. The big question is to identify whether this relative velocity refers to the resultant velocity or the velocity of the object in still medium. We can understand the importance of reference to relative velocity by interpreting some of the problems as given here (we shall work these problems subsequently) :

Problem 1 : An aircraft flies with a wind velocity of 200 km/hr blowing from south. If the relative velocity of aircraft with respect to wind is 1000 km/hr, then find the direction in which aircraft should fly such that it reaches a destination in north – east direction.

What does this relative velocity of aircraft with respect to wind mean? Is it the resultant velocity of the aircraft or is it the velocity of aircraft in still air?

Problem 2 A girl, starting from a point “P”, wants to reach a point “Q” on the opposite side of the bank of a river. The line PQ forms an angle 45° with the stream direction. If the velocity of the stream be “u”, then at what minimum speed relative to stream should the girl swim and what should be her direction?

What does this relative speed of girl with respect to stream mean? Is it the resultant speed of the girl or is it the speed of girl in still water?

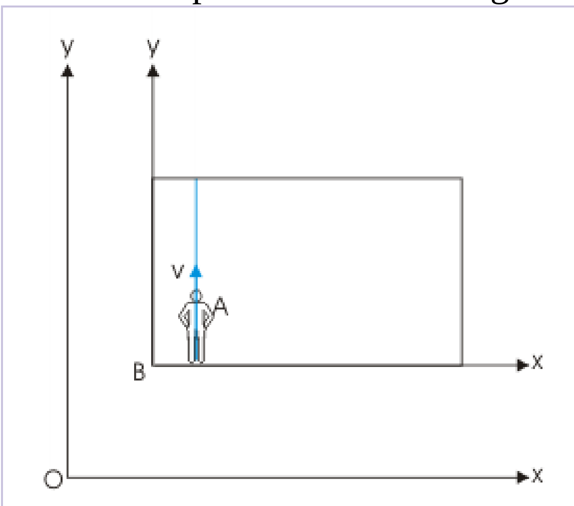
In this module, we shall learn to know the meaning of each term exactly. As a matter of fact, the most critical aspect of understanding motion in a medium is to develop skill to assign appropriate velocities to different entities.

Relative velocity with respect to a medium

We consider here an example, involving uniform motion of a person in a train, which is itself moving with constant velocity. Let us consider that the person moves across (perpendicular) the length of the compartment with constant speed “v”.

Uniform motion of the person is viewed differently by the observers on ground and on the train. Let us first think about his motion across the width of the train, when the train is not moving.

Motion of a person on a standing train

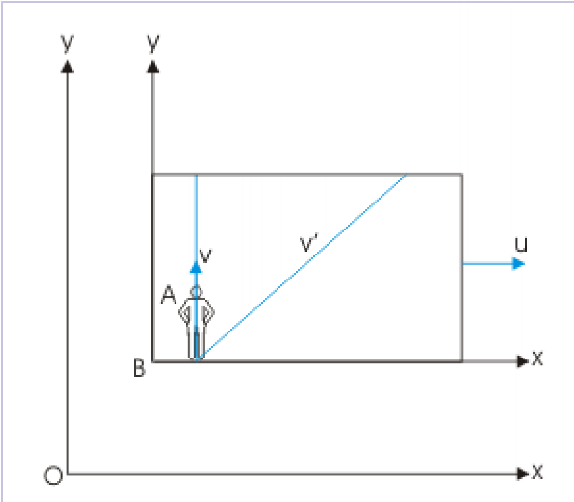


The motion of the person is in y-direction when train is standing.

The observers in two positions find that the motion of the person is exactly alike. In the figure, we have shown his path across the floor of the train (“B”). Evidently, the path of the motion of the person is a straight line for both observers attached to ground ("O") and train ("B").

Now let us consider that the train moves with a speed “u” to the right. Now, situation for the observer on the train has not changed a bit. The observer still finds that the speed of the person walking across the floor is still “v” and the path of motion is a straight line across the compartment. With our experience, we can imagine other persons walking on a train. Does the motion of co-passengers is different than what we see their motion on the ground? Our biological capacity to move remains same as on the ground. It does not matter whether we are walking on the ground or train.

Motion of a person in a train



The motion of the person, as seen from the ground, is along an inclined straight line when train is moving.

The motion of the person, however, has changed for the observer on the ground. By the time the person puts his next step on the floor in the y-direction, the train has moved in the x-direction. The position of the person after completing the step, thus, shifts right for the observer on the ground. Since, the velocities of person and train are uniform, the observer on the ground finds that the person is moving along a straight path making certain angle with the y-direction.

For the observer on the ground, the person walks longer distance. As the time interval involved in two reference systems are same, the speed (v') with which person appears to move in ground's reference is greater.

Identifying velocities

Identification of velocities and assigning them the right subscript are critical to analyze situation. We consider the earlier example of a person moving on a moving train. Let us refer person with "A" and train with "B". Then, the meaning of different notations are :

- $\mathbf{v}_{AB}$: velocity of person (“A”) with respect to train (“B”)
- $\mathbf{v}_A$: velocity of the person ("A") with respect to ground
- $\mathbf{v}_B$: velocity of train ("B") with respect to ground

From the explanation given earlier, we can say that :

$$v_{AB} = v$$
$$v_A = v\prime$$
$$v_B = u$$

The second and third assignments are evident as they are measurements with respect to ground. What is notable here is that the velocity of the person ("A") with respect to train ("B") is actually equal to the velocity of the person on the ground "v" (when train is stationary with respect to ground). This velocity is his inherent ability to walk on earth, but as explained, it is also his relative velocity with respect to a medium, when the medium (train) is moving. After all, why should biological capacity of the person change on the train?

Similar is the situation in the case of a boat sailing in a stream of certain width. The inherent mechanical speed in the still water is equal to the relative velocity of the boat with respect to the moving stream. Thus, we conclude that relative velocity of a body with respect to a moving medium is equal to its velocity in the still medium.

$\mathbf{v}_{AB} = \text{velocity of the body (”A”) with respect to moving medium (”B”)}$
$= \text{velocity of the body (”A”) in the still medium}$

This is the key aspect of learning for the motion in a medium. Rest is simply assigning values into the equation of relative velocity and evaluating the same as before.

$$\Rightarrow \mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

A closer look at the equation of relative velocity says it all. Remember, we interpreted relative velocity as the velocity of the body when reference body is stationary. Extending the interpretation to the case in hand, we can say that relative velocity of the body "A" is the velocity when the reference medium is stationary i.e. still. Thus, there is no contradiction in the equivalence of two meanings expressed in the equation above. Let us now check our

understanding and try to identify velocities in one of the questions considered earlier.

Problem 1 : An aircraft flies with a wind velocity of 200 km/hr blowing from south. If the relative velocity of aircraft with respect to wind is 1000 km/hr, then find the direction in which aircraft should fly such that it reaches a destination in north – east direction.

Can we answer the questions raised earlier - What does this relative velocity of aircraft with respect to wind mean? Yes, the answer is that the relative velocity of aircraft with respect to wind is same as velocity of aircraft in still air.

Resultant velocity and relative velocity are equivalent concepts

The concepts of resultant and relative velocities are equivalent. Rearranging the equation of relative velocity, we have :

$$\Rightarrow \mathbf{v}_A = \mathbf{v}_{AB} + \mathbf{v}_B$$

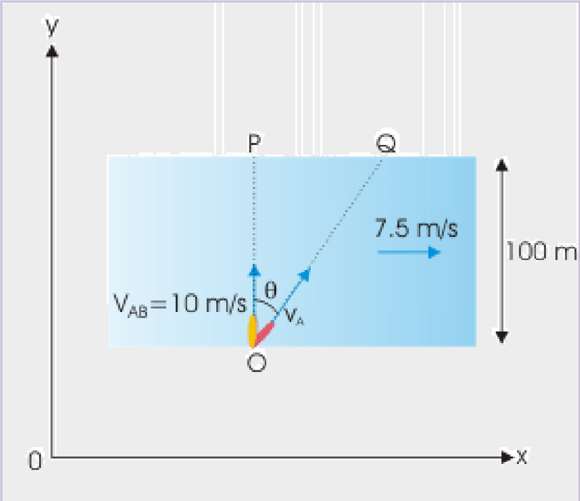
This means that resultant velocity of person ($\mathbf{v}_A$) is equal to the resultant of velocity of body in still medium ($\mathbf{v}_{\mathrm{AB}}$) and velocity of the medium ($\mathbf{v}_B$).

Example

Example:
Problem : A boat, which has a speed of 10 m/s in still water, points directly across the river of width 100 m. If the stream flows with the velocity 7.5 m/s in a linear direction, then find the velocity of the boat relative to the bank.
Solution : Let the direction of stream be in x-direction and the direction across stream is y-direction. We further denote boat with “A” and stream with “B”. Now, from the question, we have :

$$v_{AB} = 10\ m/s$$
$$v_B = 7.5\ m/s$$
$$v_A = ?$$

Resultant velocity



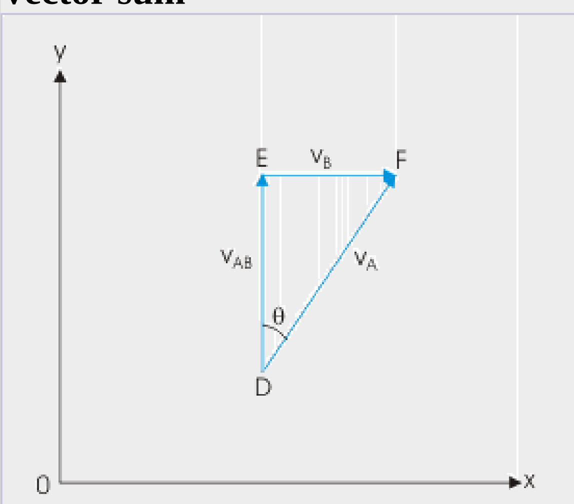
The boat moves at an angle with vertical.

Now, using $\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$, we have :

$$\Rightarrow \mathbf{v}_A = \mathbf{v}_{AB} + \mathbf{v}_B$$

We need to evaluate the right hand side of the equation. We draw the vectors and apply triangle law of vector addition. The closing side gives the sum of two vectors. Using Pythagoras theorem :

Vector sum



The velocity of boat is vector

sum of two velocities at right angles.

$$\Rightarrow v_A = \mathrm{DF} = \sqrt{\left(\mathrm{DE}^2 + \mathrm{EF}^2\right)} = \sqrt{\left(10^2 + 7.5^2\right)} = 12.5\, m/s$$

$$\Rightarrow \tan\theta = \frac{v_B}{v_{AB}} = \frac{7.5}{10} = \frac{3}{4} = \tan 37^0$$

$$\Rightarrow \theta = 37^0$$

Analysing motion in a medium
Resultant motion follows the principle of independence of motion in mutually perpendicular directions.

The motion of an object body in a medium is composed of two motions : (i) motion of the object and (ii) motion of the medium. The resulting motion as observed by an observer in the reference frame is the resultant of two motions. The basic equation that governs the context of study is the equation of relative motion in two dimensions :
Equation:

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

As discussed in the previous module, the motion of the body can alternatively be considered as the resultant of two motions. This concept of resultant motion is essentially an equivalent way of stating the concept of relative motion. Rearranging the equation, we have :
Equation:

$$\mathbf{v}_A = \mathbf{v}_{AB} + \mathbf{v}_B$$

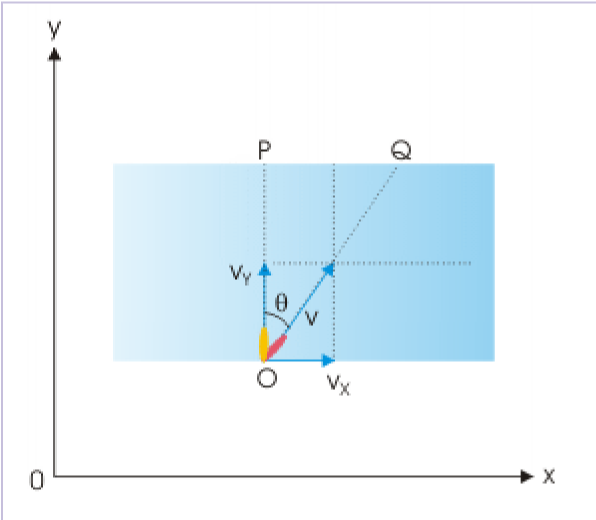
Both these equations are a vector equations, which can be broadly dealt in two ways (i) analytically (using graphics) and (ii) algebraically (using unit vectors). The use of a particular method depends on the inputs available and the context of motion.

Concept of independence of motion

Analysis of the motion of a body in a medium, specifically, makes use of independence of motions in two perpendicular directions. On a rectilinear coordinate system, the same principle can be stated in terms of component velocities. For example, let us consider the motion of a boat in a river, which tries to reach a point on the opposite bank of the river. The boat sails in the direction perpendicular to the direction of stream. Had the water been still, the boat would have reached the point exactly across the river with a velocity (v_y). But, water body is not still. It has a velocity in x-direction.

The boat, therefore, drifts in the direction of the motion of the water stream (v_x).

Motion of a Boat



Motions in two mutually perpendicular directions are independent of each other.

The important aspect of this motion is that the drift (x) depends on the component of velocity in x-direction. This drift (x) is independent of the component of velocity in y-direction. Why? Simply because, it is an experimental fact, which is fundamental to natural phenomena. We shall expand on this aspect while studying projectile motion also, where motions in vertical and horizontal directions are independent of each other.

In the case of boat, the displacements in the mutually perpendicular coordinate directions are :

Equation:

$$x = v_x t$$

Equation:

$$y = v_y t$$

Motion of boat in a stream

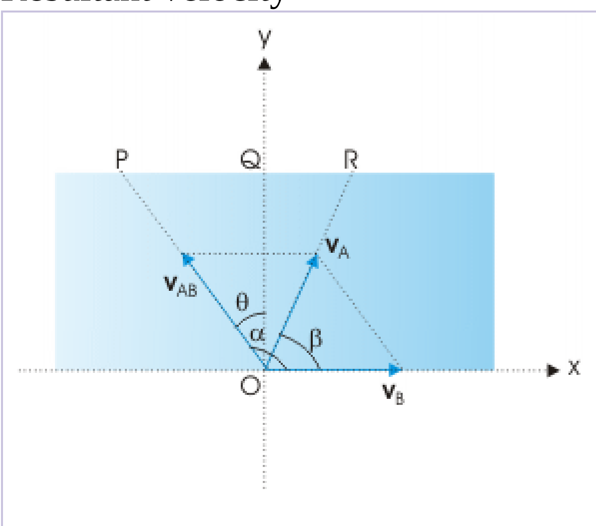
In this section, we shall consider a general situation of sailing of a boat in a moving stream of water. In order to keep our context simplified, we consider that stream is unidirectional in x-direction and the width of stream, “d”, is constant.

Let the velocities of boat (A) and stream (B) be “$\mathbf{v}_A$ “ and “ $\mathbf{v}_B$ “ respectively with respect to ground. The velocity of boat (A) with respect to stream (B), therefore, is :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

$$\Rightarrow \mathbf{v}_A = \mathbf{v}_{AB} + \mathbf{v}_B$$

Resultant velocity



The boat moves with the resultant velocity.

These velocities are drawn as shown in the figure. This is clear from the figure that boat sails in the direction, making an angle “θ” with y-direction, but reaches destination in different direction. The boat obviously is carried

along in the stream in x-direction. The displacement in x-direction (x = QR) from the directly opposite position to actual position on the other side of the stream is called the drift of the boat.

Resultant velocity

The magnitude of resultant velocity is obtained, using parallelogram theorem,

Equation:

$$v_A = \sqrt{\left(v_{AB}^2 + v_B^2 + 2v_{AB}v_B \cos\alpha\right)}$$

where "α" is the angle between $\mathbf{v}_B$ and $\mathbf{v}_{\mathrm{AB}}$ vectors. The angle "β" formed by the resultant velocity with x-direction is given as :

Equation:

$$\tan\beta = \frac{v_{AB}\sin\alpha}{v_B + v_{AB}\cos\alpha}$$

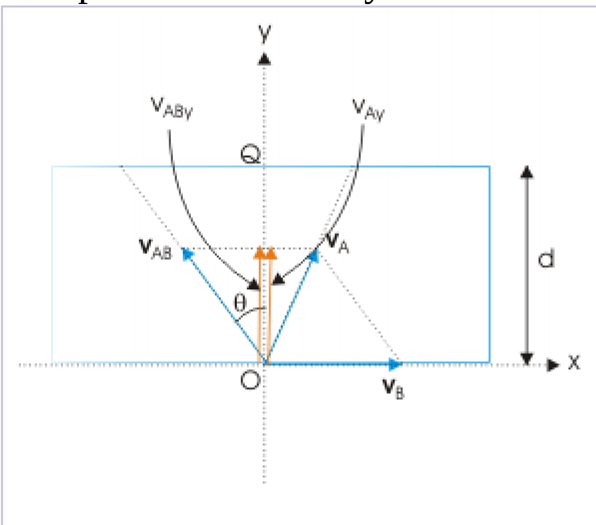
Time to cross the stream

The boat covers a linear distance equal to the width of stream "d" in time "t" in y-direction Applying the concept of independence of motions in perpendicular direction, we can say that boat covers a linear distance "OQ = d" with a speed equal to the component of resultant velocity in y-direction.

Now, resultant velocity is composed of (i) velocity of boat with respect to stream and (ii) velocity of stream. We observe here that velocity of stream is perpendicular to y – direction. As such, it does not have any component in y – direction. We, therefore, conclude that the component of resultant velocity is equal to the component of the velocity of boat with respect to stream in y -direction,. Note the two equal components shown in the figure.

They are geometrically equal as they are altitudes of same parallelogram. Hence,

Components of velocity



Components of velocity of boat w.r.t stream and velocity of stream in y-direction are equal.

$$v_{Ay} = v_{ABy} = v_{AB}\cos\theta$$

where "θ" is the angle that relative velocity of boat w.r.t stream makes with the vertical.

Equation:

$$t = \frac{d}{v_{Ay}} = \frac{d}{v_{AB}\cos\theta}$$

We can use either of the two expressions to calculate time to cross the river, depending on the inputs available.

Drift of the boat

The displacement of the boat in x-direction is independent of motion in perpendicular direction. Hence, displacement in x-direction is achieved with the component of resultant velocity in x-direction,

$$x = (v_{Ax})t = (v_B - v_{\mathrm{AB}\,x})t = (v_B - v_{AB}\sin\theta)t$$

Substituting for time “t”, we have :

Equation:

$$x = (v_B - v_{AB}\sin\theta)\frac{d}{v_{AB}\cos\theta}$$

Special cases

Shortest interval of time to cross the stream

The time to cross the river is given by :

$$t = \frac{d}{v_{Ay}} = \frac{d}{v_{AB}\cos\theta}$$

Clearly, time is minimum for greatest value of denominator. The denominator is maximum for θ = 0°. For this value,

Equation:

$$t_{\mathrm{min}} = \frac{d}{v_{AB}}$$

This means that the boat needs to sail in the perpendicular direction to the stream to reach the opposite side in minimum time. The drift of the boat for this condition is :

Equation:

$$x = \frac{v_B d}{v_{AB}}$$

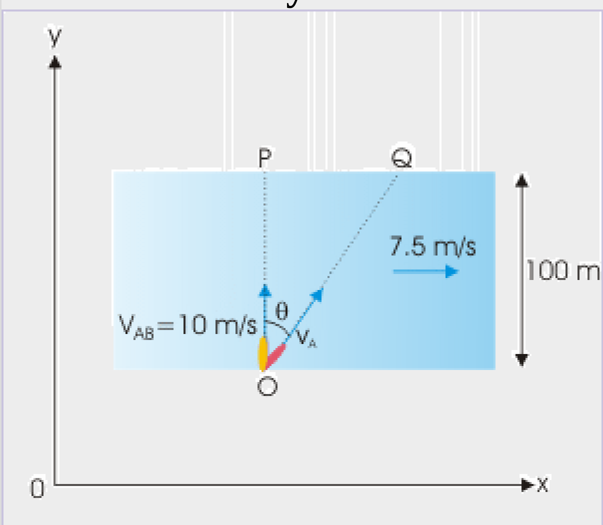
Example:

Problem : A boat, which has a speed of 10 m/s in still water, points directly across the river of width 100 m. If the stream flows with the velocity 7.5 m/s in a linear direction, then how far downstream does the boat touch on the opposite bank.

Solution : Let the direction of stream be in x-direction and the direction across stream is y-direction. We further denote boat with "A" and stream with "B". Now, from the question, we have :

$$v_{AB} = 10\, m/s$$
$$v_B = 7.5\, m/s$$

Resultant velocity



The boat moves at an angle with vertical.

The motions in two mutually perpendicular directions are independent of each other. In order to determine time (t), we consider motion in y – direction,

$$\Rightarrow t = \frac{\text{OP}}{v_{AB}} = \frac{100}{10} = 10\, s$$

The displacement in x-direction is :

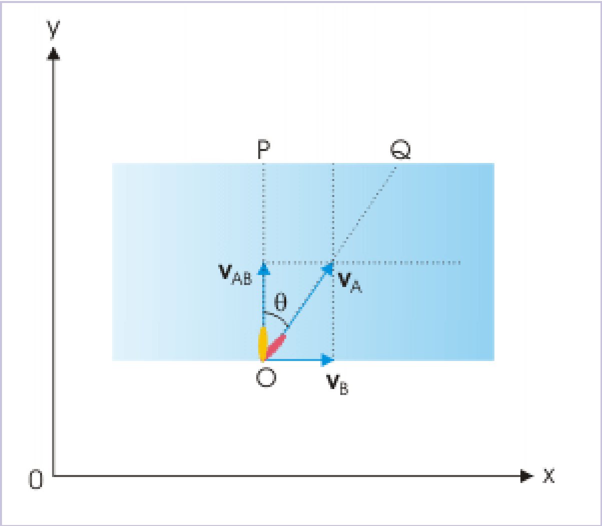
$$\mathrm{PQ} = v_B \text{ x } t$$

Putting this value, we have :

$$x = \mathrm{PQ} = v_B \text{ x } t = 7.5 \text{ x } 10 = 75\,m$$

The velocity of the boat w.r.t stream and the stream velocity are perpendicular to each other in this situation of shortest time as shown here in the figure. Magnitude of resultant velocity in this condition, therefore, is given as :

Resultant velocity



The boat moves at an angle with y-direction.

Equation:

$$v_A = \sqrt{\left(v_{AB}^2 + v_B^2\right)}$$

The angle that the resultant makes with y-direction (perpendicular to stream direction) is :

$$\tan\theta = \frac{v_B}{v_{AB}}$$

Time to cross the river, in terms of linear distance covered during the motion, is :

Equation:

$$t = \frac{OQ}{\sqrt{\left(v_{AB}^2 + v_B^2\right)}}$$

Direction to reach opposite point of the stream

If the boat is required to reach a point directly opposite, then it should sail upstream. In this case, the resultant velocity of the boat should be directed in y –direction. The drift of the boat is zero here. Hence,

$$x = \frac{(v_B - v_{AB}\sin\theta)d}{v_{AB}\cos\theta} = 0$$

$$\Rightarrow v_B - v_{AB}\sin\theta = 0$$

Equation:

$$\Rightarrow \sin\theta = \frac{v_B}{v_{AB}}$$

Thus. the boat should sail upstream at an angle given by above expression to reach a point exactly opposite to the point of sailing.

The velocity to reach opposite point of the stream

The angle at which boat sails to reach the opposite point is :

$$\sin\theta = \frac{v_B}{v_{AB}}$$

This expression points to a certain limitation with respect to velocities of boat and stream. If velocity of boat in still water is equal to the velocity of stream, then

$$\sin\theta = \frac{v_B}{v_{AB}} = 1 = \sin 90^0$$

$$\Rightarrow \theta = 90^0$$

It means that boat has to sail in the direction opposite to the stream to reach opposite point. This is an impossibility from the point of physical reality. Hence, we can say that velocity of boat in still water should be greater than the velocity of stream ($v_{AB} > v_B$) in order to reach a point opposite to the point of sailing.

In any case, if $v_{AB} < v_B$, then the boat can not reach the opposite point as sine function can not be greater than 1.

Shortest path

The magnitude of linear distance covered by the boat is given by :
Equation:

$$s = \sqrt{(d^2 + x^2)}$$

It is evident from the equation that linear distance depends on the drift of the boat, “x”. Thus, shortest path corresponds to shortest drift. Now, there are two situations depending on the relative magnitudes of velocities of boat and stream.

Note: We should be aware that though the perpendicular distance to stream (width of the river) is the shortest path, but boat may not be capable to follow this shortest path in the first place.

1: $v_{AB} > v_B$

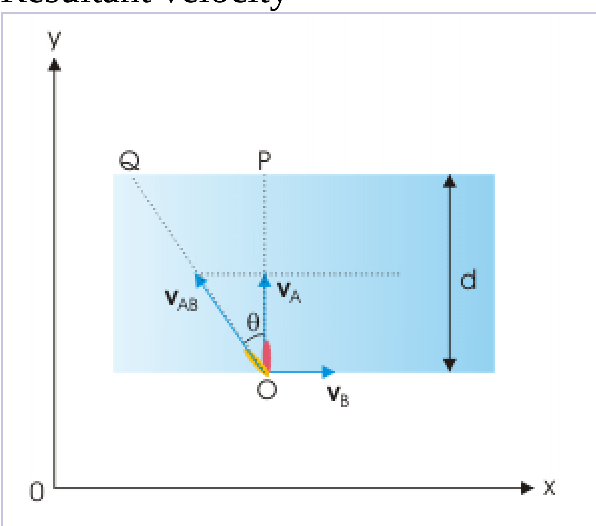
We have seen that when stream velocity (v_B) is less than the velocity of boat in still water, the boat is capable to reach the opposite point across the stream. For this condition, drift (x) is zero and represents the minimum value. Accordingly, the shortest path is :

Equation:

$$s_{\text{min}} = d$$

The boat needs to sail upstream at the specified angle. In this case, the resultant velocity is directed across the river in perpendicular direction and its magnitude is given by :

Resultant velocity



The boat moves perpendicular to stream.

Equation:

$$v_A = \sqrt{\left(v_{AB}^2 - v_B^2\right)}$$

The time taken to cross the river is :
Equation:

$$t = \frac{d}{\sqrt{\left(v_{AB}^2 - v_B^2\right)}}$$

2: $v_{AB} < v_B$

In this case, the boat is carried away from the opposite point in the direction of stream. Now, the drift "x" is given as :

$$x = \frac{(v_B - v_{AB}\sin\theta)d}{v_{AB}\cos\theta}$$

For minimum value of "x", first time derivative of "x" is equal to zero,
Equation:

$$\frac{đx}{đt} = 0$$

We need to find minimum drift and corresponding minimum length of path, subject to this condition.

Motion of an object in a medium

We have discussed the motion in the specific reference of boat in water stream. However, the consideration is general and is applicable to the motion of a body in a medium. For example, the discussion and analysis

can be extended to the motion of an aircraft, whose velocity is modified by the motion of the wind.

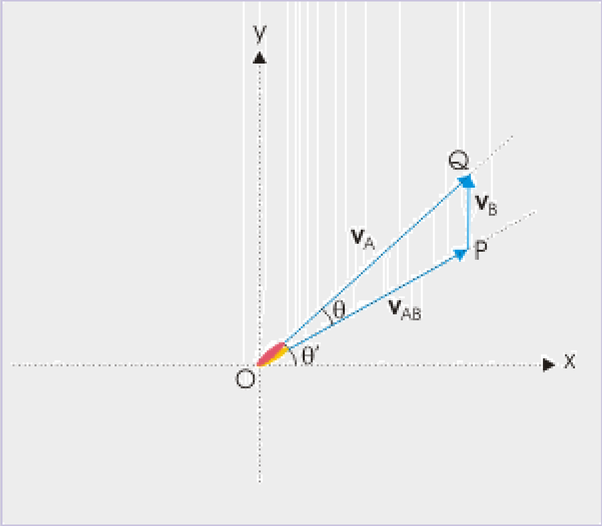
Example:

Problem : An aircraft flies with a wind velocity of $200\sqrt{2}$ km/hr blowing from south. If the relative velocity of aircraft with respect to wind is 1000 km/hr, then find the direction in which aircraft should fly such that it reaches a destination in north – east direction.

Solution : The figure here shows the velocities. OP denotes the velocity of the aircraft in the still air or equivalently it represents the relative velocity of aircraft with respect to air in motion; PQ denotes the velocity of the wind and OQ denotes the resultant velocity of the aircraft. It is clear that the aircraft should fly in the direction OP so that it is ultimately led to follow the north-east direction.

We should understand here that one of the velocities is resultant velocity of the remaining two velocities. It follows then that three velocity vectors are represented by the sides of a closed triangle.

Motion of an aircraft



The aircraft flies such a way that it keeps a north – east course.

We can get the direction of OP, if we can find the angle “θ”. The easiest technique to determine the angle between vectors composing a triangle is to apply sine law,

$$\frac{OP}{\sin 45^0} = \frac{PQ}{\sin \theta}$$

Putting values, we have :

$$\sin \theta = \frac{PQ \sin 45^0}{OP} = \frac{200\sqrt{2}}{1000x\sqrt{2}} = \frac{1}{5} = 0.2$$

$$\theta = \sin^{-1}(0.2)$$

Hence the aircraft should steer in the direction, making an angle with east as given by :

$$\theta\prime = 45^0 - \sin^{-1}(0.2)$$

Resultant motion (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

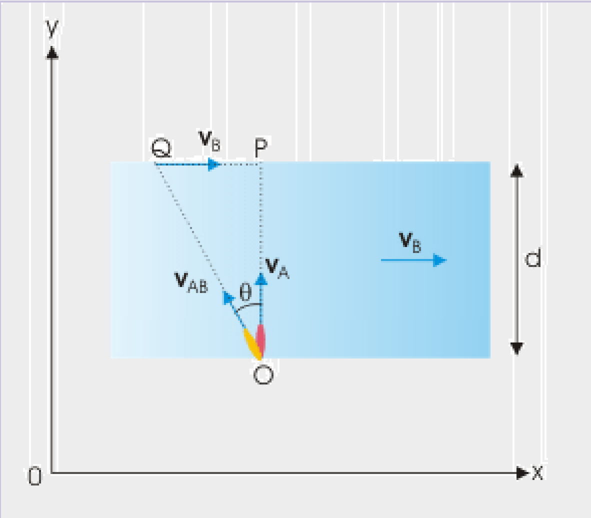
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the resultant velocity. The questions are categorized in terms of the characterizing features of the subject matter :

- Velocity of the object
- Time to cross the stream
- Multiple references
- Minimum time, distance and speed

Velocity of the object

Example:
Problem : A person can swim at 1 m/s in still water. He swims to cross a river of width 200 m to a point exactly opposite to his/her initial position. If the water stream in river flows at 2 m/s in a linear direction, then find the time taken (in seconds) to reach the opposite point.
Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. Let us also denote person with “A” and water stream with “B”.
To reach the point across, the person has to swim upstream at an angle such that the velocity of the person with respect to ground($\mathbf{v}_A$) is across the direction of water stream. The situation is shown in the figure.
Relative velocity



Here,

$$\text{Speed of the person (A) with respect to stream (B)}: v_{AB} = 1\,m/s$$
$$\text{Speed of water stream (B) with respect to ground}: v_B = 2\,m/s$$
$$\text{Speed of the person (A) with respect to ground}: v_A = ?$$
$$d = 200\,m$$

From the ΔOAB,

$$\mathrm{OQ}^2 = \mathrm{OP}^2 + \mathrm{PQ}^2$$
$$\Rightarrow \mathrm{OP} = \surd\{\mathrm{OQ}^2 - \mathrm{PQ}^2\}$$
$$\Rightarrow t = \frac{d}{\surd\{\mathrm{OB}^2 - \mathrm{AB}^2\}}$$

It is clear from the denominator of the expression that for finite time, OB > AB. From the values as given in the question, OB < AB and the denominator becomes square root of negative number. The result is interpreted to mean that the physical event associated with the expression is not possible.The swimmer, therefore, can not reach the point, which is exactly opposite to his position. The speed of the swimmer should be greater than that of the stream to reach the point lying exactly opposite. Note that we had explained the same situation in the module on the subject with the help of the value of "sinθ", which can not be greater than 1. We have taken a different approach here to illustrate the same limitation of the

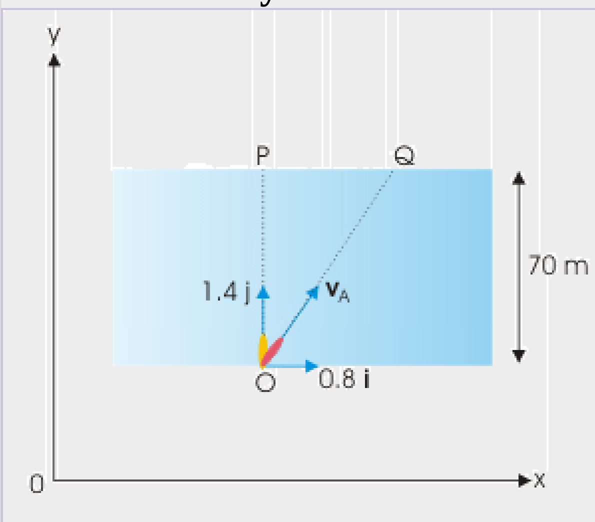
swimmer's ability to cross the water stream, showing that the interpretation is consistent and correct.

Example:

Problem : The direction of water stream in a river is along x – direction of the coordinate system attached to the ground. A swimmer swims across the river with a velocity ($0.8\mathbf{i} + 1.4\mathbf{j}$) m/s, as seen from the ground. If the river is 70 m wide, how long (in seconds) does he take to reach the river bank on the other side ?

Solution : We recognize here that the given velocity represents the resultant velocity ($\mathbf{v}_A$) of the swimmer (A). The time to reach the river bank on the other side is a function of component velocity in y-direction.

Relative velocity



Here,

$$v_x = 0.8\, m/s$$
$$v_y = 1.4\, m/s$$

$$t = \frac{\text{Width of the river}}{v_y}$$

$$t = \frac{70}{1.4} = 50\ s$$

Example:

Problem : A person can swim at a speed "u" in still water. He points across the direction of water stream to cross a river. The water stream flows with a speed "v" in a linear direction. Find the direction in which he actually swims with respect to the direction of stream.

Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. Let us also denote person with "A" and water stream with "B".

Here,

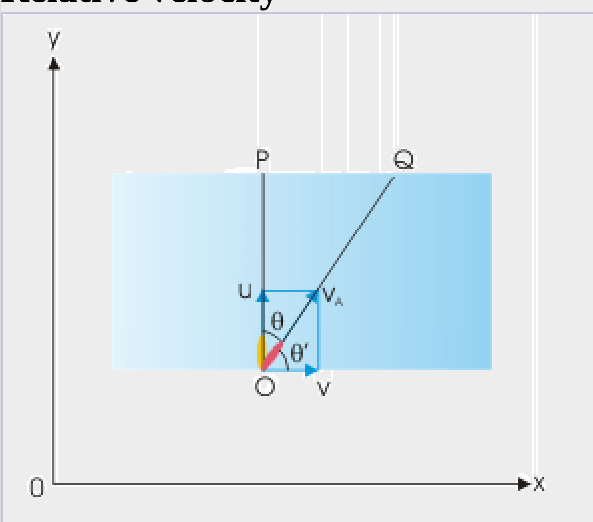
$$\text{Speed of the person (A) with respect to stream (B) :} v_{AB} = u$$
$$\text{Speed of stream (B) with respect to ground :} v_B = v$$
$$\text{Speed of the person (A) with respect to ground :} v_C = ?$$

Using equation, $\mathbf{v}_{\mathrm{AB}} = \mathbf{v}_A - \mathbf{v}_B$,

$$\Rightarrow \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{\mathrm{AB}}$$

From the figure,

Relative velocity



$$\tan \theta' = \frac{v_{AB}}{v_B} = \frac{u}{v}$$

The direction in which he actually swims with respect to the direction of stream is

$$\theta' = \tan^{-1}\left(\frac{u}{v}\right)$$

Time to cross the river

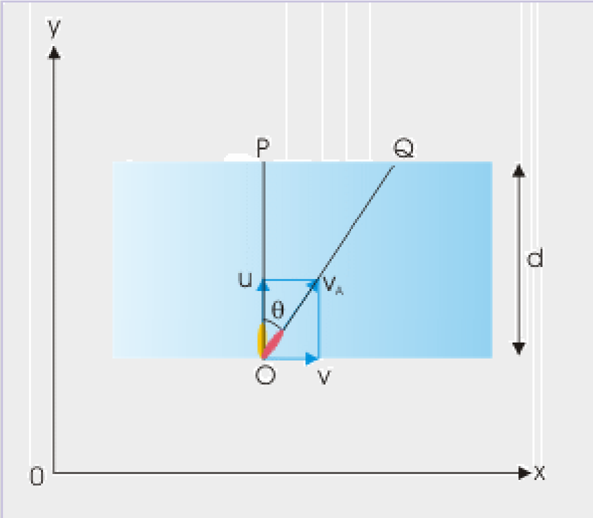
Example:
Problem : A person can swim at a speed 1 m/s in still water. He swims perpendicular to the direction of water stream, flowing at the speed 2 m/s. If the linear distance covered during the motion is 300 m, then find the time taken to cross the river.
Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. Let us also denote ground person with "A" and water stream with “B”. This is clearly the situation corresponding to the least time for crossing the river.
Here,

Speed of the person (A) with respect to stream (B) : $v_{AB} = u = 1\,m/s$
Speed of water stream (B) with respect to ground : $v_B = v = 2\,m/s$
Speed of the person (A) with respect to ground : $v_A = ?$

We note here that the perpendicular linear distance i.e. the width of river is not given. Instead, the linear distance covered during the motion is given. Hence, we need to find the resultant speed in the direction of motion to find time. Using equation for the resultant velocity,

$$\Rightarrow \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{\mathrm{AB}}$$

From the figure, we have :
Relative velocity

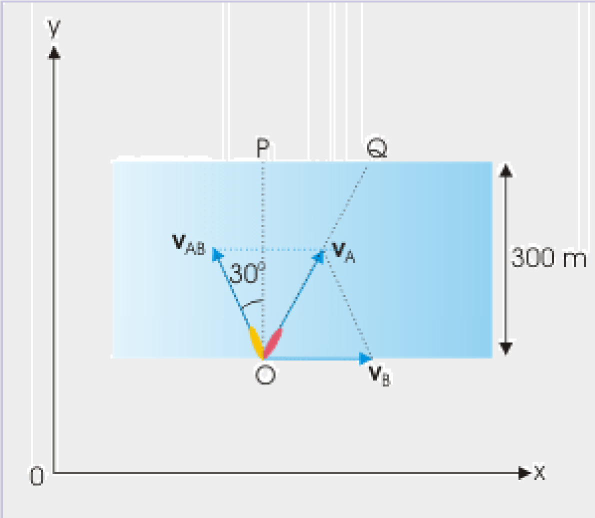


$$v_A = \sqrt{\{v_{\text{AB}}{}^2 + v_B{}^2\}} = \sqrt{\{u^2 + v^2\}}$$

$$v_A = \sqrt{\{1^2 + 2^2\}} = \sqrt{5}\, m/s$$

$$\Rightarrow t = \frac{500}{\sqrt{5}} = 100\sqrt{5}\ s$$

Example:
Problem : A person can swim at a speed of √3 m/s in still water. He swims at an angle of 120° from the stream direction while crossing a river. The water stream flows with a speed of 1 m/s. If the river width is 300 m, how long (in seconds) does he take to reach the river bank on the other side ?
Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. Here, we need to know the component of the resultant velocity in the direction perpendicular to the stream.
Relative velocity



This approach, however, would be tedious. We shall use the fact that the component of "$\mathbf{v}_A$ " in any one of the two mutually perpendicular directions is equal to the sum of the components of $\mathbf{v}_{AB}$ and $\mathbf{v}_B$ in that direction.

$$\Rightarrow v_{\text{Ay}} = v_{AB} \cos 30^0$$
$$\Rightarrow v_{\text{Ay}} = \sqrt{3} \cos 30^0 = \frac{3}{2}\, m/s$$

Thus, time taken to cross the river is :

$$t = \frac{\text{Width of the river}}{v_{\text{Ay}}}$$
$$t = \frac{\text{300x2}}{3} = 200\ s$$

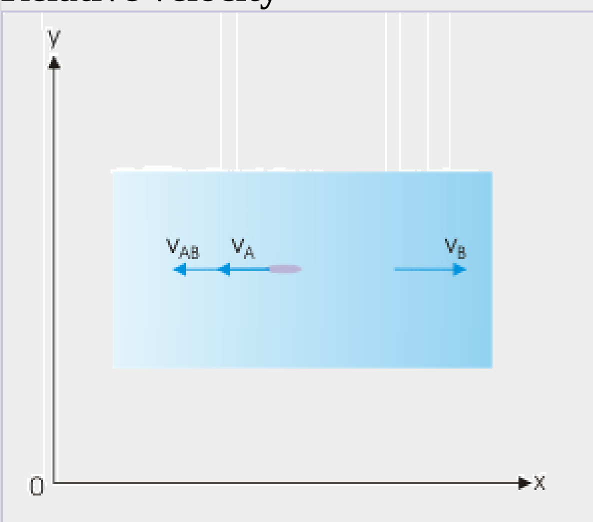
Multiple references

Example:
Problem : A boat, capable of sailing at 2 m/s, moves upstream in a river. The water stream flows at 1 m/s. A person walks from the front end to the rear end of the boat at a speed of 1 m/s along the linear direction. What is the speed of the person (m/s) with respect to ground ?

Solution : Let the direction of stream be x-direction and the direction across stream be y-direction. We further denote boat with “A”, stream with “B”, and person with “C”.
We shall work out this problem in two parts. In the first part, we shall find out the velocity of boat (A) with respect to ground and then we shall find out the velocity of person (C) with respect to ground.
Here,

$$\begin{aligned} &\text{Velocity of boat (A) with respect to stream (B)} : v_{\mathrm{AB}} = -2\, m/s \\ &\text{Velocity of the stream (A) with respect to ground} : v_A = 1\, m/s \\ &\text{Velocity of the person (C) with respect to boat (A)} : v_{\mathrm{CA}} = 1\, m/s \\ &\text{Velocity of the person (C) with respect to ground} : v_C = ? \end{aligned}$$

Relative velocity



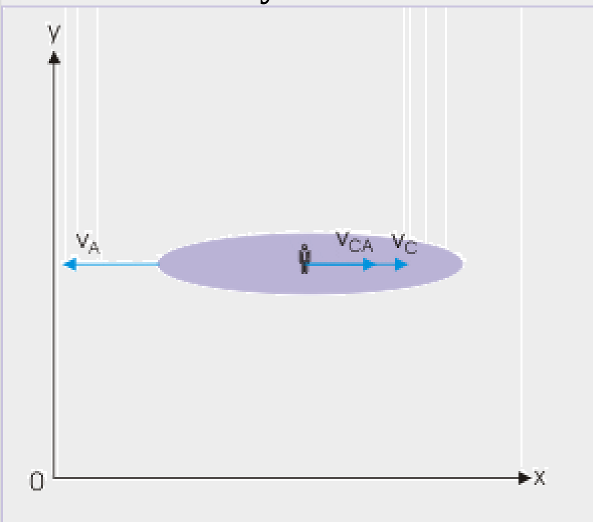
The velocity of boat with respect to ground is equal to the resultant velocity of the boat as given by :

$$\begin{aligned} &v_A = v_{AB} + v_B \\ &\Rightarrow v_A = -2 + 1 = -1\, m/s \end{aligned}$$

For the motion of person and boat, the velocity of the person with respect to ground is equal to the resultant velocity of (i) velocity of the person (C) with respect to boat (A) and (ii) velocity of the boat (A) with respect to ground. We note here that relative velocity of person with respect to boat is

given and that we have already determined the velocity of boat (A) with respect to ground in the earlier step. Hence,

Relative velocity



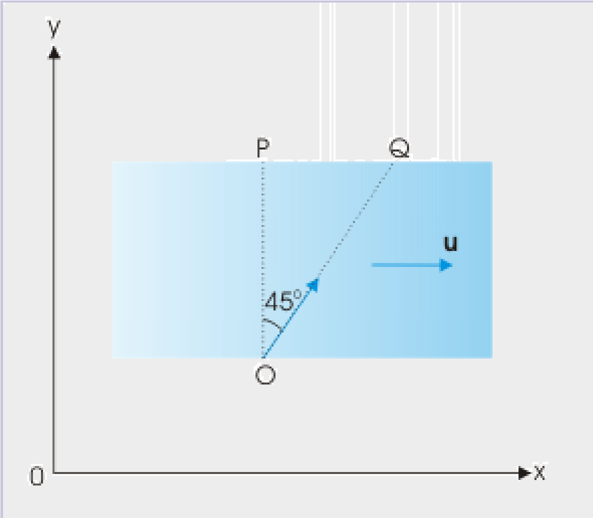
$$v_C = v_{CA} + v_A$$
$$\Rightarrow v_C = 1 + (-1) = 0$$

Minimum time, distance and speed

Example:

Problem : A boy swims to reach a point "Q" on the opposite bank, such that line joining initial and final position makes an angle of 45 with the direction perpendicular to the stream of water. If the velocity of water stream is "u", then find the minimum speed with which the boy should swim to reach his target.

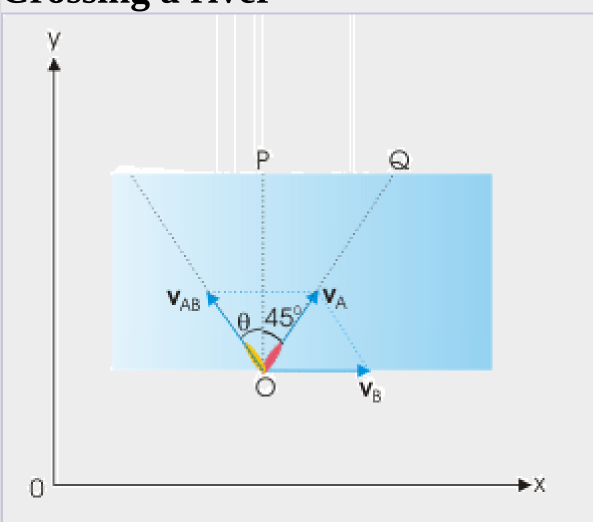
Crossing a river



The boy swims to reach point "Q".

Solution : Let "A" and "B" denote the boy and the stream respectively. Here, we are required to know the minimum speed of boy, v_{AB} (say "v") such that he reaches point "Q". Now, he can adjust his speed with the direction he swims. Let the boy swims at an angle "θ" with a speed "v".

Crossing a river



The boy swims to reach point "Q".

Looking at the figure, it can be seen that we can make use of the given angle by taking trigonometric ratio such as tangent, which will involve speed of boy in still water (v) and the speed of water stream (u). This expression may then be used to get an expression for the minimum speed as required.

The slope of resultant velocity, v_A , is :

$$\tan 45^0 = \frac{v_{Ax}}{v_{Ay}} = 1$$

$$\Rightarrow v_{Ax} = v_{Ax}$$

Now, the components of velocity in “x” and “y” directions are :

$$v_{Ax} = u - v\sin\theta$$

$$v_{Ay} = v\cos\theta$$

Putting in the equation we have :

$$u - v\sin\theta = v\cos\theta$$

Solving for “v”, we have :

$$\Rightarrow v = \frac{u}{(\sin\theta + \cos\theta)}$$

The velocity is minimum for a maximum value of denominator. The denominator is maximum for a particular value of the angle, θ; for which :

$$\frac{đ}{đ\theta}(\sin\theta + \cos\theta) = 0$$

$$\Rightarrow \cos\theta - \sin\theta = 0$$

$$\Rightarrow \tan\theta = 1$$

$$\Rightarrow \theta = 45^0$$

It means that the boy swims with minimum speed if he swims in the direction making an angle of 45 with y-direction. His speed with this angle is :

$$v = \frac{u}{\left(\sin 45^0 + \cos 45^0\right)} = \frac{\sqrt{2}u}{2} = \frac{u}{\sqrt{2}}$$

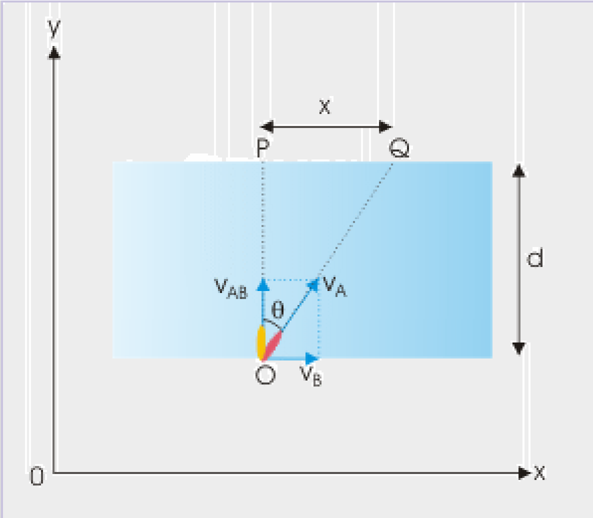
Example:

Problem : A boat crosses a river in minimum time, taking 10 minutes during which time the it drifts by 120 m in the direction of stream. On the other hand, boat takes 12.5 minutes while moving across the river. Find (i) width of the river (ii) velocity of boat in still water and (iii) speed of the stream.

Solution : There are three pieces of information about "minimum time", "drift" and "time along shortest path". Individually each of these values translate into three separate equations, which can be solved to find the required values.

The boat takes minimum time, when it sails in the direction perpendicular to the stream (current). The time to cross the river is given by dividing width with component of resultant velocity (v_{Ay}). The boat, in this case, sails in the perpendicular direction. Hence, the component of resultant velocity is equal to the velocity of boat in still water (v_{AB}). The time to cross the river in this case is :

Crossing a river



The swims towards “P”.

$$t_{\min} = \frac{d}{v_{AB}} = \frac{d}{v_{AB}} = 10$$

$$d = 10 v_{AB}$$

The drift in this time is given by :

$$x = v_B t_{\min}$$

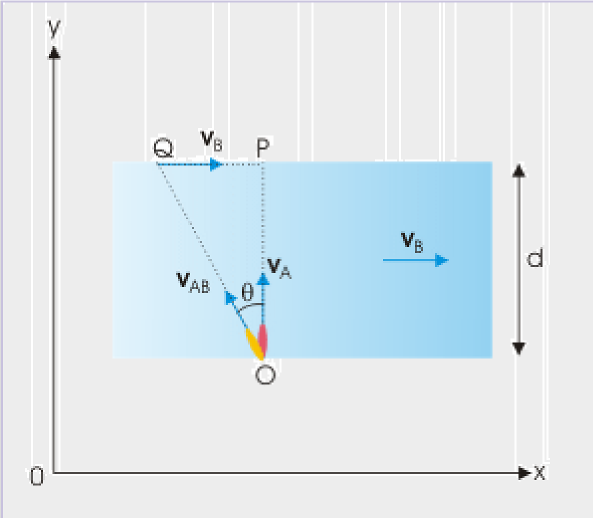
Putting values,

$$120 = v_B \quad x \quad 10$$

$$v_B = 12 \quad \text{meter / minute}$$

Now we need to use the information on shortest path. It is given that the boat moves across stream in 12.5 minutes. For this boat has to sail upstream at certain angle. The resultant speed is given by :

Crossing a river



The swims towards “P”.

$$v_A = \sqrt{v_{AB}^2 - v_B^2}$$

and the time taken is :

$$\frac{d}{\sqrt{v_{AB}^2 - v_B^2}} = 12.5$$

Substituting for “d” and “ v_B ” and squaring on both sides, we have :

$$\left(v_{AB}^2 - 12^2\right)12.5^2 = d^2 = 10^2 v_{AB}^2$$

$$v_{AB}^2\left(12.5^2 - 10^2\right) = 12^2 x 12.5^2$$

$$v_{AB} = \frac{12 \quad x \quad 12.5}{7.5} = 20 \, \text{meter} \, / \, \text{minute}$$

and

$$d = 10 v_{AB} = 10 \quad x \quad 20 = 200 \quad m$$

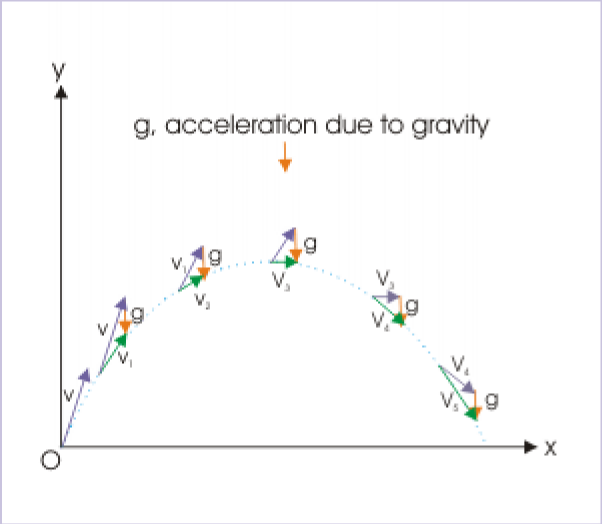
Projectile motion

Projectile motion is a special case of two dimensional motion with constant acceleration. Here, force due to gravity moderates linear motion of an object thrown at certain angle to the vertical direction. The resulting acceleration is a constant, which is always directed in vertically downward direction.

The projectile motion emphasizes one important aspect of constant acceleration that even constant acceleration, which is essentially unidirectional, is capable to produce two dimensional motion. The basic reason is that force and initial velocity of the object are not along the same direction. The linear motion of the projected object is continuously worked upon by the gravity, which results in the change of both magnitude and direction of the velocity. A change in direction of the velocity ensures that motion is not one dimensional.

The change in magnitude and direction of the velocity is beautifully managed so that time rate of change in velocity is always directed in vertically downward direction i.e. in the direction of gravity. This aspect is shown qualitatively for the motion in the figure below as velocity change successively at the end of every second from v_1 to v_2 to v_3 and so on….. by exactly a vector, whose magnitude is equal to acceleration due to gravity "g".

Projectile motion



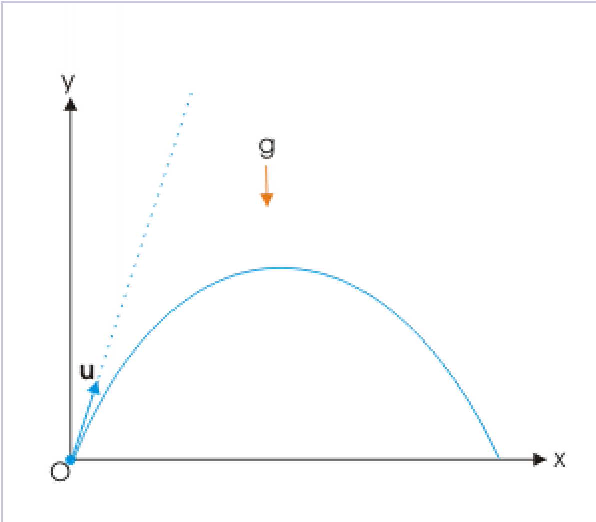
Velocity of the projectile changes by acceleration vector in unit time.

Force(s) in projectile motion

Flight of base ball, golf ball etc. are examples of projectile motion. In these cases, the projectile is projected with certain force at certain angle to vertical direction. The force that initiates motion is a contact force. Once the motion of the ball is initiated, the role of contact force is over. It does not subsequently affect or change the velocity of the ball as the contact is lost.

In order to emphasize, we restate three important facts about projectile motion. First, we need to apply force at the time of projection. This force as applied by hand or by any other mechanical device, accelerates projectile briefly till it is in contact with "thrower". The moment the projectile is physically disconnected with the throwing device, it moves with a velocity, which it gained during brief contact period. The role of force responsible for imparting motion is over. Second, motion of projectile is maintained if there is no net external force (Newton's laws of motion). This would be the case for projection in force free space. The projectile is initiated into the motion with certain initial velocity, say **u**. Had there been no other force(s), then the ball would have moved along the dotted straight line (as shown in figure below) and might have been lost in to the space.

Projectile motion



Path of a projectile projected at an angle with horizontal direction.

Third, the projectile, once out in the space, is acted upon by the force due to gravity and air resistance. We, however, neglect the effect of air resistance for the time being and confine our study of the motion which is affected by force due to gravity acting downwards. The motion or velocity of projectile is then moderated i.e. accelerated (here, acceleration means change of speed or change of direction or both) by gravity. This is the only force. Hence, acceleration due to gravity is the only acceleration involved in the motion. This downward acceleration is a constant and is the acceleration in any projectile motion near Earth, which is not propelled or dragged.

Analysis of projectile motion

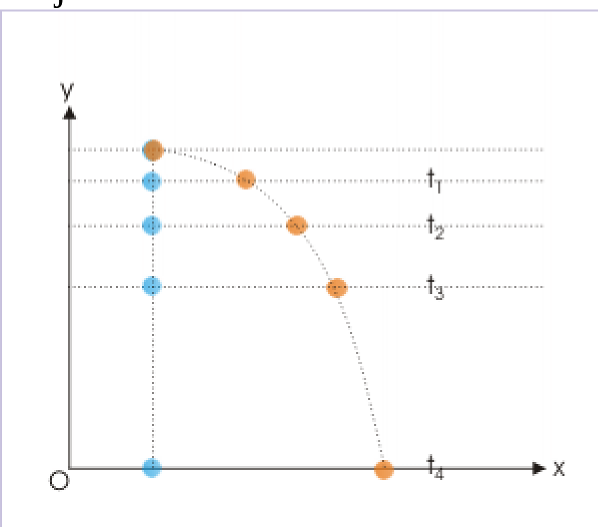
There is a very useful aspect of two dimensional motion that can be used with great effect. Two dimensional motion can be resolved in to two linear motions in two mutually perpendicular directions : (i) one along horizontal direction and (ii) the other along vertical direction. The linear motion in each direction can, then, be analyzed with the help of equivalent scalar system, in which signs of the attributes of the motion represent direction.

We can analyze the projectile motion with the help of equations of motion. As the motion occurs in two dimensions, we need to use vector equations and interpret them either graphically or algebraically as per the vector rules. We know that algebraic methods consisting of component vectors render vectors analysis in relatively simpler way. Still, vector algebra requires certain level of skills to manipulate vector components in two directions.

In the nutshell, the study of projectile motion is equivalent to two independent linear motions. This paradigm simplifies the analysis of projectile motion a great deal. Moreover, this equivalent construct is not merely a mathematical construct, but is a physically verifiable fact. The motions in vertical and horizontal directions are indeed independent of each other.

To illustrate this, let us consider the flight of two identical balls, which are initiated in motion at the same time. One ball is dropped vertically and another is projected in horizontal direction with some finite velocity from the same height. It is found that both balls reach the ground at the same time and also their elevations above the ground are same at all times during the motion.

Projectile motion



Comparing vertical and projectile motion

The fact that two balls reach the ground simultaneously and that their elevations from the ground during the motion at all times are same, point to the important aspect of the motion that vertical motion in either of the two motions are identical. This implies that the horizontal motion of the second ball does not interfere with its vertical motion. By extension, we can also say that the vertical motion of the second ball does not interfere with its horizontal motion.

Projectile motion and equations of motion

Here, we describe the projectile motion with the help of a two dimensional rectangular coordinate system such that (This not not a requirement. One can choose reference coordinate system to one's convenience):

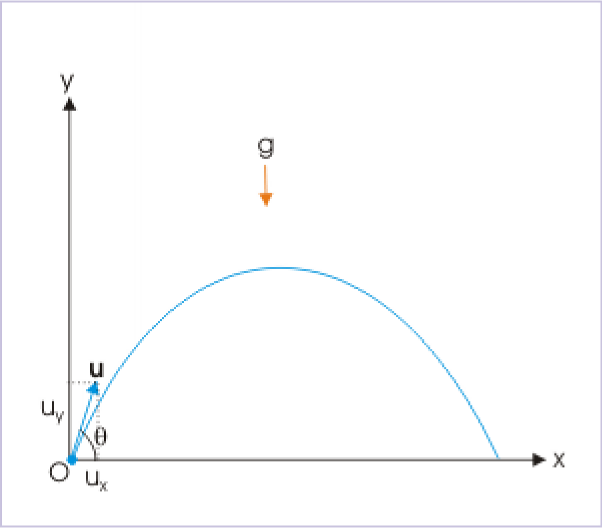
- Origin of the coordinate system coincides with point of projection.
- The x – axis represents horizontal direction.
- The y – axis represents vertical direction.

Initial velocity

Let us consider that the projectile is thrown with a velocity “u” at an angle θ from the horizontal direction as shown in the figure. The component of initial velocity in the two directions are :
Equation:

$$u_x = u\cos\theta$$
$$u_y = u\sin\theta$$

Projectile motion



Component projection velocities in x and y directions

Equations of motion in vertical direction

Motion in vertical direction is moderated by the constant force due to gravity. This motion, therefore, is described by one dimensional equations of motion for constant acceleration.

Velocity

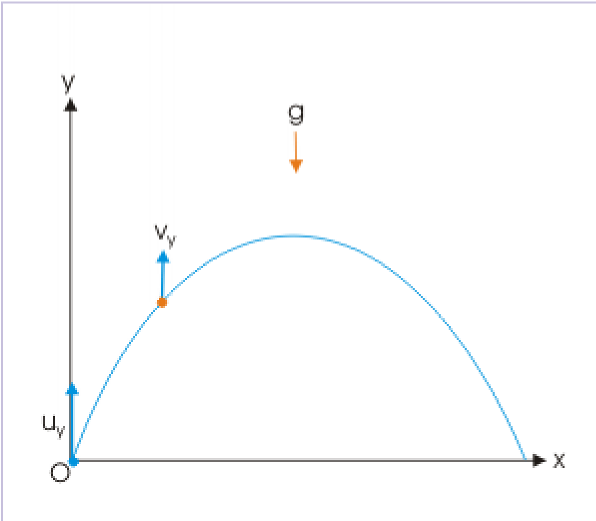
The velocity in the vertical direction is given by :
Equation:

$$v_y = u_y - gt$$

An inspection of equation - 2 reveals that this equation can be used to determine velocity in vertical direction at a given time “t” or to determine time of flight “t”, if final vertical velocity is given. This assumes

importance as we shall see that final vertical velocity at the maximum height becomes zero.

Projectile motion



Vertical component of velocity during motion

The equation for velocity further reveals that the magnitude of velocity is reduced by an amount "gt" after a time interval of "t" during upward motion. The projectile is decelerated in this part of motion (velocity and acceleration are in opposite direction). The reduction in the magnitude of velocity with time means that it becomes zero corresponding to a particular value of "t". The vertical elevation corresponding to the position, when projectile stops, is maximum height that projectile attains. For this situation ($v_y = 0$), the time of flight "t" is obtained as :

Equation:

$$v_y = u_y - gt$$
$$\Rightarrow 0 = u_y - gt$$
$$\Rightarrow t = \frac{u_y}{g}$$

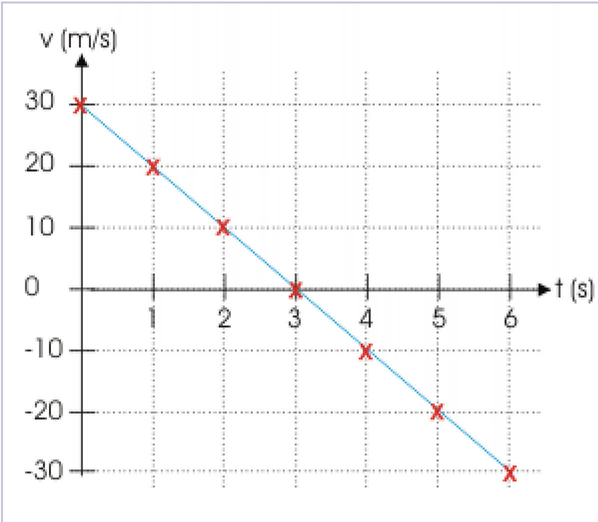
Immediately thereafter, projectile is accelerated in vertically downward direction with increasing speed. In order to appreciate variation of speed and velocity during projectile motion, we calculate the values of a projectile for successive seconds, which is projected with an initial velocity of 60 m/s making an angle of 30^0 with the horizontal. Here, vertical component of velocity is 60 sin 30^0 " i.e. 30 m/s.

```
--------------------------------------------------
------ Time gt Velocity Magnitude of velocity (s)
(m/s) (m/s) (m/s) --------------------------------
------------------------ 0 0 30 30 1 10 20 20 2 20
10 10 3 30 0 0 4 40 -10 10 5 50 -20 20 6 60 -30 30
--------------------------------------------------
------
```

Above table substantiate the observations made earlier. The magnitude of vertical velocity of the projectile first decreases during upward flight; becomes zero at maximum height; and, thereafter, picks up at the same rate during downward flight.

It is also seen from the data that each of the magnitude of vertical velocity during upward motion is regained during downward motion. In terms of velocity, for every vertical velocity there is a corresponding vertical velocity of equal magnitude, but opposite in direction.

The velocity – time plot of the motion is a straight line with negative slope. The negative slope here indicates that acceleration i.e acceleration due to gravity is directed in the opposite direction to that of positive y- direction.
Velocity – time plot



The velocity – time plot for constant acceleration in vertical direction

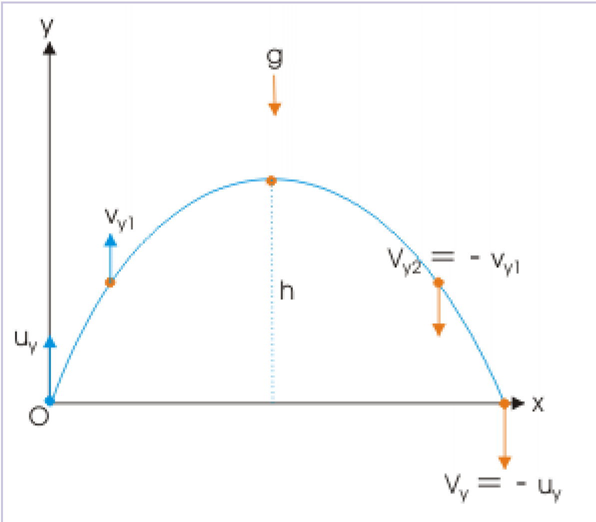
From the plot, we see that velocity is positive and acceleration is negative for upward journey, indicating deceleration i.e. decrease in speed.

The time to reach maximum height in this case is :

$$\Rightarrow t = \frac{u_y}{g} = \frac{30}{10} = 3\ s$$

The data in the table confirms this. Further, we know that vertical motion is independent of horizontal motion and time of flight for vertical motion is equal for upward and downward journey. This means that total time of flight is 2t i.e. 2x3 = 6 seconds. We must, however, be careful to emphasize that this result holds if the point of projection and point of return to the surface are on same horizontal level.

Projectile motion



Initial and final velocities are equal in magnitude but opposite in direction.

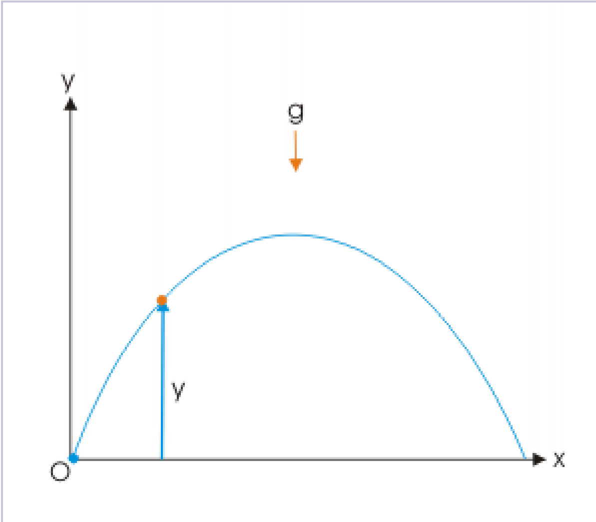
There is yet another interesting feature that can be drawn from the data set. The magnitude of vertical velocity of the projectile (30 m/s) at the time it hits the surface on return is equal to that at the time of the start of the motion. In terms of velocity, the final vertical velocity at the time of return is inverted initial velocity.

Displacement

The displacement in vertical direction is given by :
Equation:

$$y = u_y t - \frac{1}{2} g t^2$$

Projectile motion



Vertical displacement at a given time

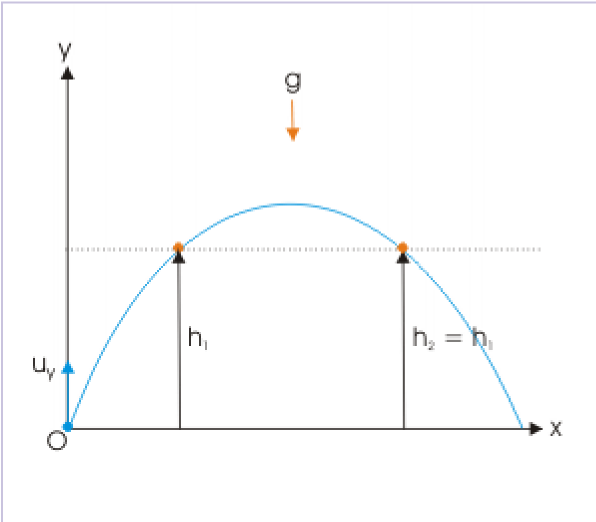
This equation gives vertical position or displacement at a given time. It is important to realize that we have simplified the equation $\Delta y = y_2 - y_1 = u_y t - \frac{1}{2} g t^2$ by selecting origin to coincide by the point of projection so that,

$$\Delta y = y_2 - y_1 = y\ (\text{say})$$

Thus, “y” with this simplification represents position or displacement.

The equation for position or displacement is a quadratic equation in time “t”. It means that solution of this equation yields two values of time for every value of vertical position or displacement. This interpretation is in fine agreement with the motion as projectile retraces all vertical displacement as shown in the figure.

Projectile motion



All vertical displacement is achieved twice by the projectile except the point of maximum height.

Time of flight

The time taken to complete the journey from the point of projection to the point of return is the time of the flight for the projectile. In case initial and final points of the journey are on the same horizontal level, then the net displacement in vertical direction is zero i.e. y = 0. This condition allows us to determine the total time of flight "T" as :

$$y = u_y T - \frac{1}{2} g T^2$$
$$\Rightarrow 0 = u_y T - \frac{1}{2} g T^2$$
$$\Rightarrow T\left(u_y - \frac{1}{2} g T\right) = 0$$
$$\Rightarrow T = 0 \text{ or } T = \frac{2u_y}{g}$$

T = 0 corresponds to the time of projection. Hence neglecting the first value, the time of flight is :

Equation:

$$\Rightarrow T = \frac{2u_y}{g}$$

We see that total time of flight is twice the time projectile takes to reach the maximum height. It means that projectile takes equal times in "up" and "down" motion. In other words, time of ascent equals time of descent.

Example:
Problem : A ball is thrown upwards with a speed of 10 m/s making an angle 30° with horizontal and returning to ground on same horizontal level. Find (i) time of flight and (ii) and time to reach the maximum height
Solution : Here, component of initial velocity in vertical direction is :

$$u_y = u \sin\theta = 10 \sin 30^\circ = 10 \text{ x } \frac{1}{2} = 5\, m/s$$

(i) The time of flight, T, is :

$$\Rightarrow T = \frac{2u_y}{g} = 2 \text{ x } \frac{5}{10} = 1\, s$$

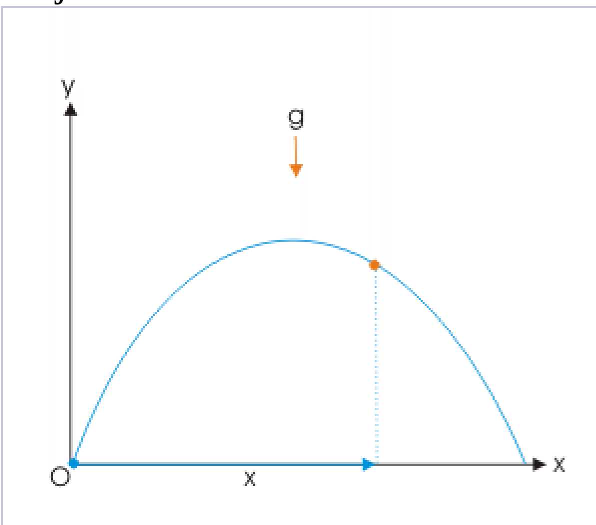
(ii) Time to reach the maximum height is half of the total flight when starting and end points of the projectile motion are at same horizontal level. Hence, the time to reach the maximum height is 0.5 s.

Equations of motion in horizontal direction

The force due to gravity has no component in horizontal direction. Since gravity is the only force acting on the projectile, this means that the motion in horizontal direction is not accelerated. Therefore, the motion in horizontal direction is an uniform motion. This implies that the component of velocity in x-direction is constant. As such, the position or displacement in x-direction at a given time “t” is :
Equation:

$$x = u_x t$$

Projectile motion



Horizontal displacement at a given time

This equation gives the value of horizontal position or displacement at any given instant.

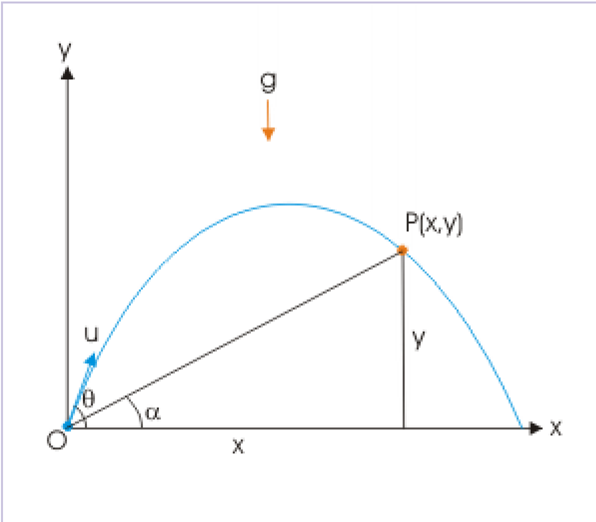
Displacement of projectile

The displacement of projectile is obtained by vector addition of displacements in x and y direction. The magnitude of displacement of the projectile from the origin at any given instant is :

Equation:

$$\text{Displacement, OP} = \sqrt{(x^2 + y^2)}$$

Displacement in projectile motion



The angle that displacement vector subtends on x-axis is :
Equation:

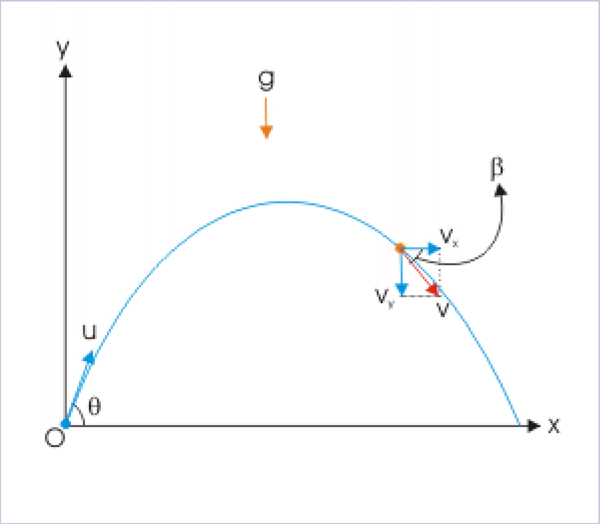
$$\tan\alpha = \frac{y}{x}$$

Velocity of projectile

The velocity of projectile is obtained by vector addition of velocities in x and y direction. Since component velocities are mutually perpendicular to each other, we can find magnitude of velocity of the projectile at any given instant, applying Pythagoras theorem :
Equation:

$$v = \sqrt{({v_x}^2 + {v_y}^2)}$$

Velocity of a projectile



The angle that the resultant velocity subtends on x-axis is :
Equation:

$$\tan\beta = \frac{v_y}{v_x}$$

Example:
Problem : A ball is projected upwards with a velocity of 60 m/s at an angle 60° to the vertical. Find the velocity of the projectile after 1 second.
Solution : In order to find velocity of the projectile, we need to know the velocity in vertical and horizontal direction. Now, initial velocities in the two directions are (Note that the angle of projection is given in relation to vertical direction.):

$$u_x = u\sin\theta = 60\sin 60^\circ = 60 \text{ x } \frac{\sqrt{3}}{2} = 30\sqrt{3}\, m/s$$
$$u_y = u\cos\theta = 60\cos 60^\circ = 60 \text{ x } \frac{1}{2} = 30\, m/s$$

Now, velocity in horizontal direction is constant as there is no component of acceleration in this direction. Hence, velocity after "1" second is :

$$v_x = u_x = 30\sqrt{3}\, m/s$$

On the other hand, the velocity in vertical direction is obtained, using equation of motion as :

$$v_y = u_y - gt$$
$$\Rightarrow v_y = 30 - 10 \text{ x } 1$$
$$\Rightarrow v_y = 20\, m/s$$

The resultant velocity, v, is given by :

$$v = \sqrt{(v_x{}^2 + v_y{}^2)}$$
$$\Rightarrow v = \sqrt{\left\{\left(30\sqrt{3}\right)^2 + (20)^2\right\}} = \sqrt{(900 \text{ x } 3 + 400)} = 55.68\, m/s$$

Equation of the path of projectile

Equation of projectile path is a relationship between “x” and “y”. The x and y – coordinates are given by equations,

$$y = u_y t - \frac{1}{2} g t^2$$
$$x = u_x t$$

Eliminating “t” from two equations, we have :
Equation:

$$y = \frac{u_y x}{u_x} - \frac{g x^2}{2 u_x{}^2}$$

For a given initial velocity and angle of projection, the equation reduces to the form of $y = Ax + Bx^2$, where A and B are constants. The equation of “y” in “x” is the equation of parabola. Hence, path of the projectile motion is a parabola. Also, putting expressions for initial velocity components $u_x = u\cos\theta$ and $u_y = u\sin\theta$, we have :
Equation:

$$\Rightarrow y = \frac{(u\sin\theta)x}{u\cos\theta} - \frac{gx^2}{2u^2\cos^2\theta}$$
$$\Rightarrow y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

Some other forms of the equation of projectile are :
Equation:

$$\Rightarrow y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}$$

Equation:

$$\Rightarrow y = x\tan\theta - \frac{gx^2\left(1+\tan^2\theta\right)}{2u^2}$$

Exercises

Exercise:

Problem:

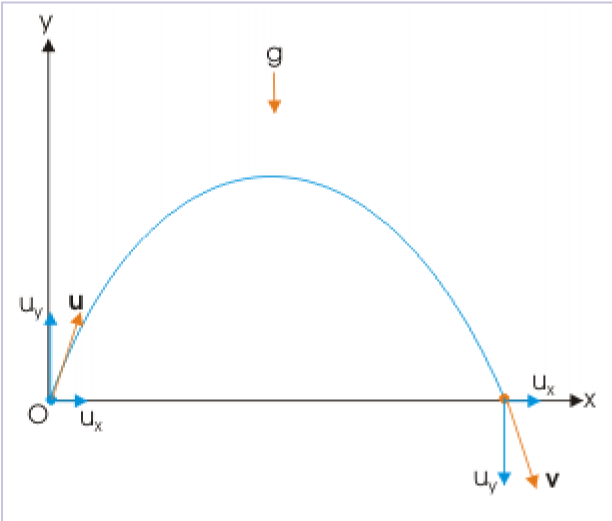
A projectile with initial velocity 2**i** + **j** is thrown in air (neglect air resistance). The velocity of the projectile before striking the ground is (consider g = 10 m/s^2) :

(a) **i** + 2**j** (b) 2**i** – **j** (c) **i** – 2**j** (d) 2**i** – 2**j**

Solution:

The vertical component of velocity of the projectile on return to the ground is equal in magnitude to the vertical component of velocity of projection, but opposite in direction. On the other hand, horizontal component of velocity remains unaltered. Hence, we can obtain velocity on the return to the ground by simply changing the sign of vertical component in the component expression of velocity of projection.

Projectile motion



Components of velocities

$\mathbf{v} = 2\mathbf{i} - \mathbf{j}$

Hence, option (b) is correct.

Exercise:

Problem:

Which of the following quantities remain unaltered during projectile motion :

(a) vertical component of velocity and vertical component of acceleration

(b) horizontal component of velocity and horizontal component of acceleration

(c) vertical component of velocity and horizontal component of acceleration

(d) horizontal component of velocity and vertical component of acceleration

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Solution:

The horizontal component of acceleration is zero. As such, horizontal component of velocity is constant. On the other hand, vertical component of acceleration is “g”, which is a constant. Clearly, vertical component of velocity is not constant.

Hence, options (b) and (d) are correct.

Exercise:

Problem:

Motion of a projectile is described in a coordinate system, where horizontal and vertical directions of the projectile correspond to x and y axes. At a given instant, the velocity of the projectile is 2**i** + 3**j** m/s. Then, we can conclude that :

(a) the projectile has just started its motion

(b) the projectile is about to hit the ground

(c) the projectile is descending from the maximum height

(d) the projectile is ascending to the maximum height

Solution:

Here, the vertical component of the velocity (3 m/s) is positive. Therefore, projectile is moving in positive y-direction. It means that the projectile is still ascending to reach the maximum height (the vertical component of velocity at maximum height is zero). It is though possible that the given velocity is initial velocity of projection. However, the same can not be concluded.

Hence, option (d) is correct.

Exercise:

Problem:

A projectile, thrown at angle " θ " with an initial velocity "u", returns to the same horizontal ground level. If "x" and "y" coordinates are in horizontal and vertical directions, the equation of projectile in x and y coordinates has the form :

$$(a) \quad x = Ay - By^2$$

$$(b) \quad y = Ax - Bx^2$$

$$(c) \quad (1 - A)x = By^2$$

$$(d) \quad x = (A + B)y^2$$

Solution:

The equation of projectile is given as :

$$y = x \tan\theta - \frac{gx^2}{u^2 \cos^2\theta}$$

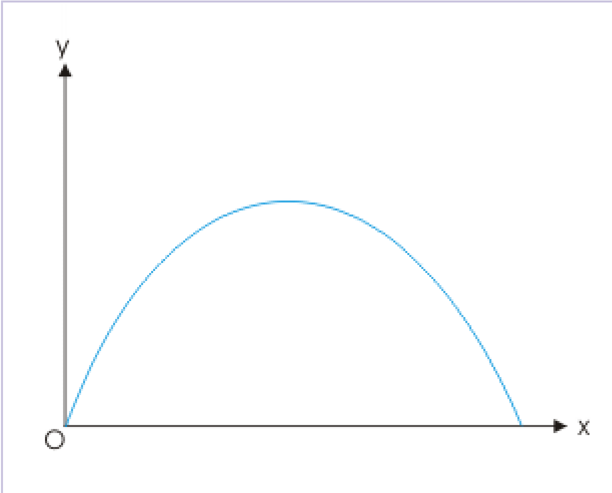
This is a quadratic equation in x. The correct choice is (b).

Exercise:

Problem:

Select correct observation(s) about the "xy" - plot of the projectile motion from the following :

Projectile motion



Projectile motion

(a) it covers greater horizontal distance during the middle part of the motion.

(b) it covers greater vertical distance during the middle part of the motion.

(c) it covers lesser horizontal distance near the ground.

(d) it covers greater vertical distance near the ground.

Solution:

There is no component of acceleration in horizontal direction. The motion in this direction is a uniform motion and, therefore, covers equal horizontal distance in all parts of motion. It means that options (a) and (c) are incorrect.

In the vertical direction, projectile covers maximum distance when vertical component of velocity is greater. Now, projectile has greater vertical component of velocity near ground at the time of projection and at the time of return. As such, it covers maximum distance near the ground.

The correct choice is (d).

Exercise:

Problem:

A projectile is projected with an initial speed "u", making an angle " θ " to the horizontal direction along x-axis. Determine the average velocity of the projectile for the complete motion till it returns to the same horizontal plane.

Solution:

Average velocity is defined as the ratio of displacement and time. Since we treat projectile motion as two dimensional motion, we can find average velocity in two mutually perpendicualr directions and then find the resultant average velocity. Projectile motion, however, is a unique case of uniform acceleration and we can find components of average velcity by averaging initial and final values.

If v_1 and v_2 be the initial and final velocities, then the average velocity for linear motion under constant acceleration is defined as :

$$v_{avg} = \frac{v_1 + v_2}{2}$$

Employing this relation in x-direction and making use of the fact that motion in horizontal direction is uniform motion, we have :

$$\Rightarrow v_{avgx} = \frac{u_x + v_x}{2}$$

$$\Rightarrow v_{avgx} = \frac{u\cos\theta + u\cos\theta}{2} = u\cos\theta$$

Similarly, applying the relation of average velocity in y-direction and making use of the fact that component of velocity in vertical direction reverses its direction on return, we have :

$$\Rightarrow v_{avgy} = \frac{u_y + v_y}{2}$$

$$\Rightarrow v_{avgy} = \frac{u \sin\theta - u \sin\theta}{2} = 0$$

Hence, the resultant average velocity is :

$$\Rightarrow v_{avg} = v_{avgx} = u \cos\theta$$

Exercise:

Problem:

A projectile is projected with an initial speed "u", making an angle "θ " to the horizontal direction along x-axis. Determine change in speed of the projectile for the complete motion till it returns to the same horizontal plane.

Solution:

Both initial and final speeds are equal. Hence, there is no change in speed during the motion of projectile.

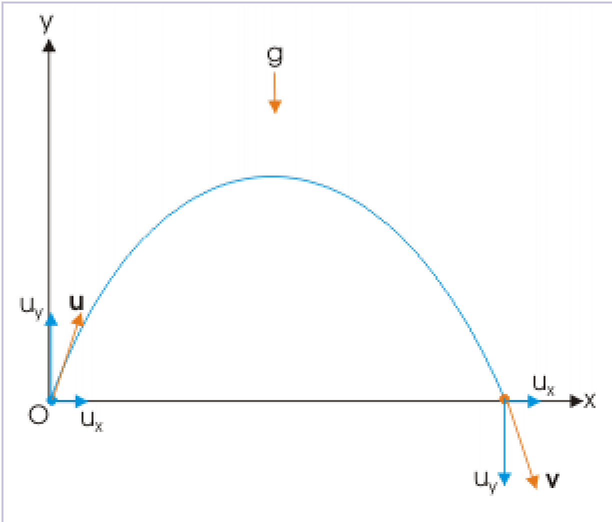
Exercise:

Problem:

A projectile is projected with an initial speed "u", making an angle "θ" to the horizontal direction along x-axis. Determine the change in velocity of the projectile for the complete motion till it returns to the same horizontal plane.

Solution:

Initial velocity is :
Projectile motion



Components of velocities

$$\mathbf{u} = u\cos\theta\mathbf{i} + u\sin\theta\mathbf{j}$$

Final velocity is :

$$\mathbf{v} = u\cos\theta\mathbf{i} - u\sin\theta\mathbf{j}$$

Change in velocity is given by :

$$\Rightarrow \Delta\mathbf{v} = u\cos\theta\mathbf{i} - u\sin\theta\mathbf{j} - u\cos\theta\mathbf{i} - u\sin\theta\mathbf{j}$$

$$\Rightarrow \Delta\mathbf{v} = -2u\sin\theta\mathbf{j}$$

where **j** is unit vector in y-direction.

Exercise:

Problem:

A projectile is projected at 60° to the horizontal with a speed of 10 m/s. After some time, it forms an angle 30° with the horizontal. Determine the speed (m/s) at this instant.

Solution:

We shall make use of the fact that horizontal component of projectile velocity does not change with time. Therefore, we can equate horizontal components of velocities at the time of projection and at the given instants and find out the speed at the later instant as required. Then,

$$u_x = v_x$$

$$\Rightarrow u\cos 60^0 = v\cos 30^0$$

$$\Rightarrow \frac{10}{2} = \frac{\sqrt{3}v}{2}$$

$$\Rightarrow v = \frac{10}{\sqrt{3}} \quad m/s$$

Exercise:

Problem:

The horizontal and vertical components of a projectile at a given instant after projection are v_x and v_y respectively at a position specified as x (horizontal),y (vertical). Then,

(a) The "x - t" plot is a straight line passing through origin.

(b) The "y - t" plot is a straight line passing through origin.

(c) The "$v_x - t$ " plot is a straight line passing through origin.

(d) The "$v_y - t$ " plot is a straight line.

Solution:

The displacement in x-direction is given as :

$$x = v_x t$$

Since v_x is a constant, the "x-t" plot is a straight line passing through origin. Hence, option (a) is correct.

The displacement in y-direction is :

$$y = u_y t - \frac{1}{2} g t^2$$

This is an equation of parabola. The option (b), therefore, is incorrect.

The motion in horizontal direction is uniform motion. This means that component of velocity in x-direction is a constant. Therefore, "$v_x - t$ ” plot should be a straight line parallel to time axis. It does not pass through origin. Hence, option (c) is incorrect.

The motion in vertical direction has acceleration due to gravity in downward direction. The component of velocity in y-direction is :

$$v_y = u_y - gt$$

This is an equation of straight line having slope of "-g" and intercept " u_y ". The "$v_y - t$ ” plot, therefore, is a straight line. Hence, option (d) is correct.

Projectile motion (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

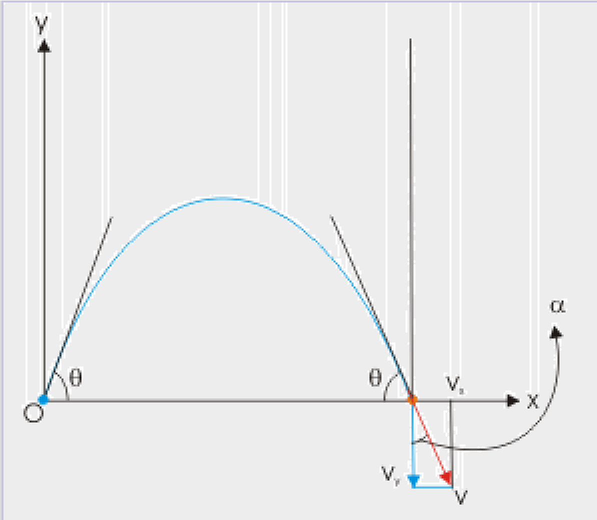
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the projectile motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Direction of motion on return
- Maximum height
- Equation of projectile motion
- Change in angles during motion
- Kinetic energy of a projectile
- Change in the direction of velocity vector

Direction of motion on return

Example:
Problem : A projectile is thrown with a speed of 15 m/s making an angle 60° with horizontal. Find the acute angle, "α", that it makes with the vertical at the time of its return on the ground (consider g = 10 m/s^2).
Solution : The vertical component of velocity of the projectile at the return on the ground is equal in magnitude, but opposite in direction. On the other hand, horizontal component of velocity remains unaltered. The figure, here, shows the acute angle that the velocity vector makes with vertical.
Projectile motion



The trajectory is symmetric about the vertical line passing through point of maximum height. From the figure, the acute angle with vertical is :

$$\alpha = 90^0 - \theta = 90^0 - 60^0 = 30^0$$

Maximum height

Example:
Problem : Motion of a projectile is described in a coordinate system, where horizontal and vertical directions of the projectile correspond to x and y axes. The velocity of the projectile is 12**i** + 20**j** m/s at an elevation of 15 m from the point of projection. Find the maximum height attained by the projectile (consider g = 10 m/s^2).
Solution : Here, the vertical component of the velocity (20 m/s) is positive. It means that it is directed in positive y-direction and that the projectile is still ascending to reach the maximum height. The time to reach the maximum height is obtained using equation of motion in vertical direction :

$$v_y = u_y - gt$$

$$\Rightarrow 0 = 20 - 10t$$

$$\Rightarrow t = 2s$$

Now, the particle shall rise to a vertical displacement given by :

$$y^{`} = u_y t - \frac{1}{2} g t^2 = 20 \text{ x } 2 - 5 \text{ x } 2^2 = 20\, m$$

The maximum height, as measured from the ground, is :

$$H = 15 + 20 = 35 \;\; m$$

Equation of projectile motion

Example:
Problem : The equation of a projectile is given as :

$$y = \sqrt{3} x - \frac{1}{2} g x^2$$

Then, find the speed of the projection.
Solution : The general equation of projectile is :

$$y = x \tan \theta - \frac{g x^2}{2 u^2 \cos^2 \theta}$$

On the other hand, the given equation is :

$$y = \sqrt{3} x - \frac{1}{2} g x^2$$

Comparing two equations, we have :

$$\tan \theta = \sqrt{3}$$
$$\Rightarrow \theta = 60^0$$

Also,

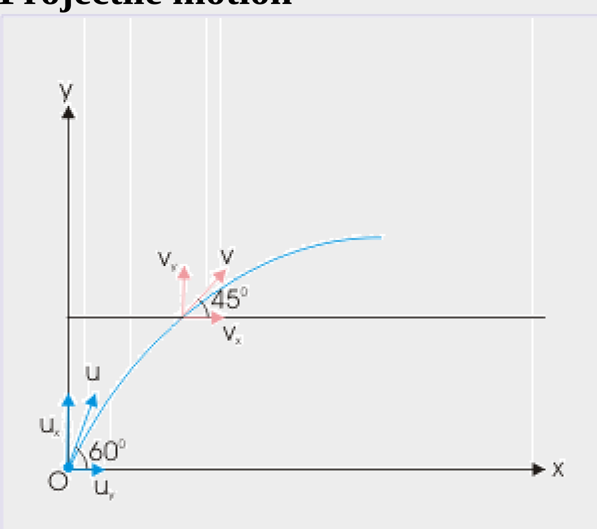
$$u^2\cos^2\theta = 1$$
$$\Rightarrow u^2 = \frac{1}{\cos^2\theta}$$
$$\Rightarrow u^2 = \frac{1}{\cos^2 60} = 4$$
$$\Rightarrow u = 2\,m/s$$

Change in angles during motion

Example:
Problem : A projectile is projected at an angle 60° from the horizontal with a speed of ($\sqrt{3}+1$) m/s. The time (in seconds) after which the inclination of the projectile with horizontal becomes 45° is :
Solution : Let "u" and "v" be the speed at the two specified angles. The initial components of velocities in horizontal and vertical directions are :

$$u_x = u\cos 60^0$$
$$u_y = u\sin 60^0$$

Projectile motion



Similarly, the components of velocities, when projectile makes an angle 45 with horizontal, in horizontal and vertical directions are :

$$v_x = v\cos 45^0$$
$$v_y = v\sin 45^0$$

But, we know that horizontal component of velocity remains unaltered during motion. Hence,

$$v_x = u_x$$
$$\Rightarrow v\cos 45^0 = u\cos 60^0$$
$$\Rightarrow v = \frac{u\cos 60^0}{\cos 45^0}$$

Here, we know initial and final velocities in vertical direction. We can apply v = u +at in vertical direction to know the time as required :

$$v\sin 45^0 = u + at = u\sin 60^0 - gt$$
$$\Rightarrow v\cos 45^0 = u\cos 60^0$$
$$\Rightarrow t = \frac{u\sin 60^0 - v\sin 45^0}{g}$$

Substituting value of "v" in the equation, we have :

$$\Rightarrow t = \frac{u\sin 60^0 - u\left(\frac{\cos 60^0}{\cos 45^0}\right)\mathrm{X}\sin 45^0}{g}$$
$$\Rightarrow t = \frac{u}{g}\left(\sin 60^0 - \cos 60^0\right)$$
$$\Rightarrow t = \frac{\left(\sqrt{3}+1\right)}{10}\left\{\frac{\left(\sqrt{3}-1\right)}{2}\right)$$
$$\Rightarrow t = \frac{2}{20} = 0.1\ s$$

Kinetic energy of a projectile

Example:

Problem : A projectile is thrown with an angle θ from the horizontal with a kinetic energy of K Joule. Find the kinetic energy of the projectile (in Joule), when it reaches maximum height.

Solution : At the time of projection, the kinetic energy is given by :

$$K = \frac{1}{2} m u^2$$

At the maximum height, vertical component of the velocity is zero. On the other hand, horizontal component of the velocity of the particle does not change. Thus, the speed of the particle, at the maximum height, is equal to the magnitude of the horizontal component of velocity. Hence, speed of the projectile at maximum height is :

$$v = u \cos \theta$$

The kinetic energy at the maximum height, therefore, is :

$$K^{‘} = \frac{1}{2} m (u \cos \theta)^2$$

Substituting value of "u" from the expression of initial kinetic energy is :

$$K^{‘} = \frac{m \text{x} 2 \text{x} K}{2m} \cos^2 \theta$$

$$K^{‘} = K \cos^2 \theta$$

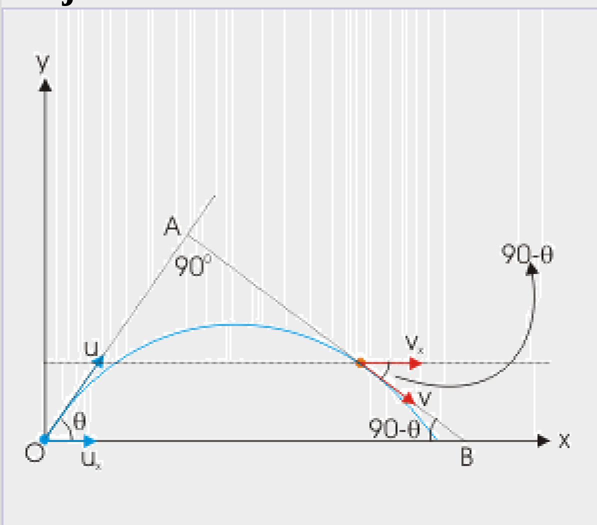
Change in the direction of velocity vector

Example:

Problem : A projectile with a speed of “u” is thrown at an angle of “θ” with the horizontal. Find the speed (in m/s) of the projectile, when it is perpendicular to the direction of projection.

Solution : We need to visualize the direction of the projectile, when its direction is perpendicular to the direction of projection. Further, we may look to determine the direction of velocity in that situation.
The figure, here, shows the direction of velocity for the condition, when the direction of projectile is perpendicular to the direction of projection.
From ΔOAB,

Projectile motion



$$\angle \mathrm{OBA} = 180^0 - \left(90^0 + \theta\right) = 90^0 - \theta$$

Thus, the acute angle between projectile and horizontal direction is 90- θ for the given condition. Now, in order to determine the speed, we use the fact that horizontal component of velocity does not change.

$$v \cos\left(90^0 - \theta\right) = u \cos\theta$$
$$\Rightarrow v \sin\theta = u \cos\theta$$
$$\Rightarrow v = u \cot\theta$$

Features of projectile motion

The span of projectile motion in the vertical plane is determined by two factors, namely the speed of projection and angle of projection with respect to horizontal. These two factors together determine (i) how long does the projectile remain in air (time of flight, T) (ii) how far does the projectile go in the horizontal direction (range of projectile, R) and (iii) how high does the projectile reach (maximum height, H).

Further, the trajectory of the projectile is symmetric about a vertical line passing through the point of maximum height if point of projection and point of return fall on the same horizontal surface.

Time of flight, T

We have already determined the time of flight, which is given by :
Equation:

$$T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}$$

This equation was derived in the earlier module Projectile motion with the assumption that both point of projection and point of return of the projectile lie on same horizontal level. It may be also be recalled that the equation of motion in vertical direction was evaluated for the condition that net displacement during the entire motion is zero. Hence, if the points are not on the same level, then above equation will not be valid and must be determined by equation of motion for the individual case with appropriate values.

From the above equation, we see that time of flight depends on initial speed and the angle of projection (θ). We must realize here that the range of θ is 0° ≤ θ ≤ 90°. For this range, sinθ is an increasing function. As such, we can say that a projection closer to vertical direction stays longer in the air for a given initial velocity. As a matter of fact, a vertical projectile for which θ = 90° and sinθ = 1, stays in the air for the maximum period.

Exercise:

Problem:

If the points of projection and return are on same level and air resistance is neglected, which of the following quantities will enable determination of the total time of flight (T) :

(a) horizontal component of projection velocity

(b) projection speed and angle of projection

(c) vertical component of projection velocity

(d) speed at the highest point

Solution:

The total time of flight is given by :

$$T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}$$

We can see that the total time of flight can be determined if vertical component of the velocity (u_y) is given. Hence option (c) is correct. The vertical component of the velocity (u_y) , in turn, is determined by the projection speed (u) and angle of projection (θ). Hence option (b) is correct.

Now speed at the highest point is equal to the horizontal component of projection velocity (u cosθ). We can not, however, determine vertical component (u sinθ) from this value, unless either "u" or "θ " is also given.

Hence, options (b) and (c) are correct.

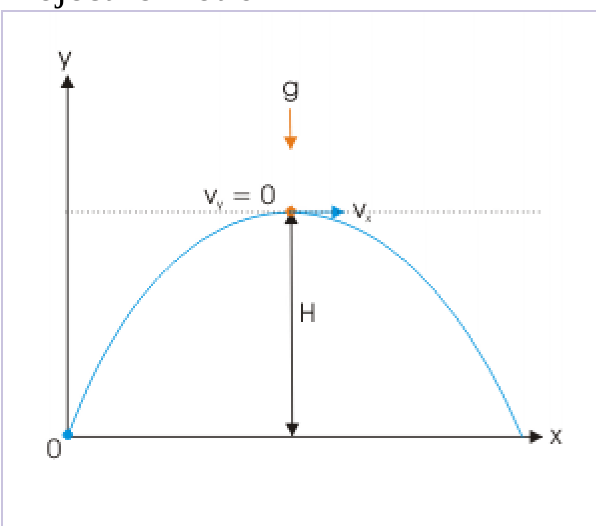
Note: We have noticed that time of flight is derived considering vertical motion. Horizontal part of the motion is not considered. Thus, the time of flight (t) at any point during the projectile motion is

dependent on vertical component of velocity or vertical part of the motion and is independent of horizontal part of the motion.

Maximum height reached by the projectile, H

The vertical component velocity of the projectile reduces to zero as motion decelerates while going up against the force due to gravity. The point corresponding to this situation, when vertical component of velocity is zero, $v_y = 0$, is the maximum height that a projectile can reach. The projectile is accelerated downward under gravity immediately thereafter. Now, considering upward motion in y-direction,

Projectile motion



Maximum height attained by projectile

$$v_y = u_y - gt$$
$$\Rightarrow 0 = u_y - gt$$
$$\Rightarrow \mathrm{t} = \frac{u_y}{g}$$

Now, using equation for displacement in the vertical direction, we can find out the vertical displacement i.e. the height as :

$$y = u_y t - \frac{1}{2} g t^2$$
$$\Rightarrow H = u_y \frac{u_y}{g} - \frac{1}{2} \frac{u_y}{g}^2$$
$$\Rightarrow H = \frac{{u_y}^2}{g} - \frac{{u_y}^2}{2g}$$
$$\Rightarrow H = \frac{{u_y}^2}{2g}$$

Putting expression of component velocity ($u_y = u \sin\theta$), we have :

Equation:

$$\Rightarrow H = \frac{u^2 \sin^2\theta}{2\text{g}}$$

We can also obtain this expression, using relation ${v_y}^2 = {u_y}^2 + 2gy$. Just like the case of the time of flight, we see that maximum height reached by the projectile depends on both initial speed and the angle of projection (θ). Greater the initial velocity and greater the angle of projection from horizontal direction, greater is the height attained by the projectile.

It is important to realize here that there is no role of the horizontal component of initial velocity as far as maximum height is concerned. It is logical also. The height attained by the projectile is purely a vertical displacement; and as motions in the two mutually perpendicular directions are independent of each other, it follows that the maximum height attained by the projectile is completely determined by the vertical component of the projection velocity.

Exercise:

Problem:

The maximum height that a projectile, thrown with an initial speed "v_0 ", can reach :

$$(a)\frac{v_0}{2g} \quad (b)\frac{v_0^2}{2g} \quad (c)\frac{v_0^2}{g} \quad (d)\frac{v_0}{g}$$

Solution:

The question only specifies speed of projection - not the angle of projection. Now, projectile rises to greatest maximum height for a given speed, when it is thrown vertically. In this case, vertical component of velocity is equal to the speed of projection itself. Further, the speed of the projectile is zero at the maximum height. Using equation of motion, we have :

$$0 = v_0^2 - 2gH$$

$$\Rightarrow H = \frac{v_0^2}{2g}$$

The assumption for the maximum height as outlined above can also be verified from the general formula of maximum height of projectile as given here :

$$H = \frac{u_y^2}{2g} = \frac{u^2 \sin^2\theta}{2g} = \frac{v_0^2 \sin^2\theta}{2g}$$

The numerator of the above is maximum when the angle is 90°.

$$\Rightarrow H = \frac{v_0^2}{2g}$$

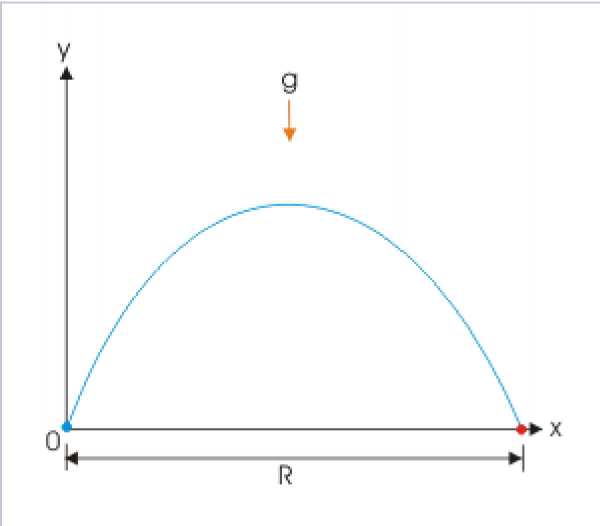
We should note that the formula of maximum height for a projectile projected at certain angle represents the maximum height of the projectile for the given angle of projection and spped. The question here, however, refers to maximum height for any angle of projection at given speed of projection. As such, we should consider an angle of projection for which projectile reaches the greatest height. This point should be kept in mind.

Hence, option (b) is correct.

Range of projectile, R

The horizontal range is the displacement in horizontal direction. There is no acceleration involved in this direction. Motion is an uniform motion. It follows that horizontal range is transversed with the horizontal component of the projection velocity for the time of flight (T). Now,

Projectile motion



The range of a projectile

$$x = u_x t$$
$$\Rightarrow R = u_x T = u \cos\theta \text{ x } T$$

where "T" is the time of flight. Putting expression for the time of flight,

$$\Rightarrow R = \frac{u \cos\theta \text{x} 2u \sin\theta}{g}$$

Equation:

$$\Rightarrow R = \frac{u^2 \sin 2\theta}{g}$$

Horizontal range, like time of flight and maximum height, is greater for greater projection speed.

For projection above ground surface, the range of the angle of projection with respect to horizontal direction, θ, is 0° ≤ θ ≤ 90° and the corresponding range of 2θ is 0° ≤ 2θ ≤ 180°. The "sin 2θ", as appearing in the numerator for the expression of the horizontal range, is an increasing function for 0° ≤ θ ≤ 45° and a decreasing function for 45° ≤ θ ≤ 90° . For θ = 45°, sin 2θ = sin 90° = 1 (maximum).

In the nutshell, the range,R, increases with increasing angle of projection for 0° ≤ θ < 45°; the range,R, is maximum when θ = 45°; the range,R, decreases with increasing angle of projection for 45° < θ ≤ 90°.

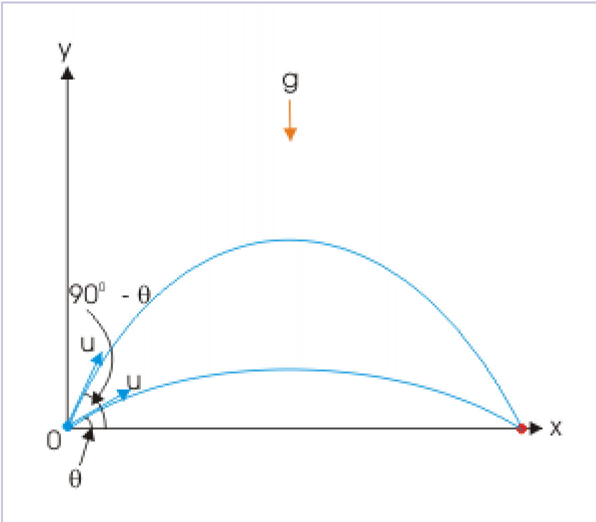
The maximum horizontal range for a given projection velocity is obtained for θ = 45° as :

$$\Rightarrow R = \frac{u^2}{g}$$

There is an interesting aspect of "sin 2θ" function that its value repeats for component angle i.e (90° – θ). For, any value of θ,

$$\sin 2\left(90^0 - \theta\right) = \sin\left(180^0 - 2\theta\right) = \sin 2\theta$$

Horizontal range



Horizontal range is same for a pair of projection angle.

It means that the range of the projectile with a given initial velocity is same for a pair of projection angles θ and 90° – θ. For example, if the range of the projectile with a given initial velocity is 30 m for an angle of projection, θ = 15°, then the range for angle of projection, θ = 90° - 15° = 75° is also 30m.

Equation of projectile motion and range of projectile

Equation of projectile motion renders to few additional forms in terms of characteristic features of projectile motion. One such relation incorporates range of projectile (R) in the expression. The equation of projectile motion is :

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

The range of projectile is :

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 2 \sin\theta \cos\theta}{g}$$

Solving for u^2 ,

$$\Rightarrow u^2 = \frac{Rg}{2\sin\theta\cos\theta}$$

Substituting in the equation of motion, we have :

$$\Rightarrow y = x\tan\theta - \frac{2\sin\theta\cos\theta g x^2}{2Rg\cos^2\theta}$$

$$\Rightarrow y = x\tan\theta - \frac{\tan\theta x^2}{R}$$

$$\Rightarrow y = x\tan\theta\left(1 - \frac{x}{R}\right)$$

It is a relatively simplified form of equation of projectile motion. Further, we note that a new variable “R” is introduced in place of “u”.

Exercise:

Problem:

If points of projection and return are on same level and air resistance is neglected, which of the following quantities will enable determination of the range of the projectile (R) :

(a) horizontal component of projection velocity

(b) projection speed and angle of projection

(c) vertical component of projection velocity

(d) speed at the highest point

Solution:

The horizontal range is determined using formulae,

$$R = \frac{u^2 \sin 2\theta}{g}$$

Hence, horizontal range of the projectile can be determined when projection speed and angle of projection are given. The inputs required in this equation can not be made available with other given quantities.

Hence, option (b) is correct.

Note : Horizontal range (R) unlike time of flight (T) and maximum height (H), depends on both vertical and horizontal motion. This aspect is actually concealed in the term "sin2θ". The formula of horizontal range (R) consists of both "u sinθ " (for vertical motion) and "u cosθ " (for horizontal motion) as shown here :

$$\Rightarrow R = \frac{u^2 \sin 2\theta}{g} = \frac{2u^2 \sin\theta \cos\theta}{g} = \frac{2(u\cos\theta)(u\sin\theta)}{g}$$

$$\Rightarrow R = \frac{2u_x u_y}{g}$$

Exercise:

Problem:

A projectile is thrown with a velocity $6\mathbf{i} + 20\mathbf{j} \quad m/s$. Then, the range of the projectile (R) is :

$$(a) 12 \quad m \quad (b) \quad 20 \quad m \quad (c) 24 \quad m \quad (d) 26 \quad m$$

Solution:

We shall not use the standard formulae as it would be difficult to evaluate angle of projection from the given data. Now, the range of the projectile (R) is given by :

$$R = u_x T$$

Here,

$$u_x = 6 \quad m/s$$

We need to know the total time of flight, T. For motion in vertical direction, the vertical displacement is zero. This consideration gives the time of flight as :

$$T = \frac{2u_y}{g} = \frac{2X20}{10} = 4 \quad s$$

Hence, range of the flight is :

$$\Rightarrow R = u_x T = 6X4 = 24$$

Hence, option (c) is correct.

Impact of air resistance

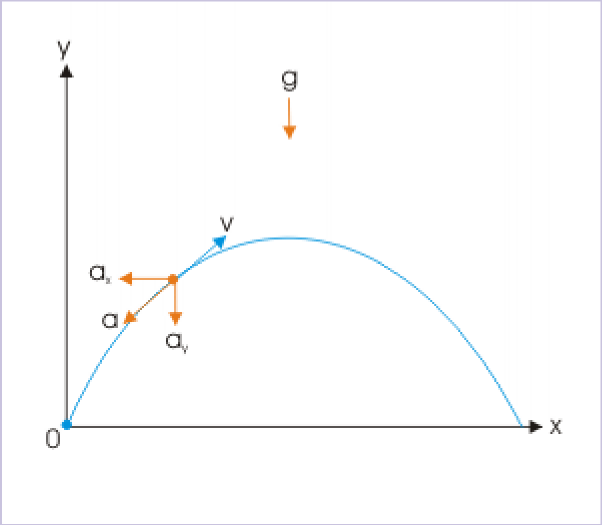
We have so far neglected the effect of air resistance. It is imperative that if air resistance is significant then the features of a projectile motion like time of flight, maximum height and range are modified. As a matter of fact, this is the case in reality. The resulting motion is generally adversely affected as far as time of flight, maximum height and the range of the projectile are concerned.

Air resistance is equivalent to friction force for solid (projectile) and fluid (air) interface. Like friction, air resistance is self adjusting in certain ways. It adjusts to the relative speed of the projectile. Generally, greater the speed greater is air resistance. Air resistance also adjusts to the direction of motion such that its direction is opposite to the direction of relative velocity of two entities. In the nutshell, air resistance opposes motion and is

equivalent to introducing a variable acceleration (resistance varies with the velocity in question) in the direction opposite to that of velocity.

For simplicity, if we consider that resistance is constant, then the vertical component of acceleration (a_y) due to resistance acts in downward direction during upward motion and adds to the acceleration due to gravity. On the other hand, vertical component of air resistance acts in upward direction during downward motion and negates to the acceleration due to gravity. Whereas the horizontal component of acceleration due to air resistance (a_x)changes the otherwise uniform motion in horizontal direction to a decelerated motion.

Projectile motion with air resistance



Acceleration due to air resistance

With air resistance, the net or resultant acceleration in y direction depends on the direction of motion. During upward motion, the net or resultant vertical acceleration is "$-g - a_y$ ". Evidently, greater vertical acceleration acting downward reduces speed of the particle at a greater rate. This, in turn, reduces maximum height. During downward motion, the net or resultant vertical acceleration is "$-g + a_y$ ". Evidently, lesser vertical acceleration acting downward increases speed of the particle at a slower

rate. Clearly, accelerations of the projectile are not equal in upward and downward motions. As a result, projection velocity and the velocity of return are not equal.

On the other hand, the acceleration in x direction is "$-a_x$ ". Clearly, the introduction of horizontal acceleration opposite to velocity reduces the range of the projectile (R).

Situations involving projectile motion

There are classic situations relating to the projectile motion, which needs to be handled with appropriate analysis. We have quite a few ways to deal with a particular situation. It is actually the nature of problem that would determine a specific approach from the following :

1. We may approach a situation analyzing as two mutually perpendicular linear motions. This is the basic approach.
2. In certain cases, the equations obtained for the specific attributes of the projectile motion such as range or maximum height can be applied directly.
3. In some cases, we would be required to apply the equation of path, which involves displacements in two directions (x and y) simultaneously in one equation.
4. We always have the option to use composite vector form of equations for two dimensional motion, where unit vectors are involved.

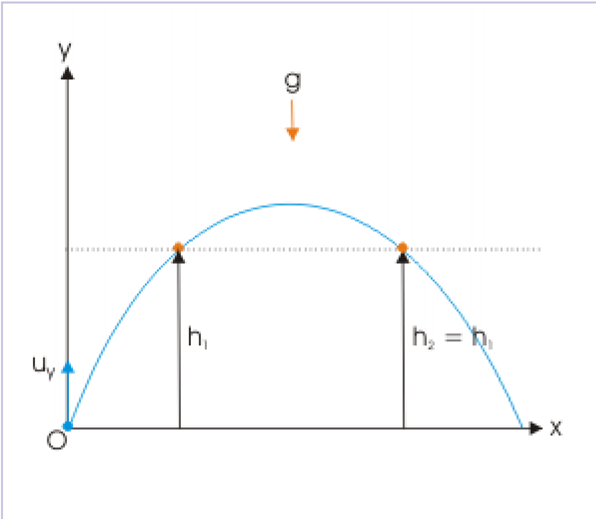
Besides, we may require combination of approached as listed above. In this section, we shall study these classic situations involving projectile motion.

Clearing posts of equal height

A projectile can clear posts of equal height, as projectile retraces vertical displacement attained during upward flight while going down. We can approach such situation in two alternative ways. The equation of motion for displacement yields two values for time for a given vertical displacement (height) : one corresponds to the time for upward flight and other for the

downward flight as shown in the figure below. Corresponding to these two time values, we determine two values of horizontal displacement (x).

Projectile motion



The projectile retraces vertical displacement.

Alternatively, we may use equation of trajectory of the projectile. The y coordinate has a quadratic equation in "x". it again gives two values of "x" for every value of "y".

Example:

Problem : A projectile is thrown with a velocity of $25\sqrt{2}$ m/s and at an angle 45° with the horizontal. The projectile just clears two posts of height 30 m each. Find (i) the position of throw on the ground from the posts and (ii) separation between the posts.

Solution : Here, we first use the equation of displacement for the given height in the vertical direction to find the values of time when projectile reaches the specified height. The equation of displacement in vertical direction (y) under constant acceleration is a quadratic equation in time (t). Its solution yields two values for time. Once two time instants are known,

we apply the equation of motion for uniform motion in horizontal direction to determine the horizontal distances as required. Here,

$$u_y = 25\sqrt{2} \text{ X } \sin 45^0 = 25 \text{ m / s}$$

$$\begin{aligned} & y = u_y t - \tfrac{1}{2} g t^2 \\ & \Rightarrow 30 = 25t - \tfrac{1}{2} \text{ X } 10 \text{ X } t^2 \\ & \Rightarrow t^2 - 5t + 6 = 0 \\ & \Rightarrow t = 2\, s \text{ or } 3\, s \end{aligned}$$

Horizontal motion (refer the figure) :

$$\text{OA} = u_x t == 25\sqrt{2} \text{ x } \cos 45^0 \text{ X } 2 = 50 \text{ m}$$

Thus, projectile needs to be thrown from a position 50 m from the pole. Now,

$$\text{OB} = u_x t == 25\sqrt{2} \text{ x } \cos 45^0 \text{ X } 3 = 75 \text{ m}$$

Hence, separation, d, is :

$$d = \text{OB} - \text{OA} = 75 - 50 = 25 \text{ m}$$

Alternatively
Equation of vertical displacement (y) is a quadratic equation in horizontal displacement (x). Solution of equation yields two values of "x" corresponding to two positions having same elevation. Now, equation of projectile is given by :

$$y = x \tan\theta - \frac{g x^2}{2u^2 \cos^2\theta}$$

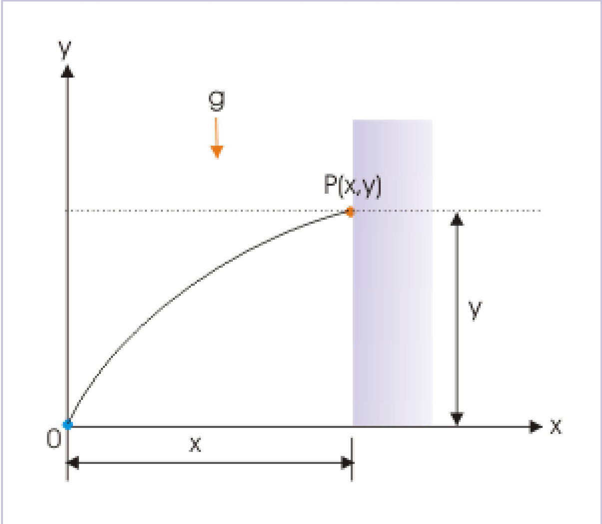
Putting values,

$$30 = x \tan 45^\circ - \frac{10x^2}{2\sqrt{25}^2 \cos^2 45^\circ}$$
$$\Rightarrow 30 = x - \frac{10x^2}{225\sqrt{2}^2 \cos^2 45^\circ}$$
$$\Rightarrow 30 = x - \frac{x^2}{125}$$
$$\Rightarrow x^2 - 125x + 3750 = 0$$
$$\Rightarrow (x-50)(x-75) = 0$$
$$\Rightarrow x = 50 \text{ or } 75\, m$$
$$\Rightarrow d = 75 - 50 = 25\, m$$

Hitting a specified target

An archer aims a bull's eye; a person throws a pebble to strike an object placed at height and so on. The motion involved in these situations is a projectile motion – not a straight line motion. The motion of the projectile (arrow or pebble) has an arched trajectory due to gravity. We need to aim higher than line of sight to the object in order to negotiate the loss of height during flight.

Projectile motion



The projectile hits the wall at a given point.

Hitting a specified target refers to a target whose coordinates (x,y) are known. There are two different settings of the situation. In one case, the angle of projection is fixed. We employ equation of the projectile to determine the speed of projectile. In the second case, speed of the projectile is given and we need to find the angle(s) of projection. The example here illustrates the first case.

Example:

Problem : A projectile, thrown at an angle 45° from the horizontal, strikes a building 30 m away at a point 15 above the ground. Find the velocity of projection.

Solution : As explained, the equation of projectile path suits the description of motion best. Here,

x = 30 m, y = 15 m and θ = 45°. Now,

$$y = x \tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

$$\Rightarrow 15 = 30\tan 45^0 - \frac{10\text{x}30^2}{2u^2\cos^2 45^0}$$

$$\Rightarrow 15 = 30 \text{ x } 1 - \frac{10\text{x}2\text{x}30^2}{2u^2}$$

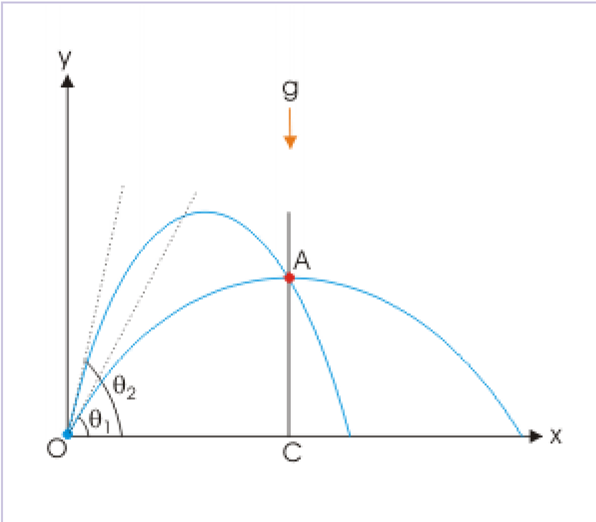
$$\Rightarrow 15 = 30 - \frac{9000}{u^2}$$

$$\Rightarrow u^2 = 600$$

$$\Rightarrow u = 24.49\, m/s$$

As pointed out earlier, we may need to determine the angle(s) for a given speed such that projectile hits a specified target having known coordinates. This presents two possible angles with which projectile can be thrown to hit the target. This aspect is clear from the figure shown here :

Projectile motion



Projectile motion

Clearly, we need to use appropriate form of equation of motion which yields two values of angle of projection. This form is :

$$y = x \tan\theta - \frac{gx^2\left(1+\tan^2\theta\right)}{2u^2}$$

This equation, when simplified, form a quadratic equation in "tanθ". This in turn yields two values of angle of projection. Smaller of the angles gives the projection for least time of flight.

Example:
Problem : A person standing 50 m from a vertical pole wants to hit the target kept on top of the pole with a ball. If the height of the pole is 13 m and his projection speed is 10√g m/s, then what should be the angle of projection of the ball so that it strikes the target in minimum time?
Solution : Equation of projectile having square of "tan θ" is :

$$\Rightarrow y = x \tan\theta - \frac{gx^2}{2u^2}\left(1 + \tan^2\theta\right)$$

Putting values,

$$\Rightarrow 13 = 50 \tan\theta - \frac{10x50^2}{2x\left(10\sqrt{g}\right)^2}\left(1 + \tan^2\theta\right)$$

$$\Rightarrow 25 \tan^2\theta - 100 \tan\theta + 51 = 0$$

$$\Rightarrow \tan\theta = 17/5 \quad or \quad 3/5$$

Taking the smaller angle of projection to hit the target,

$$\Rightarrow \theta = \tan^{-1}\left(\frac{3}{5}\right)$$

Determining attributes of projectile trajectory

A projectile trajectory under gravity is completely determined by the initial speed and the angle of projection or simply by the initial velocity (direction is implied). For the given velocity, maximum height and the range are unique – notably independent of the mass of the projectile.

Thus, a projectile motion involving attributes such as maximum height and range is better addressed in terms of the equations obtained for the specific attributes of the projectile motion.

Example:
Problem : Determine the angle of projection for which maximum height is equal to the range of the projectile.
Solution : We equate the expressions of maximum height and range (H = R) as :

$$\frac{u^2 \sin^2\theta}{2\mathrm{g}} = \frac{u^2 \sin 2\theta}{g}$$

$$u^2 \sin^2\theta = 2u^2 \sin 2\theta$$
$$\Rightarrow \sin^2\theta = 2\sin 2\theta = 4\sin\theta \text{ x } \cos\theta$$
$$\Rightarrow \sin\theta = 4\cos\theta$$
$$\Rightarrow \tan\theta = 4$$
$$\Rightarrow \theta = \tan^{-1}(4)$$

Exercises

Exercise:

Problem:

Which of the following is/ are independent of the angle of projection of a projectile :

(a) time of flight

(b) maximum height reached

(c) acceleration of projectile

(d) horizontal component of velocity

Solution:

The time of flight is determined by considering vertical motion. It means that time of flight is dependent on speed and the angle of projection.

$$T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}$$

Maximum height is also determined, considering vertical motion. As such, maximum height also depends on the angle of projection.

$$H = \frac{u_y^2}{2g} = \frac{u^2 \sin^2 \theta}{2g}$$

Horizontal component, being component of velocity, depends on the angle of projection.

$$u_y = u \cos \theta$$

It is only the acceleration of projectile, which is equal to acceleration due to gravity and is, therefore, independent of the angle of projection. Hence, option (c) is correct.

Exercise:

Problem:

Two particles are projected with same initial speeds at 30° and 60° with the horizontal. Then

(a) their maximum heights will be equal

(b) their ranges will be equal

(c) their time of flights will be equal

(d) their ranges will be different

Solution:

The maximum heights, ranges and time of lights are compared, using respective formula as :

(i)Maximum Height

$$\Rightarrow \frac{H_1}{H_2} = \frac{u^2 \sin^2\theta_1}{u^2 \sin^2\theta_2} = \frac{\sin^2 30^0}{\sin^2 60^0} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{3}$$

Thus, the maximum heights attained by two projectiles are unequal.

(ii) Range :

$$\Rightarrow \frac{R_1}{R_2} = \frac{u^2 \sin 2\theta_1}{u^2 \sin 2\theta_2} = \frac{\sin^2 60^0}{\sin^2 120^0} = \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{1}$$

Thus, the ranges of two projectiles are equal.

(iii) Time of flight

$$\Rightarrow \frac{T_1}{T_2} = \frac{2u \sin \theta_1}{2u \sin \theta_2} = \frac{\sin 30^0}{\sin 60^0} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}}$$

Thus, the times of flight of two projectiles are unequal.

Hence, option (b) is correct.

Exercise:

Problem:

The velocity of a projectile during its flight at an elevation of 8 m from the ground is 3**i** - 5**j** in the coordinate system, where x and y directions represent horizontal and vertical directions respectively. The maximum height attained (H) by the particle is :

$$(a) 10.4 \quad m \quad (b) 8.8 \quad m \quad (c) 9.25 \quad m \quad (d) 9 \quad m$$

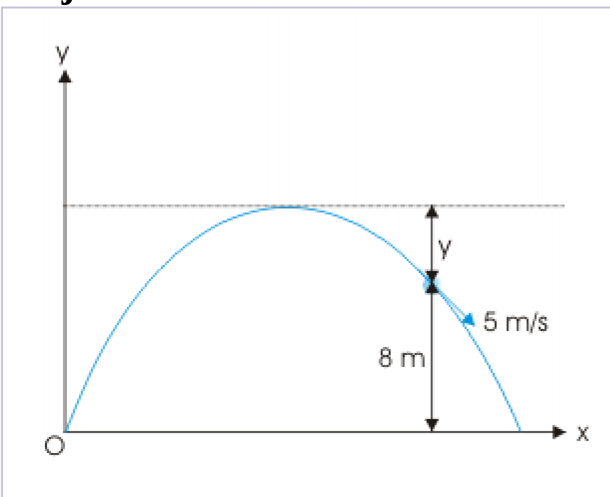
Solution:

We note that vertical component is negative, meaning that projectile is moving towards the ground. The vertical component of velocity 8 m above the ground is

$$v_y = -5 \quad m/s$$

The vertical displacement (y) from the maximum height to the point 8 m above the ground as shown in the figure can be obtained, using equation of motion.

Projectile motion



Projectile motion

$$v_y^2 = u_y^2 + 2ay$$

Considering the point under consideration as origin and upward direction as positive direction.

$$\Rightarrow (-5)^2 = 0 + 2X - 10Xh$$

$$\Rightarrow h = -25/20 = -1.25 \quad m$$

Thus, the maximum height, H, attained by the projectile is :

$$\Rightarrow H = 8 + 1.25 = 9.25 \quad m$$

Hence, option (c) is correct.

Exercise:

Problem:

A projectile is thrown with a given speed so as to cover maximum range (R). If "H" be the maximum height attained during the throw, then the range "R" is equal to :

$(a) 4H \quad (b) 3H \quad (c) \sqrt{2}H \quad (d) H$

Solution:

The projectile covers maximum range when angle of projection is equal to 45°. The maximum range "R" is given by :

$$\Rightarrow R = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin 90^0}{g} = \frac{u^2}{g}$$

On the other hand, the maximum height attained by the projectile for angle of projection, 45°, is :

$$\Rightarrow H = \frac{u_y^2}{2g} = \frac{u^2 \sin^2 45^0}{2g} = \frac{u^2}{4g}$$

Comparing expressions of range and maximum height, we have :

$$\Rightarrow R = 4H$$

Hence, option (a) is correct.

Exercise:

Problem:

The speed of a projectile at maximum height is half its speed of projection, "u". The horizontal range of the projectile is :

$$(a) \frac{\sqrt{3}u^2}{g} \quad (b) \frac{\sqrt{3}u^2}{2g} \quad (c) \frac{u^2}{4g} \quad (d) \frac{u^2}{2g}$$

Solution:

The horizontal range of the projectile is given as :

$$R = \frac{u^2 \sin 2\theta}{g}$$

In order to evaluate this expression, we need to know the angle of projection. Now, the initial part of the question says that the speed of a projectile at maximum height is half its speed of projection, "u". However, we know that speed of the projectile at the maximum height is equal to the horizontal component of projection velocity,

$$u \cos \theta = \frac{u}{2}$$

$$\Rightarrow \cos \theta = \frac{1}{2} = \cos 30^0$$

$$\Rightarrow \theta = 30^0$$

The required range is :

$$\Rightarrow R = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin 60^0}{g} = \frac{\sqrt{3}u^2}{2g}$$

Hence, option (b) is correct.

Exercise:

Problem:

Let "T_1 ” and “T_2 ” be the times of flights of a projectile for projections at two complimentary angles for which horizontal range is "R". The product of times of flight, " $T_1 T_2$ ”, is equal to :

$$(a)\frac{R}{g} \quad (b)\frac{R^2}{g} \quad (c)\frac{2R}{g} \quad (d)\frac{R}{2g}$$

Solution:

Here, we are required to find the product of times of flight. Let "θ " and "$90^0 - \theta$ " be two angles of projections. The times of flight are given as :

$$T_1 = \frac{2u\sin\theta}{g}$$

and

$$T_2 = \frac{2u\sin\left(90^0 - \theta\right)}{g} = \frac{2u\cos\theta}{g}$$

Hence,

$$\Rightarrow T_1 T_2 = \frac{4u^2\sin\theta\cos\theta}{g^2}$$

But,

$$\Rightarrow R = \frac{2u^2\sin\theta\cos\theta}{g}$$

Combining two equations,

$$\Rightarrow T_1 T_2 = \frac{2R}{g}$$

Hence, option (c) is correct.

Exercise:

Problem:

A projectile is projected with a speed "u" at an angle "θ " from the horizontal. The magnitude of average velocity between projection and the time, when projectile reaches the maximum height.

$$(a) u\cos^2\theta \quad (b)\sqrt{(3u^2\cos\theta+1)}$$

$$(c)\frac{\sqrt{(3u^2\cos\theta+1)}}{2u} \quad (d)\frac{u\sqrt{(3u^2\cos\theta+1)}}{2}$$

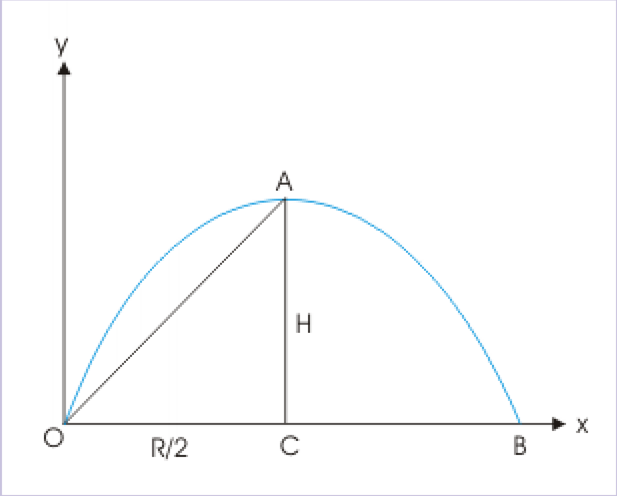
Solution:

The magnitude of average velocity is given as :

$$v_{avg} = \frac{\text{Displacement}}{\text{Time}}$$

Now the displacement here is OA as shown in the figure. From right angle triangle OAC,

Projectile motion



Projectile motion

$$OA = \sqrt{(OC^2 + BC^2)} = \sqrt{\left\{\left(\frac{R}{2}\right)^2 + H^2\right\}}$$

$$\Rightarrow OA = \sqrt{\left(\frac{R^2}{4} + H^2\right)}$$

Hence, magnitude of average velocity is :

$$\Rightarrow v_{avg} = \frac{\sqrt{\left(\frac{R^2}{4} + H^2\right)}}{T}$$

Substituting expression for each of the terms, we have :

$$\Rightarrow v_{avg} = \frac{\sqrt{\left(\frac{u^4 \sin^2 2\theta}{4g^2} + \frac{u^4 \sin^4\theta}{4g^2}\right)}}{\frac{2u\sin\theta}{g}}$$

vavg = u (3 u2cos + 1)/2

$$\Rightarrow v_{avg} = \frac{u\sqrt{(3u^2\cos\theta + 1)}}{2}$$

Hence, option (d) is correct.

Exercise:

Problem:

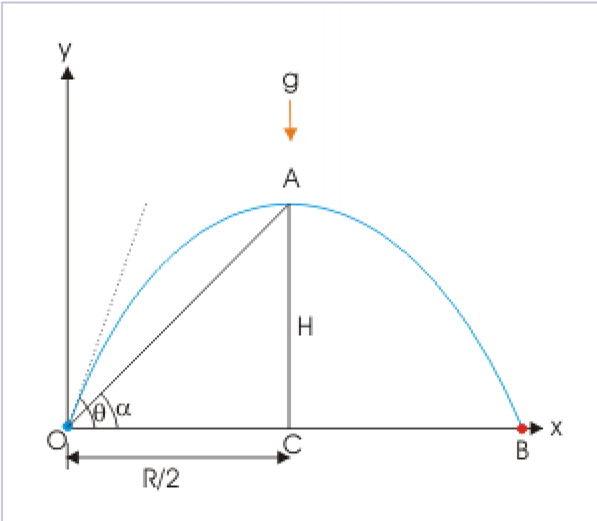
A projectile is projected at an angle "θ " from the horizon. The tangent of angle of elevation of the highest point as seen from the position of projection is :

$$(a)\frac{\tan\theta}{4} \quad (b)\frac{\tan\theta}{2} \quad (c)\tan\theta \quad (d)\frac{3\tan\theta}{2}$$

Solution:

The angle of elevation of the highest point "θ " is shown in the figure. Clearly,

Projectile motion



Projectile motion

$$\tan\alpha = \frac{AC}{OC} = \frac{H}{\frac{R}{2}} = \frac{2H}{R}$$

Putting expressions of the maximum height and range of the flight, we have :

$$\Rightarrow \tan\alpha = \frac{2u^2\sin^2\theta X g}{2gXu^2\sin 2\theta} = \frac{\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\tan\theta}{2}$$

Hence, option (b) is correct.

Exercise:

Problem:

In a firing range, shots are taken at different angles and in different directions. If the speed of the bullets is "u", then find the area in which bullets can spread.

$$(a)\frac{2\pi u^4}{g^2} \quad (b)\frac{u^4}{g^2} \quad (c)\frac{\pi u^4}{g^2} \quad (d)\frac{\pi u^2}{g^2}$$

Solution:

The bullets can spread around in a circular area of radius equal to maximum horizontal range. The maximum horizontal range is given for angle of projection of 45°.

$$R_{\text{max}} = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin 2X45^0}{g} = \frac{u^2 \sin 90^0}{g} = \frac{u^2}{g}$$

The circular area corresponding to the radius equal to maximum horizontal range is given as :

$$\Rightarrow A = \pi R_{\text{max}}^2 = \pi X \left(\frac{u^2}{g}\right)^2 = \frac{\pi u^4}{g^2}$$

Hence, option (c) is correct.

More exercises

Check the module titled " Features of projectile motion (application) to work out more problems.

Features of projectile motion (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1: In general, we should rely on analysis in two individual directions as linear motion.

2: Wherever possible, we should use the formula directly as available for time of flight, maximum height and horizontal range.

3: We should be aware that time of flight and maximum height are two attributes of projectile motion, which are obtained by analyzing motion in vertical direction. For determining time of flight, the vertical displacement is zero; whereas for determining maximum height, vertical component of velocity is zero.

4: However, if problem has information about motion in horizontal direction, then it is always advantageous to analyze motion in horizontal direction. It is so because motion in horizontal direction is uniform motion and analysis in this direction is simpler.

5: The situation, involving quadratic equations, may have three possibilities : (i) quadratic in time "t" (ii) quadratic in displacement or position "x" and (iii) quadratic in "tanθ" i.e."θ". We should use appropriate equations in each case as discussed in the module titled " Features of projectile motion ".

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the features of projectile motion. The questions are categorized in terms of

the characterizing features of the subject matter :

- Time of flight
- Horizontal range
- Maximum height
- Height attained by a projectile
- Composition of motion
- Projectile motion with wind/drag force

Time of flight

Example:
Problem : The speed of a particle, projected at 60°, is 20 m/s at the time of projection. Find the time interval for projectile to loose half its initial speed.
Solution : Here, we see that final and initial speeds (not velocity) are subject to given condition. We need to use the given condition with appropriate expressions of speeds for two instants.

$$u = \sqrt{\left(u_x^2 + u_y^2\right)}$$

$$v = \sqrt{\left(v_x^2 + v_y^2\right)}$$

According to question,

$$u = 2v$$

$$\Rightarrow u^2 = 4v^2$$

$$\Rightarrow u_x^2 + u_y^2 = 4\left(v_x^2 + v_y^2\right)$$

But, horizontal component of velocity remains same. Hence,

$$\Rightarrow u_x^2 + u_y^2 = 4\left(u_x^2 + v_y^2\right)$$

Rearranging for vertical velocity :

$$v_y^2 = u_y^2 - 3u_x^2 = \left(20\sin 60^0\right)^2 - 3X\left(20\cos 60^0\right)^2$$

$$v_y^2 = 20X\frac{3}{4} - 3X20X\frac{1}{4} = 0$$

The component of velocity in vertical direction becomes zero for the given condition. This means that the projectile has actually reached the maximum height for the given condition. The time to reach maximum height is half of the time of flight :

$$t = \frac{u\sin\theta}{g} = \frac{20X\sin 60^0}{10} = \sqrt{3} \quad s$$

Horizontal range

Example:
Problem : A man can throw a ball to a greatest height denoted by "h". Find the greatest horizontal distance that he can throw the ball (consider g = 10 m/s^2).
Solution : The first part of the question provides the information about the initial speed. We know that projectile achieves greatest height in vertical throw. Let "u" be the initial speed. We can, now, apply equation of motion " $v^2 = u^2 + 2gy$ " for vertical throw. We use this form of equation as we want to relate initial speed with the greatest height.
Here, v = 0; a = -g

$$\Rightarrow 0 = u^2 - 2gh$$
$$u^2 = 2gh$$

The projectile, on the other hand, attains greatest horizontal distance for the angle of projection, θ = 45°. Accordingly, the greatest horizontal distance is :

$$R_{\text{max}} = \frac{u^2 \sin 2X45^\circ}{g} = \frac{u^2 \sin 90^\circ}{g} = \frac{u^2}{g} = \frac{2gh}{g} = 2h$$

Example:

Problem : A bullet from a gun is fired at a muzzle speed of 50 m/s to hit a target 125 m away at the same horizontal level. At what angle from horizontal should the gun be aimed to hit the target (consider g = 10 m/s^2) ?

Solution : Here, horizontal range is given. We can find out the angle of projection from horizontal direction, using expression of horizontal range :

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\Rightarrow \sin 2\theta = \frac{gR}{u^2} = \frac{10\text{x}125}{50^2} = \frac{1}{2}$$

$$\Rightarrow \sin 2\theta = \sin 30^0$$

$$\Rightarrow \theta = 15^0$$

Example:

Problem : A projectile has same horizontal range for a given projection speed for the angles of projections θ_1 and θ_2 $(\theta_2 > \theta_1)$ with the horizontal. Find the ratio of the times of flight for the two projections.

Solution : The ratio of time of flight is :

$$\frac{T_1}{T_2} = \frac{u \sin \theta_1}{u \sin \theta_2} = \frac{\sin \theta_1}{\sin \theta_2}$$

For same horizontal range, we know that :

$$\theta_2 = \left(90^0 - \theta_1\right)$$

Putting this, we have :

$$\frac{T_1}{T_2} = \frac{\sin\theta_1}{\sin\left(90^0 - \theta_1\right)} = \tan\theta_1$$

Example:
Problem : A projectile, thrown at an angle 15° with the horizontal, covers a horizontal distance of 1000 m. Find the maximum distance the projectile can cover with the same speed (consider g = 10 m/s^2).
Solution : The range of the projectile is given by :

$$R = \frac{u^2 \sin 2\theta}{g}$$

Here, R = 1000 m, g = 10 m/s^2 , θ = 15°

$$\Rightarrow 1000 = \frac{u^2 \sin\left(2x15^0\right)}{g} = \frac{u^2}{2g}$$
$$\Rightarrow \frac{u^2}{2g} = 1000\, m$$

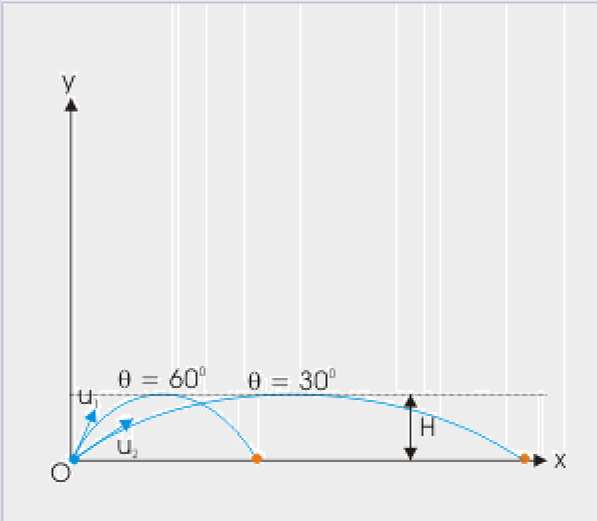
Now, the maximum range (for θ = 45°) is :

$$\Rightarrow R_{\text{max}} = \frac{u^2}{g} = 2000\, m$$

Maximum height

Example:
Problem : Two balls are projected from the same point in the direction inclined at 60° and 30° respectively with the horizontal. If they attain the same height, then the ratio of speeds of projection is :
Solution : Since the projectiles attain same height,

Projectile motion



$$H_1 = H_2 = H$$

$$\frac{{u_1}^2 \sin^2 60^0}{2g} = \frac{{u_2}^2 \sin^2 30^0}{2g}$$

$$\frac{{u_1}^2 X3}{4} = \frac{{u_2}^2 X1}{4}$$

$${u_1}^2 : {u_2}^2 = 1 : 3$$

$$\Rightarrow u_1 : u_2 = 1 : \sqrt{3}$$

Example:

Problem : A projectile is thrown vertically up, whereas another projectile is thrown at an angle θ with the vertical. Both of the projectiles stay in the air for the same time (neglect air resistance). Find the ratio of maximum heights attained by two projectiles.

Solution : Let u_1 and u_2 be the speeds of projectiles for vertical and non-vertical projections. The times of the flight for vertical projectile is given by :

$$T_1 = \frac{2u_1}{g}$$

We note here that the angle is given with respect to vertical - not with respect to horizontal as the usual case. As such, the expression of time of

flight consists of cosine term :

$$T_2 = \frac{2u_2 \cos\theta}{g}$$

As, time of flight is same,

$$\frac{2u_1}{g} = \frac{2u_2 \cos\theta}{g}$$

$$u_1 = u_2 \cos\theta$$

On the other hand, the maximum heights attained in the two cases are :

$$H_1 = \frac{{u_1}^2}{2g}$$

$$H_2 = \frac{{u_2}^2 \cos^2\theta}{2g}$$

Using the relation $u_1 = u_2 \cos\theta$ as obtained earlier, we have :

$$H_2 = \frac{{u_1}^2}{2g}$$

$$\Rightarrow H_1 = H_2$$

Note: The result is intuitive about the nature of projectile. The time of flight and vertical height both are consideration of motion in vertical direction. Since times of flight in both cases are same, the vertical components of two projectiles should be same. Otherwise, times of flight will be different. Now, if vertical component are same, then maximum heights have to be same.

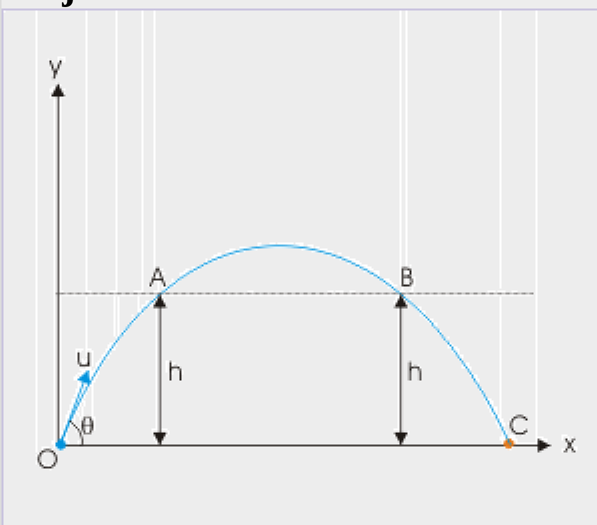
Height attained by a projectile

Example:

Problem : The times for attaining a particular vertical elevation during projectile motion are t_1 and t_2 . Find time of flight, T, in terms of t_1 and t_2 .

Solution : We can answer this question analytically without using formula. Let the positions of the projectile at two time instants be "A" and "B", as shown in the figure. The time periods t_1 and t_2 denotes time taken by the projectile to reach points "A" and "B" respectively. Clearly, time of flight, T, is equal to time taken to travel the curve OAB (t_2) plus the time taken to travel the curve BC.

Projectile motion



Now, projectile takes as much time to travel the curve OA, as it takes to travel curve BC. This is so, because the time of travel of equal vertical displacement in either direction (up or down) in vertical motion under gravity is same. Since the time of travel for curve OA is t_1 , the time of travel for curve BC is also t_1 . Thus, the total time of flight is :

$$T = t_1 + t_2$$

Alternatively,

As the heights attained are equal,

$$h_1 = h_2$$
$$\Rightarrow u_y t_1 - \frac{1}{2} g {t_1}^2 = u_y t_2 - \frac{1}{2} g {t_2}^2$$
$$\Rightarrow u_y \left(t_2 - t_1\right) = \frac{1}{2} g \left({t_2}^2 - {t_1}^2\right) = \frac{1}{2} g \left(t_2 + t_1\right)\left(t_2 - t_1\right)$$
$$\Rightarrow t_2 + t_1 = \frac{2u_y}{g} = T$$

Composition of motion

Example:
Problem : The position of a projectile projected from the ground is :

$$x = 3t$$
$$y = \left(4t - 2t^2\right)$$

where “x” and “y” are in meters and “t” in seconds. The position of the projectile is (0,0) at the time of projection. Find the speed with which the projectile hits the ground.
Solution : When the projectile hits the ground, y = 0,

$$0 = \left(4t - 2t^2\right)$$
$$\Rightarrow 2t^2 - 4t = t\left(2t - 2\right) = 0$$
$$\Rightarrow t = 0,\ t = 2\ s$$

Here t = 0 corresponds to initial condition. Thus, projectile hits the ground in 2 s. Now velocities in two directions are obtained by differentiating given functions of the coordinates,

$$v_x = \frac{đx}{đt} = 3$$
$$v_y = \frac{đy}{đt} = 4 - 4t$$

Now, the velocities for t = 2 s,

$$v_x = \frac{đx}{đt} = 3\, m/s$$

$$v_y = \frac{đy}{đt} = 4 - 4 \text{ x } 2 = -4\, m/s$$

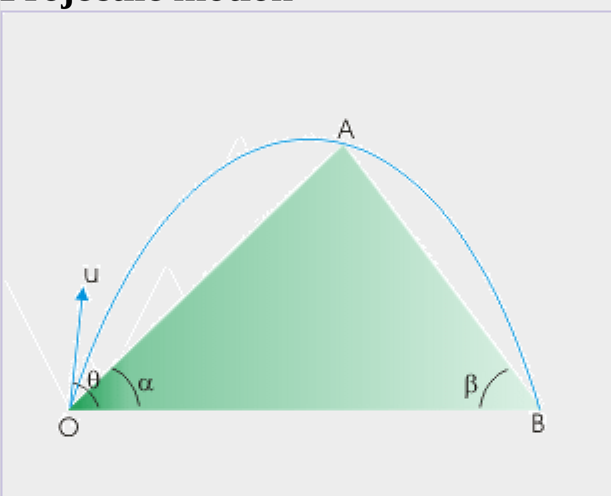
The resultant velocity of the projectile,

$$v = \sqrt{({v_x}^2 + {v_y}^2)} = \sqrt{\left\{3^2 + (-4)^2\right\}} = 5\, m/s$$

Example:

Problem : A projectile, thrown from the foot of a triangle, lands at the edge of its base on the other side of the triangle. The projectile just grazes the vertex as shown in the figure. Prove that :

Projectile motion

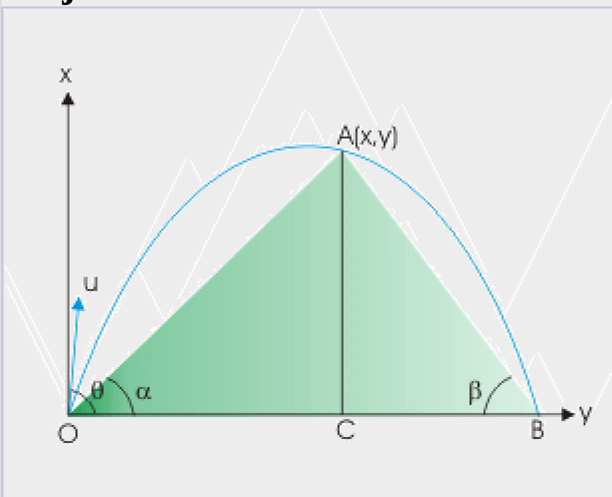


The projectile grazes the vertex of the triangle.

$$\tan\alpha + \tan\beta = \tan\theta$$

where “θ” is the angle of projection as measured from the horizontal.
Solution : In order to expand trigonometric ratio on the left side, we drop a perpendicular from the vertex of the triangle “A” to the base line OB to meet at a point C. Let x,y be the coordinate of vertex “A”, then,

Projectile motion



The projectile grazes the vertex of the triangle.

$$\tan\alpha = \frac{AC}{OC} = \frac{y}{x}$$

and

$$\tan\beta = \frac{AC}{BC} = \frac{y}{(R-x)}$$

Thus,

$$\tan\alpha + \tan\beta = \frac{y}{x} + \frac{y}{(R-x)} = \frac{yR}{x(R-x)}$$

Intuitively, we know the expression is similar to the expression involved in the equation of projectile motion that contains range of projectile,

$$\Rightarrow y = x\tan\theta\left(1-\frac{x}{R}\right)$$

$$\Rightarrow \tan\theta = \frac{yR}{x(R-x)}$$

Comparing equations,

$$\Rightarrow \tan\alpha + \tan\beta = \tan\theta$$

Projectile motion with wind/drag force

Example:
Problem : A projectile is projected at angle “θ” from the horizontal at the speed “u”. If an acceleration of “g/2” is applied to the projectile due to wind in horizontal direction, then find the new time of flight, maximum height and horizontal range.
Solution : The acceleration due to wind affects only the motion in horizontal direction. It would, therefore, not affect attributes like time of flight or maximum height that results exclusively from the consideration of motion in vertical direction. The generic expressions of time of flight, maximum height and horizontal range of flight with acceleration are given as under :

$$T = \frac{2u_y}{g}$$

$$H = \frac{u_y^2}{2g} = \frac{gT^2}{4}$$

$$R = \frac{u_x u_y}{g}$$

The expressions above revalidate the assumption made in the beginning. We can see that it is only the horizontal range that depends on the component of motion in horizontal direction. Now, considering accelerated motion in horizontal direction, we have :

$$x = R\prime = u_x T + \frac{1}{2} a_x T^2$$

$$\Rightarrow R\prime = u_x T + \frac{1}{2}\left(\frac{g}{2}\right) T^2$$

$$R\prime = R + H$$

Example:

Problem : A projectile is projected at angle "θ" from the horizontal at the speed "u". If an acceleration of g/2 is applied to the projectile in horizontal direction and a deceleration of g/2 in vertical direction, then find the new time of flight, maximum height and horizontal range.

Solution : The acceleration due to wind affects only the motion in horizontal direction. It would, therefore, not affect attributes resulting exclusively from the consideration in vertical direction. It is only the horizontal range that will be affected due to acceleration in horizontal direction. On the other hand, deceleration in vertical direction will affect all three attributes.

1: Time of flight

Let us work out the effect on each of the attribute. Considering motion in vertical direction, we have :

$$y = u_y T + \frac{1}{2} a_y T^2$$

For the complete flight, y = 0 and t = T. Also,

$$a_y = -\left(g + \frac{g}{2}\right) = -\frac{3g}{2}$$

Putting in the equation,

$$\Rightarrow 0 = u_y T - \frac{1}{2} X \frac{3g}{2} X T^2$$

Neglecting T = 0,

$$\Rightarrow T = \frac{4u_y}{3g} = \frac{4u \sin\theta}{3g}$$

2: Maximum height
For maximum height, $v_y = 0$,

$$0 = u_y^2 - 2X\frac{3g}{2}XH$$

$$H = \frac{u_y^2}{3g} = \frac{u^2 \sin^2\theta}{3g}$$

2: Horizontal range
Now, considering accelerated motion in horizontal direction, we have :

$$x = R = u_x T + \frac{1}{2} a_x T^2$$

$$R = u_x\left(\frac{4u_y}{g}\right) + \frac{1}{2}\left(\frac{g}{2}\right)\left(\frac{4u_y}{g}\right)^2$$

$$\Rightarrow R = \left(\frac{4u_y}{g}\right)\left[u_x + \frac{1}{2}\left(\frac{g}{2}\right)\left(\frac{4u_y}{g}\right)\right]$$

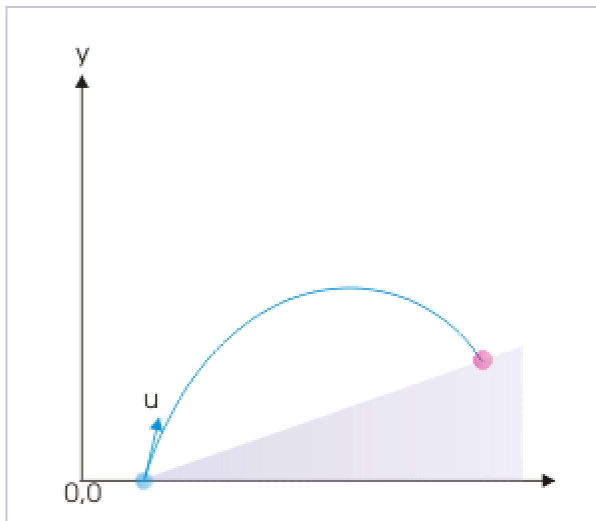
$$\Rightarrow R = \left(\frac{4u_y}{g}\right)\left\{u_x + u_y\right\}$$

$$\Rightarrow R = \left(\frac{4u^2 \sin\theta}{g}\right)\left[\cos\theta + \sin\theta\right]$$

Projectile motion types

So far, we have limited our discussion to the classic mode of a projectile motion, where points of projection and return are on the same horizontal plane. This situation, however, may be altered. The projection level may be at an elevation with respect to the plane where projectile returns or the projection level may be at a lower level with respect to the plane where projectile returns. The two situations are illustrated in the figures below.

Projection and return at different levels

Projection from higher elevation	The return of projectile at higher elevation
[missing_resource: vpm1.gif]	

The two variants are basically the same parabolic motion. These motion types are inherently similar to the one where points of projection and return are on same level. If we look closely, then we find that the motion of the projectile from an elevation and from a lower level are either an extension or a curtailment of normal parabolic motion.

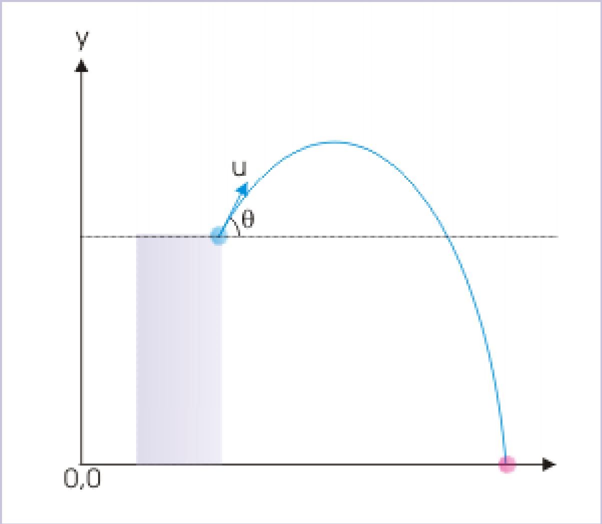
The projection on an incline (where point of return is on higher level) is a shortened projectile motion as if projectile has been stopped before returning to the normal point of return. This motion is also visualized as if the projectile is thrown over an incline or a wedge as shown in the figure. Again, there are two possibilities : (i) the projectile can be thrown up the

incline or (ii) the projectile can be thrown down the incline. For the sake of convenience and better organization, we shall study projectile motion on an incline in a separate module. In this module, we shall restrict ourselves to the first case in which projectile is projected from an elevation.

Projection from a higher level

The projection of a projectile from a higher point results in a slightly different parabolic trajectory. We can visually recognize certain perceptible differences from the normal case as listed here :

Projection from higher level



The trajectory extends beyond the normal point of return.

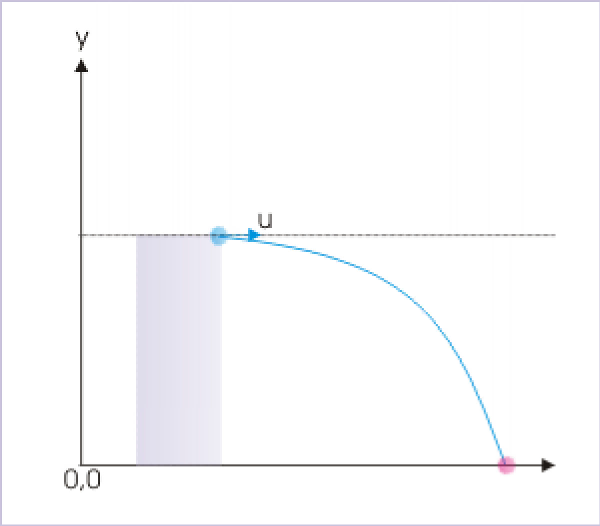
- The upward trajectory is smaller than downward trajectory.
- Time of ascent is smaller than the time of descent.
- The speed of projection is not equal to speed of return on the ground.
- The velocity of return is more aligned to vertical as the motion progresses.

It is evident that the expressions derived earlier for time of flight (T), maximum height (H) and range (R) are not valid in the changed scenario.

But the basic consideration of the analysis is necessarily same. The important aspect of projectile motion that motions in two mutually perpendicular directions are independent of each other, still, holds. Further, the nature of motion in two directions is same as before : the motion in vertical direction is accelerated due to gravity, whereas motion in horizontal direction has no acceleration.

Now, there are two important variations of this projectile motion, when projected from an elevated level. The projectile may either be projected at certain angle (up or down) with the horizontal or it may be projected in the horizontal direction.

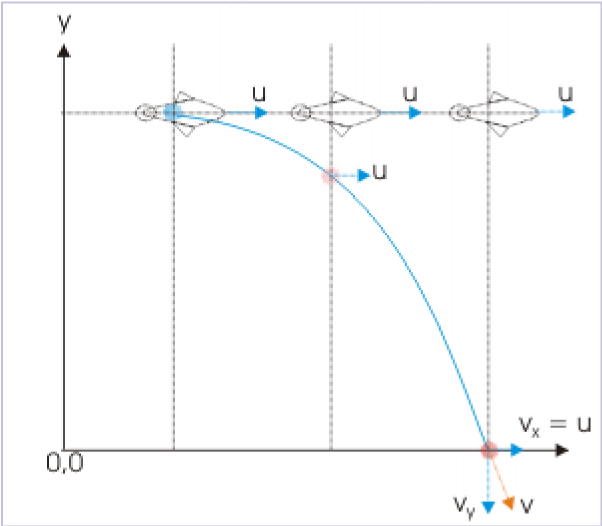
The projection from higher elevation

Projection from higher elevation at an angle	Projection from higher elevation in horizontal direction
[missing_resource: vpm1.gif]	

There are many real time situations that resemble horizontal projection. When an object is dropped from a plane flying parallel to the ground at certain height, then the object acquires horizontal velocity of the plane when the object is released. As the object is simply dropped, the velocity in vertical direction is zero. This horizontal velocity of the object, as acquired from the plane, is then modified by the force of gravity, whereby the object follows a parabolic trajectory before hitting the ground.

This situation is analogous to projection from ground except that we track motion from the highest point. Note that vertical velocity is zero and horizontal velocity is tangential to the path at the time of projection. This is exactly the same situation as when projectile is projected from the ground and reaches highest point. In the nutshell, the description of motion here is same as the description during descent when projected from the ground.

An object dropped from a plane moving in horizontal direction

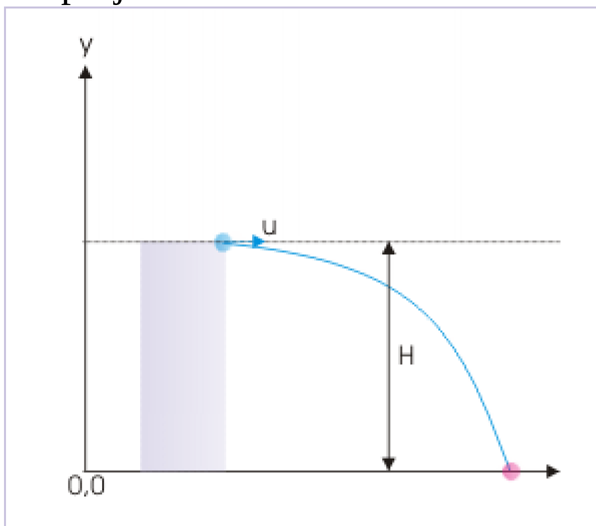


The position of plane is above object as both moves with same velocity in horizontal direction.

The interesting aspect of the object dropped from plane is that both plane and object are moving with same horizontal velocity. Hence, plane is always above the dropped object, provided plane maintains its velocity.

The case of projection from a higher level at certain angle (up or down) to the horizontal is different to the one in which projectile is projected horizontally. The projectile has a vertical component of initial velocity when thrown at an angle with horizontal. This introduces the difference between two cases. The projectile thrown up attains a maximum height above the projection level. On the return journey downward, it travels past its level of projection. The difference is visually shown in the two adjoining figures below.

Maximum height attained by the projectile

[missing_resource: vpm5.gif]

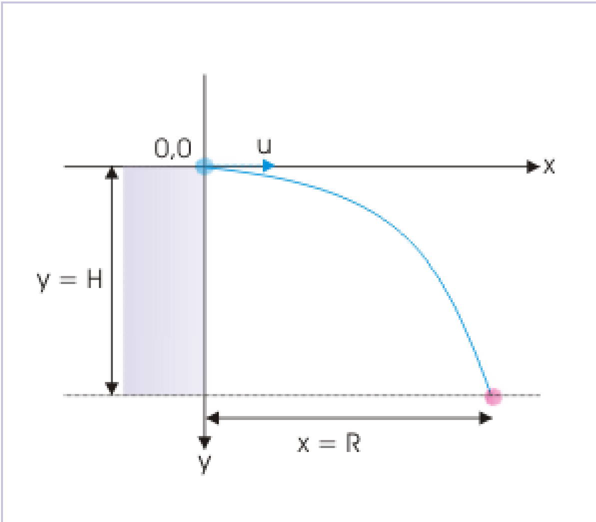
The resulting trajectory in the first case has both upward and downward motions. On the other hand, the motion in upward direction is completely missing in the horizontal projection as the projectile keeps loosing altitude all the time.

Projectile thrown in horizontal direction

We can easily analyze projectile motion following the technique of component motions in two mutually perpendicular directions (horizontal and vertical). Typically, we consider vertical component of motion to determine time of flight (T). The initial velocity in vertical direction is zero.

We consider point of projection as origin of coordinate system. Further, we choose x-axis in horizontal direction and y-axis in the vertically downward direction for the convenience of analysis. Then,

An object projected in horizontal direction



$$y = H = u_y T + \frac{1}{2} g T^2$$

But $u_y = 0$,

$$\Rightarrow H = \frac{1}{2} g T^2$$

$$\Rightarrow T = \sqrt{\frac{2H}{g}}$$

Note the striking similarity here with the free fall of a body under gravity from a height “H”. The time taken in free fall is same as the time of flight of projectile in this case. Now, the horizontal range of the projectile is given as :

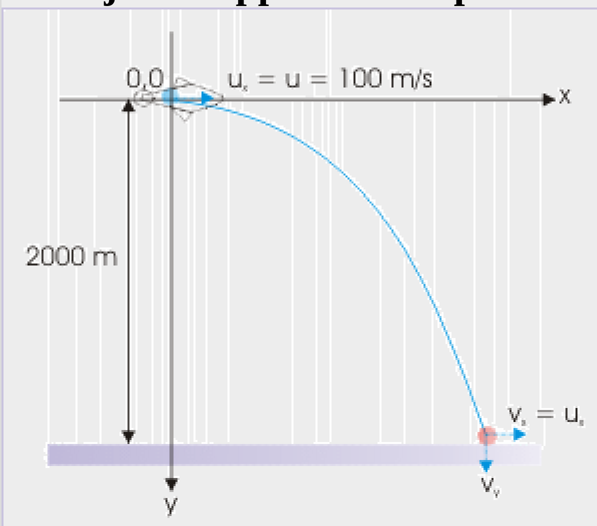
$$x = R = u_x T = u \sqrt{\frac{2H}{g}}$$

Example:
Problem : A plane flying at the speed of 100 m/s parallel to the ground drops an object from a height of 2 km. Find (i) the time of flight (ii)

velocity of the object at the time it strikes the ground and (iii) the horizontal distance traveled by the object.

Solution : The basic approach to solve the problem involves consideration of motion in two mutually perpendicular direction. Here, we consider a coordinate system with the point of release as the origin and down ward direction as the positive y-direction.

An object dropped from a plane moving in horizontal direction



(i) Time of flight, T

In vertical direction :

$$u_y = 0 \, , \; a = 10 \, m/s^2 \, , \; y = 2000 \, m \, , \; T = ?$$

Using equation, $y = u_y T + \frac{1}{2} g T^2$, we have :

$$\Rightarrow y = \frac{1}{2} g t^2$$

$$\Rightarrow T = \sqrt{\frac{2y}{g}}$$

$$\Rightarrow T = \sqrt{\frac{2\text{x}2000}{10}} = 20 \text{ s}$$

(ii) Velocity at the ground

We can find the velocity at the time of strike with ground by calculating component velocities at that instant in the two mutually perpendicular directions and finding the resultant (composite) velocity as :

$$v = \sqrt{(v_x{}^2 + v_y{}^2)}$$

Initial vertical component of initial velocity (uy) is zero and the object is accelerated down with the acceleration due to gravity. Hence,

$$v_y = u_y + gt = 0 + gt = gt$$

The component of velocity in the horizontal direction remains unchanged as there is no acceleration in this direction.

$$v_x = u_x = u$$

$$\Rightarrow v = \sqrt{(v_x{}^2 + v_y{}^2)} = \sqrt{(u^2 + g^2 t^2)}$$

Putting values,

$$\Rightarrow v = \sqrt{100^2 + 10^2 \text{ x } 20^2} = \sqrt{50000} = 100\sqrt{5}\, m/s$$

(iii) Horizontal distance traveled
From consideration of uniform motion in horizontal direction, we have :

$$x = u_x t = ut$$

Putting values,

$$\Rightarrow x = R = 100 \text{ x } 20 = 2000\, m$$

Exercise:

Problem:

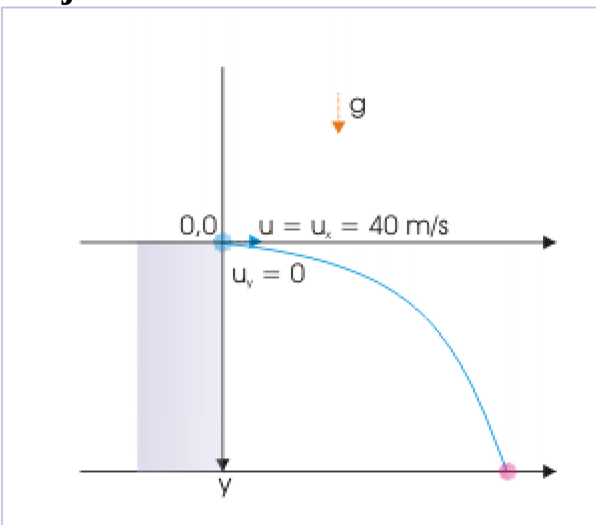
A ball is thrown horizontally from a tower at a speed of 40 m/s. The speed of the projectile (in m/s) after 3 seconds, before it touches the ground, is (consider g = 10 m/s^2) :

$$(a)30 \quad (b)40 \quad (c)50 \quad (d)60$$

Solution:

Here, we consider a reference system whose origin coincides with the point of projection. The downward direction is along y - direction as shown in the figure.

Projectile motion



Projectile motion

The initial speed of the projectile is equal to the horizontal component of the velocity, which remains unaltered during projectile motion. On the other hand, vertical component of velocity at the start of motion is zero. Thus,

$$u_x = 40 \quad m/s, \quad u_y = 0$$

Using equation of motion, we have :

$$v_y = u_y + a_y t$$

$$\Rightarrow v_y = 0 + gt = 10X3 = 30 \quad m/s$$

Since the horizontal component of velocity remains unaltered, the speed, after 3 second, is :

$$\Rightarrow v = \sqrt{v_x^2 + v_y^2} = \sqrt{40^2 + 30^2} = 50 \quad m/s$$

Hence, option (c) is correct.

Exercise:

Problem:

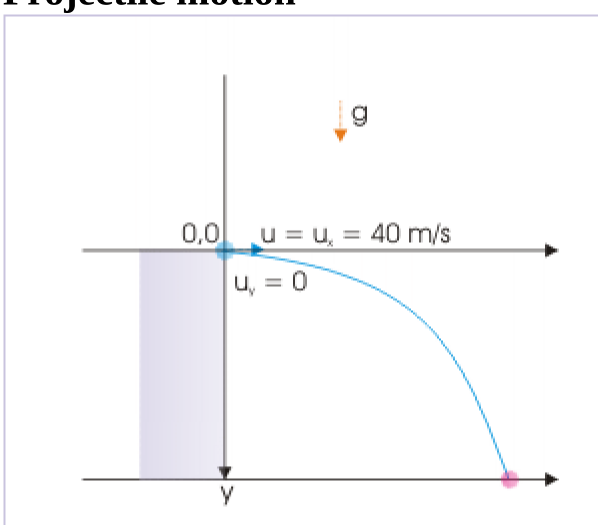
A ball is projected horizontally from a height at a speed of 30 m/s. The time after which the vertical component of velocity becomes equal to horizontal component of velocity is : (consider g = 10 m/s^2) :

$$(a) 1s \quad (b) 2s \quad (c) 3s \quad (d) 4s$$

Solution:

The ball does not have vertical component of velocity when projected. The ball, however, is accelerated downward and gains speed in vertical direction. At certain point of time, the vertical component of velocity equals horizontal component of velocity. At this instant, the angle that the velocity makes with the horizontal is :

Projectile motion



$$\tan\theta = \frac{v_y}{v_x} = 1$$

$$\Rightarrow \theta = 45^0$$

We should note that this particular angle of 45° at any point during the motion, as a matter of fact, signifies that two mutually perpendicular components are equal.

But we know that horizontal component of velocity does not change during the motion. It means that vertical component of velocity at this instant is equal to horizontal component of velocity i.e.

$$v_y = v_x = u_x = 30 \quad m/s$$

Further, we know that we need to analyze motion in vertical direction to find time as required,

$$v_y = u_y + a_y t$$

$$\Rightarrow 30 = 0 + 10t$$

$$\Rightarrow t = 3s$$

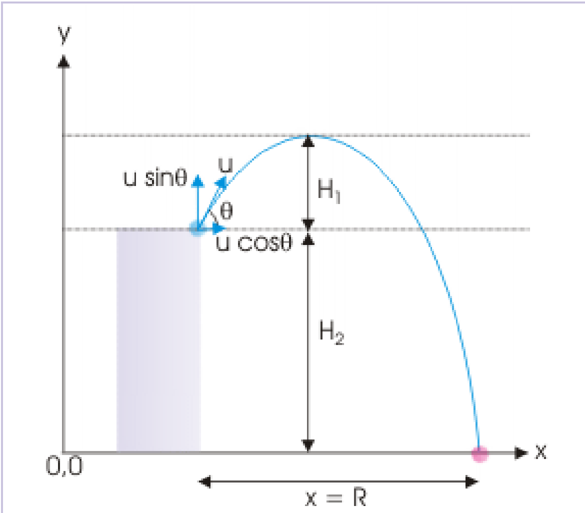
Hence, option (c) is correct.

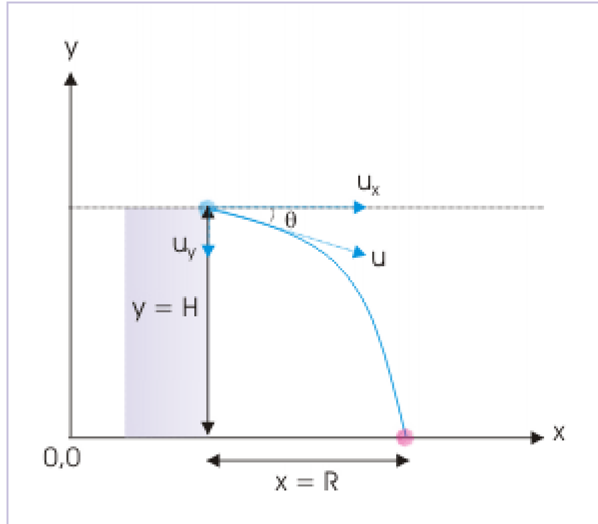
Projectile thrown at an angle with horizontal direction

There are two possibilities. The projectile can be projected up or down as shown in the figure here :

Projection from an elevated level

Projection upwards from an elevated level	Projection downwards from an elevated level





Projectile thrown up at an angle with horizontal direction

The time of flight is determined by analyzing motion in vertical direction. The net displacement during the motion is equal to the elevation of point of projection above ground i.e. H_2 . To analyze the motion, we consider point of projection as origin, horizontal direction as x-axis and upward vertical direction as y-axis.

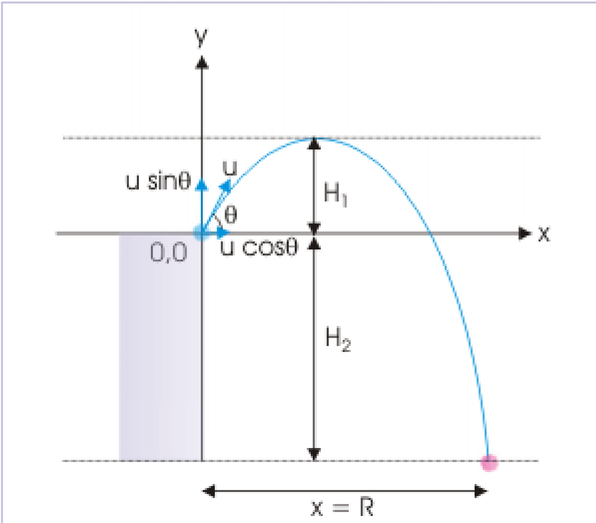
$$y = -H_2 = u_y T - \frac{1}{2} g T^2$$

Rearranging,

$$T^2 - \frac{2u_y}{g} T + \frac{2H_2}{g} = 0$$

This is a quadratic equation in “T”. Solving we get two values of T, one of which gives the time of flight.

An object projected up at an angle from an elevation



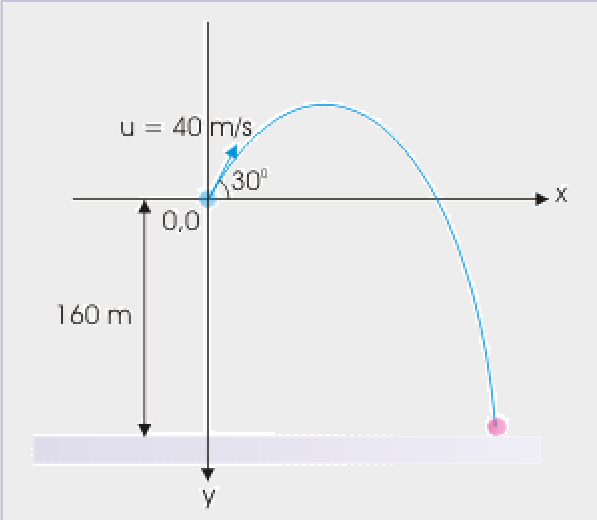
Horizontal range is given by analyzing motion in horizontal direction as :

$$x = R = u_x T$$

While calculating maximum height, we can consider motion in two parts. The first part is the motion above the projection level. On the other hand second part is the projectile motion below projection level. The total height is equal to the sum of the magnitudes of vertical displacements :

$$H = |H_1| + |H_2|$$

Example:
Problem : A projectile is thrown from the top of a building 160 m high, at an angle of 30° with the horizontal at a speed of 40 m/s. Find (i) time of flight (ii) Horizontal distance covered at the end of journey and (iii) the maximum height of the projectile above the ground.
Solution : Unlike horizontal projection, the projectile has a vertical component of initial velocity. This vertical component is acting upwards, which causes the projectile to rise above the point of projection.
Here, we choose the point of projection as the origin and downward direction as the positive y – direction.
An object projected at an angle from an elevation



(i) Time of flight, T
Here,

$$u_y = u\sin\theta = -40\sin 30^0 = -20\,m/s;$$
$$y = 160\,m;$$
$$g = 10\,m/s^2$$

Using equation, $y = u_y t + \frac{1}{2}gt^2$, we have :

$$\Rightarrow 160 = -20t + \frac{1}{2}10t^2$$
$$\Rightarrow 5t^2 - 20t - 160 = 0$$
$$\Rightarrow t^2 - 4t - 32 = 0$$
$$\Rightarrow t^2 - 8t + 4t - 32 = 0$$
$$\Rightarrow t(t-8) + 4(t-8) = 0$$
$$\Rightarrow t = -4\,s \text{ or } t = 8\,s$$

Neglecting negative value of time, T = 8 s.
(ii) Horizontal distance, R
There is no acceleration in horizontal direction. Using equation for uniform motion,

$$x = u_x T$$

Here,

$$u_x = u\cos\theta = 40\cos 30^0 = 20\sqrt{3}\,m/s;$$
$$T = 8\,s$$

$$\Rightarrow x = u_x T = 20\sqrt{3}\text{ x } 8 = 160\sqrt{3}\,m$$

(iii) Maximum height, H
The maximum height is the sum of the height of the building (H_2) and the height attained by the projectile above the building (H_1).

$$H = H_1 + H_2$$

We consider vertical motion to find the height attained by the projectile above the building (H_1).

$$H_1 = \frac{u^2 \sin^2\theta}{2g}$$

$$\Rightarrow H_1 = \frac{(40)^2 \sin^2 30^0}{2X10}$$

$$\Rightarrow H_1 = \frac{(40)^2 X\ \frac{1}{4}}{2X10} = 20 \quad m$$

Thus maximum height, H, is :

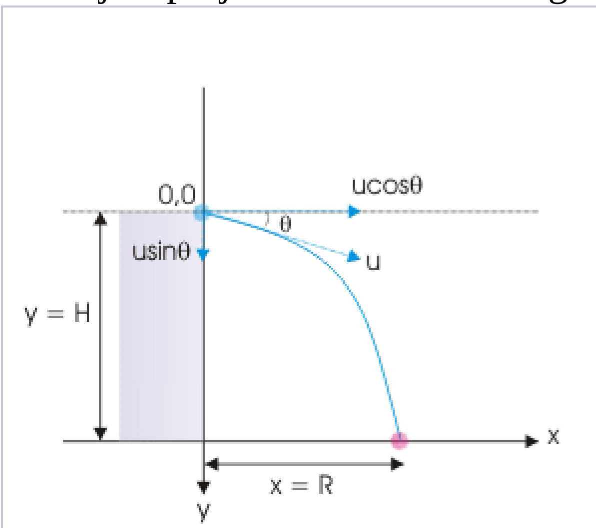
$$H = H_1 + H_2 = 20 + 160 = 180\,m$$

Projectile thrown down at an angle with horizontal direction

Projectile motion here is similar to that of projectile thrown horizontally. The only difference is that the projectile has a finite component of velocity in downward direction against zero vertical velocity.

For convenience, the point of projection is considered as origin of reference and the positive x and y directions of the coordinate system are considered

in horizontal and vertically downward directions.
An object projected down at an angle from an elevation



The time of flight is obtained considering motion in vertical direction as :

$$y = H = u_y T + \frac{1}{2} g T^2$$

Rearranging, we get a quadratic equation in “T”,

$$T^2 + \frac{2u_y}{g} T - \frac{2H_1}{g} = 0$$

One of the two values of "T" gives the time of flight in this case. The horizontal range, on the other hand, is given as :

$$x = R = u_x T$$

Exercises

Exercise:

Problem:

Two balls of masses "m_1 ” and “m_2 ”are thrown from a tower in the horizontal direction at speeds "u_1 ” and “u_2 ” respectively at the same time. Which of the two balls strikes the ground first?

(a) the ball thrown with greater speed

(b) the ball thrown with lesser speed

(c) the ball with greater mass

(d) the balls strike the ground simultaneously

Solution:

The time to strike the ground is obtained by considering motion in vertical direction.

Here, u_y = 0 and y=T (total time of flight)

$$y = \frac{1}{2}gT^2$$

$$\Rightarrow T = \sqrt{\frac{2y}{g}}$$

Thus, we see that time of flight is independent of both mass and speed of the projectile in the horizontal direction. The two balls, therefore, strike the ground simultaneously.

Hence, option (d) is correct.

Exercise:

Problem:

A body dropped from a height "h" strikes the ground with a velocity 3 m/s. Another body of same mass is projected horizontally from the same height with an initial speed of 4 m/s. The final velocity of the second body (in m/s), when it strikes the earth will be :

$$(a)3 \quad (b)4 \quad (c)5 \quad (d)7$$

Solution:

We can use the fact that velocities in vertical direction, attained by two bodies through same displacement, are equal. As such, velocity of the second body , on reaching the ground, would also be 3 m/s. Now, there is no acceleration in horizontal direction. The horizontal component of velocity of the second body, therefore, remains constant.

$$v_x = 4 \quad m/s$$

and

$$v_y = 3 \quad m/s$$

The resultant velocity of two component velocities in mutually perpendicular directions is :

$$\Rightarrow v = \sqrt{v_x^2 + v_y^2} = \sqrt{4^2 + 3^2} = 5 \quad m/s$$

Exercise:

Problem:

A projectile (A) is dropped from a height and another projectile (B) is projected in horizontal direction with a speed of 5 m/s from the same height. Then the correct statement(s) is(are) :

(a) The projectile (A) reach the ground earlier than projectile (B).

(b) Both projectiles reach the ground with the same speed.

(c) Both projectiles reach the ground simultaneously.

(d) The projectiles reach the ground with different speeds.

Solution:

First three options pertain to the time of flight. The time of flight depends only the vertical displacement and vertical component of projection velocity. The vertical components of projection in both cases are zero. The time of flight (T) in either case is obtained by considering motion in vertical direction as :

$$y = u_y T + \frac{1}{2} a_y T^2$$

$$\Rightarrow H = 0 + \frac{1}{2} g T^2$$

$$\Rightarrow T = \sqrt{\frac{2H}{g}}$$

Hence, both projectiles reach the ground simultaneously. On the other hand, component of velocity in vertical direction, on reaching the ground is :

$$v_y = u_y + a_y$$

$$\Rightarrow T = 0 + gT$$

Putting value of T, we have :

$$\Rightarrow v_y = \sqrt{2gH}$$

Projectile (A) has no component of velocity in horizontal direction, whereas projectile (B) has finite component of velocity in horizontal direction. As such, velocities of projectiles and hence the speeds of the particles on reaching the ground are different.

Hence, options (c) and (d) are correct.

Exercise:

Problem:

Four projectiles, "A","B","C" and "D" are projected from top of a tower with velocities (in m/s) 10 **i** + 10 **j**, 10 **i** - 20 **j** , -10 **i** - 10 **j** and -20 **i** + 10**j** in the coordinate system having point of projection as origin. If "x" and "y" coordinates are in horizontal and vertical directions respectively, then :

(a)Time of flight of C is least.

(b) Time of flight of B is least.

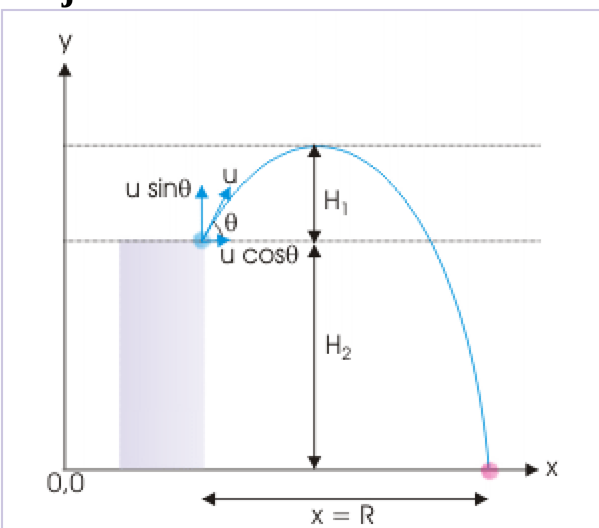
(c) Times of flights of A and D are greatest.

(d) Times of flights of A and D are least.

Solution:

The time of flight of a projectile solely depends on vertical component of velocity.

Projectile motion



Projectile motion

The projectile "A" is thrown up from an elevated point with vertical component of velocity 10 m/s. It travels to the maximum height (H_1) and the elevation from the ground(H_2).

The projectile "B" is thrown down from an elevated point with vertical component of velocity 20 m/s. It travels only the elevation from the ground (H_2).

The projectile "C" is thrown down from an elevated point with vertical component of velocity 10 m/s. It travels only the elevation from the ground (H_2).

The projectile "D" is thrown up from an elevated point with vertical component of velocity 15 m/s. It travels to the maximum height (H_1) and the elevation from the ground(H_2).

Clearly, projectile "B" and "C" travel the minimum vertical displacement. As vertical downward component of velocity of "B" is greater than that of "C", the projectile "B" takes the least time.

The projectiles "A" and "D" are projected up with same vertical components of velocities i.e. 10 m/s and take same time to travel to reach the ground. As the projectiles are projected up, they take more time than projectiles projected down.

Hence, options (b) and (c) are correct.

Acknowledgment

Author wishes to thank Scott Kravitz, Programming Assistant, Connexions for making suggestion to remove syntax error in the module.

Projectile motion types (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

1: Identify projectile motion types. The possible variants are :

- Projectile is thrown in horizontal direction. In this case, initial vertical component of velocity is zero. Consider horizontal direction as positive x-direction and vertically downward direction as positive y-direction.
- Projectile is thrown above horizontal level. The projectile first goes up and then comes down below the level of projection
- Projectile is thrown below horizontal level. Consider horizontal direction as positive x-direction and vertically downward direction as positive y-direction.

2: We can not use standard equations of time of flight, maximum height and horizontal range. We need to analyze the problem in vertical direction for time of flight and maximum height. Remember that determination of horizontal range will involve analysis in both vertical (for time of flight) and horizontal (for the horizontal range) directions.

3: However, if problem has information about motion in horizontal direction, then it is always advantageous to analyze motion in horizontal direction.

Representative problems and their solutions

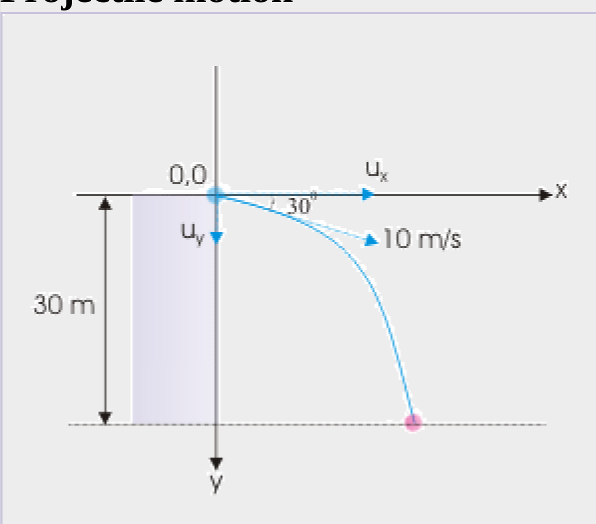
We discuss problems, which highlight certain aspects of the study leading to the projectile motion types. The questions are categorized in terms of the

characterizing features of the subject matter :

- Time of flight
- Range of flight
- Initial velocity
- Final velocity

Time of flight

Example:
Problem : A ball from a tower of height 30 m is projected down at an angle of 30° from the horizontal with a speed of 10 m/s. How long does ball take to reach the ground? (consider g = 10 m/s^2)
Solution : Here, we consider a reference system whose origin coincides with the point of projection. Further, we consider that the downward direction is positive y - direction.
Projectile motion



Motion in vertical direction :
Here, $u_y = u\sin\theta = 10\sin 30^0 = 5\,m/s; y = 30\,m;$. Using $y = u_y t + \frac{1}{2}a_y t^2$, we have :

$$\Rightarrow 30 = 5t + \frac{1}{2}10t^2$$

$$\Rightarrow t^2 + t - 6 = 0$$

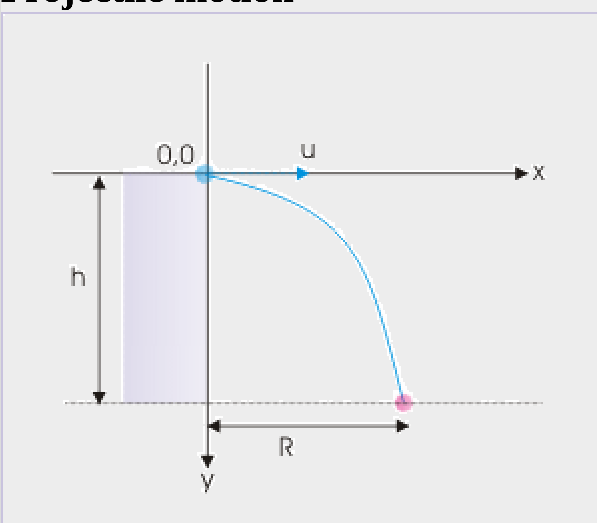
$$\Rightarrow t(t+3) - 2(t+3) = 0$$

$$\Rightarrow t = -3\ s \text{ or } t = 2\ s$$

Neglecting negative value of time, t = 2 s

Range of flight

Example:
Problem : A ball is thrown from a tower of height "h" in the horizontal direction at a speed "u". Find the horizontal range of the projectile.
Solution : Here, we consider a reference system whose origin coincides with the point of projection. we consider that the downward direction is positive y - direction.
Projectile motion



$$\Rightarrow x = R = u_x T = uT$$

Motion in the vertical direction :
Here, $u_y = 0$ and t = T (total time of flight)

$$\Rightarrow h = \frac{1}{2} g T^2$$

$$\Rightarrow T = \sqrt{\frac{2h}{g}}$$

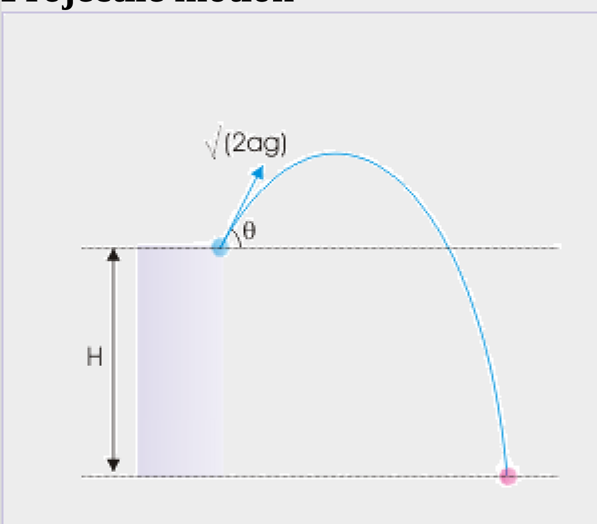
Putting expression of total time of flight in the expression for horizontal range, we have :

$$\Rightarrow R = u \sqrt{\frac{2h}{g}}$$

Example:

Problem : A projectile is projected up with a velocity √(2ag) at an angle "θ" from an elevated position as shown in the figure. Find the maximum horizontal range that can be achieved.

Projectile motion

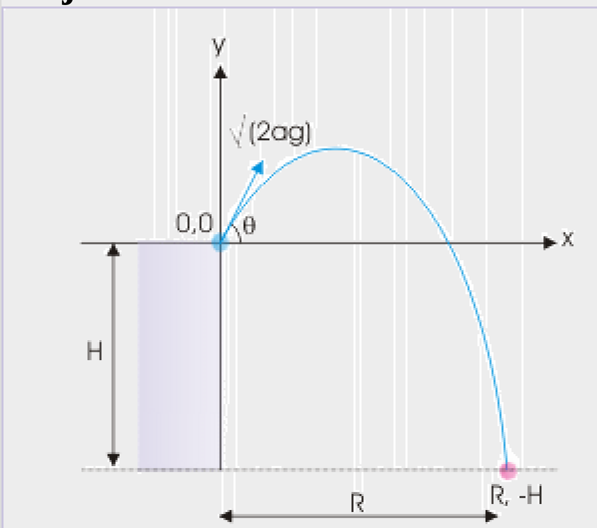


Projectile projected from an elevated point .

Solution : In order to determine the maximum horizontal range, we need to find an expression involving horizontal range. We shall use the equation

of projectile as we have the final coordinates of the motion as shown in the figure below :

Projectile motion



Projectile projected from an elevated point .

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

Substituting and changing trigonometric ratio with the objective to create a quadratic equation in “tan θ” :

$$\Rightarrow -H = R\tan\theta - \frac{gR^2}{2\left(\sqrt{(2ag)}\right)^2}\left(1+\tan^2\theta\right)$$

Rearranging, we have :

$$R^2\tan^2\theta - 4aR\tan\theta + \left(R^2 - 4aH\right) = 0$$

$$\Rightarrow \tan\theta = \frac{4aR \pm \sqrt{\left\{(4aR)^2 - 4R^2(R^2-4aH)\right\}}}{2R^2}$$

For tan θ to be real, it is required that

$$16a^2R^2 \geq 4R^2 \;\; R^2 - 4aH$$

$$\Rightarrow 4a^2 \geq \;\; R^2 - 4aH$$

$$\Rightarrow R^2 \leq 4a(a+H)$$

$$\Rightarrow R \leq \pm 2 \sqrt{a(a+H)}$$

Hence, maximum possible range is :

$$\Rightarrow R = 2 \sqrt{a(a+H)}$$

Initial velocity

Example:
Problem : A ball is thrown horizontally from the top of the tower to hit the ground at an angle of 45° in 2 s. Find the speed of the ball with which it was projected.
Solution : The question provides the angle at which the ball hits the ground. A hit at 45° means that horizontal and vertical speeds are equal.

$$\tan 45^0 = \frac{v_y}{v_x} = 1$$

$$\Rightarrow v_x = v_y$$

However, we know that horizontal component of velocity does not change with time. Hence, final velocity in horizontal direction is same as initial velocity in that direction.

$$\Rightarrow v_x = v_y = u_x$$

We can now find the vertical component of velocity at the time projectile hits the ground by considering motion in vertical direction. Here, $u_y = 0,\ t = 2s.$
Using equation of motion in vertical direction, assuming downward direction as positive :

$$v_y = u_y + at$$

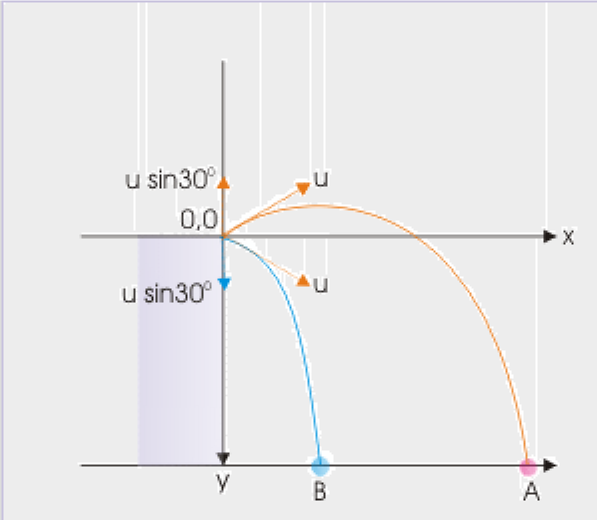
$$\Rightarrow v_y = 0 + 10X2 = 20 \quad m/s$$

Hence, the speed with which the ball was projected in horizontal direction is :

$$\Rightarrow u_x = v_y = 20 \quad m/s$$

Final velocity

Example:
Problem : A ball “A” is thrown from the edge of building “h”, at an angle of 30° from the horizontal, in upward direction. Another ball ”B” is thrown at the same speed from the same position, making same angle with horizontal, in vertically downward direction. If "u" be the speed of projection, then find their speed at the time of striking the ground.
Solution : The horizontal components of velocity for two projectiles are equal. Further, horizontal component of velocity remains unaltered during projectile motion. The speed of the projectile at the time of striking depends solely on vertical component of velocity. For the shake of convenience of analysis, we consider point of projection as the origin of coordinate system and vertically downward direction as positive y - direction.
Projectile motion



Motion in vertical direction :
For A, $u_{\mathrm{yA}} = -u\sin 30^0 = -\frac{u}{2}$
For B, $u_{\mathrm{yB}} = u\sin 30^0 = \frac{u}{2}$
Thus, velocities in vertical direction are equal in magnitude, but opposite in direction. The ball,"A", which is thrown upward, returns after reaching the maximum vertical height. For consideration in vertical direction, the ball returns to the point of projection with same speed it was projected. What it means that the vertical component of velocity of ball "A" on return at the point projection is "u/2". This further means that two balls "A" and "B", as a matter of fact, travel down with same downward vertical component of velocity.
In other words, the ball "A" returns to its initial position acquiring same speed "u/2" as that of ball "B" before starting its downward journey. Thus, speeds of two balls are same i.e "u/2" for downward motion. Hence, two balls strike the ground with same speed. Let the final speed is "v".
Now, we apply equation of motion to determine the final speed in the vertical direction.

$$v_y^2 = u_y^2 + 2gh$$

Putting values, we have :

$$v_y^2 = {\frac{u}{2}}^2 + 2gh = \frac{u^2 + 8gh}{4}$$

$$v_y = \sqrt{\frac{(u^2 + 8gh)}{2}}$$

The horizontal component of velocity remains same during the journey. It is given as :

$$vx = u\cos 30^0 = \frac{\sqrt{3}u}{2}$$

The resultant of two mutually perpendicular components is obtained, using Pythagoras theorem :

$$\Rightarrow v^2 = v_x^2 + v_y^2 = \frac{3u^2}{4} + \frac{u^2 + 8gh}{4}$$

$$\Rightarrow v^2 = \frac{3u^2 + u^2 + 8gh}{4} = u^2 + 2gh$$

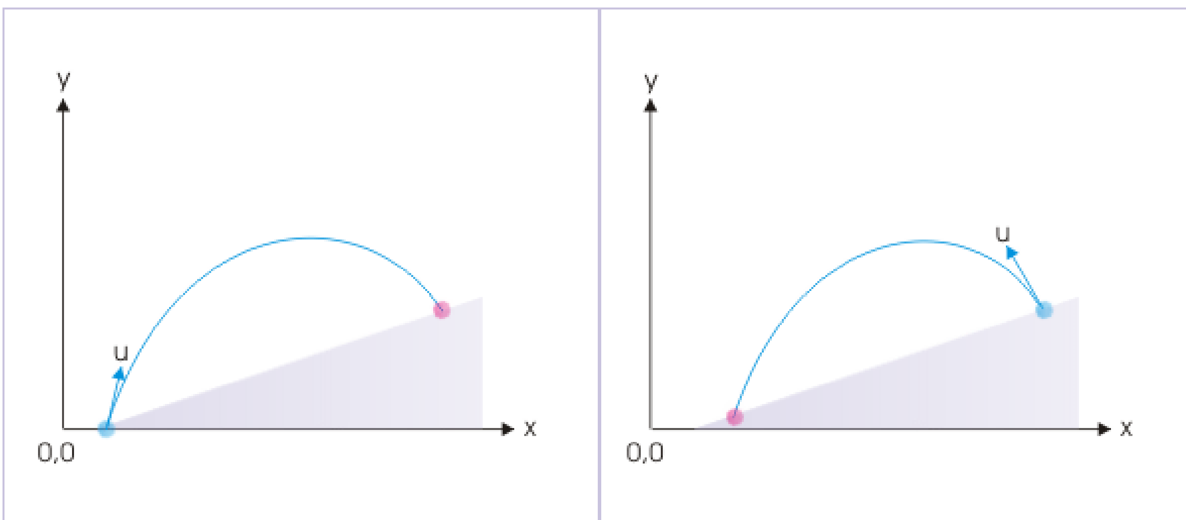
$$\Rightarrow v = \overline{(u^2 + 2gh)}$$

Projectile motion on an incline

Projectile motion on an incline plane is one of the various projectile motion types. The main distinguishing aspect is that points of projection and return are not on the same horizontal plane. There are two possibilities : (i) the point of return is at a higher level than the point of projection i.e projectile is thrown up the incline and (ii) Point of return is at a lower level than point of projection i.e. projectile is thrown down the incline.

Projection on the incline

Projection up the incline Projection down the incline



We have so far studied the projectile motion, using technique of component motions in two mutually perpendicular directions – one which is horizontal and the other which is vertical. We can simply extend the methodology to these types of projectile motion types as well. Alternatively, we can choose coordinate axes along the incline and in the direction of perpendicular to the incline. The analysis of projectile motion in two coordinate systems differs in the detail of treatment.

For convenience of comparison, we shall refer projectile motion on a horizontal surface as the “normal case”. The reference to “normal case” enables us to note differences and similarities between “normal case” and the case of projectile motion on an incline plane.

Analyzing alternatives

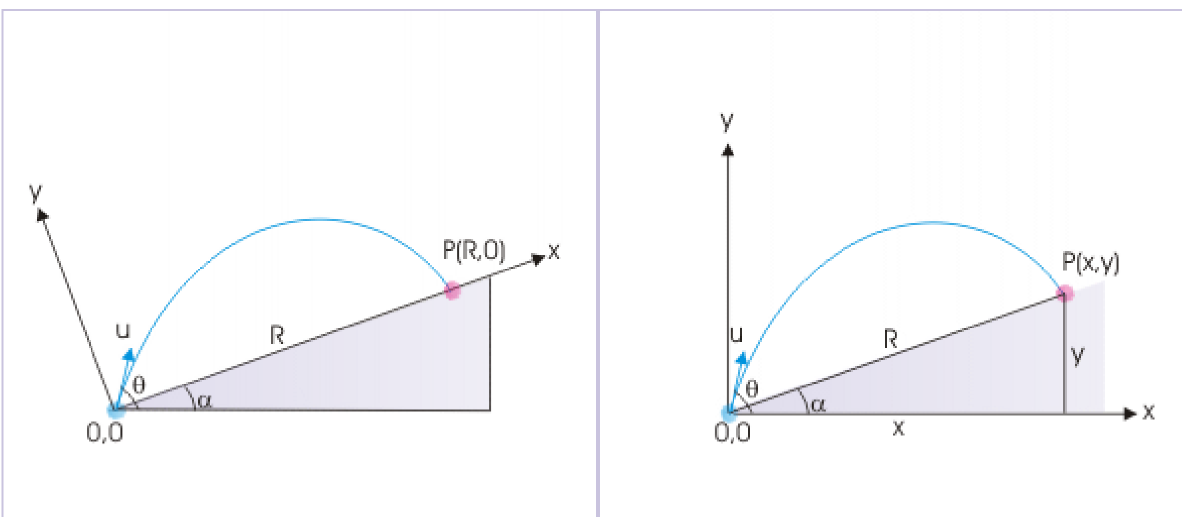
As pointed out, there are two different approaches of analyzing projectile motion on an incline plane. The first approach could be to continue analyzing motion in two mutually perpendicular horizontal and vertical directions. The second approach could be to analyze motion by changing the reference orientation i.e. we set up our coordinate system along the incline and a direction along the perpendicular to incline.

The analysis alternatives are, therefore, distinguished on the basis of coordinate system that we choose to employ :

- planar coordinates along incline (x) and perpendicular to incline (y)
- planar coordinates in horizontal (x) and vertical (y) directions

Coordinate systems

With reference to incline With reference to horizontal



The two alternatives, as a matter of fact, are entirely equivalent. However, we shall study both alternatives separately for the simple reason that they provide advantage in analyzing projectile motion in specific situation.

Projection up the incline

As pointed out, the projection up the incline can be studied in two alternative ways. We discuss each of the approach, highlighting intricacies of each approach in the following sub-section.

Coordinates along incline (x) and perpendicular to incline (y)

This approach is typically superior approach in so far as it renders measurement of time of flight in a relatively simpler manner. However, before we proceed to analyze projectile motion in this new coordinate set up, we need to identify and understand attributes of motion in mutually perpendicular directions.

Measurement of angle of projection is one attribute that needs to be handled in a consistent manner. It is always convenient to follow certain convention in referring angles involved. We had earlier denoted the angle of projection as measured from the horizontal and denoted the same by the symbol "θ". It is evident that it would be reasonable to extend the same convention and also retain the same symbol for the angle of projection.

It also follows that we measure other angles from the horizontal – even if we select x-coordinate in any other direction like along the incline. This convention avoids confusion. For example, the angle of incline "α" is measured from the horizontal. The horizontal reference, therefore, is actually a general reference for measurement of angles in the study of projectile motion.

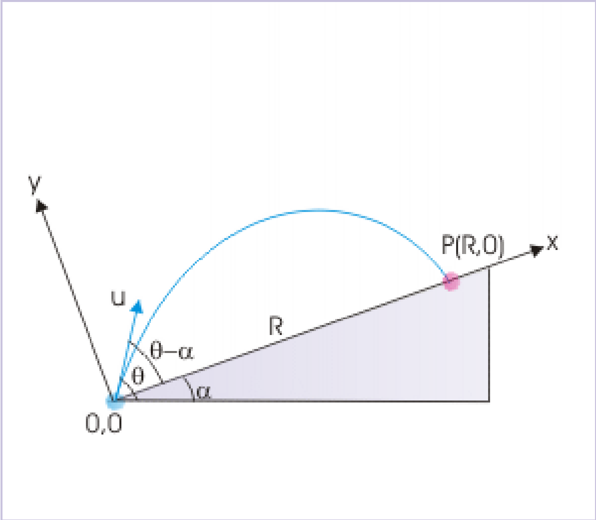
Now, let us have a look at other characterizing aspects of new analysis set up :

1: The coordinate "x" is along the incline – not in the horizontal direction; and the coordinate "y" is perpendicular to incline – not in the vertical direction.

2: Angle with the incline

From the figure, it is clear that the angle that the velocity of projection makes with x-axis (i.e. incline) is "θ – α".

Projectile motion up an incline



The projection from lower level.

3: The point of return

The point of return is specified by the coordinate R,0 in the coordinate system, where “R” is the range along the incline.

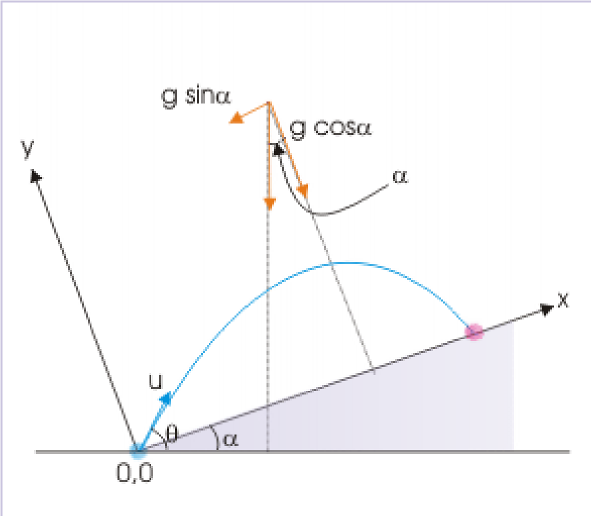
4: Components of initial velocity

$$u_x = u\cos(\theta - \alpha)$$

$$u_y = u\sin(\theta - \alpha)$$

5: The components of acceleration

In order to determine the components of acceleration in new coordinate directions, we need to know the angle between acceleration due to gravity and y-axis. We see that the direction of acceleration is perpendicular to the base of incline (i.e. horizontal) and y-axis is perpendicular to the incline.
Components of acceleration due to gravity



The acceleration due to gravity forms an angle with y-axis, which is equal to angle of incline.

Thus, the angle between acceleration due to gravity and y – axis is equal to the angle of incline i.e. “α”. Therefore, components of acceleration due to gravity are :

$$a_x = -g\sin\alpha$$

$$a_y = -g\cos\alpha$$

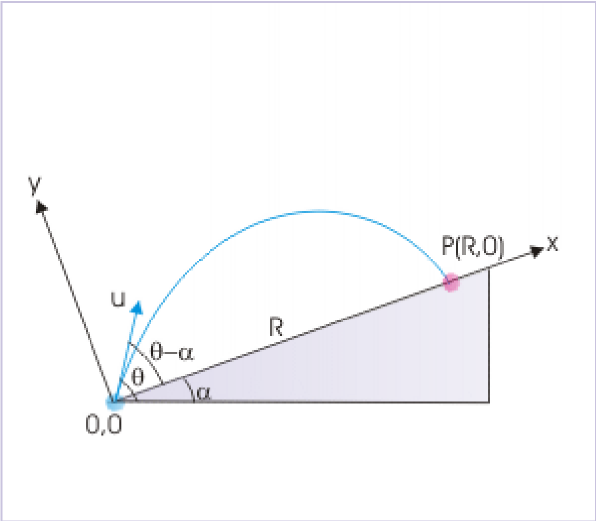
The negative signs precede the expression as two components are in the opposite directions to the positive directions of the coordinates.

6: Unlike in the normal case, the motion in x-direction i.e. along the incline is not uniform motion, but a decelerated motion. The velocity is in positive x-direction, whereas acceleration is in negative x-direction. As such, component of motion in x-direction is decelerated at a constant rate “gsin α”.

Time of flight

The time of flight (T) is obtained by analyzing motion in y-direction (which is no more vertical as in the normal case). The displacement in y-direction after the projectile has returned to the incline, however, is zero as in the normal case. Thus,

Projectile motion up an incline



The projection from lower level.

$$y = u_y T + \frac{1}{2} a_y T^2 = 0$$

$$\Rightarrow u \sin(\theta - \alpha) T + \frac{1}{2}(-g \cos \alpha) T^2 = 0$$

$$\Rightarrow T \left\{ u \sin(\theta - \alpha) + \frac{1}{2}(-g \cos \alpha) T \right\} = 0$$

Either,

$$T = 0$$

or,

$$\Rightarrow T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

The first value represents the initial time of projection. Hence, second expression gives us the time of flight as required. We should note here that the expression of time of flight is alike normal case in a significant manner.

In the generic form, we can express the formula of the time of flight as :

$$T = \left| \frac{2u_y}{a_y} \right|$$

In the normal case, $u_y = u \sin \theta$ and $a_y = -g$. Hence,

$$T = \frac{2u \sin \theta}{g}$$

In the case of projection on incline plane, $u_y = u \sin(\theta - \alpha)$ and $a_y = -g \cos \alpha$. Hence,

$$T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

This comparison and understanding of generic form of the expression for time of flight helps us write the formula accurately in both cases.

Range of flight

First thing that we should note that we do not call “horizontal range” as the range on the incline is no more horizontal. Rather we simply refer the displacement along x-axis as “range”. We can find range of flight by considering motion in both “x” and “y” directions. Note also that we needed the same approach even in the normal case. Let “R” be the range of projectile motion.

The motion along x-axis is no more uniform, but decelerated. This is the major difference with respect to normal case.

$$x = u_x T - \frac{1}{2} a_x T^2$$

Substituting value of “T” as obtained before, we have :

$$R = \frac{u\cos(\theta - \alpha)X2u\sin(\theta - \alpha)}{g\cos\alpha} - \frac{g\sin\alpha X4u^2\sin^2(\theta - \alpha)}{2g^2\cos^2\alpha}$$

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{2\cos(\theta - \alpha)\sin(\theta - \alpha)\cos\alpha - \sin\alpha X2\sin^2(\theta - \alpha)\right\}$$

Using trigonometric relation, $2\sin^2(\theta - \alpha) = 1 - \cos 2(\theta - \alpha)$,

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left[\sin 2(\theta - \alpha)\cos\alpha - \sin\alpha\left\{1 - \cos 2(\theta - \alpha)\right\}\right]$$

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{\sin 2(\theta - \alpha)\cos\alpha - \sin\alpha + \sin\alpha\cos 2(\theta - \alpha)\right\}$$

We use the trigonometric relation,
$\sin(A + B) = \sin A\cos B + \cos A\sin B$,

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{\sin(2\theta - 2\alpha + \alpha) - \sin\alpha\right\}$$

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{\sin(2\theta - \alpha) - \sin\alpha\right\}$$

This is the expression for the range of projectile on an incline. We can see that this expression reduces to the one for the normal case, when $\alpha = 0$,

$$\Rightarrow R = \frac{u^2\sin 2\theta}{g}$$

Maximum range

The range of a projectile thrown up the incline is given as :

$$R = \frac{u^2}{g\cos^2\alpha}\left\{\sin(2\theta - \alpha) - \sin\alpha\right\}$$

We see here that the angle of incline is constant. The range, therefore, is maximum for maximum value of "sin(2θ – α)". Thus, range is maximum for the angle of projection as measured from horizontal direction, when :

$$\sin(2\theta - \alpha) = 1$$

$$\Rightarrow \sin(2\theta - \alpha) = \sin\pi/2$$

$$\Rightarrow 2\theta - \alpha = \pi/2$$

$$\Rightarrow \theta = \pi/2 + \alpha/2$$

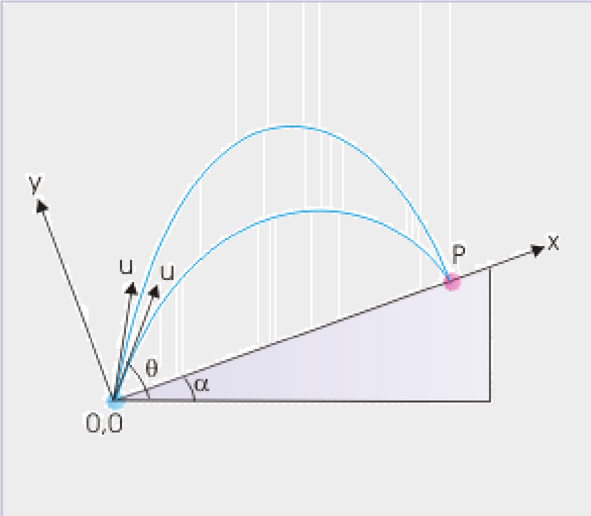
The maximum range, therefore, is :

$$\Rightarrow R_{\text{max}} = \frac{u^2}{g\cos^2\alpha}(1 - \sin\alpha)$$

Example:

Problem : Two projectiles are thrown with same speed, "u", but at different angles from the base of an incline surface of angle "α". The angle of projection with the horizontal is "θ" for one of the projectiles. If two projectiles reach the same point on incline, then determine the ratio of times of flights for the two projectiles.

Projectile motion up an incline



Two projectiles reach the same point on the incline.

Solution : We need to find the ratio of times of flights. Let T_1 and T_2 be the times of fights. Now, the time of flight is given by :

$$T = \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}$$

Here, the angle of projection of one of the projectiles, “θ”, is given. However, angle of projection of other projectile is not given. Let “θ’” be the angle of projection of second projectile.

$$\Rightarrow \frac{T_1}{T_2} = \frac{2u\sin(\theta - \alpha)}{2u\sin(\theta' - \alpha)}$$

We need to know “θ’” to evaluate the above expression. For this, we shall make use of the fact that projectiles have same range for two angles of projections. We can verify this by having a look at the expression of range, which is given as :

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{\sin(2\theta - \alpha) - \sin\alpha\right\}$$

Since other factors remain same, we need to analyze motions of two projectiles for same range in terms of angle of projection only. We have noted in the case of normal projectile motion that there are complimentary angle for which horizontal range is same. Following the same line of argument and making use of the trigonometric relation sinθ = sin (π -θ), we analyze the projectile motions of equal range. Here,

$$\sin\left(2\theta' - \alpha\right) = \sin\left\{\pi - (2\theta - \alpha)\right\} = \sin(\pi - 2\theta + \alpha)$$

$$2\theta' - \alpha = \pi - 2\theta + \alpha$$

$$\Rightarrow 2\theta' = \pi - 2\theta + 2\alpha$$

$$\Rightarrow \theta' = \frac{\pi}{2} - \theta + \alpha$$

Putting this value in the expression for the ratio of times of flights, we have :

$$\Rightarrow \frac{T_1}{T_2} = \frac{2u\sin(\theta - \alpha)}{2u\sin(\pi/2 - \theta + \alpha - \alpha)}$$

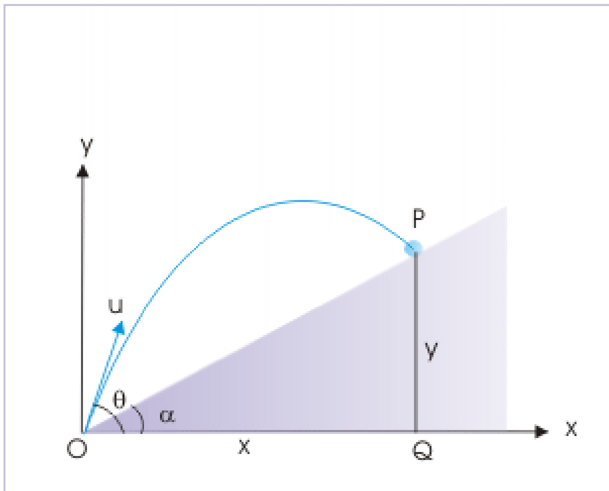
$$\Rightarrow \frac{T_1}{T_2} = \frac{\sin(\theta - \alpha)}{\sin(\pi/2 - \theta)}$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{\sin(\theta - \alpha)}{\cos\theta}$$

Coordinates in horizontal (x) and vertical (y) directions

This approach retains the coordinates used in the normal case (in which projectile returns to the same horizontal level). In this consideration, the description of projectile motion is same as normal case except that motion is aborted in the mid-air by the incline. Had incline been not there, the projectile would have continued with its motion as shown in the figure.

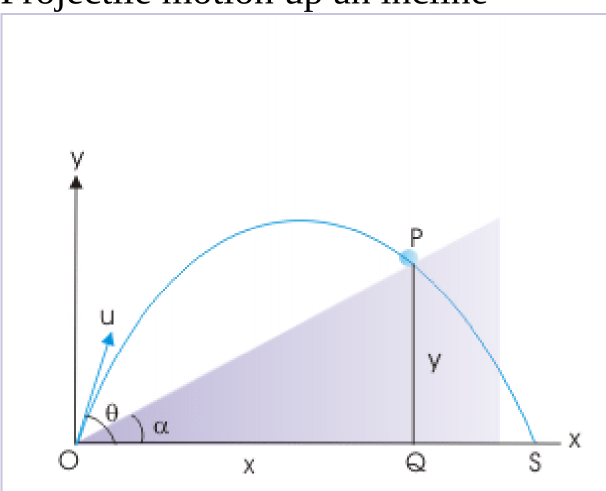
Projectile motion up an incline



Projetile motion up is curtailed by incline.

When the projectile is allowed to return to the projection level, then the point of return is (OQ,0), where OQ is the horizontal range. This position of point of return changes to a new point (x,y), specified by the angle of elevation "α" of the wedge with respect to horizontal as shown in the figure.

Projectile motion up an incline



Projetile motion descrbed with x-axis in horizontal direction and y-axis in vertical direction.

From the triangle OPQ,

$$\cos\alpha = \frac{x}{OP} = \frac{x}{R}$$

The range of the projectile is given by :

$$\Rightarrow R = \frac{x}{\cos\alpha}$$

The strategy here is to determine “x” i.e. “OQ” considering the motion as normal projectile motion. Thus, we shall first determine “x” and then using above relation, we obtain the relation for the range of flight along the incline. Now, considering motion in horizontal direction , we have :

$$x = u\cos\theta X T$$

where “T” is the time of flight of projectile motion on the incline. It is given as determined earlier :

$$T = \frac{2u\sin(\theta-\alpha)}{g\cos\alpha}$$

Substituting in the epression of “x”, we have :

$$\Rightarrow x = \frac{u\cos\theta X 2u\sin(\theta-\alpha)}{g\cos\alpha}$$

$$\Rightarrow x = \frac{u^2 2\sin(\theta-\alpha)\cos\theta}{g\cos\alpha}$$

We simplify this relation, using trigonometric relation as given here :

$$\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$$

Comparing right hand side of the equation with the expression in the numerator of the equation of “x”, we have :

$$C - D = 2\theta - 2\alpha$$

$$C + D = 2\theta$$

Adding, we have :

$$\Rightarrow C = 2\theta - \alpha$$

$$\Rightarrow D = \alpha$$

Thus, we can write :

$$\Rightarrow 2\sin(\theta - \alpha)\cos\theta = \sin(2\theta - \alpha) - \sin\alpha$$

Substituting the expression in the equation of “x”,

$$\Rightarrow x = \frac{u^2}{g\cos\alpha}\left\{\sin(2\theta - \alpha) - \sin\alpha\right\}$$

Using the relation connecting horizontal range “x” with the range on incline, “R”, we have :

$$R = \frac{x}{\cos\alpha}$$

$$\Rightarrow R = \frac{u^2}{g\cos^2\alpha}\left\{\sin(2\theta - \alpha) - \sin\alpha\right\}$$

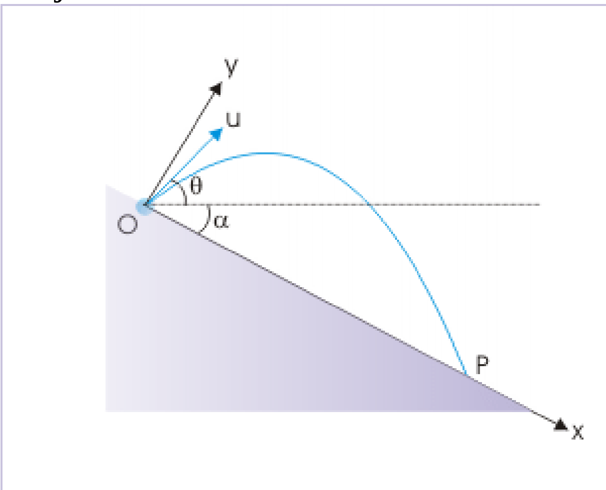
Thus, we get the same expression for range as expected. Though the final expressions are same, but the understanding of two approaches is important as they have best fit application in specific situations.

Projection down the incline

A typical projection down the incline is shown with a reference system in which “x” and “y” axes are directions along incline and perpendicular to incline. The most important aspect of the analysis of this category of

projectile motion is the emphasis that we put on the convention for measuring angles.

Projectile motion down an incline



Projetile motion thrown from a higher point.

The angle of projection and angle of incline both are measured from a horizontal line. The expression for the time of flight is obtained by analyzing motion in vertical directions. Here we present the final results without working them out as the final forms of expressions are suggestive.

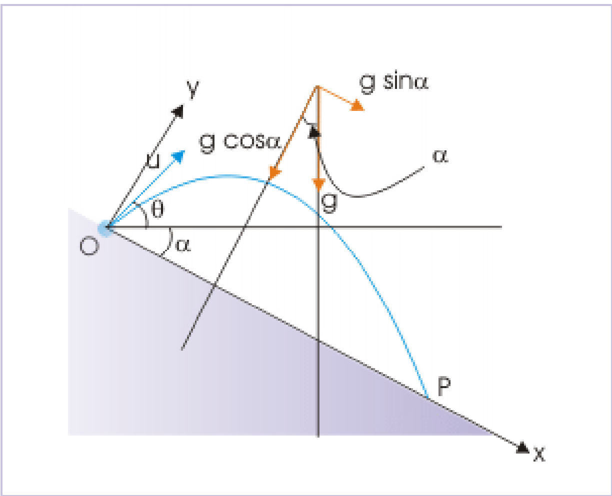
1: Components of initial velocity

$$u_x = u\cos(\theta + \alpha)$$

$$u_y = u\sin(\theta + \alpha)$$

2: Components of acceleration

Components of acceleration due to gravity



Accelration due to gravity forms an angle with y-direction, which is equal to angle of incline.

$$a_x = g \sin \alpha$$

$$a_y = -g \cos \alpha$$

3: Time of flight

The expression of time of flight differs only with respect to angle of sine function in the numerator of the expression :

$$T = \frac{2u \sin(\theta + \alpha)}{g \cos \alpha}$$

4: Range of flight

The expression of range of flight differs only with respect to angle of sine function :

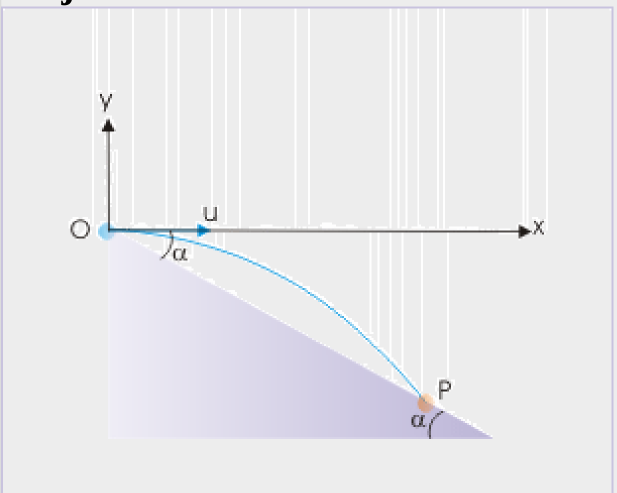
$$R = \frac{u^2}{g \cos^2 \alpha} \left\{ \sin(2\theta + \alpha) + \sin \alpha \right\}$$

It is very handy to note that expressions have changed only with respect of the sign of “α” for the time of flight and the range. We only need to exchange “α” by “-α” .

Example:

Problem : A ball is projected in horizontal direction from an elevated point “O” of an incline of angle “α” with a speed “u”. If the ball hits the incline surface at a point “P” down the incline, find the coordinates of point “P”.

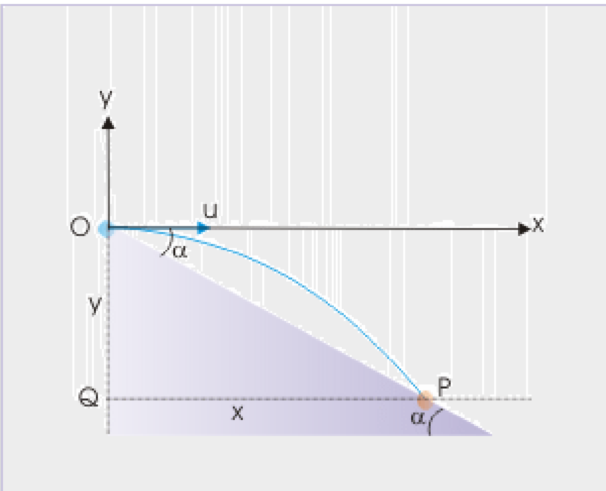
Projectile motion down an incline



A ball is projected in horizontal direction.

Solution : We can answer this question, using the relation of coordinates with range “R” as :

Projectile motion down an incline



A ball is projected in horizontal direction.

$$x = R\cos\alpha$$

$$y = -R\sin\alpha$$

Now, range of the flight for the downward flight is given as :

$$R = \frac{u^2}{g\cos^2\alpha}\left\{\sin(2\theta + \alpha) + \sin\alpha\right\}$$

The important thing to realize here is that the ball is projected in horizontal direction. As we measure angle from the horizontal line, it is evident that the angle of projection is zero. Hence,

$$\theta = 0^0$$

Putting in the equation for the range of flight, we have :

$$R = \frac{2u^2\sin\alpha}{g\cos^2\alpha}$$

Therefore, coordinates of the point of return, “P”, is :

$$\Rightarrow x = R\cos\alpha = \frac{2u^2\sin\alpha}{g\cos\alpha}$$

$$\Rightarrow x = \frac{2u^2\tan\alpha}{g}$$

Similarly,

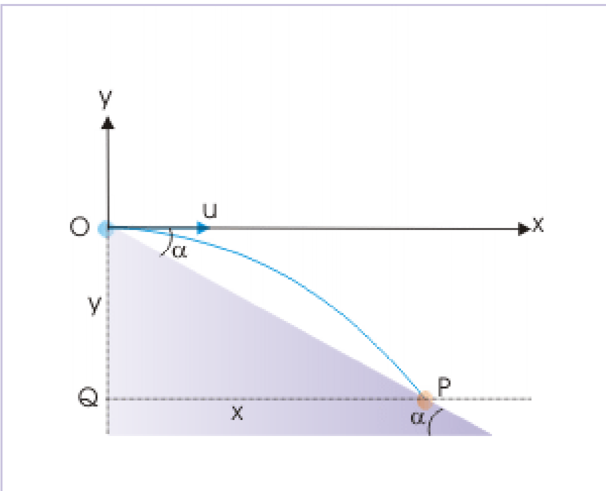
$$\Rightarrow y = -R\sin\alpha = -\frac{2u^2\sin\alpha}{g\cos^2\alpha}\sin\alpha$$

$$\Rightarrow y = -\frac{2u^2\tan^2\alpha}{g}$$

This example illustrated how to use formulae of the range of flight. We should, however, know that actually, we have the options to analyze projectile motion down an incline without using derived formula.

As a matter of fact, we can consider projectile motion down an incline as equivalent to projectile motion from an elevated point as studied in the previous module with out any reference to an incline or wedge. We need to only shift the horizontal base line to meet the point of return. The line joining the point of projection and point of return, then, represents the incline surface.

Projectile motion down an incline



A ball is projected in horizontal direction.

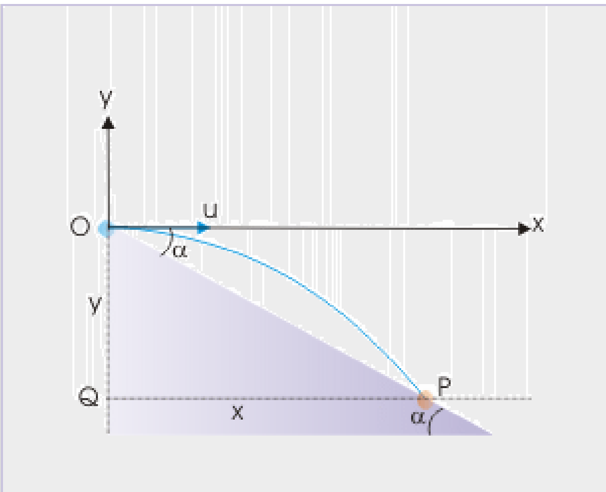
Observe the projectile motion from a height as shown in the figure. Let the projectile returns to a point “P”. The line “OP” then represents the incline surface. We can analyze this motion in rectangular coordinates “x” and “y” in horizontal and vertical direction, using general technique of analysis in component directions.

Here, we work with the same example as before to illustrate the working in this alternative manner.

Example:

Problem : A ball is projected in horizontal direction from an elevated point “O” of an incline of angle “α” with a speed “u”. If the ball hits the incline surface at a point “P” down the incline, find the coordinates of point “P”.

Projectile motion down from a height



A ball is projected in horizontal direction.

Solution : We draw or shift the horizontal base and represent the same by the line QP as shown in the figure below.
From the general consideration of projectile motion, the vertical displacement, "y", is :

$$y = u_y T + \frac{1}{2} a_y T^2$$

$$\Rightarrow y = 0 - \frac{1}{2} g T^2$$

Considering the magnitude of vertical displacement only, we have :

$$\Rightarrow y = \frac{1}{2} g T^2$$

On the other hand, consideration of motion in x-direction yields,

$$x = u_x T = uT$$

Since, we aim to find the coordinates of point of return "P", we eliminate "T" from the two equations. This gives us :

$$y = \frac{gx^2}{2u^2}$$

From the triangle OPQ, we have :

$$\tan\alpha = \frac{y}{x}$$

$$\Rightarrow y = x\tan\alpha$$

Combining two equations, we have :

$$y = x\tan\alpha = \frac{gx^2}{2u^2}$$

$$\Rightarrow \tan\alpha = \frac{gx}{2u^2}$$

$$\Rightarrow x = \frac{2u^2\tan\alpha}{g}$$

and

$$\Rightarrow y = x\tan\alpha = \frac{2u^2\tan^2\alpha}{g}$$

The y-coordinate, however, is below origin and is negative. Thus, we put a negative sign before the expression :

$$\Rightarrow y = -\frac{2u^2\tan^2\alpha}{g}$$

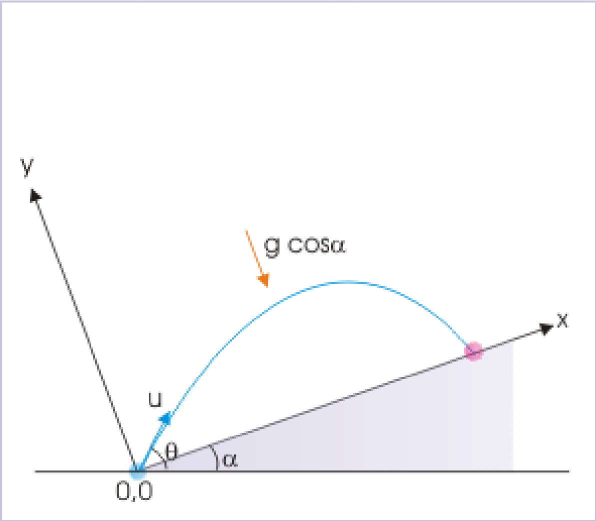
Exercises

Exercise:

Problem:

A projectile is thrown from the base of an incline of angle 30° as shown in the figure. It is thrown at an angle of 60° from the horizontal direction at a speed of 10 m/s. The total time of flight is (consider g = 10 m/s^2) :

Projectile motion on an incline



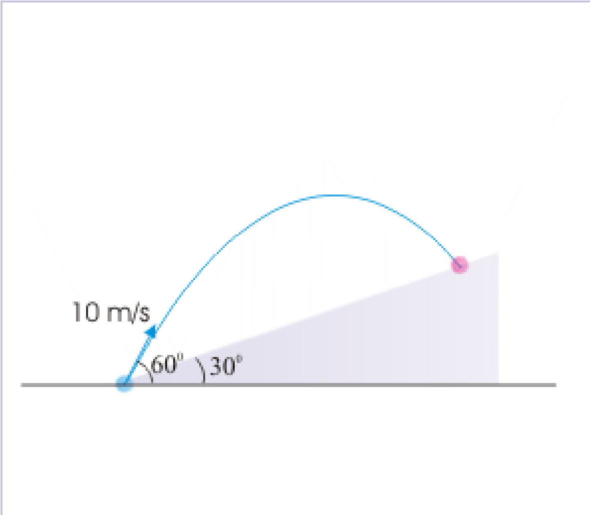
Projectile motion on an incline

$$(a)2 \quad (b)\sqrt{3} \quad (c)\frac{\sqrt{3}}{2} \quad (d)\frac{2}{\sqrt{3}}$$

Solution:

This situation can be handled with a reoriented coordinate system as shown in the figure. Here, angle of projection with respect to x - direction is (θ-α) and acceleration in y - direction is "g cosα". Now, total time of flight for projectile motion, when points of projection and return are on same level, is :

Projectile motion on an incline



Projectile motion on an incline

$$T = \frac{2u \sin \theta}{g}$$

Replacing "θ" by "(θ-α)" and "g" by "gcosα", we have formula of time of flight over the incline :

$$\Rightarrow T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

Now, θ = 60° , α = 30°, u = 10 m/s. Putting these values,

$$\Rightarrow T = \frac{2X10 \sin\left(60^0 - 30^0\right)}{g \cos 30^0} = \frac{20 \sin 30^0}{10 \cos 30^0} = \frac{2}{\sqrt{3}}$$

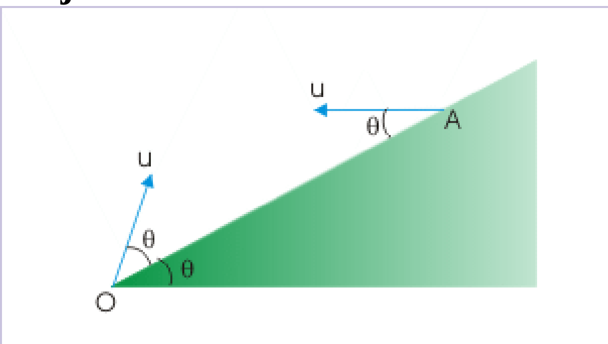
Hence, option (d) is correct.

Exercise:

Problem:

Two projectiles are thrown with the same speed from point "O" and "A" so that they hit the incline. If t_O and t_A be the time of flight in two cases, then :

Projectile motion on an incline



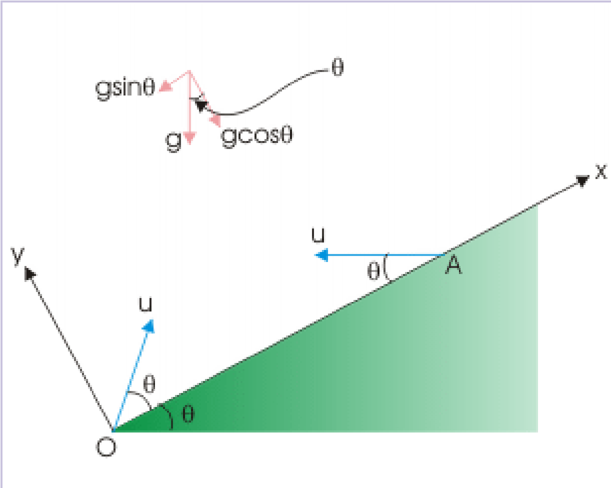
Projectile motion on an incline

$$(a) t_O = t_A \quad (b) t_O < t_A \quad (c) t_O > t_A \quad (d) t_O = t_A = \frac{u \tan\theta}{g}$$

Solution:

We have discussed that projectile motion on an incline surface can be rendered equivalent to projectile motion on plane surface by reorienting coordinate system as shown here :

Projectile motion on an incline



Projectile motion on an incline

In this reoriented coordinate system, we need to consider component of acceleration due to gravity along y-direction. Now, time of flight is given by :

$$t = \frac{2u_y}{g\cos\theta}$$

Let us first consider the projectile thrown from point "O". Considering the angle the velocity vector makes with the horizontal, the time of flight is given as :

$$\Rightarrow t_O = \frac{2u\sin(2\theta - \theta)}{g\cos\theta}$$

$$\Rightarrow t_O = \frac{2u\tan\theta}{g}$$

For the projectile thrown from point "A", the angle with horizontal is zero. Hence, the time of flight is :

$$\Rightarrow t_A = \frac{2u\sin(2X0 + \theta)}{g\cos\theta} = \frac{2u\tan\theta}{g}$$

Thus, we see that times of flight in the two cases are equal.

$$\Rightarrow t_A = t_O$$

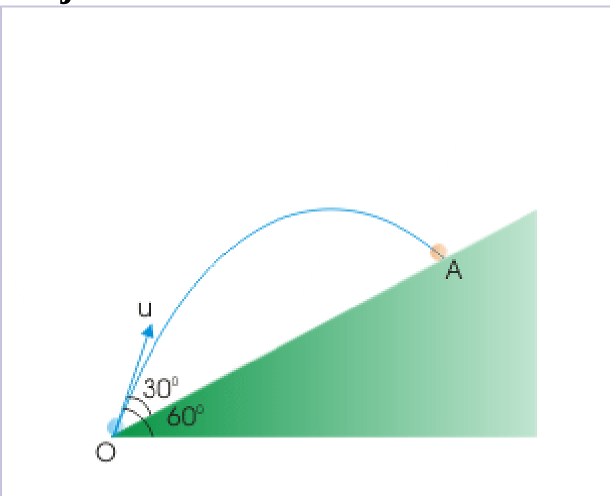
Hence, option (a) is correct.

Exercise:

Problem:

A ball is projected on an incline of 30° from its base with a speed 20 m/s, making an angle 60° from the horizontal. The magnitude of the component of velocity, perpendicular to the incline, at the time ball hits the incline is :

Projectile motion on an incline



Projectile motion on an incline

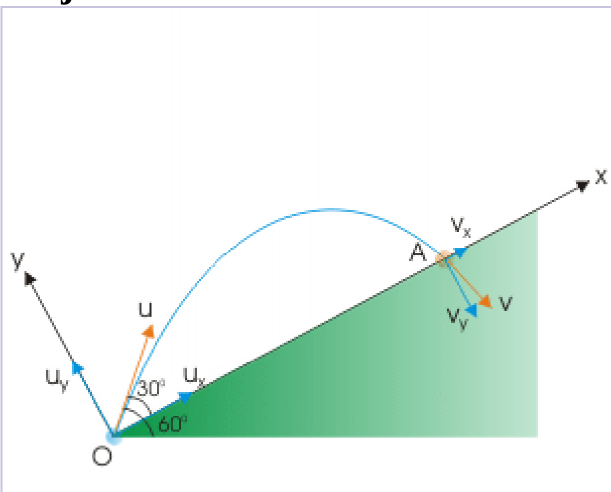
$$(a) 10 \quad m/s \quad (b) 10\sqrt{3} \quad m/s \quad (c) 20\sqrt{3} \quad m/s \quad (d) 20\sqrt{3} \quad m/s$$

Solution:

The velocity in y-direction can be determined making use of the fact that a ball under constant acceleration like gravity returns to the ground

with the same speed, but inverted direction. The component of velocity in y-direction at the end of the journey, in this case, is :

Projectile motion on an incline



Projectile motion on an incline

$$v_y = u_y = -20 \sin 30^0 = -20X\frac{1}{2} = -10 \quad m/s$$

$$|v_y| = 10 \quad m/s$$

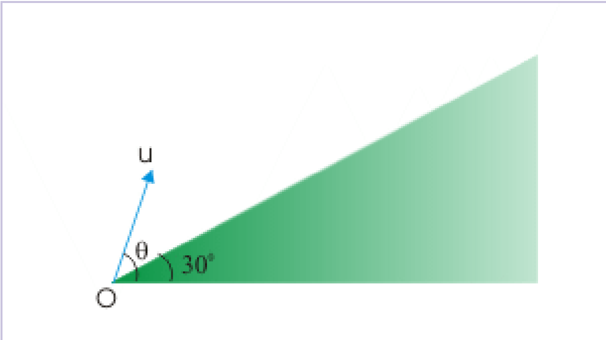
Hence, option (a) is correct.

Exercise:

Problem:

A projectile is projected from the foot of an incline of angle 30°. What should be the angle of projection, as measured from the horizontal direction so that range on the incline is maximum?

Projectile motion on an incline



Projectile motion on an incline

$$(a)45^\circ \quad (b)60^\circ \quad (c)75^\circ \quad (d)90^\circ$$

Solution:

We can interpret the equation obtained for the range of projectile :

$$R = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta - \alpha) - \sin\alpha\right]$$

The range is maximum for the maximum value of "sin(2θ-α) “ :

$$\sin(2\theta - \alpha) = 1 = \sin 90^0$$

$$\Rightarrow 2\theta - 30^0 = 90^0$$

$$\Rightarrow \theta = 60^0$$

Hence, option (b) is correct.

Exercise:

Problem:

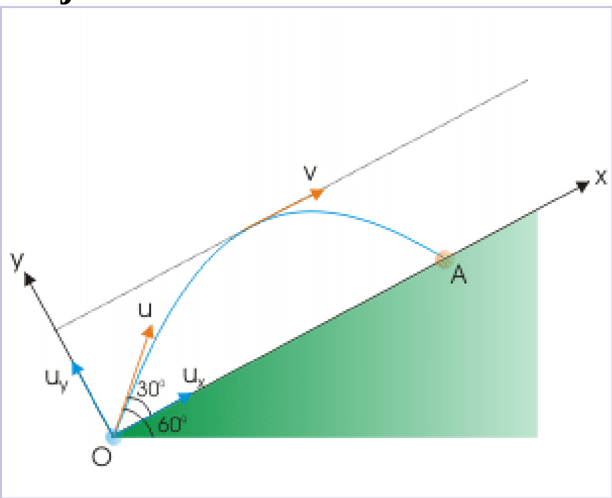
A projectile is projected from the foot of an incline of angle 30° with a velocity 30 m/s. The angle of projection as measured from the horizontal is 60°. What would be its speed when the projectile is parallel to the incline?

$$(a) 10 \quad m/s \quad (b) 2\sqrt{3} \quad m/s \quad (c) 5\sqrt{3} \quad m/s \quad (d) 10\sqrt{3} \quad m/s$$

Solution:

In the coordinate system of incline and perpendicular to incline, motion parallel to incline denotes a situation when component of velocity in y-direction is zero. Note that this is an analogous situation to the point of maximum height in the normal case when projectile returns to same level.

Projectile motion on an incline



Projectile motion on an incline

We shall analyze the situation, taking advantage of this fact. Since component of velocity in y - direction is zero, it means that velocity of

projectile is same as that of component velocity in x-direction. For consideration of motion in y -direction, we have :

$$v_y = u_y + a_y t$$

$$\Rightarrow 0 = 30 \sin 30^0 - g \cos 30^0 X t$$

$$\Rightarrow t = \frac{15X2}{10X\sqrt{3}} = \sqrt{3} s$$

For consideration of motion in x -direction, we have :

$$v_x = u_x + a_x t = 30 \cos 30^0 - g \sin 30^0 X t$$

$$\Rightarrow v_x = 30X\frac{\sqrt{3}}{2} - 10X\frac{1}{2}X\sqrt{3}$$

$$\Rightarrow v_x = 15X\sqrt{3} - 5X\sqrt{3} = 10\sqrt{3}$$

$$\Rightarrow v_x = 10\sqrt{3} \quad m/s$$

But, component of velocity in y-direction is zero. Hence,

$$v = v_A = 10\sqrt{3} \quad m/s$$

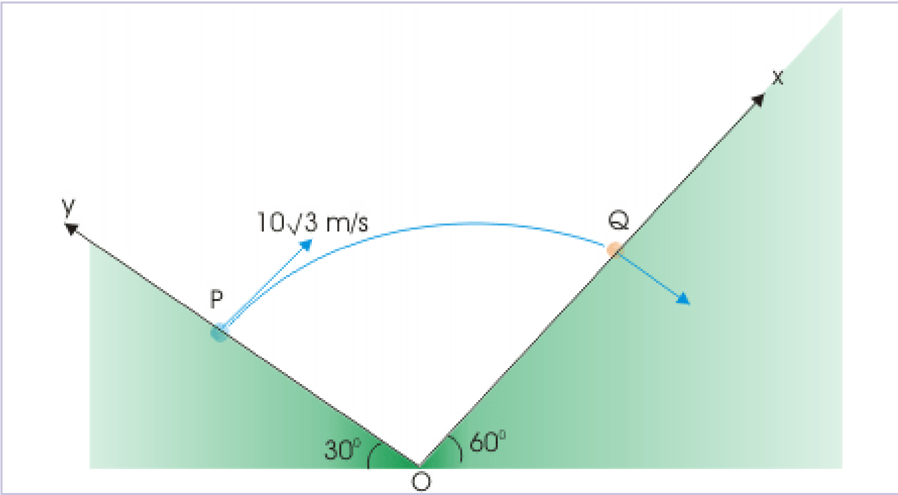
Hence, option (d) is correct.

Exercise:

Problem:

Two incline planes of angles 30° and 60° are placed touching each other at the base as shown in the figure. A projectile is projected at right angle with a speed of $10\sqrt{3}$ m/s from point "P" and hits the other incline at point "Q" normally. If the coordinates are taken along the inclines as shown in the figure, then

Projectile motion on an incline



Projectile motion on an incline

(a) component of acceleration in x-direction is $-5\sqrt{3}m/s^2$

(b) component of acceleration in x-direction is $-10\sqrt{3}m/s^2$

(c) component of acceleration in y-direction is $-5\sqrt{3}m/s^2$

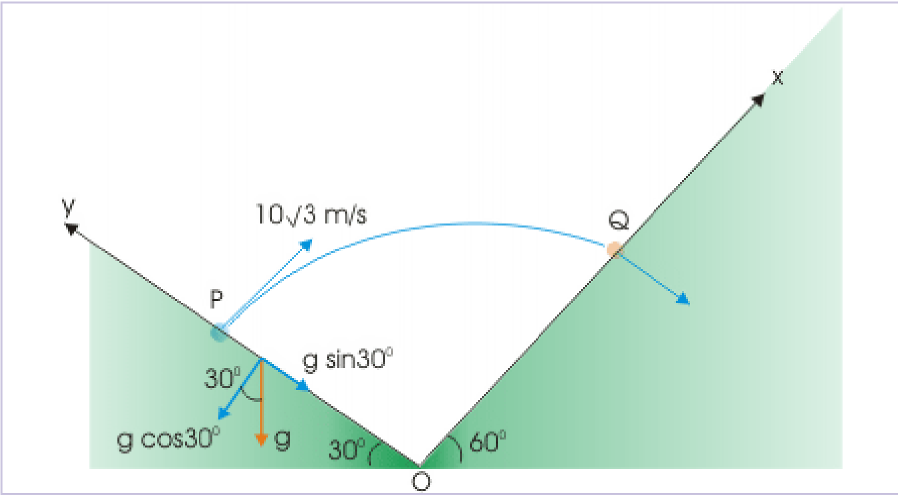
(d) component of acceleration in y-direction is $-5m/s^2$

Solution:

This arrangement is an specific case in which incline plane are right angle to each other. We have actually taken advantage of this fact in assigning our coordinates along the planes, say y-axis along first incline and x-axis against second incline.

The acceleration due to gravity is acting in vertically downward direction. We can get the component accelerations either using the angle of first or second incline. Either of the considerations will yield same result. Considering first incline,

Projectile motion on an incline



Projectile motion on an incline

$$a_x = -g\cos 30^0 = -10X\frac{\sqrt{3}}{2} = -5\sqrt{3}m/s^2$$

$$a_y = -g\sin 30^0 = -10X\frac{1}{2} = -5 \quad m/s^2$$

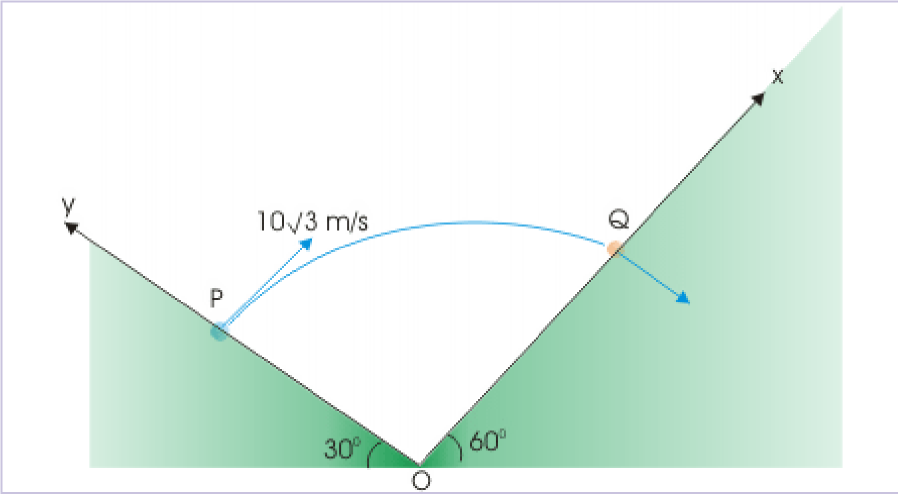
Hence, options (a) and (d) are correct.

Exercise:

Problem:

Two incline planes of angles 30° and 60° are placed touching each other at the base as shown in the figure. A projectile is projected at right angle with a speed of $10\sqrt{3}$ m/s from point "P" and hits the other incline at point "Q" normally. Then, the time of flight is :

Projectile motion on an incline



Projectile motion on an incline

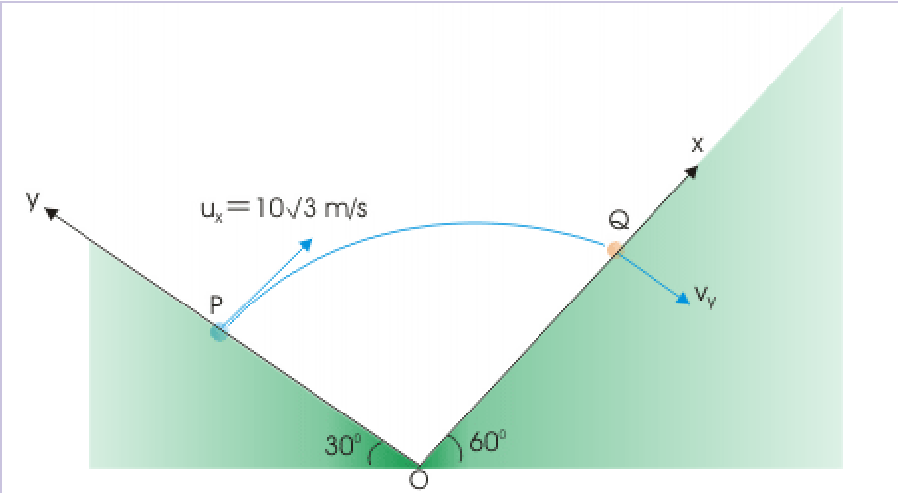
$$(a) 1s \quad (b) 2s \quad (c) 3s \quad (d) 4s$$

Solution:

This arrangement is an specific case in which incline plane are right angle to each other. We have actually taken advantage of this fact in assigning our coordinates along the planes, say y-axis along first incline and x-axis against second incline.

In order to find the time of flight, we can further use the fact that projectile hits the other plane at right angle i.e. parallel to y-axis. This means that component of velocity in x-direction i.e. along the second incline is zero. This, in turn, suggests that we can analyze motion in x-direction to obtain time of flight.

Projectile motion on an incline



Projectile motion on an incline

In x-direction,

$$v_x = u_x + a_x T$$

$$\Rightarrow 0 = u_x + a_x T$$

$$\Rightarrow T = -\frac{u_x}{a_x}$$

We know need to know "ux” and "ax” in this coordinate system. The acceleration due to gravity is acting in vertically downward direction. Considering first incline,

$$a_x = -g\cos 30^0 = -10X\frac{\sqrt{3}}{2} = -5\sqrt{3}m/s^2$$

$$a_y = -g\sin 30^0 = -10X\frac{1}{2} = -5 \quad m/s^2$$

Also, we observe that projectile is projected at right angle. Hence, component of projection velocity in x - direction is :

$$u_x = 10\sqrt{3} \quad m/s$$

Putting values in the equation and solving, we have :

$$\Rightarrow T = -\frac{u_x}{a_x} = \frac{-10\sqrt{3}}{-5\sqrt{3}} = 2 \quad s$$

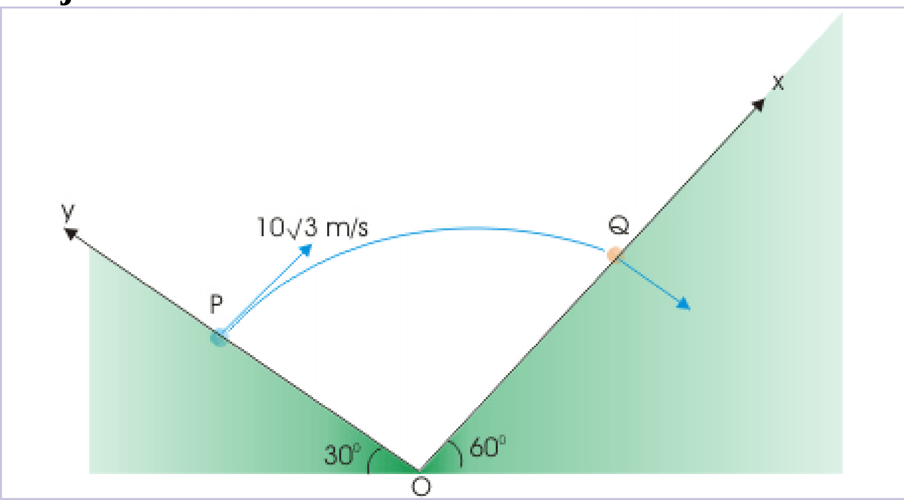
Hence, option (b) is correct.

Exercise:

Problem:

Two incline planes of angles 30° and 60° are placed touching each other at the base as shown in the figure. A projectile is projected at right angle with a speed of $10\sqrt{3}$ m/s from point "P" and hits the other incline at point "Q" normally. The speed with which the projectile hits the incline at "Q" is :

Projectile motion on an incline



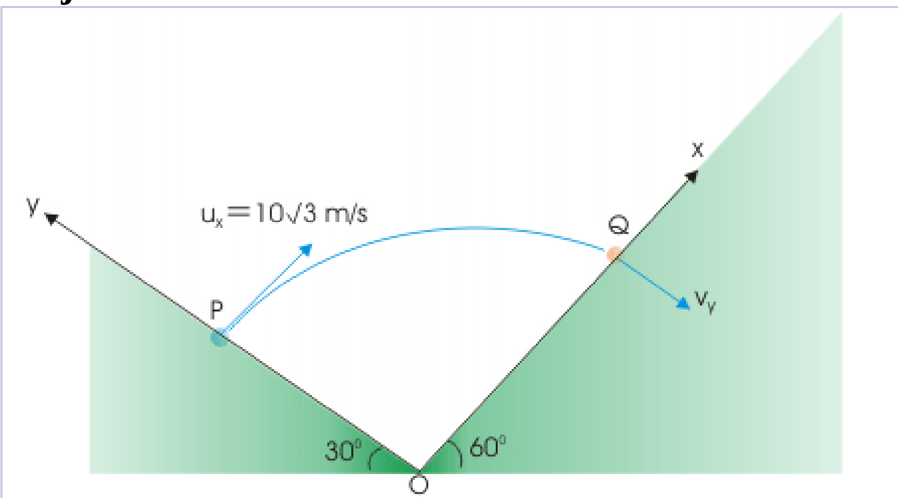
Projectile motion on an incline

$$(a) 5 \quad m/s \quad (b) 10 \quad m/s \quad (c) 10\sqrt{3} \quad m/s \quad (d) 20 \quad m/s$$

Solution:

We notice here that initial velocity in y-direction is zero. On the other hand, final velocity in the y-direction is equal to the velocity with which projectile hits at "Q". The x-component of velocity at "Q" is zero. The analysis of motion in y - direction gives us the relation for component of velocity in y-direction as :

Projectile motion on an incline



Projectile motion on an incline

In y-direction,

$$v = v_y = u_y + a_y$$

$$\Rightarrow T = 0 + a_y T$$

$$\Rightarrow v = v_y = a_y T$$

Thus, we need to know component of acceleration in y-direction and time of flight. As far as components of acceleration are concerned, the acceleration due to gravity is acting in vertically downward direction. Considering first incline, we have :

$$a_x = -g \cos 30^0 = -10X\frac{\sqrt{3}}{2} = -5\sqrt{3}m/s^2$$

$$a_y = -g\sin 30^0 = -10X\frac{1}{2} = -5 \quad m/s^2$$

In order to find the time of flight, we can further use the fact that the component of velocity in x-direction i.e. along the second incline is zero. This, in turn, suggests that we can analyze motion in x-direction to obtain time of flight.

In x-direction,

$$v_x = u_x + a_xT$$

$$\Rightarrow 0 = u_x + a_xT$$

$$\Rightarrow T = -\frac{u_x}{a_x}$$

We know need to know "ux" in this coordinate system. We observe that projectile is projected at right angle. Hence, component of projection velocity in x - direction is :

$$u_x = 10\sqrt{3} \quad m/s$$

Putting values in the equation and solving, we have :

$$\Rightarrow T = -\frac{u_x}{a_x} = \frac{-10\sqrt{3}}{-5\sqrt{3}} = 2 \quad s$$

Thus putting values for the expression for the speed of the projectile with which it hits the incline is :

$$\Rightarrow v = v_y = a_yT = -5X2 = -10 \quad m/s$$

Thus, speed is 10 m/s.

Hence, option (b) is correct.

Projectile motion on an incline (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints for solving problems

Problems based on projectile motion over an incline are slightly difficult. The analysis is complicated mainly because there are multitudes of approaches available. First there is issue of coordinates, then we might face the conflict to either use derived formula or analyze motion independently in component directions and so on. We also need to handle motion up and down the incline in an appropriate manner. However, solutions get easier if we have the insight into the working with new set of coordinate system and develop ability to assign appropriate values of accelerations, angles and component velocities etc.

Here, we present a simple set of guidelines in a very general way :

1: Analyze motion independently along the selected coordinates. Avoid using derived formula to the extent possible.

2: Make note of information given in the question like angles etc., which might render certain component of velocity zero in certain direction.

3: If range of the projectile is given, we may try the trigonometric ratio of the incline itself to get the answer.

4: If we use coordinate system along incline and in the direction perpendicular to it, then always remember that component motion along both incline and in the direction perpendicular to it are accelerated motions. Ensure that we use appropriate components of acceleration in the equations of motion.

Representative problems and their solutions

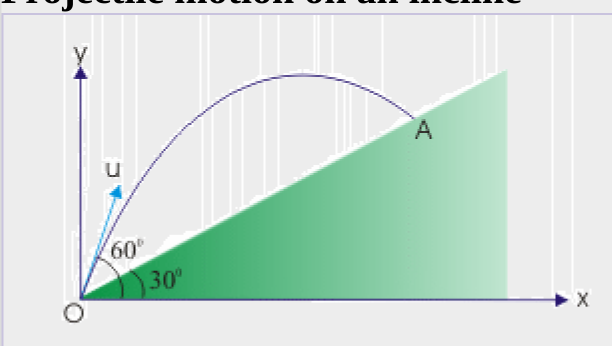
We discuss problems, which highlight certain aspects of the study leading to the concept of projectile motion on an incline. The questions are categorized in terms of the characterizing features of the subject matter :

- Range of the flight
- Angle of projection
- Final Speed of the projectile
- Elastic collision with the incline
- Projectile motion on double inclines

Range of the flight

Example:
Problem : A projectile is thrown with a speed "u" at an angle 60° over an incline of 30°. If the time of flight of the projectile is "T", then find the range of the flight.

Projectile motion on an incline

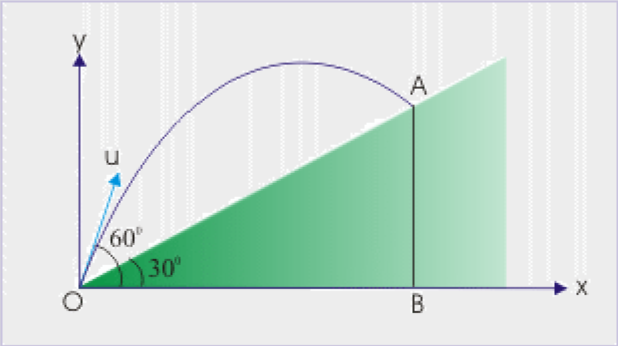


The time of flight is "T".

Solution : We can see here that time of flight is already given. We can find range considering projectile motion in the coordinates of horizontal and vertical axes. The range of the projectile "R" is obtained by using

trigonometric ratio in triangle OAB. The range is related to horizontal base "OB" as :

Projectile motion on an incline



The time of flight is "T".

$$\cos 30^0 = \frac{OB}{OA}$$

$$R = OA = \frac{OB}{\cos 30^0} = OB \sec 30^0$$

Now, we can find OB by considering motion in horizontal direction :

$$\Rightarrow R = OB = u_x T = u \cos 60^0 T = \frac{uT}{2}$$

Thus, the range of the projectile, OA, is :

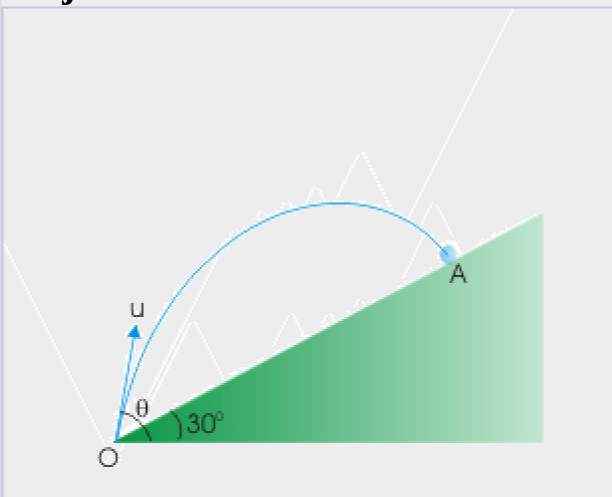
$$\Rightarrow R = OA = \frac{uT \sec 30^0}{2} = \frac{uT}{\sqrt{3}}$$

Angle of projection

Example:

Problem : A particle is projected from the foot of an incline of angle “30°” at a certain velocity so that it strikes the incline normally. Find the angle of projection (θ) as measured from the horizontal.

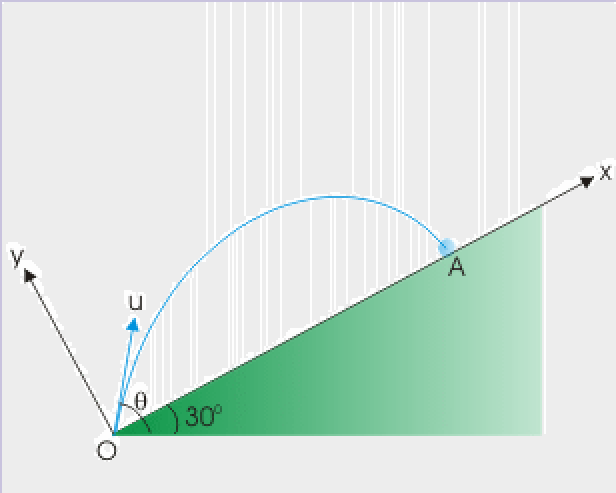
Projectile motion on an incline



The projectile hits the incline normally.

Solution : Here, projectile hits the incline normally. It means that component of velocity along the incline is zero. We should remember that the motion along the incline is not uniform motion, but a decelerated motion. In order to take advantage of the fact that final component velocity along incline is zero, we consider motion in a coordinate system along the incline and along a direction perpendicular to it.

Projectile motion on an incline



The projectile hits the incline normally.

$$v_x = u_x + a_x T$$

$$\Rightarrow 0 = u\cos\left(\theta - 30^0\right) - g\sin 30^0 T$$

$$T = \frac{u\cos\left(\theta - 30^0\right)}{g\sin 30^0}$$

Now, time of flight is also given by the formulae :

$$\Rightarrow T = \frac{2u\sin\left(\theta - 30^0\right)}{g\cos 30^0}$$

Equating two expressions for time of flight, we have :

$$\Rightarrow \frac{2u\sin\left(\theta - 30^0\right)}{g\cos 30^0} = \frac{u\cos\left(\theta - 30^0\right)}{g\sin 30^0}$$

$$\Rightarrow \tan\left(\theta - 30^0\right) = \frac{1}{2\tan 30^0} = \frac{\sqrt{3}}{2}$$

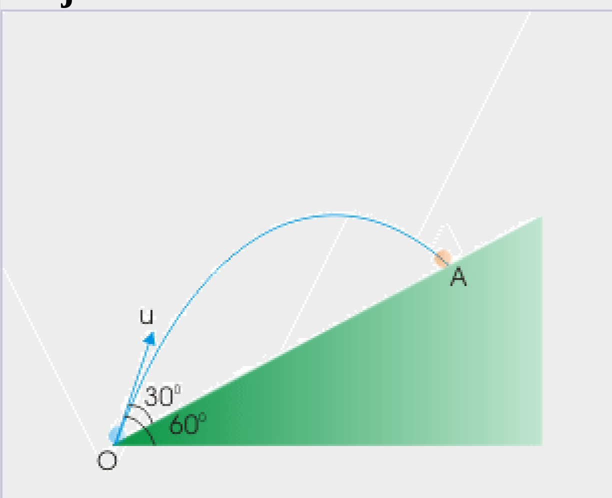
$$\Rightarrow \theta = 30^0 + \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Final speed of the projectile

Example:

Problem : A ball is projected on an incline of 30° from its base with a speed 20 m/s, making an angle 60° from the horizontal. Find the speed with which the ball hits the incline.

Projectile motion on an incline



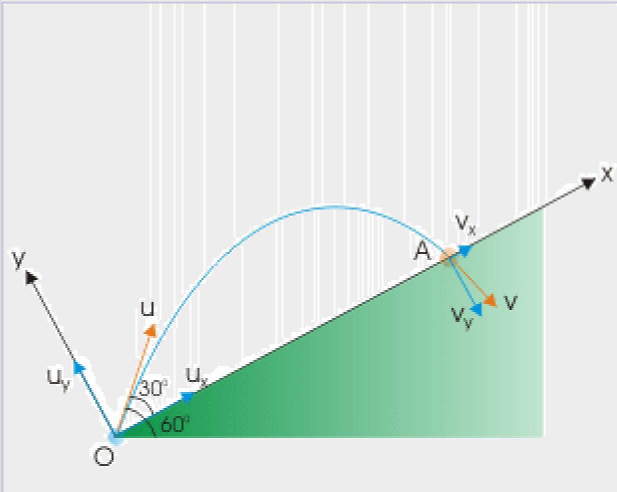
The projectile is projected from base of the incline.

Solution : We analyze this problem in the coordinates along the incline (x-axis) and in the direction perpendicular to the incline (y-axis). In order to find the speed at the end of flight, we need to find the component velocities in "x" and "y" directions.

The velocity in y-direction can be determined making use of the fact that a ball under constant acceleration like gravity returns to the ground with the

same speed, but in opposite direction. The component of velocity in y-direction at the end of the journey, therefore, is :

Projectile motion on an incline



The projectile is projected from base of the incline.

$$v_y = -u_y = -20\sin 30^0 = -20X\frac{1}{2} = -10 \quad m/s$$

Now, we should attempt to find the component of velocity in x-direction. We should, however, recall that motion in x-direction is not a uniform motion, but has deceleration of “-gsinα”. Using equation of motion in x-direction, we have :

$$v_x = u_x + a_x T$$

Putting values,

$$v_x = u\cos 30^0 - g\sin 30^0 T = 20X\frac{\sqrt{3}}{2} - 10X\frac{1}{2}XT = 10\sqrt{3} - 5T$$

Clearly, we need to know time of flight to know the component of velocity in x-direction. The time of flight is given by :

$$T = \frac{2u_y}{a_y} = \frac{2u\sin 30^0}{g\cos 30^0} = \frac{2X20}{\sqrt{3}X10} = \frac{4}{\sqrt{3}}$$

Hence, component of velocity in x-direction is :

$$\Rightarrow vx = 10\sqrt{3} - 5T = 10\sqrt{3} - 5X\frac{4}{\sqrt{3}}$$

$$vx = \frac{30-20}{\sqrt{3}} = \frac{10}{\sqrt{3}}$$

The speed of the projectile is equal to resultant of components :

$$\Rightarrow v = \sqrt{\left\{(-10)^2 + \left(\frac{10}{\sqrt{3}}\right)^2\right\}}$$

$$v = \sqrt{\left(\frac{400}{3}\right)} = \frac{20}{\sqrt{3}} \quad m/s$$

Elastic collision with the incline

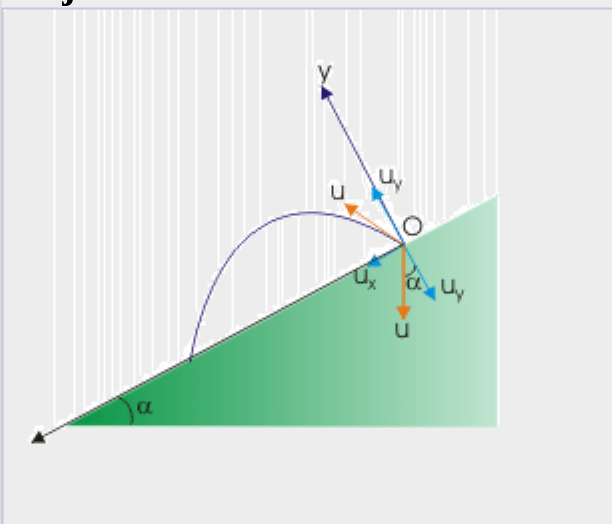
Example:
Problem : A ball falls through a height “H” and impacts an incline elastically. Find the time of flight between first and second impact of the projectile on the incline.
Solution : The ball falls vertically through a distance "H". We can get the value of initial speed by considering free fall of the ball before impacting incline.

$$\Rightarrow 0 = u^2 - 2gH$$

$$\Rightarrow u = \sqrt{(2gH)}$$

In order to answer this question, we need to identify the velocity with which projectile rebounds. Since impact is considered elastic, the projectile is rebounded without any loss of speed. The projectile is rebounded such that angle of incidence i.e. the angle with the normal is equal to angle of reflection.

Projectile motion on an incline



The ball impacts incline elastically.

The components of velocity along the incline and perpendicular to it are shown in the figure. The motion of ball, thereafter, is same as that of a projectile over an incline. Here, we shall analyze motion in y-direction (normal to the incline) to find the time of flight. We note that the net displacement between two strikes is zero in y – direction.

Applying equation of motion

$$y = u_y T + \frac{1}{2} a_y T^2$$

$$\Rightarrow 0 = u \cos \alpha T - \frac{1}{2} g \cos \alpha T^2$$

$$T = 0$$

Or

$$T = \frac{2u\cos\alpha}{g\cos\alpha} = \frac{2u}{g}$$

Putting value of initial speed in the equation of time of flight, we have :

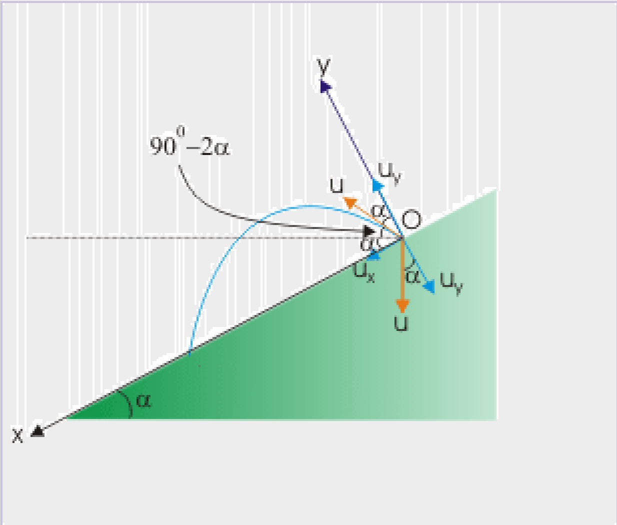
$$\Rightarrow T = \sqrt{\left(\frac{8H}{g}\right)}$$

It should be noted here that we can find the time of flight also by using standard formula of time of flight for projectile motion down the incline. The time of flight for projection down an incline is given as :

$$T = \frac{2u\sin(\theta + \alpha)}{g\cos\alpha}$$

We need to be careful while appropriating angles in the above expression. It may be recalled that all angles are measured from the horizontal. We redraw the figure to denote the value of angle of projection "θ" from the horizon.

Projectile motion on an incline



The ball impacts incline

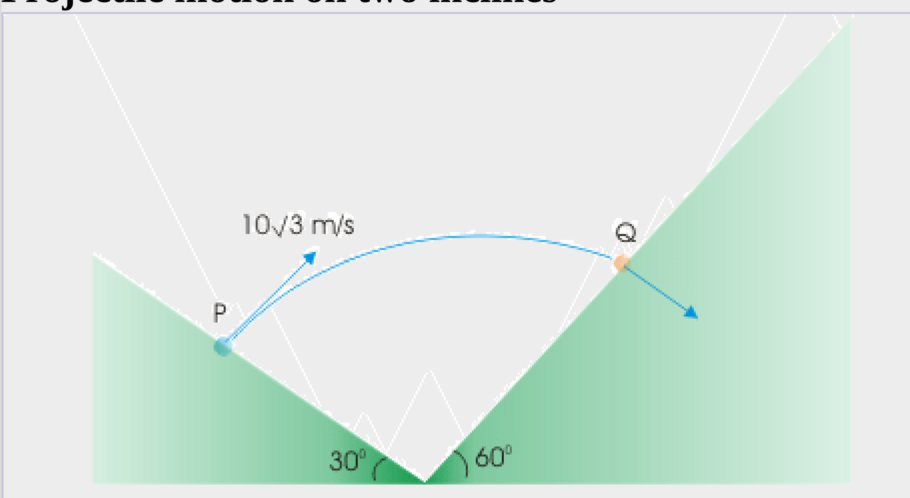
elastically.

$$\Rightarrow T=\frac{2u\sin\left(90^0-2\alpha+\alpha\right)}{g\cos\alpha}=\frac{2u\sin\left(90^0-\alpha\right)}{g\cos\alpha}$$

$$\Rightarrow T=\frac{2u\cos\alpha}{g\cos\alpha}=\frac{2u}{g}=\sqrt{\left(\frac{8H}{g}\right)}$$

Projectile motion on two inclines

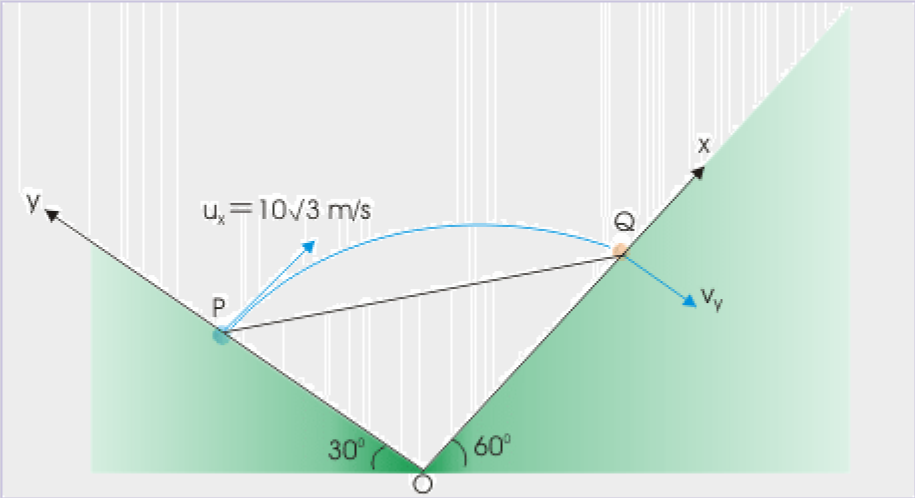
Example:
Problem : Two incline plane of angles 30° and 60° are placed touching each other at the base as shown in the figure. A projectile is projected at right angle with a speed of 10√3 m/s from point "P" and hits the other incline at point "Q" normally. Find the linear distance between PQ.
Projectile motion on two inclines



A projectile projected at right angle from an incline hits another incline at right angle.

Solution : We notice here OPQ forms a right angle triangle at “O”. The linear distance, “PQ” is related as :

Projectile motion on two inclines



A projectile projected at right angle from an incline hits another incline at right angle.

$$PQ^2 = OP^2 + OQ^2$$

In order to find “PQ”, we need to know “OP” and “OQ”. We can find “OP”, considering motion in y-direction.

$$y = OP = u_y T + \frac{1}{2} a_y T^2 = 0 + \frac{1}{2} a_y T^2 = \frac{1}{2} a_y T^2$$

Similarly,

$$x = OQ = u_x T + \frac{1}{2} a_x T^2 = 10\sqrt{3} T + \frac{1}{2} a_x T^2$$

In order to evaluate these two relations, we need to find components of accelerations and time of flight.

Considering first incline, we have :

$$a_x = -g\cos 30^0 = -10X\frac{\sqrt{3}}{2} = -5\sqrt{3} \quad m/s^2$$

$$a_y = -g\sin 30^0 = -10X\frac{1}{2} = -5 \quad m/s^2$$

In order to find the time of flight, we can further use the fact that the component of velocity in x-direction i.e. along the second incline is zero. This, in turn, suggests that we can analyze motion in x-direction to obtain time of flight.
In x-direction,

$$v_x = u_x + a_x T$$

$$\Rightarrow 0 = u_x + a_x T$$

$$\Rightarrow T = -\frac{u_x}{a_x}$$

Putting values in the equation and solving, we have :

$$\Rightarrow T = -\frac{10\sqrt{3}}{-5\sqrt{3}} = 2s$$

Now, we can evaluate “x” and “y” displacements as :

$$\Rightarrow y = OP = \frac{1}{2}a_y T^2 = -\frac{1}{2}X5X2^2 = -10 \quad m$$

$$\Rightarrow x = OQ = 10\sqrt{3}x2 + \frac{1}{2} \; -5\sqrt{3} \;\; X2^2$$

$$x = OQ = 10\sqrt{3}$$

Considering positive values, the linear distance, “PQ” is given as :

$$PQ = \quad \overline{(OP^2 + OQ^2)} = \sqrt{\left\{(10)^2 + \;\; 10\sqrt{3} \;\; ^2\right\}} = 20m$$

Relative motion of projectiles

In this module, we shall apply the concept of relative velocity and relative acceleration to the projectile motion. The description here is essentially same as the analysis of relative motion in two dimensions, which was described earlier in the course except that there is emphasis on projectile motion. Besides, we shall extend the concept of relative motion to analyze the possibility of collision between projectiles.

We shall maintain the convention of subscript designation for relative quantities for the sake of continuity. The first letter of the subscript determines the “object”, whereas the second letter determines the “other object” with respect to which measurement is carried out. Some expansion of meaning is given here to quickly recapitulate uses of subscripted terms :

$\mathbf{v}_{AB}$: Relative velocity of object “A” with respect to object “B”

v_{ABx} : Component of relative velocity of object “A” with respect to object “B” in x-direction

For two dimensional case, the relative velocity is denoted with bold type vector symbol. We shall , however, favor use of component scalar symbol with appropriate sign to represent velocity vector in two dimensions like in the component direction along the axes of the coordinate system. The generic expression for two dimensional relative velocity are :

In vector notation :
Equation:

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$$

In component scalar form :
Equation:

$$v_{ABx} = v_{Ax} - v_{Bx}$$

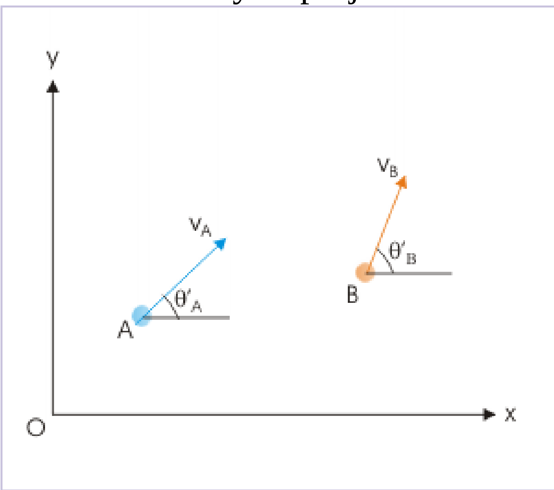
Equation:

$$v_{ABy} = v_{Ay} - v_{By}$$

Relative velocity of projectiles

The relative velocity of projectiles can be found out, if we have the expressions of velocities of the two projectiles at a given time. Let “ $\mathbf{v}_A$ ” and “ $\mathbf{v}_B$ ” denote velocities of two projectiles respectively at a given instant “t”. Then :

Relative velocity of projectiles



Velocities of projectiles.

$$\mathbf{v}_A = v_{Ax}\mathbf{i} + v_{Ay}\mathbf{j}$$

$$\mathbf{v}_B = v_{Bx}\mathbf{i} + v_{By}\mathbf{j}$$

Hence, relative velocity of projectile “A” with respect to projectile “B” is :

$$\mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B = v_{Ax}\mathbf{i} + v_{Ay}\mathbf{j} - v_{Bx}\mathbf{i} - v_{By}\mathbf{j}$$

Equation:

$$\mathbf{v}_{AB} = (v_{Ax} - v_{Bx})\mathbf{i} + (v_{Ay} - v_{By})\mathbf{j}$$

We can interpret this expression of relative velocity as equivalent to consideration of relative velocity in component directions. In the nutshell, it means that we can determine relative velocity in two mutually perpendicular directions and then combine them as vector sum to obtain the resultant relative velocity. Mathematically,

Equation:

$$\mathbf{v}_{AB} = v_{ABx}\mathbf{i} + v_{ABy}\mathbf{j}$$

where,

$$v_{ABx} = v_{Ax} - v_{Bx}$$

$$\mathbf{v}_{ABy} = v_{Ay} - v_{By}$$

This is a significant analysis simplification as study of relative motion in one dimension can be done with scalar representation with appropriate sign.

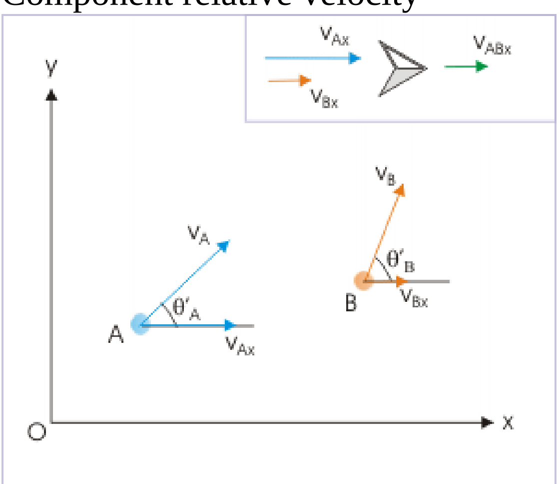
Interpretation of relative velocity of projectiles

The interpretation is best understood in terms of component relative motions. We consider motion in both horizontal and vertical directions.

Relative velocity in horizontal direction

The interpretation is best understood in terms of component relative motion. In horizontal direction, the motion is uniform for both projectiles. It follows then that relative velocity in horizontal x-direction is also a uniform velocity i.e. motion without acceleration.

Component relative velocity



Component relative velocity in x-direction.

The relative velocity in x-direction is :

$$v_{ABx} = v_{Ax} - v_{Bx}$$

As horizontal component of velocity of projectile does not change with time, we can re-write the equation of component relative velocity as :

Equation:

$$v_{ABx} = u_{Ax} - u_{Bx}$$

Significantly, relative velocity in x-direction is determined by the initial velocities or initial conditions of the projection of two projectiles. It is expected also as components of velocities in x-direction do not change with time. The initial velocity, on the other hand, is fixed for a given projectile motion. As such, horizontal component of relative velocity is constant. A plot of relative velocity in x - direction .vs. time will be a straight line parallel to time axis.

The separation between two objects in x-direction at a given time "t" depends on two factors : (i) the initial separation of two objects in x-direction and (ii) relative velocity in x - direction. The separation in x - direction is given as :

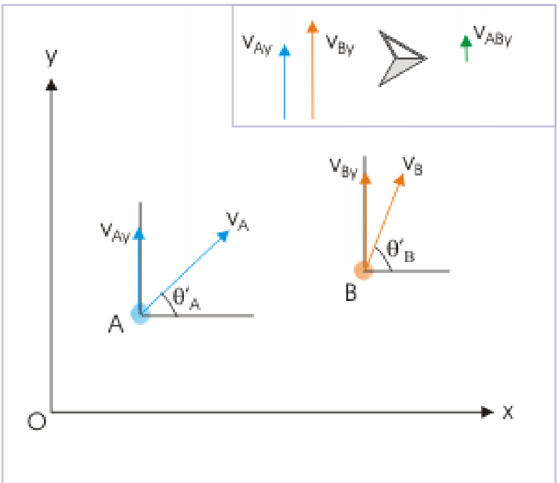
$$\Delta x = x_A - x_B = x_0 + v_{ABx} t$$

where x_0 is the initial separation between two projectiles in x-direction. Clearly, the separation in horizontal direction .vs. time plot would be a straight line with a constant slope. In physical terms, the separation between two projectiles in horizontal direction keeps increasing at a constant rate, which is equal to the magnitude of the component of relative velocity in that direction.

Relative velocity in vertical direction

The motion in the vertical direction, however, is subject to acceleration due to gravity, which always acts in vertically downward direction. The relative velocity in y-direction is :

Component relative velocity



Component relative velocity in y-direction.

$$v_{ABy} = v_{Ay} - v_{By}$$

As vertical component of motion is not a uniform motion, we can use equation of motion to determine velocity at a given time "t" as,

$$v_{Ay} = u_{Ay} - gt$$

$$v_{By} = u_{By} - gt$$

Putting in the expression of relative velocity in y-direction, we have :

$$v_{ABy} = v_{Ay} - v_{By} = u_{Ay} - gt - u_{By} + gt$$

Equation:

$$\Rightarrow v_{ABy} = u_{Ay} - u_{By}$$

The important aspect of the relative velocity in vertical y-direction is that acceleration due to gravity has not made any difference. The component relative velocity in y-direction is equal to simple difference of components of initial velocities of two projectiles in vertical direction. It is clearly due to the fact the acceleration of two projectiles in y-direction are same i.e. acceleration due to gravity and hence relative acceleration between two projectiles in vertical

direction is zero. It means that the nature of relative velocity in vertical direction is same as that in the horizontal direction. A plot of relative velocity in y -direction .vs. time will be a straight line parallel to time axis.

The separation between two objects in y-direction at a given time "t" depends on two factors : (i) the initial separation of two objects in y-direction and (ii) relative velocity in y - direction. The separation in y - direction is given as :

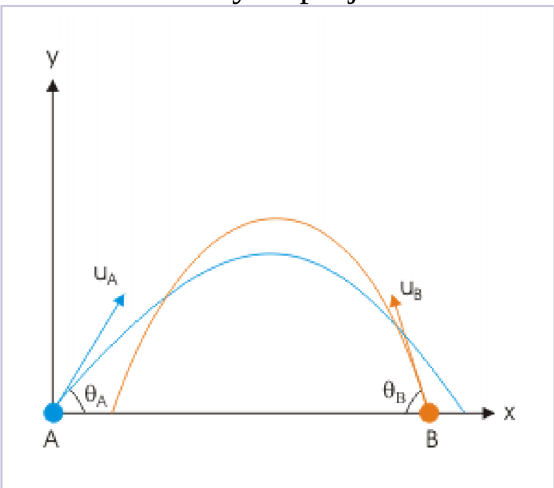
$$\Delta y = y_A - y_B = y_0 + v_{ABy} t$$

where y_0 is the initial separation between two projectiles in y-direction. Note that acceleration term has not appeared in the expression of relative velocity, because they cancel out. Clearly, the separation in vertical direction .vs. time plot would be a straight line with a constant slope. In physical terms, the separation between two projectiles in vertical direction keeps increasing at a constant rate, which is equal to the magnitude of the component of relative velocity in that direction.

Resultant relative motion

The component relative velocities in two mutually perpendicular directions have been derived in the previous section as :

Relative velocity of projectiles



Relative velocity of projectiles depends on initial velocities of projectiles.

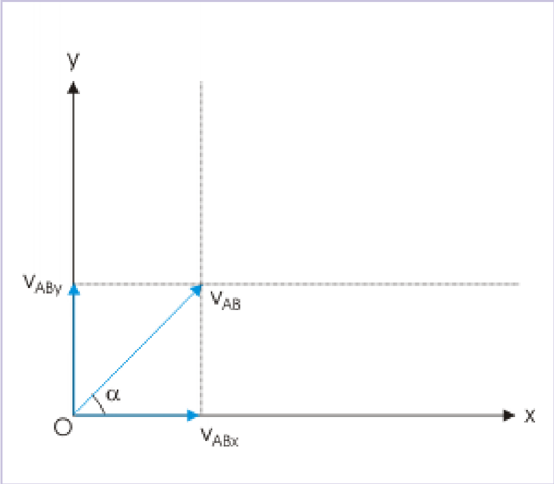
$$v_{ABx} = u_{Ax} - u_{Bx}$$

$$v_{ABy} = u_{Ay} - u_{By}$$

These equations are very important results. It means that relative velocity between projectiles is exclusively determined by initial velocities of the two projectiles i.e. by the initial conditions of the two projectiles as shown in the figure below. The component relative velocities do not depend on the subsequent motion i.e. velocities. The resultant relative velocity is vector sum of component relative velocities :

$$\mathbf{v}_{AB} = v_{ABx}\mathbf{i} + v_{ABy}\mathbf{j}$$

Resultant relative velocity of projectiles



Resultant relative velocity of projectiles is constant.

Since the component relative velocities do not depend on the subsequent motion, the resultant relative velocity also does not depend on the subsequent motion. The magnitude of resultant relative velocity is given by :

Equation:

$$\Rightarrow v_{AB} = \sqrt{v_{ABx}^2 + v_{ABy}^2}$$

The slope of the relative velocity of “A” with respect to “B” from x-direction is given as :

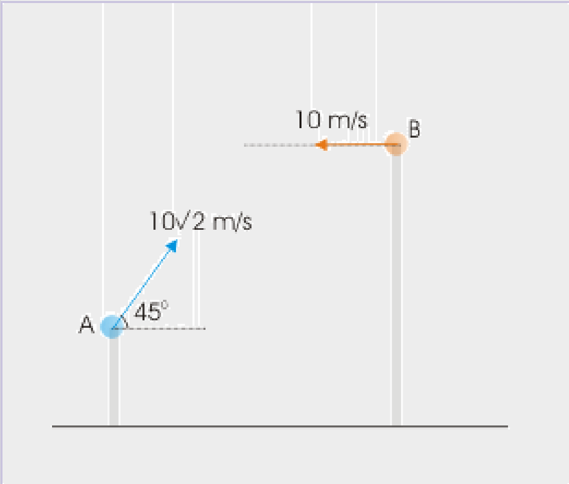
Equation:

$$\Rightarrow \tan\alpha = \frac{v_{ABy}}{v_{ABx}}$$

Example:

Problem : Two projectiles are projected simultaneously from two towers as shown in the figure. Find the magnitude of relative velocity, v_{AB}, and the angle that relative velocity makes with horizontal direction

Relative motion of projectiles



Relative motion of projectiles

Solution : We shall first calculate relative velocity in horizontal and vertical directions and then combine them to find the resultant relative velocity. Let "x" and "y" axes be in horizontal and perpendicular directions. In x-direction,

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\sqrt{2}\cos 45^0 - (-10) = 10\sqrt{2}X\frac{1}{\sqrt{2}} + 10 = 20 \quad m/s$$

In y-direction,

$$v_{ABy} = u_{Ay} - u_{By} = 10\sqrt{2}\cos 45^0 - 0 = 10\sqrt{2}X\frac{1}{\sqrt{2}} = 10 \quad m/s$$

The magnitude of relative velocity is :

$$\Rightarrow v_{AB} = \overline{\left(v_{ABx}^2 + v_{ABy}^2\right)} = \sqrt{\;20^2 + 10^2\;} = 10\sqrt{5} \quad m/s$$

The angle that relative velocity makes with horizontal is :

$$\Rightarrow \tan\theta = \frac{v_{ABy}}{v_{ABx}} = \frac{10}{20} = \frac{1}{2}$$

$$\Rightarrow \theta = \tan^{-1} \quad \frac{1}{2}$$

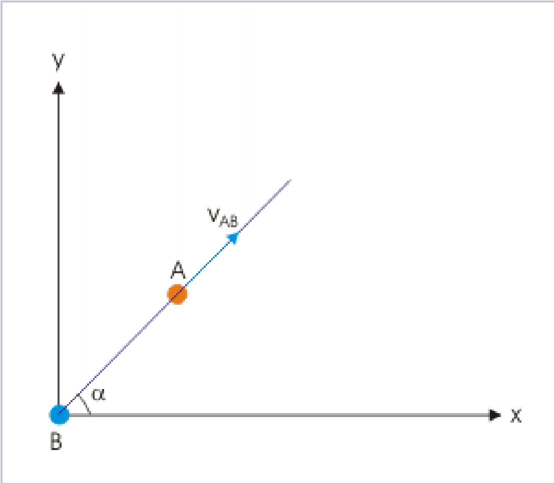
Physical interpretation of relative velocity of projectiles

The physical interpretation of the results obtained in the previous section will help us to understand relative motion between two projectiles. We recall that relative velocity can be interpreted by assuming that the reference object is stationary. Consider the expression, for example,

$$v_{ABx} = u_{Ax} - u_{Bx}$$

What it means that relative velocity “vABx” of object “A” with respect to object “B” in x-direction is the velocity of the object “A” in x-direction as seen by the stationary object “B”. This interpretation helps us in understanding the nature of relative velocity of projectiles.

Extending the reasoning, we can say that object “A” is moving with uniform motion in “x” and “y” directions as seen by the stationary object “B”. The resultant motion of “A”, therefore, is along a straight line with a constant slope. This result may be a bit surprising as we might have expected that two projectiles see (if they could) each other moving along some curve - not a straight line.



Relative velocity of projectiles is constant.

Special cases

There are two interesting cases. What if horizontal component of velocities of the two projectiles are same? In this case, relative velocity of projectiles in horizontal direction is zero. Also, it is imperative that there is no change in the initial separation between two projectiles in x-direction. Mathematically,

$$v_{ABx} = u_{Ax} - u_{Bx} = 0$$

There is no relative motion between two projectiles in horizontal direction. They may, however, move with respect to each other in y-direction. The relative velocity in y-direction is given by :

$$v_{ABy} = u_{Ay} - u_{By}$$

Since relative velocity in x-direction is zero, the relative velocity in y-direction is also the net relative velocity between two projectiles.

In the second case, components of velocities in y-direction are equal. In this case, there is no relative velocity in y-direction. The projectiles may, however, have relative velocity in x-direction. As such the relative velocity in y-direction is also the net relative velocity between two projectiles.

Exercises

Exercise:

Problem:

Two projectiles are projected simultaneously at same speeds, but with different angles of projections from a common point in a given vertical plane. The path of one projectile, as viewed from other projectile is :

(a) a straight line parallel to horizontal

(b) a straight line parallel to vertical

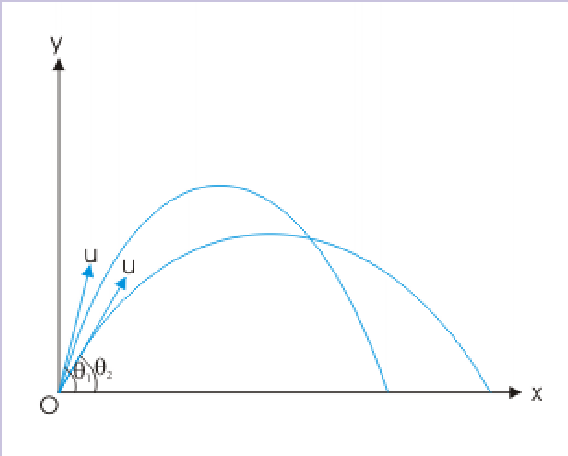
(c) a circle

(d) None of above

Solution:

The component relative velocities in horizontal and vertical directions are, defined in terms of initial velocities, which are constant for the given pair of projectiles. Therefore, the relative velocities of two projectiles in horizontal and vertical directions are constants. Let "u_A ” and “u_B ” be the initial velocities of two projectiles, then component relative velocities in "x" and "y" directions are :

Relative motion of projectiles



Relative motion of projectiles

$$v_{ABx} = u_{Ax} - u_{Bx}$$

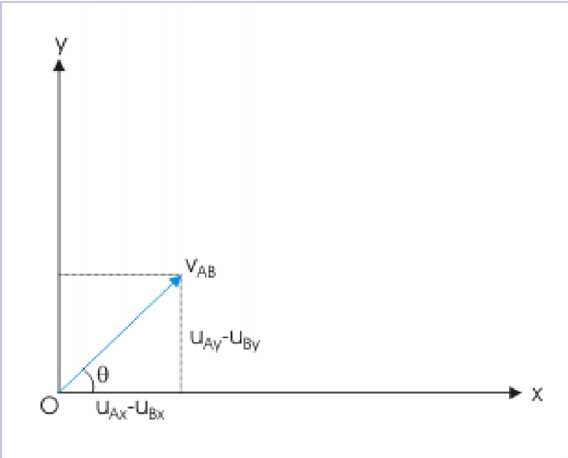
$$v_{ABy} = u_{Ay} - u_{By}$$

The resultant relative velocity is :

$$\mathbf{u}_{AB} = (u_{Ax} - u_{Bx})\mathbf{i} + (u_{Ay} - u_{By})\mathbf{j}$$

As given in the question, the initial speeds of the projectiles are same, but angles of projections are different. Since sine and cosine of two different angles are different, it follows that component velocities of two projectiles are different in either direction. This is ensured as speeds of two projectiles are same. It implies that components (horizontal or vertical) of the relative velocity are non-zero and finite constant. The resultant relative velocity is, thus, constant, making an angle "θ " with horizontal (x-axis) such that :

Relative motion of projectiles



Relative motion of projectiles

$$\tan\theta = \frac{(u_{Ay} - u_{By})}{(u_{Ax} - u_{Bx})}$$

Thus, one projectile (B) sees other projectile (A) moving in a straight line with constant velocity, which makes a constant angle with the horizontal. Hence, option (d) is correct.

Exercise:

Problem:

Two projectiles are projected simultaneously at different velocities from a common point in a given vertical plane. If the components of initial velocities of two projectiles in horizontal direction are equal, then the path of one projectile as viewed from other projectile is :

(a) a straight line parallel to horizontal

(b) a straight line parallel to vertical

(c) a parabola

(d) None of above

Solution:

The component relative velocities in horizontal and vertical directions are, defined in terms of initial velocities, which are constant for the given projectiles. Let "u_A” and “u_B” be the initial velocities of two projectiles, then component relative velocities in "x" and "y" directions are :

$$v_{ABx} = u_{Ax} - u_{Bx}$$

$$v_{ABy} = u_{Ay} - u_{By}$$

According to the question, components of initial velocities of two projectiles in horizontal direction are equal :

$$u_{Ax} - u_{Bx} = 0$$

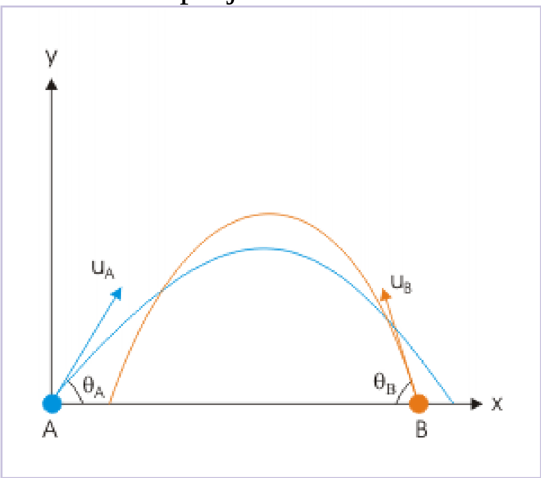
It is given that velocities of projections are different. As horizontal components of velocities in horizontal directions are equal, the components of velocities in vertical directions are different. As such, above expression evaluates to a constant vector. Thus, one projectile sees other projectile moving in a straight line parallel to vertical (i.e. direction of unit vector **j**).

Hence, option (b) is correct.

Collision of projectiles

Collision between two projectiles is a rare eventuality. The precise requirement of collision is that the two projectiles are at the same position (having same x and y coordinates) at a given time instant. This is the condition for two point objects to collide or for a collision to occur. This is possible only rarely. Consider projections of two projectiles as shown in the figure.

Collision of projectiles



The position of plane with respect to the earth keeps changing with time.

We see that paths of two projectiles cross twice. Thus, there are two positions (or corresponding time instants) when it is possible that projectiles occupy same position and thus may collide with each other. But, there are infinite possibilities that they would not. Projectile "A" may rise to the required height but "B" may be at lower or higher position. Similarly, even if two projectiles are at same height they may be horizontally separated. To top these, the projectiles may have time difference at the start of their motion. However, if two projectiles collide then we can make lot many simplifying assumptions resulting from the requirement of collision that two projectiles are at the same position at the same time. In this module, we shall examine these simplifying aspects of projectile motion under collision.

Analysis of motion

Two projectiles need to approach towards each other for collision to take place. If we look at this requirement in component form, then projectiles should approach towards each other both vertically and horizontally. It is possible that projectiles have different projection times. We can, however, extend the analysis even when projectiles are projected at different times by accounting motion for the additional time available to one of projectiles. For the sake of simplicity however, we consider that two projectiles are initiated at the same time instant.

The most important aspect of analysis of projectile motion involving collision is that we can interpret condition of collision in terms of relative velocity. We actually use the fact that components of relative velocity in either x or y direction is uniform motion - not accelerated one. This simplifies the analysis a great deal.

Relative motion in x-direction

If "x_0 " is the initial separation between projectiles, then for collision they should cover this separation with x-component of relative velocity. Since two projectiles are initiated at the same time instant, the time when collision occurs is given by :

$$t = \frac{x_0}{v_{ABx}}$$

where v_{ABx} is the relative speed of approach of A with respect B. Note that time expression evaluates to same value whether we compute it with v_{ABx} or v_{BAx} . On the other hand, if there is no initial separation in x – direction, the projectiles should cover same horizontal distance for all time intervals. It is so because projectiles have to reach same horizontal i..e x-position at the point of collision in two dimensional space. This means that relative velocity of projectiles in x-direction is zero for the condition of collision. Mathematically,

$$v_{ABx} = v_{Ax} - v_{Bx} = 0$$

Relative motion in y-direction

If "y_0 " is the initial separation between projectiles, then for collision they should cover this separation with y-component of relative velocity. Since two projectiles are initiated at the same time instant, the time when collision occurs is given by :

$$t = \frac{y_0}{v_{ABy}}$$

where v_{ABy} is the relative speed of approach of A with respect B. Note that time expression evaluates to same value whether we compute it with v_{ABy} or v_{BAy}. On the other hand, if there is no initial separation in y – direction, the projectiles should cover same vertical distance for all time intervals. It is so because projectiles have to reach same vertical i..e y-position at the point of collision in two dimensional space. This means that relative velocity of projectiles in y-direction is zero for the condition of collision. Mathematically,

$$v_{ABy} = v_{Ay} - v_{By} = 0$$

Collision of projectiles initiated without vertical separation

In this case, there is no initial vertical separation. Such is the case, when projectiles are projected from same horizontal level. Both projectiles should rise to same height for all time. Clearly, relative velocity in vertical i.e y-direction is zero :

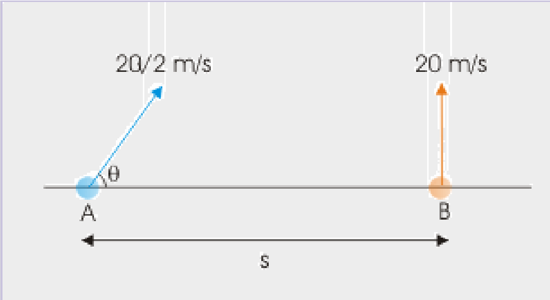
$$v_{ABy} = v_{Ay} - v_{By} = 0$$

On the other hand, time of collision is obtained by considering relative motion in x-direction :

$$t = \frac{x_0}{v_{ABx}}$$

There are different cases for projection from the same horizontal level. Some important cases are : (i) one projectile is projected at certain angle to the horizontal while the other projectile is projected vertically and (ii) Both projectiles are projected at certain angles to the horizontal. Here, we shall work out examples for each of these cases.

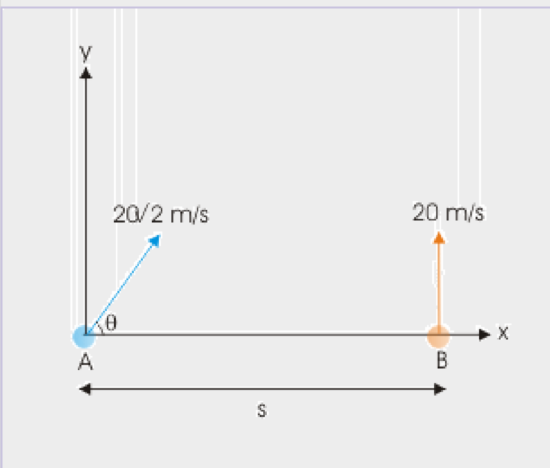
Example:
Problem : Two projectiles “A” and “B” are projected simultaneously as shown in the figure. If they collide after 0.5 s, then determine (i) angle of projection “θ” and (ii) the distance “s”.
Relative motion



The projectiles collide after 0.5 s.

Solution : We see here that projectile “A” is approaching towards projectile “B” in horizontal direction. Their movement in two component directions should be synchronized so that they are at the same position at a particular given time. There is no separation in vertical direction at the start of motion. As such, relative velocity in y-direction should be zero for collision to occur.

Relative motion



The projectiles collide after 0.5 s.

$$v_{ABy} = u_{Ay} - u_{By} = 0$$

$$\Rightarrow 20\sqrt{2} \quad \sin\theta = 20$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} = \sin 45^0$$

$$\Rightarrow \theta = 45^0$$

In the x-direction, the relative velocity is :

$$v_{ABx} = u_{Ax} - u_{Bx} = 20\sqrt{2}\cos 45^0 - 0 = 20 \quad m/s$$

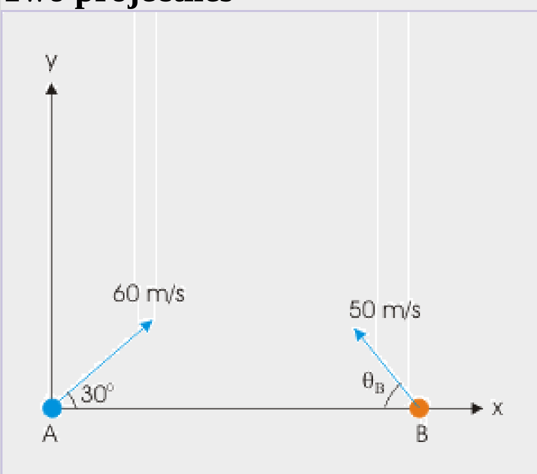
The distance "s" covered with the relative velocity in 0.5 second is :

$$s = v_{ABx} X t = 20 X 0.5 = 10m$$

Example:

Problem : Two projectiles "A" and "B" are thrown simultaneously in opposite directions as shown in the figure. If they happen to collide in the mid air, then find the time when collision takes place.

Two projectiles



Two projectiles are thrown towards each other.

Solution : There is no separation in vertical direction at the start of motion. As such, relative velocity in y-direction should be zero for collision to occur.

$$v_{ABy} = u_{Ay} - u_{By} = 0$$

$$\Rightarrow u_A \sin\theta_A = u_B \sin\theta_B$$

Putting values,

$$\Rightarrow \sin\theta_B = \frac{60}{50}\text{ X }\sin 30^0$$

$$\Rightarrow \sin\theta_B = \frac{3}{5}$$

The projectiles move towards each other with the relative velocity in horizontal direction. The relative velocity in x-direction is :

$$v_{ABx} = u_{Ax} - u_{Bx} = 60\cos 30^0 + 50\cos\theta_B$$

In order to find relative velocity in x-direction, we need to know " $\cos\theta_B$ ". Using trigonometric relation, we have :

$$\cos\theta_B = \sqrt{(1-\sin^2\theta_B)} = \sqrt{\left\{1-\left(\frac{3}{5}\right)^2\right\}} = \frac{4}{5}$$

Hence,

$$\Rightarrow v_{ABx} = 60X\frac{\sqrt{3}}{2} + 50\text{ X }\frac{4}{5} = 30\sqrt{3}+40 = 91.98 \approx 92m/s$$

We should now understand that projectiles move towards each other with a relative velocity of 92 m/s. We can interpret this as if projectile "B" is stationary and projectile "A" moves towards it with a velocity 92 m/s, covering the initial separation between two particles for collision to take place. The time of collision, therefore, is :

$$\Rightarrow t = 92/92 = 1s$$

Collision of projectiles initiated without horizontal separation

In this case, there is no initial horizontal separation. Projectiles are thrown from different levels. Both projectiles travel same horizontal distnace for all time. Clearly, relative velocity in horizontal i.e x-direction is zero :

$$v_{ABx} = v_{Ax} - v_{Bx} = 0$$

On the other hand, time of collision is obtained by considering relative motion in y-direction :

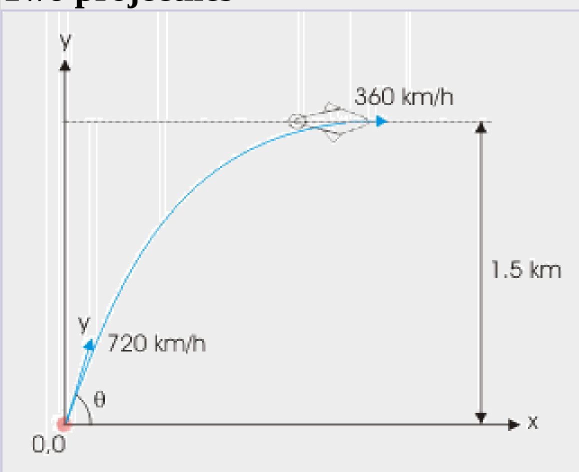
$$t = \frac{y_0}{v_{ABx}}$$

Example:

Problem : A fighter plane is flying horizontally at a speed of 360 km/hr and is exactly above an aircraft gun at a given moment. At that instant, the aircraft gun is fired to hit the plane, which is at a vertical height of 1 km. If the speed of the shell is 720 km/hr, then at what angle should the gun be aimed to hit the plane (Neglect resistance due to air)?

Solution : This is a case of collision between fighter plane and shell fired from the gun. Since plane is overhead, it is required that horizontal component of velocity of the shell be equal to that of the plane as it is flying in horizontal direction. This will ensure that shell will hit the plane whenever it rises to the height of the plane.

Two projectiles



The shell fired aircraft gun hits the fighter plane.

Now the horizontal and vertical components of shell velocity are :

$$u_x = 720 \cos\theta$$

$$\Rightarrow u_y = 720 \sin\theta$$

For collision,

$$\Rightarrow u_x = 720\cos\theta = 360$$

$$\Rightarrow \cos\theta = \frac{1}{2} = \cos 60^0$$

$$\Rightarrow \theta = 60^0$$

However, we need to check that shell is capable to rise to the height of the plane. Here,

$$720 \quad km/hr = 720X\frac{5}{18} = 200 \quad m/s$$

The maximum height achieved by the shell is obtained as here :

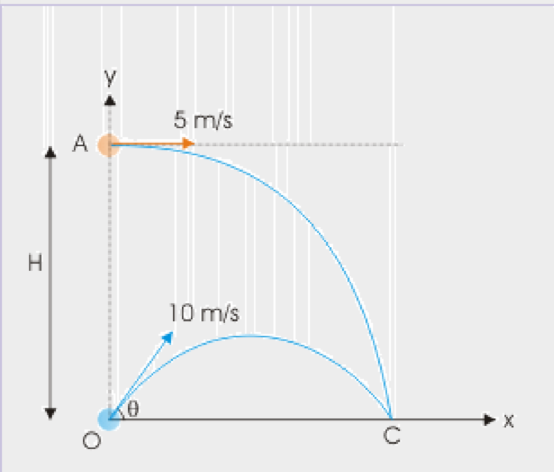
$$H = \frac{u^2\sin^2\theta}{2g} = \frac{200^2\left(\frac{\sqrt{3}}{2}\right)^2}{2X10} = 1500 \quad m$$

The shell indeed rises to the height of the plane (1000 m) and hence will hit it.

Example:

Problem : Two projectiles are projected simultaneously from a point on the ground “O” and an elevated position “A” respectively as shown in the figure. If collision occurs at the point of return of two projectiles on the horizontal surface, then find the height of “A” above the ground and the angle at which the projectile "O" at the ground should be projected.

Relative motion



The projectiles collide in the mid

The projectiles collide in the mid air.

Solution : There is no initial separation between two projectiles in x-direction. For collision to occur, the relative motion in x-direction should be zero. In other words, the component velocities in x-direction should be equal so that two projectiles cover equal horizontal distance at any given time. Hence,

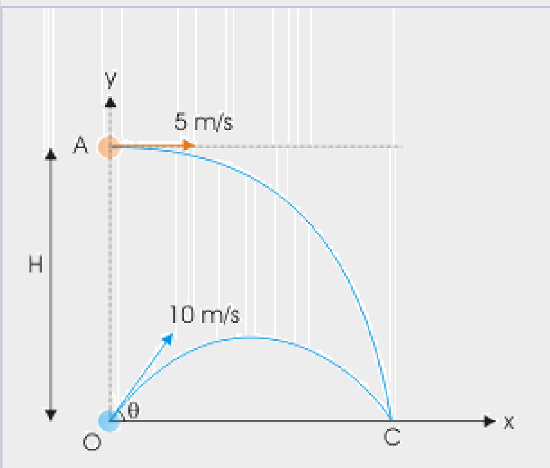
$$u_{Ox} = u_{Ax}$$

$$\Rightarrow u_O \cos\theta = u_A$$

$$\Rightarrow \cos\theta = \frac{u_A}{u_O} = \frac{5}{10} = \frac{1}{2} = \cos 60^0$$

$$\Rightarrow \theta = 60^0$$

Relative motion



The projectiles collide in the mid air.

We should ensure that collision does occur at the point of return. It means that by the time projectiles travel horizontal distances required, they should also cover vertical distances so that both projectiles are at “C” at the same time. In the nutshell, their times of flight should be equal. For projectile from "O",

$$T = \frac{2u_O \sin\theta}{g}$$

For projectile from "A",

$$T = \sqrt{\left(\frac{2H}{g}\right)}$$

For projectile from "A",

$$T = \frac{2u_O \sin\theta}{g} = \sqrt{\left(\frac{2H}{g}\right)}$$

Squaring both sides and putting values,

$$\Rightarrow H = \frac{4u_O^2 \sin^2\theta}{2g}$$

$$\Rightarrow H = \frac{4X10^2 \sin^2 60^0}{2X10}$$

$$H = 20\left(\frac{\sqrt{3}}{2}\right)^2 = 15m$$

We have deliberately worked out this problem taking advantage of the fact that projectiles are colliding at the end of their flights and hence their times of flight should be equal. We can, however, proceed to analyze in typical manner, using concept of relative velocity. The initial separation between two projectiles in the vertical direction is “H”. This separation is covered with the component of relative velocity in vertical direction.

$$\Rightarrow v_{OAy} = u_{Oy} - u_{Ay} = u_O \sin 60^0 - 0 = 10X\frac{\sqrt{3}}{2} = 5\sqrt{3} \quad m/s$$

Now, time of flight of projectile from ground is :

$$T = \frac{2u_O \sin\theta}{g} = \frac{2x10x \sin 60^0}{10} = \sqrt{3}$$

Hence, the vertical displacement of projectile from "A" before collision is :

$$\Rightarrow H = v_{OAy}XT = 5\sqrt{3}x\sqrt{3} = 15m/s$$

Collision of projectiles initiated from different horizontal and vertical levels

In this case, there are finite initial horizontal and vertical separations. For collision to occur, projectiles need to cover these separations simultaneously. Clearly, component relative velocity in x and y directions are finitie. We can obtain time of collision by consideration of relative motion in either direction,

$$t = \frac{x_0}{v_{ABx}}$$

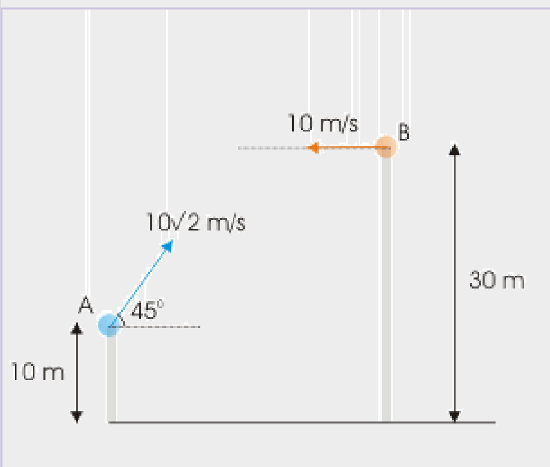
or

$$\Rightarrow t = \frac{y_0}{v_{ABx}}$$

Example:

Problem : Two projectiles are projected simultaneously from two towers as shown in the figure. If the projectiles collide in the air, then find the distance “s” between the towers.

Relative motion

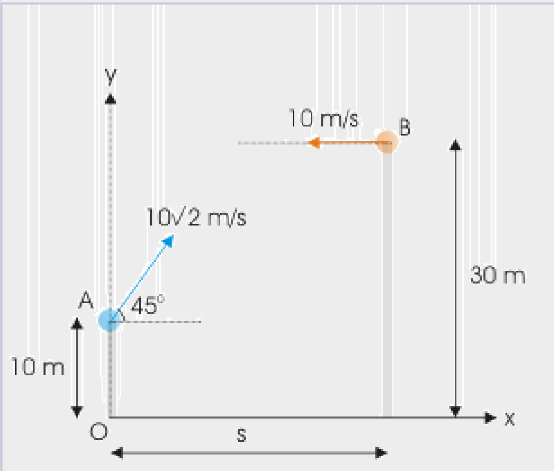


The projectiles collide in the mid air.

Solution : We see here that projectiles are approaching both horizontally and vertically. Their movement in two component directions should be synchronized so that they are at the same position at a particular given time. For collision, the necessary requirement is that relative velocity and displacement should be in the same direction.

It is given that collision does occur. It means that two projectiles should cover the displacement with relative velocity in each of the component directions.

Relative motion



The projectiles collide in the mid air.

In x-direction,

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\sqrt{2}\cos 45^0 - (-10) = 10\sqrt{2}x\frac{1}{\sqrt{2}} + 10 = 20m/s$$

If "t" is time after which collision occurs, then

$$\Rightarrow s = v_{ABx}Xt = 20t$$

Clearly, we need to know "t" to find "s". The component of relative velocity in y-direction is :

$$v_{ABy} = u_{Ay} - u_{By}$$

$$\Rightarrow v_{ABy} = u\sin 45^0 - 0 = 10\sqrt{2}X\frac{1}{\sqrt{2}} = 10m/s$$

The initial vertical distance between points of projection is 30-10 = 20 m. This vertical distance is covered with component of relative velocity in vertical direction. Hence, time taken to collide, “t”, is :

$$\Rightarrow t = \frac{20}{10} = 2$$

Putting this value in the earlier equation for “s”, we have :

$$\Rightarrow s = 20t = 20x2 = 40 \quad m$$

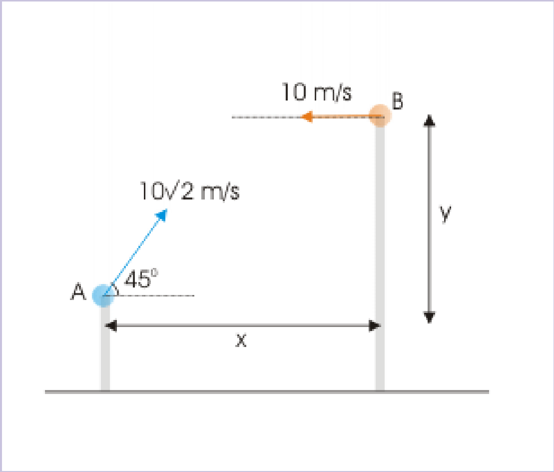
Exercises

Exercise:

Problem:

Two projectiles are projected simultaneously from two towers as shown in the figure. If collision takes place in the air, then what should be the ratio "x/y" :

Relative motion of projectiles



Relative motion of projectiles

$$(a)\frac{1}{2} \quad (b)1 \quad (c)2 \quad (d)\sqrt{3}$$

Solution:

According to question, collision occurs in the mid air. The necessary condition for collision is that direction of relative velocity and initial displacement between projectiles should be same.

In x-direction,

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\sqrt{2}\cos 45^0 - (-10) = 10\sqrt{2}X\frac{1}{\sqrt{2}} + 10 = 20 \quad m/s$$

In y-direction,

$$v_{ABy} = u_{Ay} - u_{By} = 10\sqrt{2}\cos 45^0 - 0 = 10\sqrt{2}X\frac{1}{\sqrt{2}} = 10 \quad m/s$$

The angle that relative velocity makes with horizontal is :

$$\Rightarrow \tan\theta = \frac{v_{ABy}}{v_{ABx}} = \frac{10}{20} = \frac{1}{2}$$

This is also the slope of displacement AB,

$$\Rightarrow \tan\theta = \frac{1}{2} = \frac{y}{x}$$

$$\Rightarrow \frac{x}{y} = 2$$

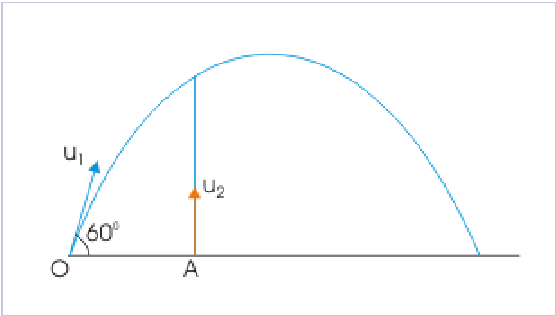
Hence, option (c) is correct.

Exercise:

Problem:

Two balls are projected simultaneously with speeds "u_1 ” and “"u_2 ” from two points "O" and "A" respectively as shown in the figure. If the balls collide, then find the ratio "$\frac{u_1}{u_2}$ ” (consider g = 10 m/s^2) :

Relative motion of projectiles



Relative motion of projectiles

$$(a)\frac{2}{\sqrt{3}} \quad (b)\frac{\sqrt{3}}{2} \quad (c)1/\sqrt{3} \quad (d)\sqrt{3}$$

Solution:

It is given that the projectiles collide. Now the initial separation in vertical direction is zero. It follows then that component of relative velocity between two projectiles in vertical direction should be zero for collision to take place. This is possible if the vertical component of velocity of projectile form "O" is equal to the speed of projectile form "A" (Remember that acceleration due to gravity applies to both projectiles and relative acceleration in vertical direction is zero).

$$u_1 \sin 60^0 = u_2$$

$$\Rightarrow \frac{u_1}{u_2} = \frac{1}{\sin 60^0} = \frac{2}{\sqrt{3}}$$

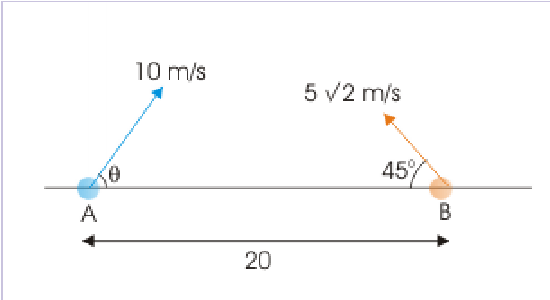
Hence, option (a) is correct.

Exercise:

Problem:

Two projectiles “A” and “B” are projected simultaneously towards each other as shown in the figure. Determine if they collide.

Relative motion

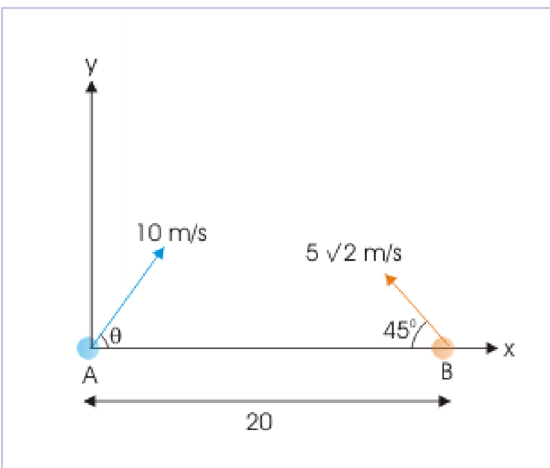


The projectiles collide in the mid air.

Solution:

There is no separation in vertical direction at the start of motion. As such, relative velocity in y-direction should be zero for collision to occur.

Relative motion



The projectiles collide in the mid air.

$$v_{ABy} = u_{Ay} - u_{By} = 0$$

$$10 \sin\theta = 5\sqrt{2} X \sin 45^0 = 5\sqrt{2} X \frac{1}{\sqrt{2}} = 5$$

$$\sin\theta = \frac{1}{2} = \sin 30^0$$

$$\theta = 30^0$$

In the x-direction, the initial separation is 15 m and the relative velocity is :

$$v_{ABx} = u_{Ax} - u_{Bx} = 10\cos 30^0 + 5\sqrt{2}x\cos 45^0$$

$$v_{ABx} = 10X\sqrt{3}\frac{}{2} + 5\sqrt{2}X\sqrt{2} = 5\sqrt{3} + 5 = 5\left(\sqrt{3}+1\right)$$

The time after which collision may occur is :

$$\Rightarrow t = \frac{20}{5\left(\sqrt{3}+1\right)} = \frac{4}{\left(\sqrt{3}+1\right)} = 1.464s$$

We should, however, check whether projectiles stay that long in the air in the first place? Now, we have seen that the projectiles should have same vertical component of velocity for collision to occur. As time of flight is a function of vertical component of velocity, the time of the flight of the projectiles are equal. For projectile “A” :

$$T = \frac{2u_A\sin 30^0}{g} = \frac{2X10X1}{10X2} = 1s$$

This means that projectiles would not stay that long in the air even though the necessary conditions (but not sufficient conditions) for the collision are fulfilled. Clearly, collision does not occur in this case. We should clearly understand that "analysis of motion with respect to collision" and "requirements of collision" are not same. In this module, we have studied the "consequence of collision" - not the "conditions for collision".

Uniform circular motion
Uniform circular motion (UCM) is the basic unit of rotational kinematics just like the uniform linear motion is the basic unit of translational kinematics.

Uniform circular motion denotes motion of a particle along a circular arc or a circle with constant speed. This statement, as a matter of fact, can be construed as the definition of uniform circular motion.

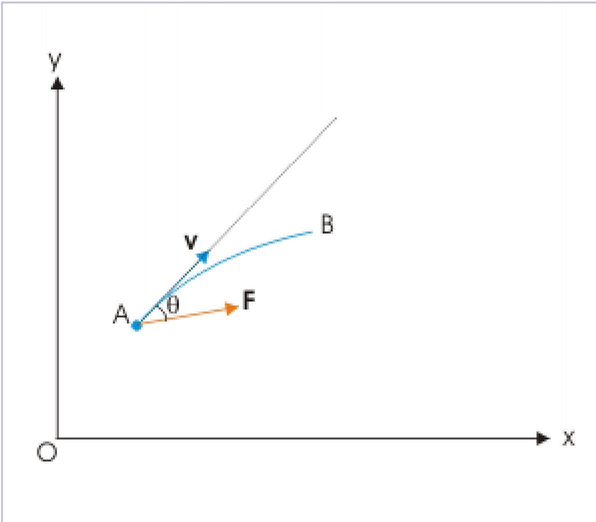
Requirement of uniform circular motion

The uniform circular motion represents the basic form of rotational motion in the same manner as uniform linear motion represents the basic form of translational motion. They, however, are different with respect to the requirement of force to maintain motion.

Uniform linear motion is the reflection of the inherent natural tendency of all natural bodies. This motion by itself is the statement of Newton's first law of motion : an object keeps moving with its velocity unless there is net external force. Thus, uniform linear motion indicates "absence" of force.

On the other hand, uniform circular motion involves continuous change in the direction of velocity without any change in its magnitude (v). A change in the direction of velocity is a change in velocity (**v**). It means that an uniform circular motion is associated with an acceleration and hence force. Thus, uniform circular motion indicates "presence" of force.

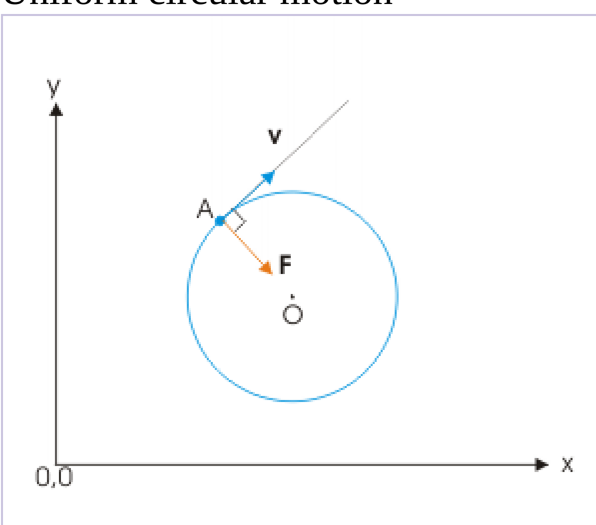
Let us now investigate the nature of force required to maintain uniform circular motion. We know that a force acting in the direction of motion changes only the magnitude of velocity. A change in the direction of motion, therefore, requires that velocity of the particle and force acting on it should be at an angle. However, such a force, at an angle with the direction of motion, would have a component along the direction of velocity as well and that would change the magnitude of the motion.
Change of direction



A change in the direction of motion requires that velocity of the particle and force should be at an angle.

In order that there is no change in the magnitude of velocity, the force should have zero component along the direction of velocity. It is possible only if the force be perpendicular to the direction of velocity such that its component in the direction of velocity is zero ($F\cos 90° = 0$). Precisely, this is the requirement for a motion to be uniform circular motion.

Uniform circular motion



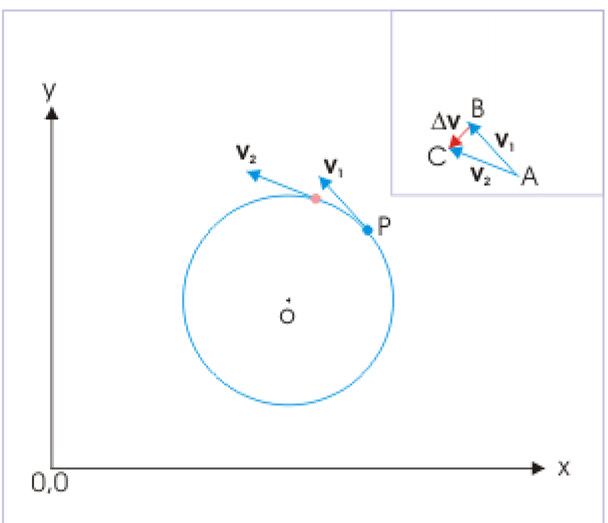
Force is perpendicular to the direction of velocity.

In plain words, uniform circular motion (UCM) needs a force, which is always perpendicular to the direction of velocity. Since the direction of velocity is continuously changing, the direction of force, being perpendicular to velocity, should also change continously.

The direction of velocity along the circular trajectory is tangential. The perpendicular direction to the circular trajectory is, therefore, radial direction. It implies that force (and hence acceleration) in uniform direction motion is radial. For this reason, acceleration in UCM is recognized to seek center i.e. centripetal (seeking center).

This fact is also validated by the fact that the difference of velocity vectors, whose time rate gives acceleration, at two instants (Δ**v**) is radial.

Uniform circular motion



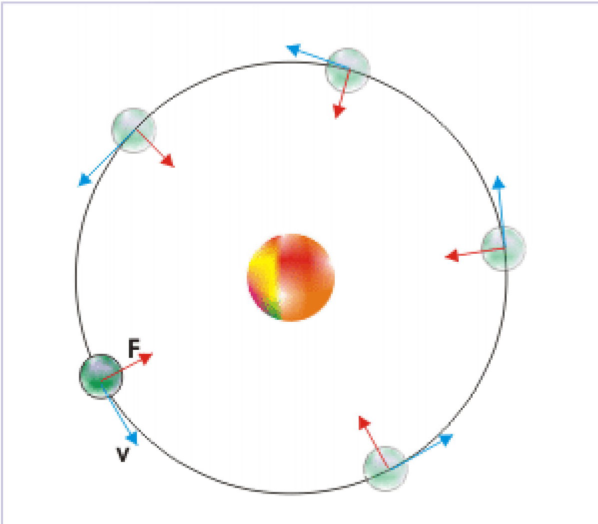
Force is centripetal.

The yet another important aspect of the UCM is that the centripetal force is radial and hence does not constitute a torque as the force is passing through the axis of rotation. A torque is force multiplied by the perpendicular distance of the line of action of the force from the axis of rotation. It must be clearly understood that the requirement of centripetal force is essentially additional or different to the force, which is required in non-uniform circular motion to accelerate the particle tangentially. The centripetal force is required to accelerate particle in radial direction and is different to one required to accelerate particle tangentially.

Irrespective of whether circular motion is uniform (constant speed) or non-uniform (varying speed), the circular motion inherently associates a radial acceleration to ensure that the direction of motion is continuously changed – at all instants. We shall learn about the magnitude of radial acceleration soon, but let us be emphatic to differentiate radial acceleration (accounting change in direction that arises from radial force) with tangential acceleration (accounting change in the speed that arises from tangential force or equivalently a torque).

Motion of natural bodies and sub-atomic particles are always under certain force system. Absence of force in the observable neighborhood is rare. Thus, uniform linear motion is rare, while uniform circular motion abounds in nature as there is availability of external force that continuously changes direction of the motion of the bodies or particles. Consider the electrostatic force between nucleus and an electron in an atom. The force keeps changing direction as electron moves. So is the case of gravitational force (indicated by red arrow in the figure), which keeps changing its direction as the planet moves around sun.

Approximated circular motion of Earth around Sun



Gravitational force is radial,
whereas velocity is tangential.

It may sometimes be perceived that a force like centripetal force should have caused the particle or body to move towards center ultimately. The important point to understand here is that a force determines initial direction of motion only when the particle is stationary. However, if the particle is already in motion, then force modifies the direction in accordance with initial inclination between velocity and force such that the resulting acceleration (change in vector velocity) is in the direction of force. We shall soon see that this is exactly the case in uniform circular motion.

Let us summarize the discussion of uniform circular motion so far :

- The trajectory of uniform circular motion is circular arc or a circle and hence planar or two dimensional.
- The speed (v) of the particle in UCM is a constant.
- The velocity (**v**) of the particle in UCM is variable and is tangential or circumferential to the path of motion.
- Centripetal force (**F**) is required to maintain uniform circular motion against the natural tendency of the bodies to move linearly.
- Centripetal force (**F**) is variable and is radial in direction.

- Centripetal force ($\mathbf{F}$) is not a torque and does not cause acceleration in tangential direction.
- Centripetal acceleration ($\mathbf{a_R}$) is variable and radial in direction.

Analysis of uniform circular motion

It has been pointed out that any motion, that changes directions, requires more than one dimension for representation. Circular motion by the geometry of the trajectory is two dimensional motion.

In the case of circular motion, it is a matter of convenience to locate origin of the two dimensional coordinate system at the center of circle. This choice ensures the symmetry of the circular motion about the origin of the reference system.

Coordinates of the particle

The coordinates of the particle is given by the "x" and "y" coordinate pair as :

Equation:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The angle "θ" is measured anti-clockwise from x-axis.

Position of the particle

The position vector of the position of the particle, **r**, is represented in terms of unit vectors as :

Equation:

$$
\begin{aligned}
&\mathbf{r} = x\mathbf{i} + y\mathbf{j} \\
&\Rightarrow \mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} \\
&\Rightarrow \mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})
\end{aligned}
$$

Velocity of the particle

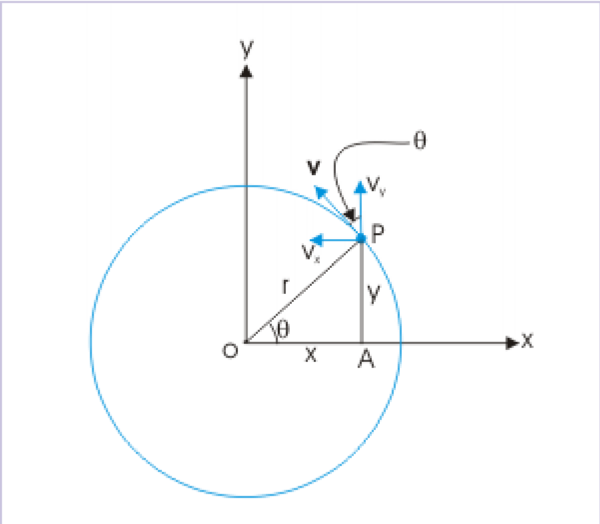
The magnitude of velocity of the particle (v) is constant by the definition of uniform circular motion. In component form, the velocity (refer to the figure) is :

Equation:

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$$

From the geometry as shown in the figure,

Uniform circular motion



Component velocities in UCM.

Equation:

$$v_x = -v \sin\theta$$
$$v_y = v \cos\theta$$
$$\Rightarrow \mathbf{v} = -v\sin\theta\mathbf{i} + v\cos\theta\mathbf{j}$$

We may emphasize here that it is always easy to find the sign of the component of a vector like velocity. Decide the sign by the direction of component (projection) with respect to positive direction of reference axis. Note from the figure that component along x-direction is opposite to the positive reference direction and hence negative.

From the ΔOAP,

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$

Putting these values in the expression for velocity, we have :
Equation:

$$\mathbf{v} = -\frac{vy}{r}\mathbf{i} + \frac{vx}{r}\mathbf{j}$$

Centripetal acceleration

We are now sufficiently equipped and familiarized with the nature of uniform circular motion and the associated centripetal (center seeking) acceleration. Now, we seek to determine this acceleration.

Knowing that speed, "v" and radius of circle, "r" are constants, we differentiate the expression of velocity with respect to time to obtain expression for centripetal acceleration as :

$$\mathbf{a} = -\frac{v}{r}\left(\frac{đy}{đt}\mathbf{i} - \frac{đx}{đt}\mathbf{j}\right)$$

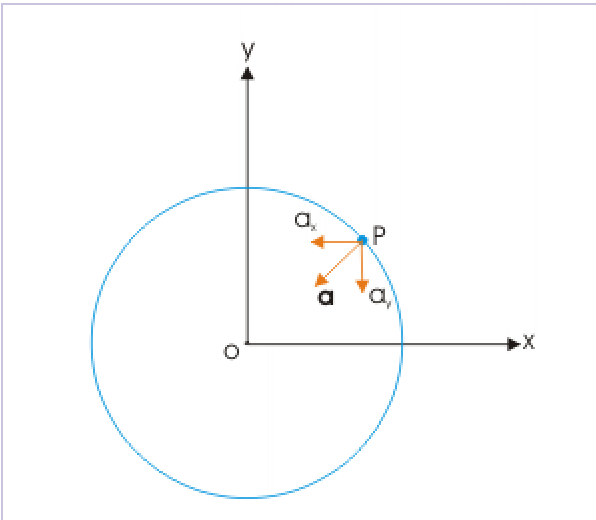
$$\Rightarrow \mathbf{a} = -\frac{v}{r}\left(v_y\mathbf{i} - v_x\mathbf{j}\right)$$

Putting values of component velocities in terms of angle,

Equation:

$$\Rightarrow \mathbf{a} = -\frac{v}{r}\left(v\cos\theta\mathbf{i} + v\sin\theta\mathbf{j}\right) = a_x\mathbf{i} + a_y\mathbf{j}$$

Uniform circular motion



Components of radial accelerations

where

$$a_x = -\frac{v^2}{r}\cos\theta$$
$$a_y = -\frac{v^2}{r}\sin\theta$$

It is evident from the equation of acceleration that it varies as the angle with horizontal, "θ" changes. The magnitude of acceleration is :

Equation:

$$a = \left|\mathbf{a}\right| = \sqrt{({a_x}^2 + {a_y}^2)}$$
$$\Rightarrow a = \left|\mathbf{a}\right| = \frac{v}{r}\sqrt{\{v^2(\cos^2\theta + \cos^2\theta)\}}$$
$$\Rightarrow a = \frac{v^2}{r}$$

The radius of the circle is constant. The magnitude of velocity i.e. speed, "v" is constant for UCM. It, then, follows that though velocity changes with the motion (angle from reference direction), but the speed of the particle in UCM is a constant.

Example:
Problem : A cyclist negotiates the curvature of 20 m with a speed of 20 m/s. What is the magnitude of his acceleration?
Solution : The speed of the cyclist is constant. The acceleration of cyclist, therefore, is the centripetal acceleration required to move the cyclist along a circular path i.e. the acceleration resulting from the change in the direction of motion along the circular path.
Here, v = 20 m/s and r = 20 m

$$\Rightarrow a = \frac{v^2}{r} = \frac{20^2}{20} = 20\, m/s^2$$

This example points to an interesting aspect of circular motion. The centripetal acceleration of the cyclist is actually two (2) times that of acceleration due to gravity (g = 10 m/s^2). This fact is used to create large acceleration in small space with appropriate values of “v” and “r” as per the requirement in hand. The large acceleration so produced finds application in particle physics and for equipments designed to segregate material on the basis of difference in density. This is also used to simulate large acceleration in a centrifuge for astronauts, who experience large acceleration at the time of take off or during entry on the return.

Generation of high magnitude of acceleration during uniform circular motion also points to a potential danger to pilots, while maneuvering circular trajectory at high speed. Since a pilot is inclined with the head leaning towards the center of motion, the blood circulation in the brain is low. If his body part, including brain, is subjected to high acceleration

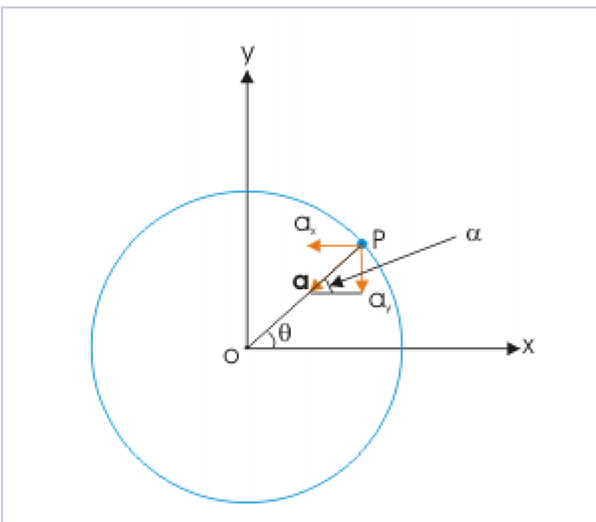
(multiple of acceleration due to gravity), then it is likely that the pilot experiences dizziness or sometimes even looses consciousness.

Circular motion has many interesting applications in real world and provides explanations for many natural events. In this module, however, we restrict ourselves till we study the dynamics of the circular motion also in subsequent modules.

Direction of centripetal acceleration

Here, we set out to evaluate the angle “α” as shown in the figure. Clearly, if this angle “α” is equal to “θ”, then we can conclude that acceleration is directed in radial direction.

Uniform circular motion



Radial acceleration

Now,

$$\tan\alpha = \frac{a_y}{a_x} = \frac{-\frac{v^2}{r}\sin\theta}{-\frac{v^2}{r}\cos\theta} = \tan\theta$$

$$\alpha = \theta$$

This proves that centripetal acceleration is indeed radial (i.e acting along radial direction).

Time period of uniform circular motion

A particle under UCM covers a constant distance in completing circular trajectory in one revolution, which is equal to the perimeter of the circle.

$$s = 2\pi r$$

Further the particle covers the perimeter with constant speed. It means that the particle travels the circular trajectory in a constant time given by its time period as :

Equation:

$$T = \frac{2\pi r}{v}$$

Exercise:

Problem:

An astronaut, executing uniform circular motion in a centrifuge of radius 10 m, is subjected to a radial acceleration of 4g. The time period of the centrifuge (in seconds) is :

(a) π (b) 2π (c) 3π (d) 4π

Solution:

From the inputs given in the question,

$$a = \frac{v^2}{r}$$
$$\Rightarrow v^2 = ar = 4 \text{ x } 10 \text{ x } 10 = 400$$
$$\Rightarrow v = 20\, m/s$$

The time period of the motion is :

$$T = \frac{2\pi r}{v} = \frac{2\pi \text{x} 10}{20} = \pi \text{ seconds}$$

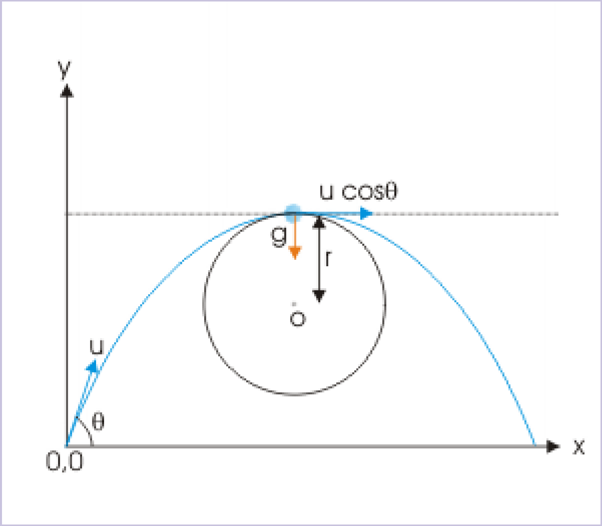
Hence, option (a) is correct.

Uniform circular motion and projectile motion

Both these motions are two dimensional motions. They are alike in the sense that motion in each case is subjected to continuous change of the direction of motion. At the same time, they are different in other details. The most important difference is that projectile motion has a constant acceleration, whereas uniform circular motion has a constant magnitude of acceleration.

Significantly, a projectile motion completely resembles uniform circular motion at one particular instant. The projectile has only horizontal component of velocity, when it is at the maximum height. At that instant, the force of gravity, which is always directed downward, is perpendicular to the direction of velocity. Thus, projectile at that moment executes an uniform circular motion.

Projectile motion



Projectile meets the condition of UCM at maximum height

Let radius of curvature of the projectile trajectory at maximum height be "r", then

$$a = g = \frac{{v_x}^2}{r} = \frac{v^2 \cos^2 \theta}{r}$$
$$\Rightarrow r = \frac{v^2 \cos^2 \theta}{g}$$

Note: The radius of curvature is not equal to maximum height – though expression appears to be same. But, they are actually different. The numerator here consists of cosine function, whereas it is sine function in the expression of maximum height. The difference is also visible from the figure.

Exercise

Exercise:

Problem:

A particle moves along a circle of radius "r" meter with a time period of "T". The acceleration of the particle is :

(a) $\frac{4\pi^2 r}{T^2}$ (b) $\frac{4\pi^2 r^2}{T^2}$ (c) $4\pi r T$ (c) $\frac{4\pi r}{T}$

Solution:

In order to determine centripetal acceleration, we need to find the speed of the particle. We can determine the speed of the particle, using expression of time period :

$$T = \frac{2\pi r}{v}$$

$$\Rightarrow v = \frac{2\pi r}{T}$$

The acceleration of the particle is :

$$a = \frac{v^2}{r} \quad = \frac{(2\pi r)^2}{T^2 r} \quad \frac{4\pi^2 r}{T^2}$$

Hence, option (a) is correct.

Exercise:

Problem:

Which of the following quantities are constant in uniform circular motion :

(a) velocity and acceleration

(b) magnitude of acceleration and radius of the circular path

(c) radius of the circular path and speed

(d) speed and acceleration

Solution:

Three quantities, namely, radius of the circular path, magnitude of acceleration and speed are constant in uniform circular motion.

Hence, options (b) and (c) are correct.

Exercise:

Problem:

Which of these is/are correct for a particle in uniform circular motion :

(a) the acceleration of the particle is tangential to the path

(b) the acceleration of the particle in the direction of velocity is zero

(c) the time period of the motion is inversely proportional to the radius

(d) for a given acceleration, velocity of the particle is proportional to the radius of circle

---

Solution:

The options (a), (c) and (d) are wrong.

Hence, option (b) is correct.

Exercise:

Problem:

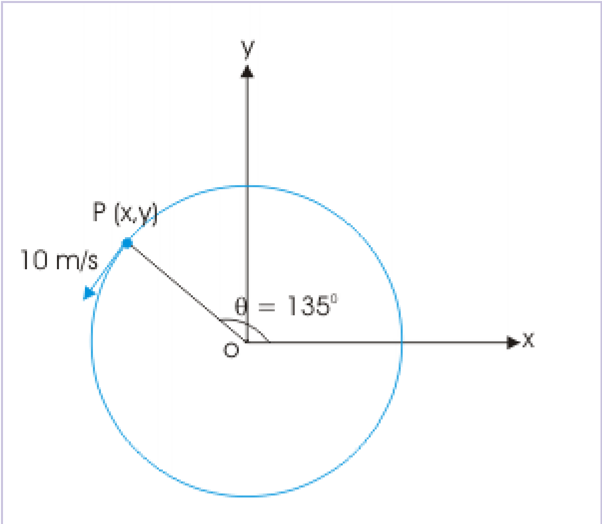
A particle moves with a speed of 10 m/s along a horizontal circle of radius 10 m in anti-clockwise direction. The x and y coordinates of the particle is 0,r at t = 0. The velocity of the particle (in m/s), when its position makes an angle 135° with x - axis, is :

(a) $-5\sqrt{2}\mathbf{i} - 5\sqrt{2}\mathbf{j}$ (b) $5\sqrt{2}\mathbf{i} - 5\sqrt{2}\mathbf{j}$ (c) $5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}$ (d) $-5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}$

Solution:

The velocity is a vector quantity. We need to know the components of the velocity when particle makes the angle 135° with x - axis.

Uniform circular motion



$$v_x = -v\sin\theta = -10\sin 135^0 = -10 \text{ x } \left(\frac{1}{\sqrt{2}}\right) = -5\sqrt{2}$$
$$v_y = v\cos\theta = 10\cos 135^0 = 10 \text{ x } \left(\frac{-1}{\sqrt{2}}\right) = -5\sqrt{2}$$

The component v_x is in the reference x - direction, whereas component v_y is in the negative reference y - direction.

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$$
$$\Rightarrow \mathbf{v} = -\left(5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}\right) m/s$$

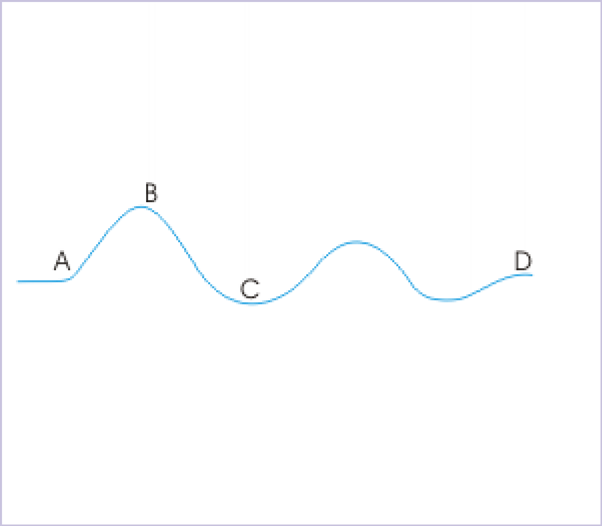
Hence, option (a) is correct.

Exercise:

Problem:

A car moves along a path as shown in the figure at a constant speed. Let a_A , a_B , a_C and a_D be the centripetal accelerations at locations A,B,C and D respectively. Then

Uniform circular motion

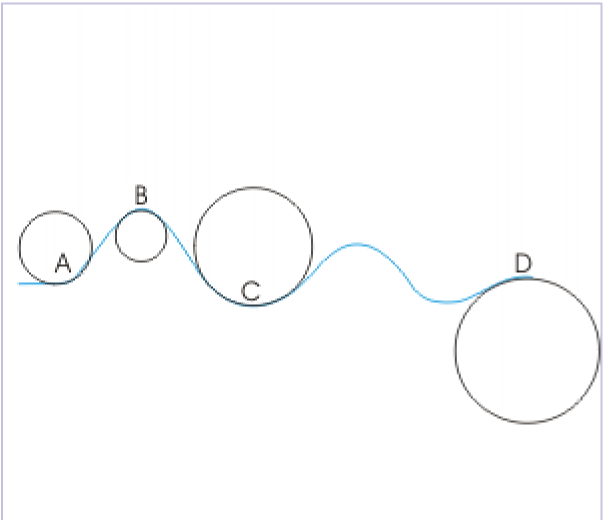


(a) $a_A < a_C$ (b) $a_A > a_D$ (c) $a_B < a_C$ (d) $a_B > a_D$

Solution:

The curvatures of path at four points are indicated with our circles drawn so that they align exactly with the curved path at these points. The radii of these circles at points A and B are smaller than that at points C and D.

Uniform circular motion



Now, the centripetal acceleration is given by :

$$a = \frac{v^2}{r}$$

The smaller the radius, greater is centripetal acceleration. Thus,

$$a_A > a_D$$

and

$$a_B > a_D$$

Hence, options (b) and (d) are correct.

Exercise:

Problem:

A particle executes uniform circular motion with a speed “v” in xy plane, starting from position (r,0), where “r” denotes the radius of the circle. Then

(a) the component of velocity along x - axis is constant

(b) the component of acceleration along x - axis is constant

(c) the angle swept by the line joining center and particle in equal time interval is constant

(d) the velocity is parallel to external force on the particle

---

Solution:

The component of velocity along x - direction is “-v sinθ” and is not a constant. The component of acceleration along x - direction is “ $a_x = -\frac{v^2}{r}\cos\theta$ ” and is not a constant. The external force in uniform circular motion is equal to centripetal force, which is directed towards the center. Thus, the velocity is perpendicular to external force.

On the other hand, the particle covers equal circular arc in equal time interval. Hence, the angle subtended by the arc at the center is also equal in equal time interval.

Hence, option (c) is correct.

Exercise:

Problem:

A particle executes uniform circular motion with a speed “v” in anticlockwise direction, starting from position (r,0), where “r” denotes the radius of the circle. Then, both components of the velocities are positive in :

(a) first quadrant of the motion

(b) second quadrant of the motion

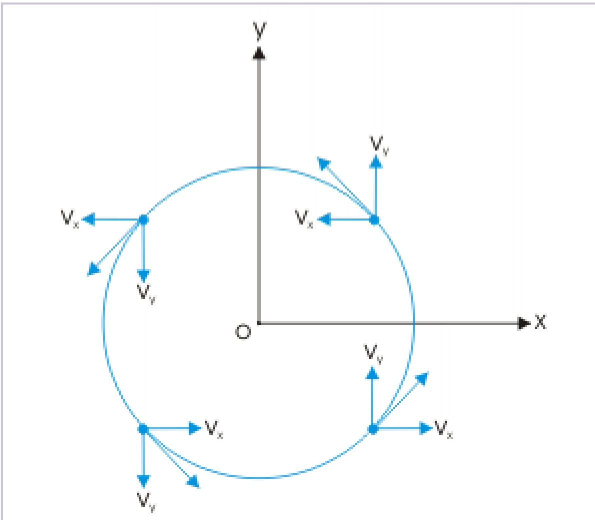
(c) third quadrant of the motion

(d) fourth quadrant of the motion

Solution:

From the figure, it is clear that the components of velocity in the fourth quadrant are both positive.

Uniform circular motion



Hence, option (d) is correct.

Exercise:

Problem:

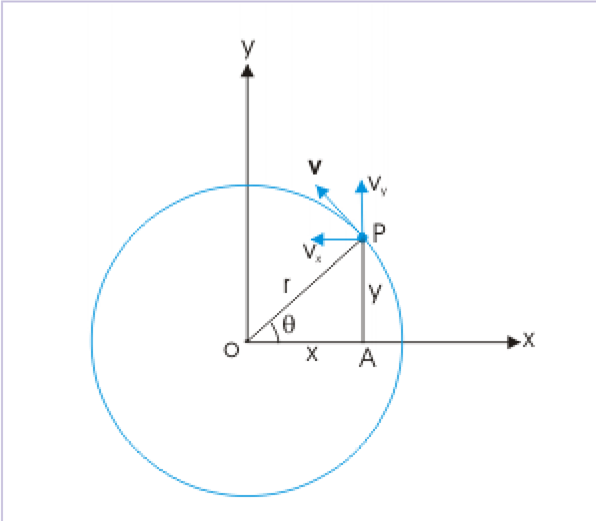
A particle executes uniform circular motion with a constant speed of 1 m/s in xy - plane. The particle starts moving with constant speed from position (r,0), where "r" denotes the radius of the circle. If center of circle is the origin of the coordinate system, then what is the velocity in "m/s" of the particle at the position specified by the coordinates (3m,-4m) ?

(a) $\frac{1}{\ }\left(\ \mathbf{i}+\ \mathbf{j}\right)$ (b) $\left(\ \mathbf{i}-\ \mathbf{j}\right)$ (c) $\frac{1}{\ }\left(\ \mathbf{i}-\ \mathbf{j}\right)$ (d) $\frac{1}{\ }\left(\ \mathbf{i}+\ \mathbf{j}\right)$

Solution:

Here, the inputs are given in terms of coordinates. Thus, we shall use the expression of velocity, which involves coordinates. Velocity of the particle in circular motion is given by :

Uniform circular motion



$$\mathbf{v} = -\frac{vy}{r}\mathbf{i} + \frac{vx}{r}\mathbf{j}$$

Here, radius of the circle is :

$$r = \sqrt{(3^2 + 4^2)} = 5\, m/s$$

$$\Rightarrow \mathbf{v} = -\frac{1\text{x}(-4)}{5}\mathbf{i} + \frac{1\text{x}3}{5}\mathbf{j}$$

$$\Rightarrow \mathbf{v} = \frac{1}{5}\left(4\mathbf{i} + 3\mathbf{j}\right)$$

Hence, option (a) is correct.

Uniform circular motion (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

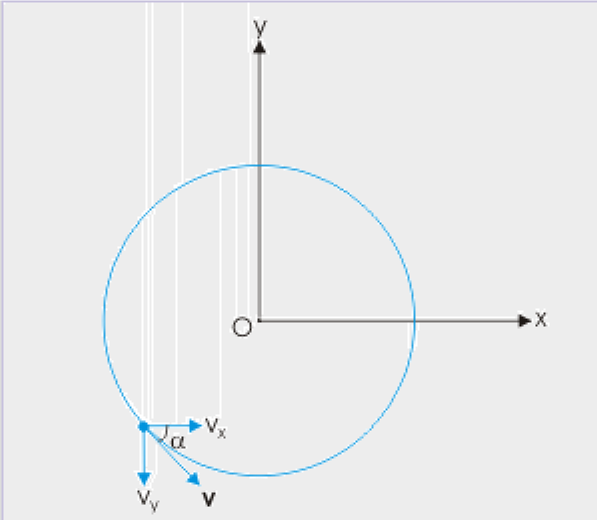
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the uniform circular motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Direction of velocity
- Direction of position vector
- Velocity
- Relative speed
- Nature of UCM

Direction of velocity

Example:
Problem : A particle moves in xy-plane along a circle of radius "r". The particle moves at a constant speed in anti-clockwise direction with center of circle as the origin of the coordinate system. At a certain instant, the velocity of the particle is $\mathbf{i} - \sqrt{3}\mathbf{j}$. Determine the angle that velocity makes with x-direction.
Solution : The sign of y-component of velocity is negative, whereas that of x-component of velocity is positive. It means that the particle is in the third quadrant of the circle as shown in the figure.
Top view of uniform circular motion in xy-plane



The acute angle formed by the velocity with x-axis is obtained by considering the magnitude of components (without sign) as :

$$\tan\alpha = \frac{v_y}{v_x} = \frac{\sqrt{3}}{1} = \sqrt{3} = \tan 60^0$$

$$\Rightarrow \alpha = 60^0$$

This is the required angle as measured in clockwise direction from x-axis. If the angle is measured in anti-clockwise direction from positive direction of x-axis, then

$$\Rightarrow \alpha\prime = 360^0 - 60^0 = 300^0$$

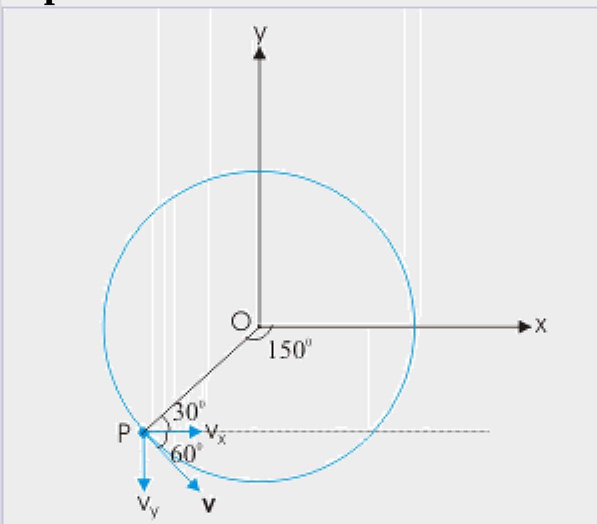
Direction of position vector

Example:
Problem : A particle moves in xy-plane along a circle of radius "r". The particle moves at a constant speed in anti-clockwise direction with center of circle as the origin of the coordinate system. At a certain instant, the

velocity of the particle is **i** – √3**j**. Determine the angle that position vector makes with x-direction.

Solution : The sign of y-component of velocity is negative, whereas that of x-component of velocity is positive. It means that the particle is in the third quadrant of the circle as shown in the figure.

Top view of uniform circular motion in xy-plane



The acute angle formed by the velocity with x-axis is obtained by considering the magnitude of components (without sign) as :

$$\tan\alpha = \frac{v_y}{v_x} = \frac{\sqrt{3}}{1} = \sqrt{3} = \tan 60^0$$

$$\Rightarrow \alpha = 60^0$$

But, we know that position vector is perpendicular to velocity vector. By geometry,

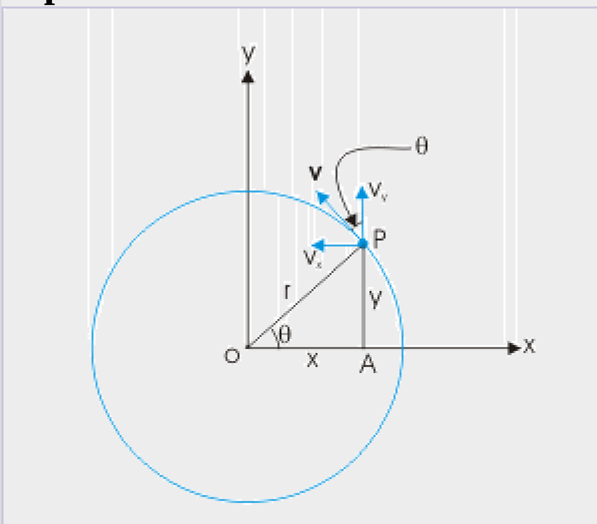
$$\theta = 180^0 - 30^0 = 150^0$$

This is the angle as measured in clockwise direction from x-axis. If the angle is measured in anti-clockwise direction from positive direction of x-axis, then

$$\Rightarrow \alpha\prime = 360^0 - 150^0 = 210^0$$

Note : Recall the derivation of the expression of velocity vector in the previous module. We had denoted “θ” as the angle that position vector makes with x-axis (not the velocity vector). See the figure that we had used to derive the velocity expression.

Top view of uniform circular motion in xy-plane



As a matter of fact “θ” is the angle that velocity vector makes with y-axis (not x-axis). We can determine the angle “θ” by considering the sign while evaluating tan θ,

$$\tan\theta = \frac{v_x}{v_y} = \frac{1}{\left(-\sqrt{3}\right)} = \tan 150^0$$

$$\theta = 150^0$$

Velocity

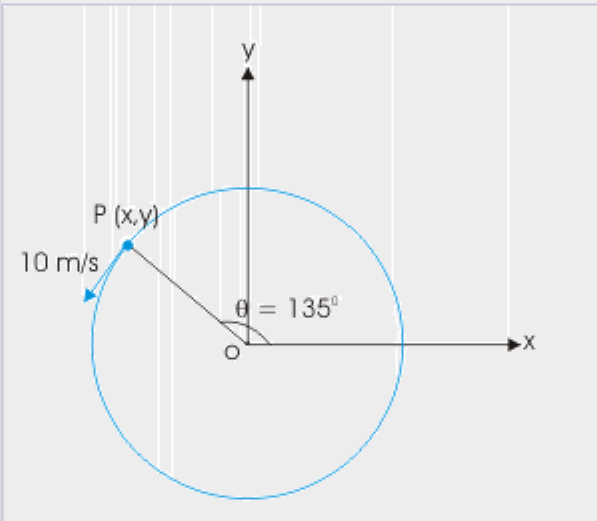
Example:

Problem : A particle moves with a speed 10 m/s in xy-plane along a circle of radius 10 m in anti-clockwise direction. The particle starts moving with constant speed from position (r,0), where "r" denotes the radius of the

circle. Find the velocity of the particle (in m/s), when its position makes an angle 135° with x – axis.

Solution : The velocity of the particle making an angle "θ" with x – axis is given as :

Uniform circular motion



$$v = v_x \mathbf{i} + v_y \mathbf{j} = -v \sin\theta \mathbf{i} + v \cos\theta \mathbf{j}$$

Here,

$$v_x = -v \sin\theta = -10 \sin 135^0 = -10 \text{ x } \left(\frac{1}{\sqrt{2}}\right) = -5\sqrt{2}$$

$$v_y = v \cos\theta = 10 \cos 135^0 = 10 \text{ x } \left(-\frac{1}{\sqrt{2}}\right) = -5\sqrt{2}$$

Here, both the components are negative.

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$$

$$\Rightarrow \mathbf{v} = -\left(5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}\right) m/s$$

Relative speed

Example:

Problem : Two particles tracing a circle of radius 10 m begin their journey simultaneously from a point on the circle in opposite directions. If their speeds are 2.0 m/s and 1.14 m/s respectively, then find the time after which they collide.

Solution : The particles approach each other with a relative speed, which is equal to the sum of their speeds.

$$v_{rel} = 2.0 + 1.14 = 3.14 \quad m/s$$

For collision to take place, the particles need to cover the initial separation with the relative speed as measured above. The time for collision is, thus, obtained as :

$$t = \frac{2\pi r}{v_{rel}} = \frac{2x3.14x10}{3.14} = 20\ s$$

Nature of UCM

Example:

Problem : Two particles “A” and “B” are moving along circles of radii " r_A " and " r_B " respectively at constant speeds. If the particles complete one revolution in same time, then prove that speed of the particle is directly proportional to radius of the circular path.

Solution : As the time period of the UCM is same,

$$T = \frac{2\pi r_A}{v_A} = \frac{2\pi r_B}{v_B}$$

$$\Rightarrow \frac{v_A}{r_A} = \frac{v_B}{r_B}$$

$$\Rightarrow \frac{v_A}{v_B} = \frac{r_A}{r_B}$$

Hence, speed of the particle is directly proportional to the radius of the circle.

Example:

Problem : Two particles "A" and "B" are moving along circles of radii " r_A " and " r_B " respectively at constant speeds. If the particles have same acceleration, then prove that speed of the particle is directly proportional to square root of the radius of the circular path.

Solution : As the acceleration of the UCM is same,

$$\frac{v_A^2}{r_A} = \frac{v_B^2}{r_B}$$

$$\Rightarrow \frac{v_A^2}{v_B^2} = \frac{r_A}{r_B}$$

$$\Rightarrow \frac{v_A}{v_B} = \frac{\sqrt{r_A}}{\sqrt{r_B}}$$

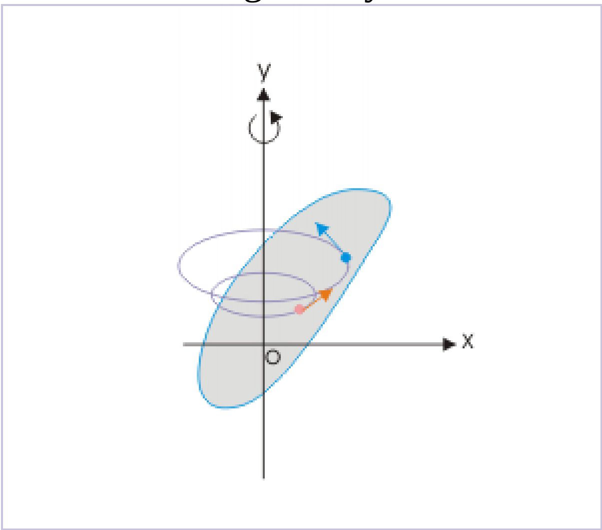
Hence, speed of the particle is directly proportional to square root of the radius of the circular path.

Circular motion and rotational kinematics

Each part of a rigid body under pure rotational motion describes a circular motion about a fixed axis.

In pure rotational motion, the constituent particles of a rigid body rotate about a fixed axis in a circular trajectory. The particles, composing the rigid body, are always at a constant perpendicular distance from the axis of rotation as their internal distances within the rigid body is locked. Farther the particle from the axis of rotation, greater is the speed of rotation of the particle. Clearly, rotation of a rigid body comprises of circular motion of individual particles.

Rotation of a rigid body about a fixed axis



Each particle constituting the body executes an uniform circular motion about the fixed axis.

We shall study these and other details about the rotational motion of rigid bodies at a later stage. For now, we confine ourselves to the aspects of rotational motion, which are connected to the circular motion as executed by a particle. In this background, we can say that uniform circular motion (UCM) represents the basic form of circular motion and circular motion, in turn, constitutes rotational motion of a rigid body.

The description of a circular and hence that of rotational motion is best suited to corresponding angular quantities as against linear quantities that we have so far used to describe translational motion. In this module, we shall introduce these angular quantities and prepare the ground work to enable us apply the concepts of angular quantities to “circular motion” in general and “uniform circulation motion” in particular.

Most important aspect of angular description as against linear description is that there exists one to one correspondence of quantities describing motion : angular displacement (linear displacement), angular velocity (linear velocity) and angular acceleration (linear acceleration).

Angular quantities

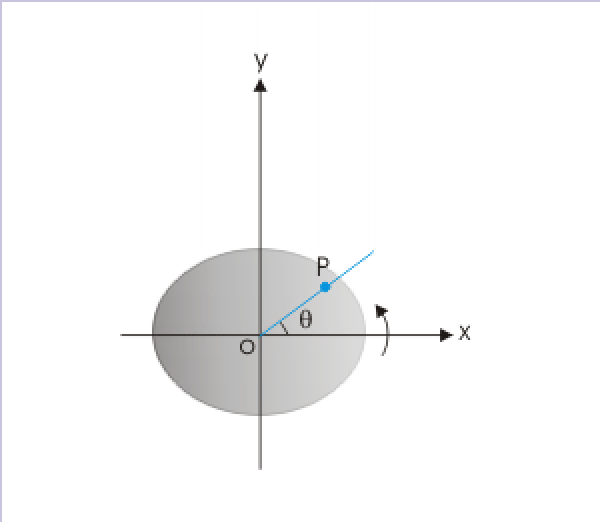
In this section, we discuss some of the defining quantities, which are used to study uniform circular motion of a particle and rotational motion of rigid bodies. These quantities are angular position, angular displacement and angular velocity. They possess directional properties. Their measurement in counter clockwise direction is considered positive, whereas quantities measured in clockwise direction is considered negative. This gives us a simplified scheme to represent an angular vector by a simple variable, whose sign indicates its direction.

Notably, we shall not discuss angular acceleration in this module. It will be discussed as a part of non-uniform circular motion in a separate module.

Angular position (θ)

We need two straight lines to measure an angle. In rotational motion, one of them represents fixed direction, while another represents the rotating arm containing the particle. Both these lines are perpendicular to the rotating axis. The rotating arm, additionally, passes through the position of the particle.

Angular position (θ)

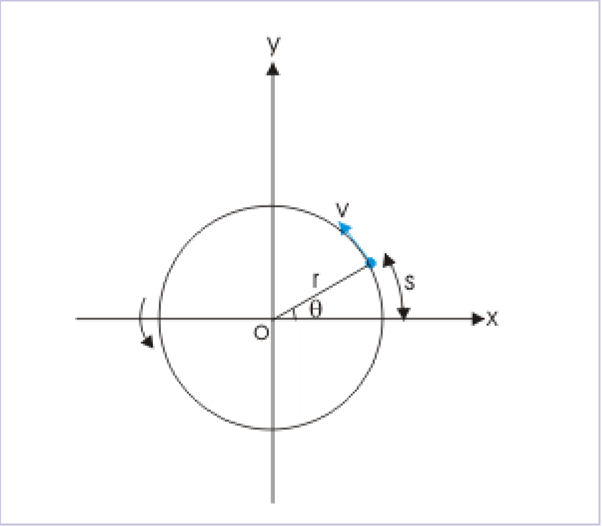


Angular position is the angle between reference direction and rotating arm.

For convenience, the reference direction like x – axis of the coordinate system serves to represent fixed direction. The angle between reference direction and rotating arm (OP) at any instant is the angular position of the particle (θ).

It must be clearly understood that angular position (θ) is an angle and does not represent the position of the particle by itself. It requires to be paired with radius of the circle (r) along which particle moves in order to specify the position of the particle. Thus, a specification of a position in the reference system will require both “r” and “θ” to be specified.

Relation between distance (s) and angle (θ)



By geometry,
Equation:

$$\theta = \frac{s}{r}$$
$$\Rightarrow s = \theta r$$

where "s" is the length of the arc subtending angle “θ” at the origin and “r” is the radius of the circle containing the position of the particle. The angular position is measured in “radian”, which has no dimension, being ratio of two lengths. One revolution contains 2π radians. The unit of radian is related to other angle measuring units "degree" and "revolution" as :

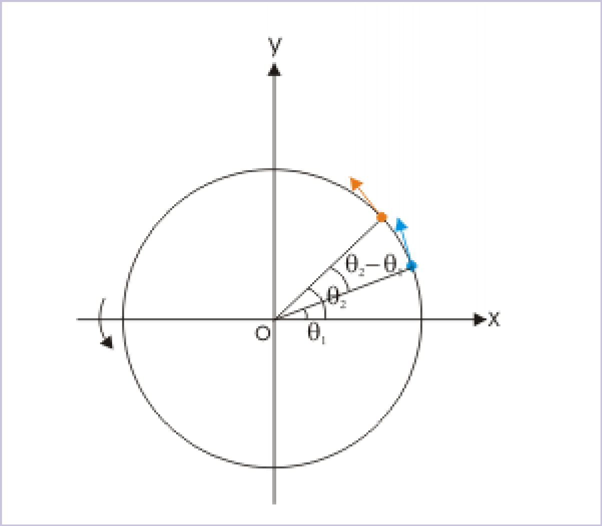
1 revolution = 360° = 2π radian

Note: The quantities related to angular motion are expressed in terms of angular position. It must be ensured that values of angular position wherever it appears in the expression be substituted in radians only. If the given value is in some other unit, then we first need to change the value into radian. It is so because, radian is a unit derived from the definition of the angle. The defining relation θ = s/r will not hold unless “θ” is in radian.

Angular displacement (Δθ)

Angular displacement is equal to the difference of angular positions at two instants of rotational motion.

Angular displacement (Δθ)



Angular displacement is equal to the difference of angular positions at two positions.

Equation:

$$\Delta\theta = \theta_2 - \theta_1$$

The angular displacement is also measured in “radian” like angular position. In case our measurement of angular position coincides with the reference direction, we can make substitution as given here :

$$\theta_1 = 0$$

$$\theta_2 = \theta$$

With these substitution, we can simply express angular displacement in terms of angle as :

$$\Rightarrow \Delta\theta = \theta_2 - \theta_1 = \theta - 0 = \theta$$

Angular velocity (ω)

Angular speed is the ratio of the magnitude of angular displacement and time interval.
Equation:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

This ratio is called average angular velocity, when it is evaluated for finite time interval; and instantaneous angular velocity, when it is evaluated for infinitesimally small period ($\Delta \rightarrow 0$).
Equation:

$$\omega = \frac{d\theta}{dt}$$

The angular velocity is measured in “rad/s”.

Description of circular motion

Circular motion is completely described when angular position of a particle is given as a function of time like :

$$\theta = f(t)$$

For example, $\theta = 2t^2 - 3t + 1$ tells us the position of the particle with the progress of time. The attributes of circular motion such as angular velocity and acceleration are first and second time derivatives of this function in time. Similarity to pure translational motion is quite obvious here. In pure translational motion, each particle constituting a rigid body follows parallel linear paths. The position of a particle is a function of time, whereby :

$$x = f(t)$$

Example:

Problem : The angular position (in radian) of a particle under circular motion about a perpendicular axis with respect to reference direction is given by the function in time (seconds) as :

$$\theta = t^2 - 0.2t + 1$$

Find (i) angular position when angular velocity is zero and (ii) determine whether rotation is clock-wise or anti-clockwise.

Solution : The angular velocity is equal to first derivative of angular position,

$$\omega = \frac{đ\theta}{đt} = \frac{đ}{đt}\left(t^2 - 0.2t + 1\right) = 2t - 0.2$$

For ω= 0, we have :

$$2t - 0.2 = 0$$
$$\Rightarrow t = 0.1\ s$$

The angular position at t = 0.1 s,

$$\theta = (0.1)^2 - 0.2 \text{ x } 0.1 + 1 = 0.99 \text{ rad}$$
$$\Rightarrow \theta = \frac{0.99\text{x}360}{2\pi} = \frac{0.99\text{x}360\text{x}7}{2\text{x}22} = 56.7^0$$

As the particle makes a positive angle with respect to reference direction, we conclude that the particle is moving in anti-clockwise direction (We shall discuss the convention regarding direction of angular quantities in detail subsequently).

Relationship between linear (v) and angular speed (ω)

In order to understand the relation, let us consider two uniform circular motions with equal time period (T) along two circular trajectories of radii r_1 and r_2 ($r_2 > r_2$). It is evident that particle along the outer circle is moving at a greater speed as it has to cover greater perimeter or distance. On the

other hand angular speeds of the two particles are equal as they transverse equal angles in a given time.

This observation is key to understand the relation between linear and angular speed. Now, we know that :

$$s = r\theta$$

Differentiating with respect to time, we have :

$$\frac{ds}{dt} = \frac{dr}{dt}\theta + r\frac{d\theta}{dt}$$

Since, “r” is constant for a given circular motion, $\frac{dr}{dt} = 0$.

$$\frac{ds}{dt} = \frac{d\theta}{dt}r = \omega r$$

Now, $\frac{ds}{dt}$ is equal to linear speed, v. Hence,

Equation:

$$v = \omega r$$

This is the relation between angular and linear speeds. Though it is apparent, but it is emphasized here for clarity that angular and linear speeds do not represent two separate individual speeds. Remember that a particle can have only one speed at a particular point of time. They are, as a matter of fact, equivalent representation of the same change of position with respect to time. They represent same speed – but in different language or notation.

Example:

Problem : The angular position (in radian) of a particle with respect to reference direction, along a circle of radius 0.5 m is given by the function in time (seconds) as :

$$\theta = t^2 - 0.2t$$

Find linear velocity of the particle at t = 0 second.
Solution : The angular velocity is given by :

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}\left(t^2 - 0.2t\right) = 2t - 0.2$$

For t = 0, the angular velocity is :

$$\Rightarrow \omega = 2 \text{ x } 0 - 0.2 = 0.2 \, \text{rad/s}$$

The linear velocity at this instant is :

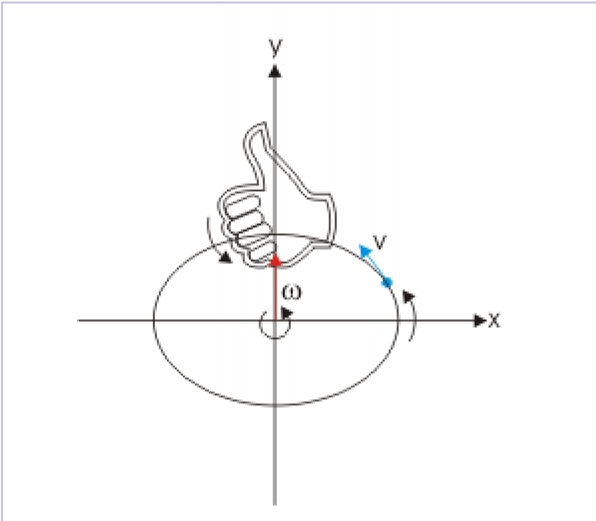
$$\Rightarrow v = \omega r = 0.2 \text{ x } 0.5 = 0.1 \, m/s$$

Vector representation of angular quantities

The angular quantities (displacement, velocity and acceleration) are also vector quantities like their linear counter parts and follow vector rules of addition and multiplication, with the notable exception of angular displacement. Angular displacement does not follow the rule of vector addition strictly. In particular, it can be shown that addition of angular displacement depends on the order in which they are added. This is contrary to the property of vector addition. Order of addition should not affect the result. We intend here to skip the details of this exception to focus on the subject matter in hand. Besides, we should know that we may completely ignore this exception if the angles involved have small values.

The vector angular quantities like angular velocity (**ω**) or ω (as scalar representation of angular vector) is represented by a vector, whose direction is obtained by applying “Right hand rule”. We just hold the axis of rotation with right hand in such a manner that the direction of the curl of fingers is along the direction of the rotation. The direction of extended thumb (along y-axis in the figure below) then represents the direction of angular velocity (**ω**).

Vector cross product



Right Hand Rule (RHR)

The important aspect of angular vector representation is that the angular vector is essentially a straight line of certain magnitude represented on certain scale with an arrow showing direction (shown in the figure as a red line with arrow) – not a curl as some might have expected. Further, the angular vector quantities are axial in nature. This means that they apply along the axis of rotation. Now, there are only two possible directions along the axis of rotation. Thus, we can work with sign (positive or negative) to indicate directional attribute of angular quantities. The angular quantities measured in counter clockwise direction is considered positive, whereas quantities measured in clockwise direction is considered negative.

This simplicity resulting from fixed axis of rotation is very useful. We can take the liberty to represent angular vector quantities in terms of signed scalar quantities as done in the case of linear quantities. The sign of the angular quantity represents the relative direction of the angular quantity with respect to a reference direction.

Analysis of angular motion involves working interchangeably between linear and angular quantities. We must understand here that the relationships essentially involve both axial (angular) and polar (linear)

vectors. In this context, it is recommended that we know the relationship between linear and angular quantities in vector forms as vector relation provide complete information about the quantities involved.

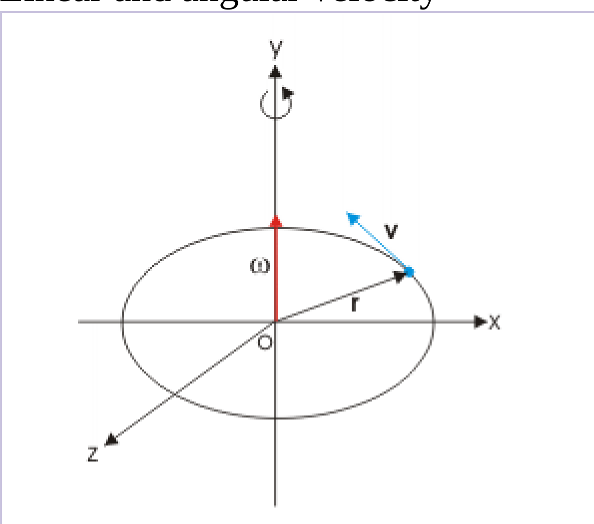
Linear and angular velocity relation in vector form

If we want to write the relation for velocities (as against the one derived for speed, v = ω r), then we need to write the relation as vector cross product :
Equation:

$$\mathbf{v} = \boldsymbol{\omega} \, \mathbf{x} \, \mathbf{r}$$

The order of quantities in vector product is important. A change in the order of cross product like ($\mathbf{r} \, \mathbf{x} \, \boldsymbol{\omega}$) represents the product vector in opposite direction. The directional relationship between thee vector quantities are shown in the figure. The vectors “**v**” and “**r**” are in the plane of “xz” plane, whereas angular velocity (**ω**), is in y-direction.

Linear and angular velocity

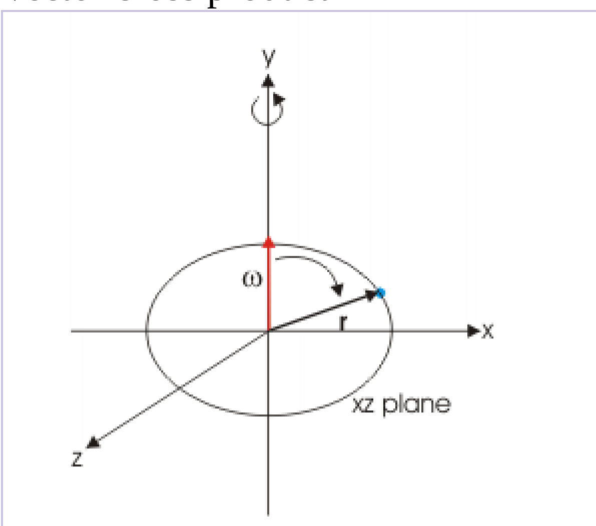


Directional relation between linear and angular velocity

Here, we shall demonstrate the usefulness of vector notation. Let us do a bit of interpretation here to establish the directional relationship among the quantities from the vector notation. It is expected from the equation ($\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$) that the vector product of angular velocity (**ω**) and radius vector (**r**) should yield the direction of velocity (**v**).

Remember that a vector cross product is evaluated by Right Hand Rule (RHR). We move from first vector (**ω**) to the second vector (**r**) of the vector product in an arc as shown in the figure.

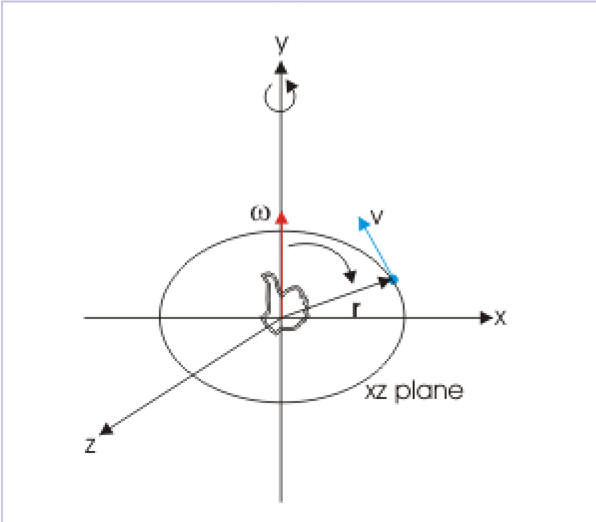
Vector cross product



Determining direction of vector cross product

We place our right hand such that the curl of fingers follows the direction of arc. The extended thumb, then, represents the direction of cross product (**v**), which is perpendicular (this fact lets us draw the exact direction) to each of the vectors and the plane containing two vectors (**ω** and **r**) whose products is being evaluated. In the case of circular motion, vectors **ω** and **r** are perpendicular to each other and vector **v** is perpendicular to the plane defined by vectors **ω** and **r**.

Vector cross product



Determining direction of vector cross product

Thus, we see that the interpretation of cross products completely defines the directions of quantities involved at the expense of developing skill to interpret vector product (we may require to do a bit of practice).

Also, we can evaluate magnitude (speed) as :

Equation:

$$v = |\mathbf{v}| = \omega r \sin\theta$$

where θ is the angle between two vectors ω and r. In the case of circular motion, θ = 90°, Hence,

$$\Rightarrow v = |\mathbf{v}| = \omega r$$

Thus, we have every detail of directional quantities involved in the equation by remembering vector form of equation.

Uniform circular motion

In the case of the uniform circular motion, the speed (v) of the particle is constant (by definition). This implies that angular velocity ($\omega = v/r$) in uniform circular motion is also constant.

$$\Rightarrow \omega = \frac{v}{r} = \text{constant}$$

Also, the time period of the uniform circular motion is :
Equation:

$$\Rightarrow T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

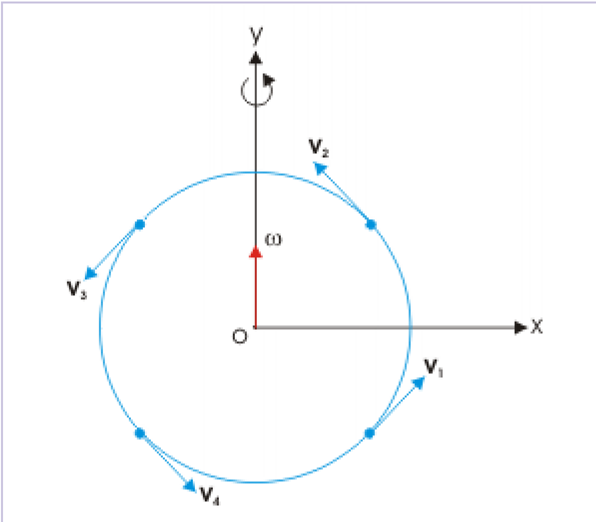
Linear .vs. angular quantity

The description of circular motion is described better in terms of angular quantity than its linear counter part.

The reasons are easy to understand. For example, consider the case of uniform circular motion. Here, the velocity of particle is changing - though the motion is "uniform". The two concepts do not go together. The general connotation of the term "uniform" indicates "constant", but the velocity is actually changing all the time.

When we describe the same uniform circular motion in terms of angular velocity, there is no contradiction. The velocity (i.e. angular velocity) is indeed constant. This is the first advantage of describing uniform circular motion in terms of angular velocity.

In other words, the vector manipulation or analysis of linear velocity along the circular path is complicated as its direction is specific to a particular point on the circular path and is basically multi-directional. On the other hand, direction of angular velocity is limited to be bi-directional at the most, along the fixed axis of rotation.
Linear and angular velocity



Second advantage is that angular velocity conveys the physical sense of the rotation of the particle as against linear velocity, which indicates translational motion. Alternatively, angular description emphasizes the distinction between two types of motion (translational and rotational).

Finally, angular quantities allow to write equations of motion as available for translational motion with constant acceleration. For illustration purpose, we can refer to equation of motion connecting initial and final angular velocities for a motion with constant angular acceleration "α" as :

$$\omega_2 = \omega_1 + \alpha t$$

We shall study detailed aspect of circular motion under constant angular acceleration in a separate module.

Exercises

Exercise:

Problem:

A flywheel of a car is rotating at 300 revolutions per minute. Find its angular speed in radian per second.

$$(a) 10 \quad (b) 10\pi \quad (c) \frac{10}{\pi} \quad (c) \frac{20}{\pi}$$

Solution:

Flywheel covers angular displacement of "2π" in one revolution. Therefore, the angular speed in "radian/second" is :

$$\omega = \frac{2\pi X 300}{60} = 10\pi \quad \text{rad}/s$$

Hence, option (b) is correct.

Exercise:

Problem:

A particle is rotating along a circle of radius 0.1 m at an angular speed of π rad/s. The time period of the rotation is :

$$(a) 1\,s \quad (b) 2\,s \quad (c) 3\,s \quad (d) 4\,s$$

Solution:

The time period of rotation is :

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\,s$$

Note : The input of radius is superfluous here.

Hence, option (b) is correct.

Exercise:

Problem:

A particle is rotating at constant speed along a circle of radius 0.1 m, having a time period of 1 second. Then, angular speed in "revolution/s" is :

$$(a)1 \quad (b)2 \quad (c)3 \quad (d)4$$

Solution:

The time period of rotation is :

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

Rearranging, the angular speed is :

$$\Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

Converting to in the unit of revolution per second, the angular speed is :

$$\Rightarrow \omega = 2\pi \quad \text{rad /s} = \frac{2\pi}{2\pi} = 1 \quad \text{revolution/s}$$

Hence, option (a) is correct.

Exercise:

Problem:

A particle is moving with a constant angular speed "ω" along a circle of radius "r" about a perpendicular axis passing through the center of the circle. If "**n**" be the unit vector in the positive direction of axis of rotation, then linear velocity is given by :

$$(a) - \mathbf{r} X \boldsymbol{\omega} \quad (b) r\omega\mathbf{n} \quad (c) - \boldsymbol{\omega} X \mathbf{r} \quad (d) \boldsymbol{\omega} X \mathbf{r}$$

Solution:

The linear velocity is given by the following vector cross product,

$$\mathbf{v} = \boldsymbol{\omega} X \mathbf{r}$$

We know that a change in the order of operands in cross product reverses the resulting vector. Hence,

$$\mathbf{v} = -\mathbf{r} X \boldsymbol{\omega}$$

Further, it is given in the question that positive axial direction has the unit vector "n". But, the plane of motion is perpendicular to axis of rotation. Hence, velocity is not aligned in the direction of unit vector **n**.

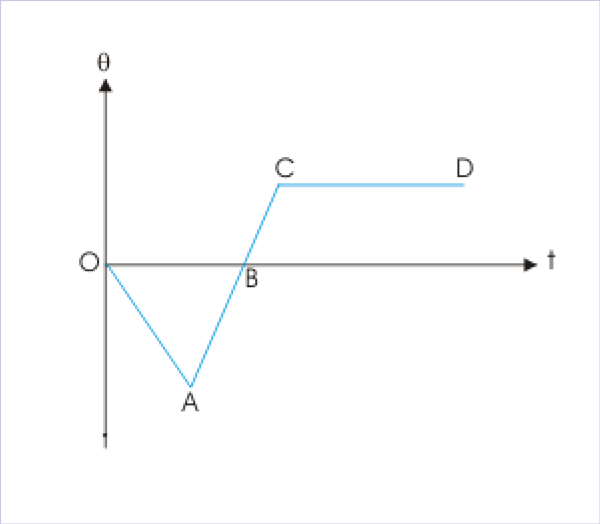
Hence, options (a) and (b) are correct.

Exercise:

Problem:

The figure shows the plot of angular displacement and time of a rotating disc. Corresponding to the segments marked on the plot, the direction of rotation is as :

Angular displacement



Angular displacement

(a) disk rotates in clockwise direction in the segments OA and AB.

(b) disk rotates in clockwise direction in the segment OA, but in anti-clockwise direction in the segment AB.

(c) disk rotates in anti-clockwise direction in the segment BC.

(d) disk rotates in anti-clockwise direction in the segment CD.

Solution:

The direction of rotation is determined by the sign of angular velocity. In turn, the sign of angular velocity is determined by the sign of the slope on angular displacement - time plot. The sign of slope is negative for line OA, positive for line AC and zero for line CD.

The positive angular velocity indicates anti-clockwise rotation and negative angular velocity indicates clockwise rotation. The disk is stationary when angular velocity is zero.

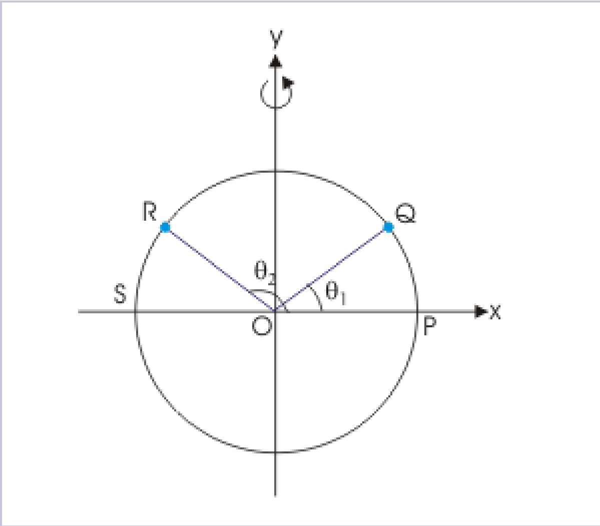
Hence, options (b) and (c) are correct.

Exercise:

Problem:

A particle moves along a circle in xy-plane with center of the circle as origin. It moves from position Q(1, √3) to R(-1, √3) as shown in the figure. The angular displacement is :

Angular displacement



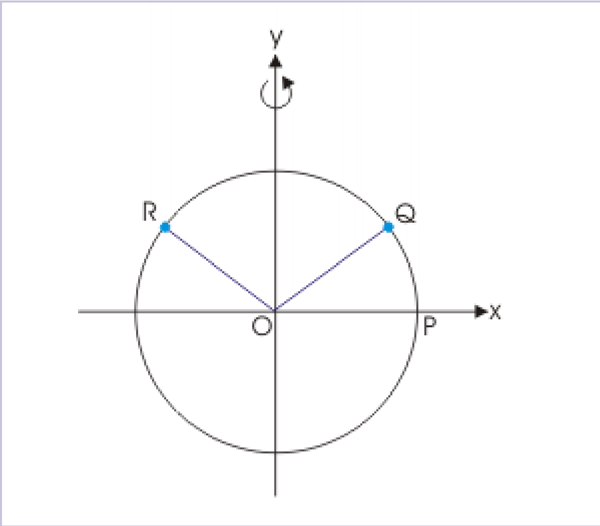
Angular displacement

$$(a) 30^\circ \quad (b) 45^\circ \quad (c) 60^\circ \quad (d) 75^\circ$$

Solution:

In order to find angular displacement, we need to find initial and final angular positions. From the geometry,

Angular displacement



Angular displacement

$$\tan\theta_1 = \frac{\sqrt{3}}{1} = \sqrt{3} = \tan 60^0$$

$$\Rightarrow \theta_1 = 60^0$$

Similarly,

$$\tan\theta_2 = -\frac{\sqrt{3}}{1} = -\sqrt{3} = \tan 120^0$$

$$\Rightarrow \theta_2 = 120^0$$

Thus, angular displacement is :

$$\Rightarrow \Delta\theta = \theta_2 - \theta_1 = 120^0 - 60^0 = 60^0$$

Hence, option (c) is correct.

Exercise:

Problem: Select the correct statement(s) :

(a) The direction of angular velocity is tangential to the circular path.

(b) The direction of angular velocity and centripetal acceleration are radial towards the center of the circle.

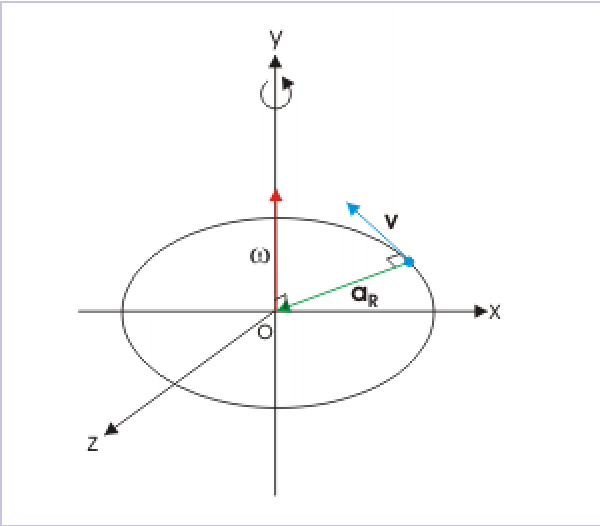
(c) The linear velocity, angular velocity and centripetal accelerations are mutually perpendicular to each other.

(d) The direction of angular velocity is axial.

Solution:

The direction of linear velocity is normal to radial direction. The direction of centripetal acceleration is radial. The direction of angular velocity is axial. See the figure.

Angular motion



Angular motion

Clearly, the three directions are mutually perpendicular to each other.

Hence, options (c) and (d) are correct.

Exercise:

Problem: The magnitude of centripetal acceleration is given by :

$$(a)\frac{v^2}{r} \quad (b)\frac{\omega^2}{r} \quad (c)\omega r \quad (d)\ |\boldsymbol{\omega} X \mathbf{v}|$$

Solution:

The centripetal acceleration is given by :

$$a_R = \frac{v^2}{r}$$

The linear speed "v" is related to angular speed by the relation,

$$\Rightarrow v = \omega r$$

Substituting this in the expression of centripetal acceleration, we have :

$$\Rightarrow a_R = \frac{\omega^2 r^2}{r} = \omega^2 r$$

We need to evaluate fourth cross product expression to know whether its modulus is equal to the magnitude of centripetal acceleration or not. Let the cross product be equal to a vector "**A**". The magnitude of vector "**A**" is :

$$A = |\boldsymbol{\omega} X \mathbf{v}| = \omega v \sin\theta$$

For circular motion, the angle between linear and angular velocity is 90°. Hence,

$$\Rightarrow A = \quad \boldsymbol{\omega} X \mathbf{v} \quad = \omega v \sin 90^0 = \omega v$$

Substituting for " ω", we have :

$$\Rightarrow A = \omega v = \frac{v^2}{r}$$

The modulus of the expression, therefore, is equal to the magnitude of centripetal acceleration.

Note : The vector cross product "| **ω** X**v**|” is the vector expression of centripetal force.

Hence, options (a) and (d) are correct.

Circular motion and rotational kinematics (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

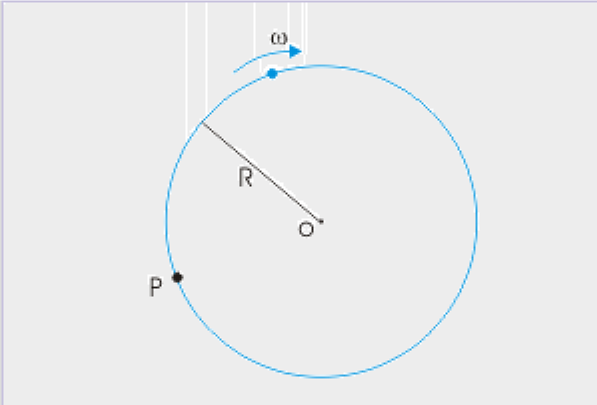
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the circular motion and rotational kinematics. The questions are categorized in terms of the characterizing features of the subject matter :

- Measurement of angular displacement
- Angular speeds
- Centripetal acceleration

Measurement of angular displacement

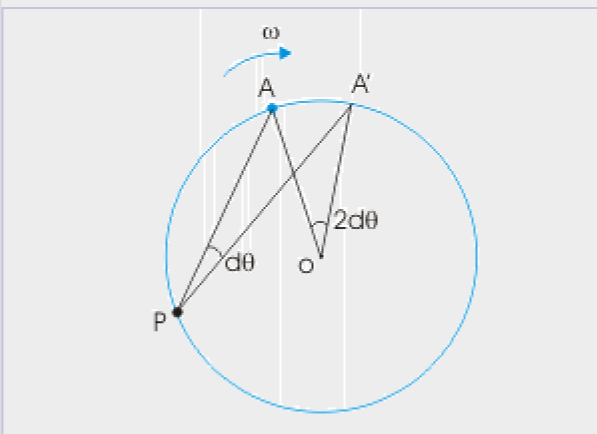
Example:
Problem : A particle in uniform circular motion about the center has angular velocity "ω ". What is its angular velocity with respect to a point "P" on the circumference of the circle ?
Uniform circular motion



Solution : Angular velocity is a measure of angle in unit time. In the question, measurement of angular velocity about the center of circle is given. It is, therefore, imperative that we seek a relation of angles formed by the motion of the particle at two points of references.

We consider a small arc AA' as shown in the figure, which is covered by the particle in time "dt". By geometry, if the arc subtends an angle "dθ" at "P", then the arc subtends an angle "2dθ" at the center.

Uniform circular motion



Let ω_P be the angular velocity of the particle with respect to point "P", then

$$\omega_P = \frac{đ\theta}{đt}$$

From the relation between angles as obtained earlier, the angular velocity of the particle with respect to center is :

$$\Rightarrow \omega = \frac{đ(2\theta)}{đt} = 2\frac{đ\theta}{đt} = 2\omega_P$$

$$\Rightarrow \omega_P = \frac{\omega}{2}$$

Angular speeds

Example:
Problem : Let ω_H and ω_E respectively be the angular speeds of the hour hand of a watch and that of the earth around its own axis. Compare the angular speeds of earth and hour hand of a watch.
Solution : The time periods of hour hand and that of the earth are 12 hours and 24 hours respectively. Now,

$$\omega_H = \frac{2\pi}{12}$$

and

$$\omega_E = \frac{2\pi}{24}$$
$$\Rightarrow \omega_H = 2\omega_E$$

Example:
Problem : A particle is kept on a uniformly rotating turn table at a radial distance of 2 m from the axis of rotation. The speed of the particle is "v". The particle is then shifted to a radial distance of 1 m from the axis of rotation. Find the speed of the particle at the new position.
Solution : The angular speed of rotation is constant. Now, linear velocities of the particle at the two positions are :

$$v_1 = \omega r_1 = \omega \text{ x } 2 = v \text{ (given)}$$
$$v_2 = \omega r_2 = \omega \text{ x } 1 = \omega$$
$$\Rightarrow v_2 = \frac{v}{2}$$

Centripetal acceleration

Example:
Problem : Find the centripetal acceleration (in $\mathrm{km} / \mathrm{hr}^2$) at a point on the equator of the earth (consider earth as a sphere of radius = 6400 km).
Solution : The centripetal acceleration in terms of angular speed is given by :

$$a = \omega^2 r$$

Now, angular speed, in terms of time period is :

$$\omega = \frac{2\pi}{T}$$

Combining two equations, we have :

$$a = \frac{4\pi^2 r}{T^2}$$

$$\Rightarrow a = \frac{4\mathrm{x}(3.14)^2 \mathrm{x}6400}{24^2} = 439 \ \mathrm{km} / \mathrm{hr}^2$$

Accelerated motion in two dimensions

Motion in two dimensions with one dimensional acceleration (projectile) is analyzed with component motions in coordinate system, whereas motion in two dimensions with two dimensional acceleration (circular motion) is analyzed with the help of component accelerations - tangential and normal accelerations.

We have already studied two dimensional motions such as projectile and uniform circular motion. These motions are the most celebrated examples of two dimensional motion, but it is easy to realize that they are specific instances with simplifying assumptions. The motions that we investigate in our surrounding mostly occur in two or three dimensions in a non-specific manner. The stage is, therefore, set to study two-dimensional motion in non-specific manner i.e. in a very general manner. This requires clear understanding of both linear and non-linear motion. As we have already studied circular motion - an instance of non-linear motion, we can develop an analysis model for a general case involving non-linear motion.

The study of two dimensional motion without any simplifying assumptions, provides us with an insight into the actual relationship among the various motional attributes, which is generally concealed in the consideration of specific two dimensional motions like projectile or uniform circular motion. We need to develop an analysis frame work, which is not limited by any consideration. In two dimensional motion, the first and foremost consideration is that acceleration denotes a change in velocity that reflects a change in the velocity due to any of the following combinations :

- change in the magnitude of velocity i.e. speed
- change in the direction of velocity
- change in both magnitude and direction of velocity

In one dimensional motion, we mostly deal with change in magnitude and change in direction limited to reversal of motion. Such limitations do not exist in two or three dimensional motion. A vector like velocity can change by virtue of even direction only as in the case of uniform circular motion. Further, a circular motion may also involve variable speed i.e. a motion in which velocity changes in both direction and magnitude.

Most importantly, the generalized consideration here will resolve the subtle differences that arises in interpreting vector quantities like displacement, velocity etc. We have noted that there are certain subtle differences in interpreting terms such as Δr and $|\Delta \mathbf{r}|$; dr/dt and $|d\mathbf{r}/dt|$; dv/dt and $|d\mathbf{v}/dt|$ etc. In words, we have seen that time rate of change in the magnitude of velocity (speed) is not equal to the magnitude of time rate of change in velocity. This is a subtle, but significant difference that we should account for. In this module, we shall find that time rate of change in the magnitude of velocity (speed), as a matter of fact, represents the magnitude of a component of acceleration known as "tangential acceleration".

Characteristics of two dimensional motion

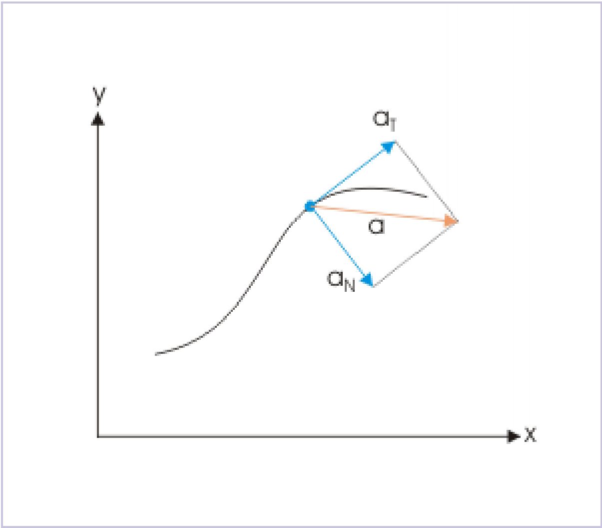
Let us have a look at two dimensional motions that we have so far studied. We observe that projectile motion is characterized by a constant acceleration, “g”, i.e. acceleration due to gravity. What it means that though the motion itself is two dimensional, but acceleration is one dimensional. Therefore, this motion presents the most simplified two dimensional motion after rectilinear motion, which can be studied with the help of consideration of motion in two component directions.

Uniform circular motion, on the other hand, involves an acceleration, which is not one dimensional. It is constant in magnitude, but keeps changing direction along the line connecting the center of the circle and the particle. The main point is that acceleration in uniform circular motion is two dimensional unlike projectile motion in which acceleration (due to gravity) is one dimensional. As a more generalized case, we can think of circular motion in which both magnitude and direction of acceleration is changing. Such would be the case when particle moves with varying speed along the circular path.

In the nutshell, we can conclude that two dimensional motion types (circular motion, elliptical motion and other non-linear motion) involve varying acceleration in two dimensions. In order to facilitate study of general class of motion in two dimensions, we introduce the concept of components of acceleration in two specific directions. Notably, these directions are not same as the coordinate directions (“x” and “y”). One of

the component acceleration is called “tangential acceleration”, which is directed along the tangent to the path of motion and the other is called “normal acceleration”, which is perpendicular to the tangent to the path of motion. Two accelerations are perpendicular to each other. The acceleration (sometimes also referred as total acceleration) is the vector sum of two mutually perpendicular component accelerations,

Two dimensional acceleration



There are tangential and normal components of acceleration.

Equation:

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_N$$

The normal acceleration is also known as radial or centripetal acceleration, $\mathbf{a}_R$, particularly in reference of circular motion.

Tangential acceleration

Tangential acceleration is directed tangentially to the path of motion. Since velocity is also tangential to the path of motion, it is imperative that

tangential acceleration is directed in the direction of velocity. This leads to an important meaning. We recall that it is only the component of force in the direction of velocity that changes the magnitude of velocity. This means that component of acceleration in the tangential direction represents the change in the magnitude of velocity (read speed). In non-uniform circular motion, the tangential acceleration accounts for the change in the speed (we shall study non-uniform circular motion in detail in a separate module).

By logical extension, we can define that tangential acceleration is time rate of change of "speed". The speed is highlighted here to underscore the character of tangential acceleration. Mathematically,
Equation:

$$a_T = \frac{đv}{đt}$$

This insight into the motion should resolve the differences that we had highlighted earlier, emphasizing that rate of change in the magnitude of velocity (dv/dt) is not equal to the magnitude of rate of change of velocity (|d**v**/dt|). What we see now that rate of change in the magnitude of velocity (dv/dt) is actually just a component of total acceleration (d**v**/dt).

It is easy to realize that tangential acceleration comes into picture only when there is change in the magnitude of velocity. For example, uniform circular motion does not involve change in the magnitude of velocity (i.e. speed is constant). There is, therefore, no tangential acceleration involved in uniform circular motion.

Normal acceleration

Normal (radial) acceleration acts in the direction perpendicular to tangential direction. We have seen that the normal acceleration, known as centripetal acceleration in the case of uniform circular motion, is given by :
Equation:

$$a_N = \frac{v^2}{r}$$

where “r” is the radius of the circular path. We can extend the expression of centripetal acceleration to all such trajectories of two dimensional motion, which involve radius of curvature. It is so because, radius of the circle is the radius of curvature of the circular path of motion.

In the case of tangential acceleration, we have argued that the motion should involve a change in the magnitude of velocity. Is there any such inference about normal (radial) acceleration? If motion is along a straight line without any change of direction, then there is no normal or radial acceleration involved. The radial acceleration comes into being only when motion involves a change in direction. We can, therefore, say that two components of accelerations are linked with two elements of velocity (magnitude and direction). A time rate of change in magnitude represents tangential acceleration, whereas a time rate of change of direction represents radial (normal) acceleration.

The above deduction has important implication for uniform circular motion. The uniform circular motion is characterized by constant speed, but continuously changing velocity. The velocity changes exclusively due to change in direction. Clearly, tangential acceleration is zero and radial acceleration is finite and acting towards the center of rotation.

Total acceleration

Total acceleration is defined in terms of velocity as :
Equation:

$$\mathbf{a} = \frac{đ\mathbf{v}}{đt}$$

In terms of component accelerations, we can write total accelerations in the following manner :

$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_N$$

The magnitude of total acceleration is given as :
Equation:

$$a = \left|\mathbf{a}\right| = \left|\frac{đ\mathbf{v}}{đt}\right| = \sqrt{\left({a_T}^2 + {a_N}^2\right)}$$

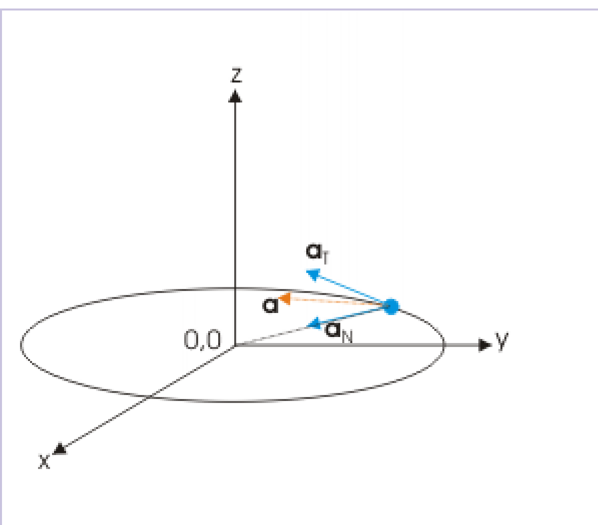
where

$$a_T = \frac{đv}{đt}$$

In the nutshell, we see that time rate of change in the speed represents a component of acceleration in tangential direction. On the other hand, magnitude of time rate of change in velocity represents the magnitude of total acceleration. Vector difference of total and tangential acceleration is equal to normal acceleration in general. In case of circular motion or motion with curvature, radial acceleration is normal acceleration.

Tangential and normal accelerations in circular motion

We consider motion of a particle along a circular path. As pointed out in the section above, the acceleration is given as vector sum of two acceleration components as :
Two dimensional circular motion



There are tangential and normal

components of acceleration.

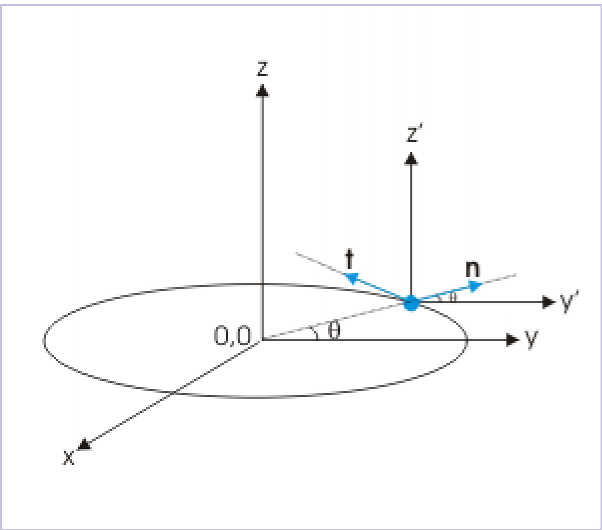
$$\mathbf{a} = \mathbf{a}_T + \mathbf{a}_N$$

Equation:

$$\mathbf{a} = a_T\mathbf{t} + a_N\mathbf{n}$$

where “**t**” and “**n**” are unit vectors in the tangential and radial directions. Note that normal direction is same as radial direction. For the motion shown in the figure, the unit vector in radial direction is :

Unit vectors



Unit vectors in tangential and normal directions.

Equation:

$$\mathbf{n} = 1 \ \text{x} \ \cos\theta\mathbf{i} + 1 \ \text{x} \ \sin\theta\mathbf{j} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

Similarly, the unit vector in tangential direction is :

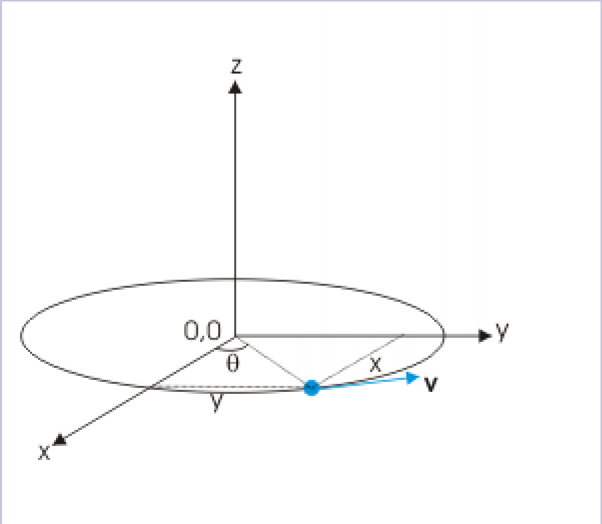
Equation:

$$\mathbf{t} = -1 \ \text{x} \ \sin\theta\mathbf{i} + 1 \ \text{x} \ \cos\theta\mathbf{j} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

Note: There is an easy way to find the sign of component, using graphical representation. Shift the vector at the origin, if the vector in question does not start from the origin. Simply imagine the component of a vector as projection on the coordinate. If the projection is on the positive side of the coordinate, then sign of component is positive; otherwise negative.

The position vector of a particle in circular motion is given in terms of components as :

Position vector



Position vector of a particle moving along a circular path.

Equation:

$$\mathbf{r} = r\mathbf{n} = x\mathbf{i} + y\mathbf{j} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$$

1: Velocity

The velocity of the particle, therefore, is obtained by differentiating with respect to time,:

$$\mathbf{v} = \frac{đ\mathbf{r}}{đt} = \left(-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}\right)\frac{đ\theta}{đt}$$

Equation:

$$\mathbf{v} = (-r\omega\sin\theta\mathbf{i} + r\omega\cos\theta\mathbf{j}) = -r\omega\mathbf{t}$$

where ω = dθ/dt is angular velocity. Also note that velocity is directed tangentially to path. For this reason, velocity vector is expressed with the help of unit vector in tangential direction.

2: Acceleration

The acceleration of the particle is obtained by differentiating the above expression of velocity with respect to time. However, as the radius of the circle is a constant, we take the same out of the differentiation,

Equation:

$$\mathbf{a} = \frac{đ\mathbf{v}}{đt} = \left\{r\omega\frac{đ}{đt}\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)\right\} + \left\{r\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)\frac{đ\omega}{đt}\right\}$$
$$\Rightarrow \mathbf{a} = \left\{r\omega\left(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right)\frac{đ\theta}{đt}\right\} + \left\{r\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)\frac{đ\omega}{đt}\right\}$$
$$\Rightarrow \mathbf{a} = -r\omega^2\mathbf{n} + r\frac{đ\omega}{đt}\mathbf{t}$$
$$\Rightarrow \mathbf{a} = -\frac{v^2}{r}\mathbf{n} + \frac{đv}{đt}\mathbf{t}$$

Thus, we see that :

$$\Rightarrow a_T = \frac{đv}{đt}$$
$$\Rightarrow a_N = -\frac{v^2}{r}$$

The above expressions, therefore, give two components of total acceleration in two specific directions. Again, we should emphasize that these directions are not the same as coordinate directions.

The derivation of acceleration components for two dimensional motion has, though, been carried out for circular motion, but the concepts of acceleration components as defined here can be applied - whenever there is curvature of path (non-linear path). In the case of rectilinear motion, normal acceleration reduces to zero as radius of curvature is infinite and as such total acceleration becomes equal to tangential acceleration.

Elliptical motion

In order to illustrate the features of two dimensional motion, we shall consider the case of elliptical motion of a particle in a plane. We shall use this motion to bring out the basic elements associated with the understanding of acceleration and its relation with other attributes of motion.

It is important that we work with the examples without any pre-notion such as “constant” acceleration etc. The treatment here is very general and intuitive of the various facets of accelerated motion in two dimensions.

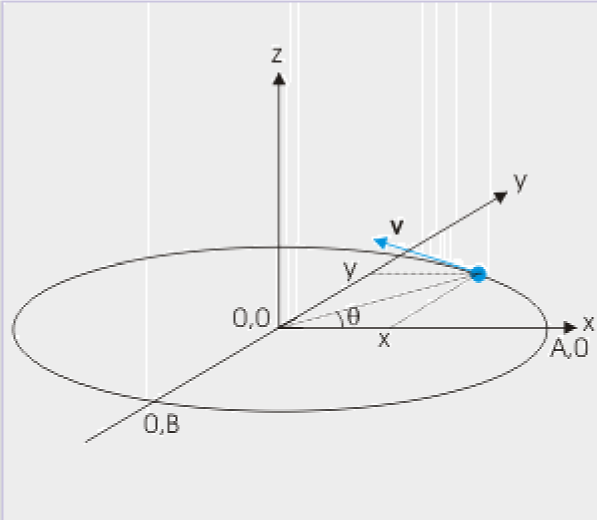
Path of motion

Example:

Problem : The coordinates of a particle moving in a plane are given by $x = A \cos(\omega t)$ and $y = B \sin(\omega t)$ where A, B (< A) and ω are positive constants. Find the nature of path of motion.

Solution : We shall use the general technique to find path of motion in two dimensional case. In order to find the path motion, we need to have an equation that connects “x” and “y” coordinates of the planar coordinate system. Note that there is no third coordinate.

Elliptical motion



Motion of a particle moving along an elliptical path path.

An inspection of the expressions of “x” and “y” suggests that we can use the trigonometric identity,

$$\sin^2\theta + \cos^2\theta = 1$$

Here, we have :

$$x = A\cos(\omega t)$$
$$\Rightarrow \cos\left(\omega t\right) = \frac{x}{A}$$

Similarly, we have :

$$y = B\sin(\omega t)$$
$$\Rightarrow \sin\left(\omega t\right) = \frac{y}{B}$$

Squaring and adding two equations,

$$\sin^2\left(\omega t\right) + \cos^2\left(\omega t\right) = 1$$

$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

This is an equation of ellipse. Hence, the particle follows an elliptical path.

Nature of velocity and acceleration

Example:
Problem : The coordinates of a particle moving in a plane are given by x = A cos(ωt) and y = B sin (ωt) where A, B (< A) and ω are positive constants. Investigate the nature of velocity and acceleration for this motion. Also, discuss the case for A = B and when "ω" is constant.
Solution : We can investigate the motion as required if we know expressions of velocity and acceleration. Therefore, we need to determine velocity and acceleration. Since components of position are given, we can find components of velocity and acceleration by differentiating the expression with respect to time.
1: Velocity
The components of velocity in “x” and “y” directions are :

$$\frac{đx}{đt} = v_x = -A\omega \sin\left(\omega t\right)$$
$$\frac{đy}{đt} = v_y = B\omega \cos\left(\omega t\right)$$

The velocity of the particle is given by :

$$\Rightarrow \mathbf{v} = \omega\{-A\sin(\omega t)\mathbf{i} + B\cos(\omega t)\mathbf{j}\}$$

Evidently, magnitude and direction of the particle varies with time.
2: Acceleration
We find the components of acceleration by differentiating again, as :

$$\frac{đ^2x}{đt^2} = a_x = -A\omega^2 \cos\left(\omega t\right)$$
$$\frac{đ^2y}{đt^2} = a_y = -B\omega^2 \sin\left(\omega t\right)$$

Both “x” and “y” components of the acceleration are trigonometric functions. This means that acceleration varies in component direction. The net or resultant acceleration is :

$$\Rightarrow \mathbf{a} = -\omega^2 \left\{ A\cos\left(\omega t\right)\mathbf{i} + B\sin\left(\omega t\right)\mathbf{j} \right\}$$

3: When A = B and "ω" is constant
When A = B, the elliptical motion reduces to circular motion. Its path is given by the equation :

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{A^2} = 1$$

$$\Rightarrow x^2 + y^2 = A^2$$

This is an equation of circle of radius “A”. The speed for this condition is given by :

$$v = \sqrt{\left\{ A^2\omega^2 \sin^2\left(\omega t\right) + A^2\omega^2 \cos^2\left(\omega t\right) \right\}}$$
$$v = A\omega$$

Thus, speed becomes a constant for circular motion, when ω = constant. The magnitude of acceleration is :

$$a = \omega^2 \sqrt{\left\{ A^2 \sin^2\left(\omega t\right) + A^2 \cos^2\left(\omega t\right) \right\}}$$
$$a = A\omega^2$$

Thus, acceleration becomes a constant for circular motion, when ω = constant.

Application

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a

verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the accelerated motion in two dimensions. The questions are categorized in terms of the characterizing features of the subject matter :

- Path of motion
- Tangential and normal accelerations
- Nature of motion
- Displacement in two dimensions

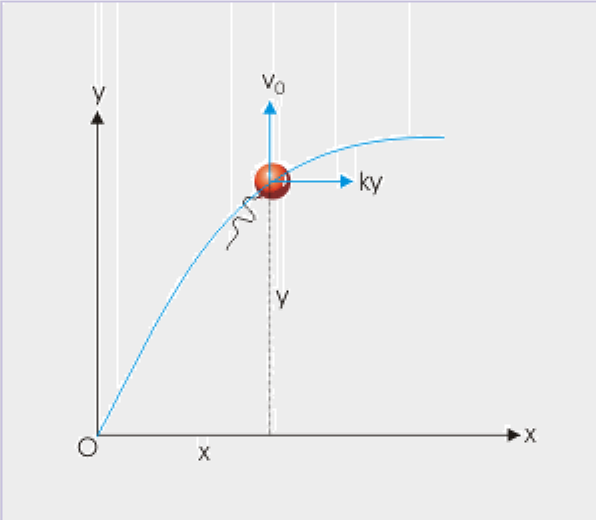
Path of motion

Example:

Problem : A balloon starts rising from the surface with a constant upward velocity, “ v_0 ”. The balloon gains a horizontal drift due to the wind. The horizontal drift velocity is given by “ky”, where “k” is a constant and “y” is the vertical height of the balloon from the surface. Derive an expression of path of the motion.

Solution : An inspection of the equation of drift velocity (v = ky) suggests that balloon drifts more with the gain in height. A suggestive x-y plot of the motion is shown here.

Motion of a balloon



The balloon moves with an acceleration in horizontal direction.

Let vertical and horizontal direction corresponds to “y” and “x” axes of the coordinate system. Here,

$$v_y = v_0$$

$$v_x = k_y$$

We are required to know the relation between vertical and horizontal components of displacement from the expression of component velocities. It means that we need to know a lower order attribute from higher order attribute. Thus, we shall proceed with integration of differential equation, which defines velocity as :

$$\Rightarrow \frac{đx}{đt} = ky$$

Similarly,

$$\frac{đy}{đt} = v_0$$

$$\Rightarrow đy = v_0 đt$$

Combining two equations by eliminating “dt”,

$$đx = \frac{kyđy}{v_0}$$

Now, integrating both sides, we have :

$$x = \int \frac{kyđy}{v_0}$$

Taking out constants out of the integral,

$$\Rightarrow x = \frac{k}{v_0} \int yđy$$

$$x = \frac{ky^2}{2v_0}$$

This is the required equation of motion, which is an equation of a parabola. Thus, the suggested plot given in the beginning, as a matter of fact, was correct.

Tangential and normal accelerations

Example:
Problem : A balloon starts rising from the surface with a constant upward velocity, “ v_0 ”. The balloon gains a horizontal drift due to the wind. The horizontal drift velocity is given by “ky”, where “k” is a constant and “y” is the vertical height of the balloon from the surface. Derive expressions for the tangential and normal accelerations of the balloon.
Solution : We can proceed to find the magnitude of total acceleration by first finding the expression of velocity. Here, velocity is given as :

$$v = ky\mathbf{i} + v_0\mathbf{j}$$

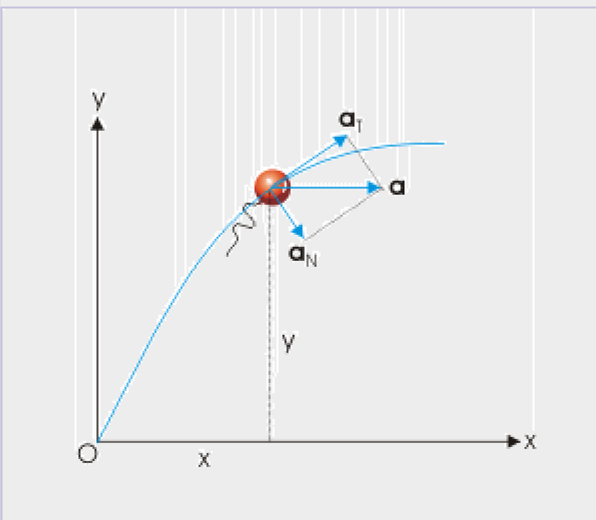
Since acceleration is higher order attribute, we obtain its expression by differentiating the expression of velocity with respect to time :

$$\Rightarrow \mathbf{a} = \frac{đv}{đt} = kv_y i = kv_0\mathbf{i}$$

It is obvious that acceleration is one dimensional. It is evident from the data given also. The balloon moving with constant vertical velocity has no acceleration in y-direction. The speed of the balloon in x-direction, however, keeps changing with height (time) and as such total acceleration of the balloon is in x-direction. The magnitude of total acceleration is :

$$a = |\mathbf{a}| = kv_y = kv_0$$

Motion of a balloon



The acceleration of the balloon has two components in mutually perpendicular directions.

Thus, we see that total acceleration is not only one dimensional, but constant as well. However, this does not mean that component accelerations viz tangential and normal accelerations are also constant. We

need to investigate their expressions. We can obtain tangential acceleration as time rate of change of the magnitude of velocity i.e. the time rate of change of speed. We, therefore, need to first know an expression of the speed. Now, speed is :

$$v=\sqrt{(ky)^2+(v_0)^2}$$

Differentiating with respect to time, we have :

$$\Rightarrow a_T=\frac{đv}{đt}=\frac{2k^2y}{2\sqrt{\left(k^2y^2+v_0^2\right)}}\quad x\quad\frac{dy}{dt}$$

$$\Rightarrow a_T=\frac{dv}{dt}=\frac{k^2yv_0}{\sqrt{\left(k^2y^2+v_0^2\right)}}$$

In order to find the normal acceleration, we use the fact that total acceleration is vector sum of two mutually perpendicular tangential and normal accelerations.

$$a^2=a_T^2+a_N^2$$

$$\Rightarrow a_N^2=a^2-a_T^2=k^2v_0^2-\frac{k_4y_2v_0^2}{\left(k^2y^2+v_0^2\right)}$$

$$\Rightarrow a_N^2=k^2v_0^2\left\{1-\frac{k^2y^2}{\left(k^2y^2+v_0^2\right)}\right\}$$

$$\Rightarrow a_N^2=k^2v_0^2\left\{\frac{k^2y^2+v_0^2-k^2y^2}{\left(k^2y^2+v_0^2\right)}\right\}$$

$$\Rightarrow a_N^2=\frac{k^2v_0^4}{\left(k^2y^2+v_0^2\right)}$$

$$\Rightarrow a_N = \frac{kv_0^2}{\sqrt{\left(k^2y^2 + v_0^2\right)}}$$

Nature of motion

Example:

Problem : The coordinates of a particle moving in a plane are given by x = A cos(ωt) and y = B sin (ωt) where A, B (< A) and "ω" are positive constants of appropriate dimensions. Prove that the velocity and acceleration of the particle are normal to each other at t = π/2ω.

Solution : By differentiation, the components of velocity and acceleration are as given under :

The components of velocity in "x" and "y" directions are :

$$\frac{đx}{đt} = v_x = -A\omega \sin \omega t$$

$$\frac{đy}{đt} = v_y = B\omega \cos \omega t$$

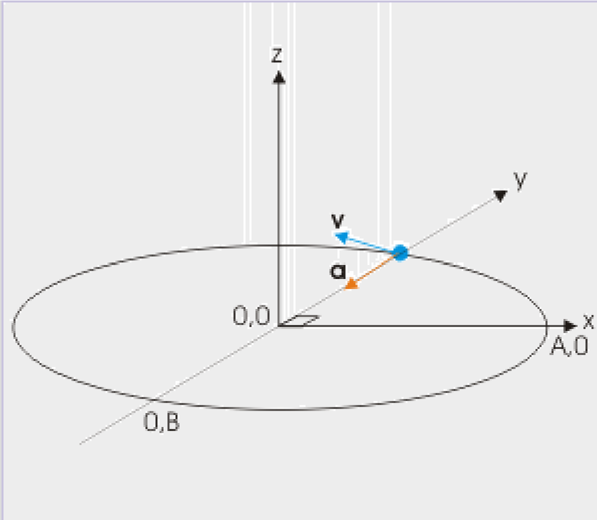
The components of acceleration in "x" and "y" directions are :

$$\frac{đ^2x}{đt^2} = a_x = -A\omega^2 \cos \omega t$$

$$\frac{đ^2y}{đt^2} = a_y = -B\omega^2 \sin \omega t$$

At time, $t = \frac{\pi}{2\omega}$ and $\theta = \omega t = \frac{\pi}{2}$. Putting this value in the component expressions, we have :

Motion along elliptical path



Velocity and acceleration are perpendicular at the given instant.

$$\Rightarrow v_x = -A\omega \sin \omega t = -A\omega \sin \pi/2 = -A\omega$$

$$\Rightarrow v_y = B\omega \cos \omega t = B\omega \cos \pi/2 = 0$$

$$\Rightarrow a_x = -A\omega^2 \cos \omega t = -A\omega^2 \cos \pi/2 = 0$$

$$\Rightarrow ay = -B\omega^2 \sin \omega t = -b\omega^2 \sin \pi/2 = -B\omega^2$$

The net velocity is in negative x-direction, whereas net acceleration is in negative y-direction. Hence at $t = \frac{\pi}{2\omega}$, velocity and acceleration of the particle are normal to each other.

Example:

Problem : Position vector of a particle is :

$$\mathbf{r} = a \cos \omega t\mathbf{i} + a \sin \omega t\mathbf{j}$$

Show that velocity vector is perpendicular to position vector.
Solution : We shall use a different technique to prove as required. We shall use the fact that scalar (dot) product of two perpendicular vectors is zero. We, therefore, need to find the expression of velocity. We can obtain the same by differentiating the expression of position vector with respect to time as :

$$\mathbf{v} = \frac{đ\mathbf{r}}{đt} = -a \sin \omega t\mathbf{i} + a \cos \omega t\mathbf{j}$$

To check whether velocity is perpendicular to the position vector, we take the scalar product of r and **v** as :

$$\Rightarrow \mathbf{r.\ v} = (a \cos \omega t\mathbf{i} + a \sin \omega t\mathbf{j}).\ (-a \sin \omega t\mathbf{i} + a \cos \omega t\mathbf{j})$$

$$\Rightarrow \mathbf{r.\ v} = -a \sin \omega t \cos \omega t + a \sin \omega t \cos \omega t = 0$$

This means that the angle between position vector and velocity are at right angle to each other. Hence, velocity is perpendicular to position vector.

Displacement in two dimensions

Example:
Problem : The coordinates of a particle moving in a plane are given by x = A cos(ω t) and y = B sin (ω t) where A, B (<A) and ω are positive constants of appropriate dimensions. Find the displacement of the particle in time interval t = 0 to t = π/2 ω.
Solution : In order to find the displacement, we shall first know the positions of the particle at the start of motion and at the given time. Now, the position of the particle is given by coordinates :

$$x = A \cos \omega t$$

and

$$y = B \sin \omega t$$

At t = 0, the position of the particle is given by :

$$\Rightarrow x = A\cos(\omega x 0) = A\cos 0 = A$$

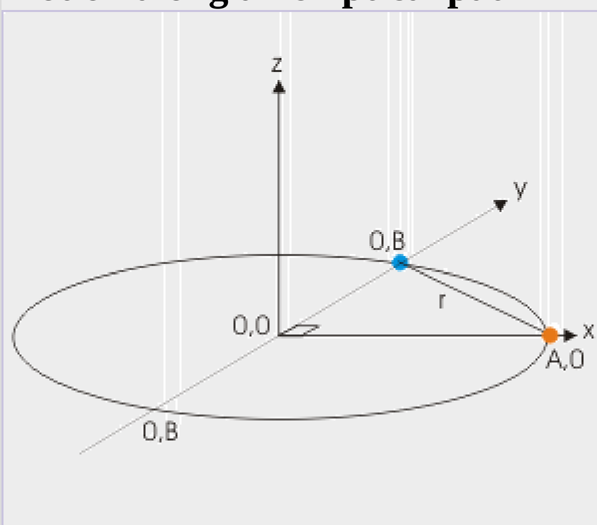
$$\Rightarrow y = B\sin(\omega x 0) = B\sin 0 = 0$$

At $t = \frac{\pi}{2\omega}$, the position of the particle is given by :

$$\Rightarrow x = A\cos(\omega x \pi/2\omega) = A\cos \pi/2 = 0$$

$$\Rightarrow y = B\sin(\omega x \pi/2\omega) = a\sin \pi/2 = B$$

Motion along an elliptical path



The linear distance equals displacement.

Therefore , the displacement in the given time interval is :

$$r = \sqrt{(A^2 + B^2)}$$

Transformation of graphs

Transformation of graphs means changing graphs. This generally allows us to draw graphs of more complicated functions from graphs of basic or simpler functions by applying different transformation techniques. It is important to emphasize here that plotting a graph is an extremely powerful technique and method to know properties of a function such as domain, range, periodicity, polarity and other features which involve differentiability of a function. Subsequently, we shall see that plotting enables us to know these properties more elegantly and easily as compared to other analytical methods.

Graphing of a given function involves modifying graph of a core function. We modify core function and its graph, applying various mathematical operations on the core function. There are two fundamental ways in which we operate on core function and hence its graph. We can either modify input to the function or modify output of function.

Broad categories of transformation

- Transformation applied by modification to input
- Transformation applied by modification to output
- Transformation applied by modulus function
- Transformation applied by greatest integer function
- Transformation applied by fraction part function
- Transformation applied by least integer function

We shall cover first transformation in this module. Others will be taken up in other modules.

Important concepts

Graph of a function

It is a plot of values of function against independent variable x. The value of function changes in accordance with function rule as x changes. Graph depicts these changes pictorially. In the current context, both core function

and modified function graphs are plotted against same independent variable x.

Input to the function

What is input to the function? How do we change input to the function? Values are passed to the function through argument of the function. The argument itself is a function in x i.e. independent variable. The simplest form of argument is "x" like in function f(x). The modified arguments are "2x" in function f(2x) or "2x-1" in function f(2x-1). This changes input to the function. Important to underline is that independent variable x remains what it is, but argument of the function changes due to mathematical operation on independent variable. Thus, we modify argument though mathematical operation on independent variable x. Basic possibilities of modifying argument i.e. input by using arithmetic operations on x are addition, subtraction, multiplication, division and negation. In notation, we write modification to the input of the function as :

$$\text{Argument/input} = bx + c; \quad b, c \in R$$

These changes are called internal or pre-composition modifications.

Output of the function

A modification in input to the graph is reflected in the values of the function. This is one way of modifying output and hence corresponding graph. Yet another approach of changing output is by applying arithmetic operations on the function itself. We shall represent such arithmetic operations on the function as :

$$af(x) + d; \quad a, d \in R$$

These changes are called external or post-composition modifications.

Arithmetic operations

Addition/subtraction operations

Addition and subtraction to independent variable x is represented as :

$$x + c; \quad c \in R$$

The notation represents addition operation when c is positive and subtraction when c is negative. In particular, we should underline that notation “bx+c” does not represent addition to independent variable. Rather it represents addition/ subtraction to “bx”. We shall develop proper algorithm to handle such operations subsequently. Similarly, addition and subtraction operation on function is represented as :

$$\mathrm{f(x)} + d; \quad d \in R$$

Again, “af(x) + d” is addition/ subtraction to “af(x)” not to “f(x)”.

Product/division operations

Product and division operations are defined with a positive constant for both independent variable and function. It is because negation i.e. multiplication or division with -1 is a separate operation from the point of graphical effect. In the case of product operation, the magnitude of constants (a or b) is greater than 1 such that resulting value is greater than the original value.

$$bx; \quad |b| > 1 \quad \text{for independent variable}$$

$$af(x); \quad |a| > 1 \quad \text{for function}$$

The division operation is eqivalent to product operation when value of multiplier is less than 1. In this case, magnitude of constants (a or b) is less than 1 such that resulting value is less than the original value.

$$bx; \quad 0 < |b| < 1 \quad \text{for independent variable}$$

$$af(x); \quad 0 < |a| < 1 \quad \text{for function}$$

Negation

Negation means multiplication or division by -1.

Effect of arithmetic operations

Addition/ subtraction operation on independent variable results in shifting of core graph along x-axis i..e horizontally. Similarly, product/division operations results in scaling (shrinking or stretching) of core graph horizontally. The change in graphs due to negation is reflected as mirroring (across y–axis) horizontally. Clearly, modifications resulting from modification to input modifies core graph horizontally. Another important aspect of these modification is that changes takes place opposite to that of operation on independent variable. For example, when “2” is added to independent variable, then core graph shifts left which is opposite to the direction of increasing x. A multiplication by 2 shrinks the graph horizontally by a factor 2, whereas division by 2 stretches the graph by a factor of 2.

On the other hand, modification in the output of function is reflected in change in graphs along y-axis i.e. vertically. Effects such as shifting, scaling (shrinking or stretching) or mirroring across x-axis takes place in vertical direction. Also, the effect of modification in output is in the direction of modification as against effects due to modifications to input. A multiplication of function by a positive constant greater than 1, for example, stretches the graph in y-direction as expected. These aspects will be clear as we study each of the modifications mentioned here.

Forms of representation

There is a bit of ambiguity about the nature of constants in symbolic representation of transformation. Consider the representation,

$$af(bx+c)+d; \quad a,b,c \in R$$

In this case "a", "b", "c" and "d" can be either positive or negative depending on the particular transformation. A positive "d" means that graph is shifted up. On the other hand, we can specify constants to be positive in the following representation :

$$\pm af(\pm bx \pm c) \pm d; \quad a,b,c > 0$$

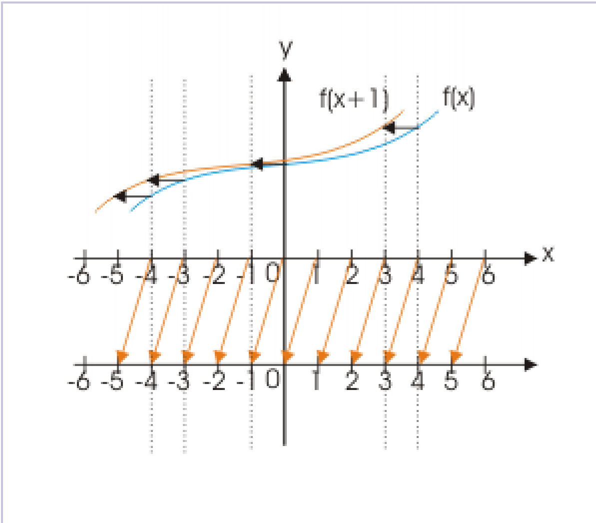
The form of representation appears to be cumbersome, but is more explicit in its intent. It delinks sign from the magnitude of constants. In this case, the signs preceding positive constants need to be interpreted for the nature of transformation. For example, a negative sign before c denotes right horizontal shift. It is, however, clear that both representations are essentially equivalent and their use depends on personal choice or context. This difference does not matter so long we understand the process of graphing.

Transformation of graph by input

Addition and subtraction to independent variable

In order to understand this type of transformation, we need to explore how output of the function changes as input to the function changes. Let us consider an example of functions f(x) and f(x+1). The integral values of inedependent variable are same as integral values on x-axis of coordinate system. Note that independent variable is plotted along x-axis as real number line. The integral x+1 values to the function f(x+1) - such that input values are same as that of f(x) - are shown on a separate line just below x-axis. The corresponding values are linked with arrow signs. Input to the function f(x+1) which is same as that of f(x) corresponds to x which is 1 unit smaller. It means graph of f(x+1) is same as graph of f(x), which has been shifted by 1 unit towards left. Else, we can say that the origin of plot (also x-axis) has shifted right by 1 unit.

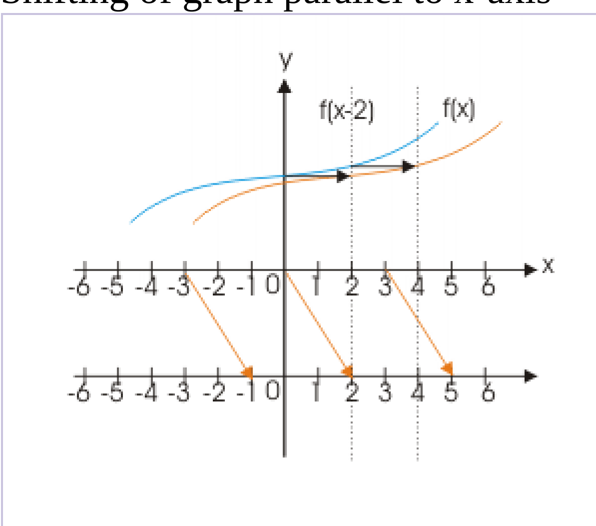
Shifting of graph parallel to x-axis



Each element of graph is shifted
left by same value.

Let us now consider an example of functions f(x) and f(x-2). Input to the function f(x-2) which is same as that of f(x) now appears 2 unit later on x-axis. It means graph of f(x-2) is same as graph of f(x), which has been shifted by 2 units towards right. Else, we can say that the origin of plot (also x-axis) has shifted left by 2 units.

Shifting of graph parallel to x-axis



Each element of graph is shifted

right by same value.

The addition/subtraction transformation is depicted symbolically as :

$$y = f(x) \quad \Rightarrow \quad y = f(x \pm |a|); \quad |a| > 0$$

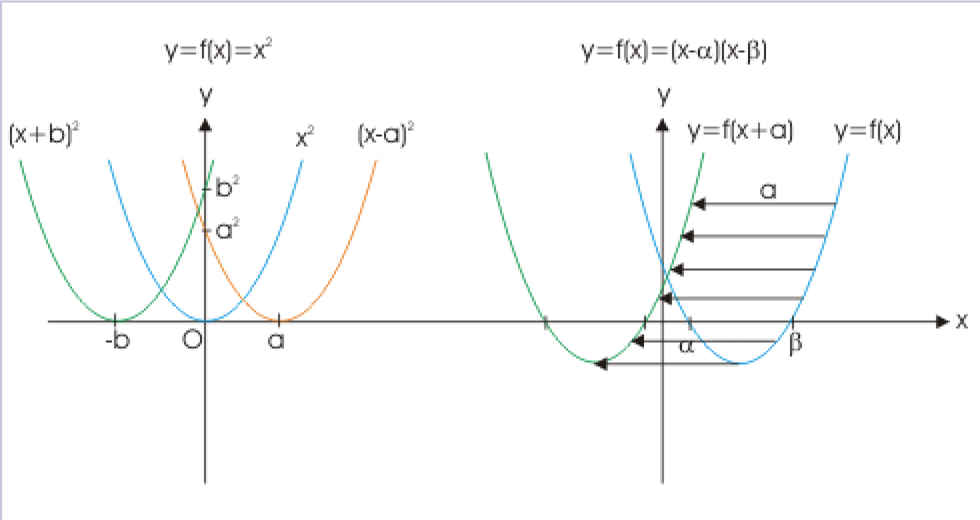
If we add a positive constant to the argument of the function, then value of y at x=x in the new function y=f(x+|a|) is same as that of y=f(x) at x=x-|a|. For this reason, the graph of f(x+|a|) is same as the graph of y=f(x) shifted left by unit "a" in x-direction. Similarly, the graph of f(x-|a|) is same as the graph of y=f(x) shifted right by unit "a" in x-direction.

1 : The plot of y=f(x+|a|); is the plot of y=f(x) shifted left by unit "|a|".

2 : The plot of y=f(x-|a|); is the plot of y=f(x) shifted right by unit "|a|".

We use these facts to draw graph of transformed function f(x±a) by shifting graph of f(x) by unit "a" in x-direction. Each point forming the plot is shifted parallel to x-axis (see quadratic graph showm in the of figure below). The graph in the center of left figure depicts monomial function $y = x^2$ with vertex at origin. It is shifted right by "a" units (a>0) and the function representing shifted graph is $y = (x - a)^2$. Note that vertex of parabola is shifted from (0,0) to (a,0). Further, the graph is shifted left by "b" units (b>0) and the function representing shifted graph is $y = (x + b)^2$. In this case, vertex of parabola is shifted from (0,0) to (-b,0).

Shifting of graph parallel to x-axis

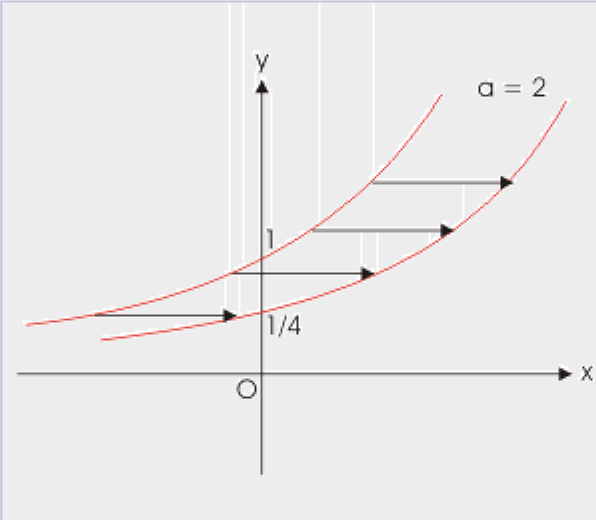


Each element of graph is shifted by same value.

Example:
Problem : Draw graph of function $4y = 2^x$.
Solution : Given function is exponential function. On simplification, we have :

$$\Rightarrow y = 2^{-2} X 2^x = 2^{x-2}$$

Here, core graph is $y = 2^x$. We draw its graph first and then shift the graph right by 2 units to get the graph of given function.
Shifting of exponential graph parallel to x-axis



Each element of graph is shifted by same value.

Note that the value of function at x=0 for core and modified functions, respectively, are :

$$\Rightarrow y = 2^x = 2^0 = 1$$

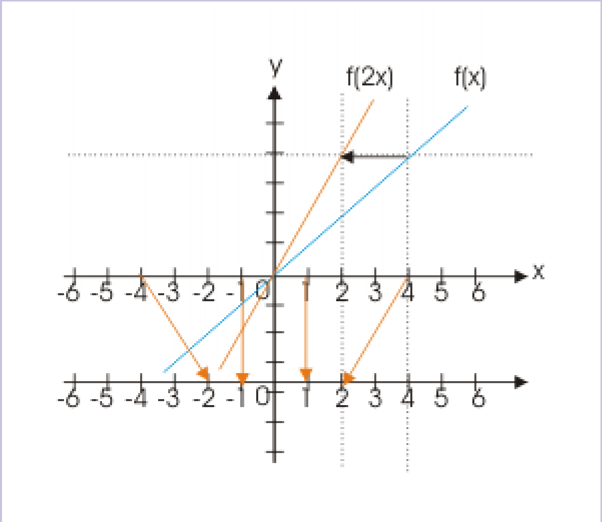
$$\Rightarrow y = 2^{x-2} = 2^{-2} = \frac{1}{4} = 0.25$$

Multiplication and division of independent variable

Let us consider an example of functions f(x) and f(2x). The integral values of independent variable are same as integral values on x-axis of coordinate system. Note that independent variable is plotted along x-axis as real number line. The integral 2x values to the function f(2x) - such that input values are same as that of f(x) - are shown on a separate line just below x-axis. The corresponding values are linked with arrow signs. Input to the function f(2x) which is same as that of f(x) now appears closer to origin by a factor of 2. It means graph of f(2x) is same as graph of f(x), which has

been shrunk by a factor 2 towards origin. Else, we can say that x-axis has been stretched by a factor 2.

Multiplication of independent variable

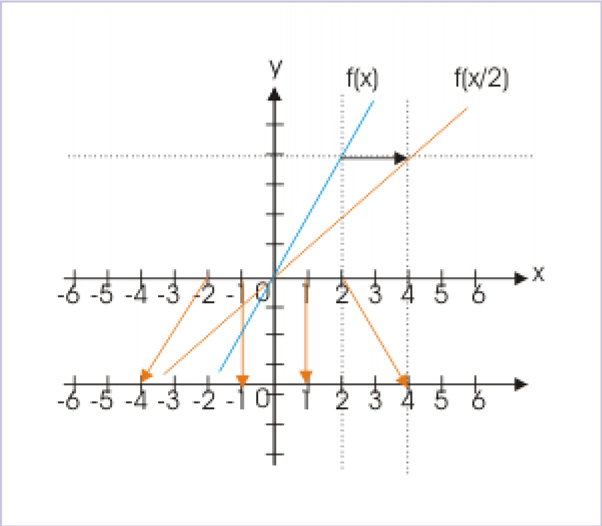


The graph shrinks towards origin.

$$y = f(x) \qquad \Rightarrow y = f(bx); \quad |b| > 1$$

Let us consider another example of functions f(x) and f(x/2). The integral values of independent variable are same as integral values on x-axis of coordinate system. Note that independent variable is plotted along x-axis as real number line. The integral x/2 values to the function f(x/2) - such that input values are same as that of f(x) - are shown on a separate line just below x-axis. The corresponding values are linked with arrow signs. Input to the function f(x/2) which is same as that of f(x) now appears away from origin by a factor of 2. It means graph of f(x/2) is same as graph of f(x), which has been stretched by a factor 2 away from origin. Else, we can say that x-axis has been shrunk by a factor 2.

Multiplication of independent variable



The graph stretches away from origin.

$$y = f(x) \qquad \Rightarrow y = f\left(\frac{x}{b}\right); \quad \left|b\right| > 1$$

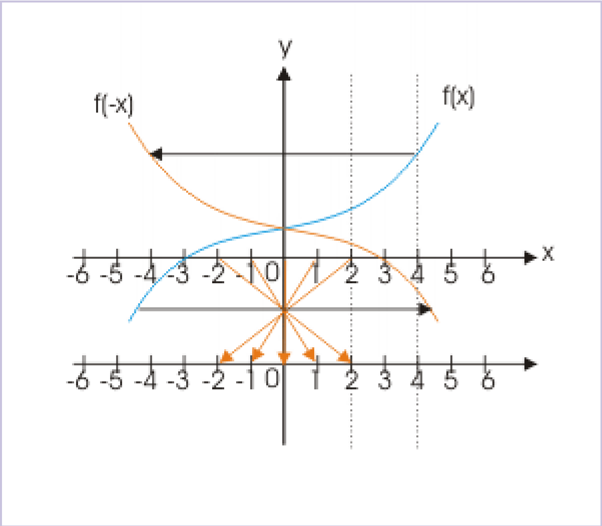
Important thing to note about horizontal scaling (shrinking or stretching) is that it takes place with respect to origin of the coordinate system and along x-axis – not about any other point and not along y-axis. What it means that behavior of graph at x=0 remains unchanged. In equivalent term, we can say that y-intercept of graph remains same and is not affected by scaling resulting from multiplication or division of the independent variable.

Negation of independent variable

Let us consider an example of functions f(x) and f(-x). The integral values of independent variable are same as integral values on x-axis of coordinate system. Note that independent variable is plotted along x-axis as real number line. The integral -x values to the function f(-x) - such that input values are same as that of f(x) - are shown on a separate line just below x-axis. The corresponding values are linked with arrow signs. Input to the

function f(-x) which is same as that of f(x) now appears to be flipped across y-axis. It means graph of f(-x) is same as graph of f(x), which is mirror image in y-axis i.e. across y-axis.

Negation of independent variable



The graph flipped across y. -axis.

The form of transformation is depicted as :

$$y = f(x) \quad \Rightarrow \quad y = f(-x)$$

A graph of a function is drawn for values of x in its domain. Depending on the nature of function, we plot function values for both negative and positive values of x. When sign of the independent variable is changed, the function values for negative x become the values of function for positive x and vice-versa. It means that we need to flip the plot across y-axis. In the nutshell, the graph of y=f(-x) can be obtained by taking mirror image of the graph of y=f(x) in y-axis.

While using this transformation, we should know about even function. For even function. f(x)=f(-x). As such, this transformation will not have any implication for even functions as they are already symmetric about y-axis. It means that two parts of the graph of even function across y-axis are

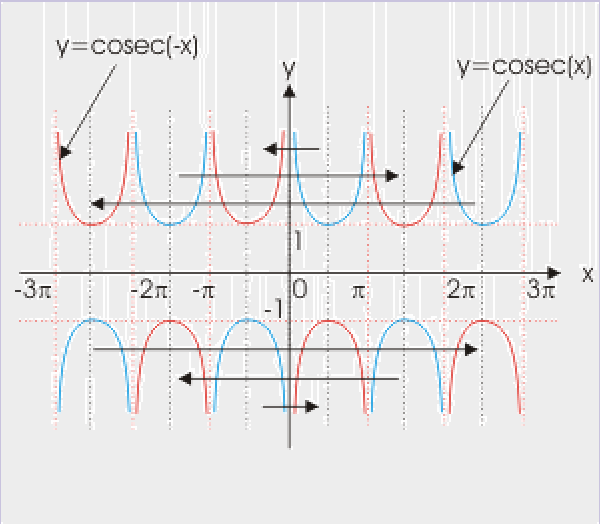
image of each other. For this reason, y=cos(-x) = cos(x), y = |-x|=|x| etc. The graphs of these even functions are not affected by change in sign of independent variable.

Example:

Problem : Draw graph of y=cosec(-x) function

Solution : The plot is obtained by plotting image of core graph y=cosec(x) in y axis.

Changing sign of the argument of graph



The transformed graph is image of core graph in y-axis.

Combined input operations

Certain function are derived from core function as a result of multiple arithmetic operations on independent variable. Consider an example :

$$f(x) = -2x - 2$$

We can consider this as a function composition which is based on identity function $f(x) = x$ as core function. From the composition, it is apparent that order of formation consists of operations as :

(i) $f(2x)$ i.e. multiply independent variable by 2 i.e. shrink the graph horizontally by half.

(ii) $f(-2x)$ i.e. negate independent variable x i.e. flip the graph across y-axis.

(iii) $f(-2x-2)$ i.e. subtract 2 from -2x.

This sequence of operation is not correct for the reason that third operation is a subtraction operation to -2x not to independent variable x, whereas we have defined transformation for subtraction from independent variable. The order of operation for transformation resulting from modifications to input can, therefore, be determined using following considerations :

1 : Order of operations for transformation due to input is opposite to the order of composition.

2 : Precedence of addition/subtraction is higher than that of multiplication/division.

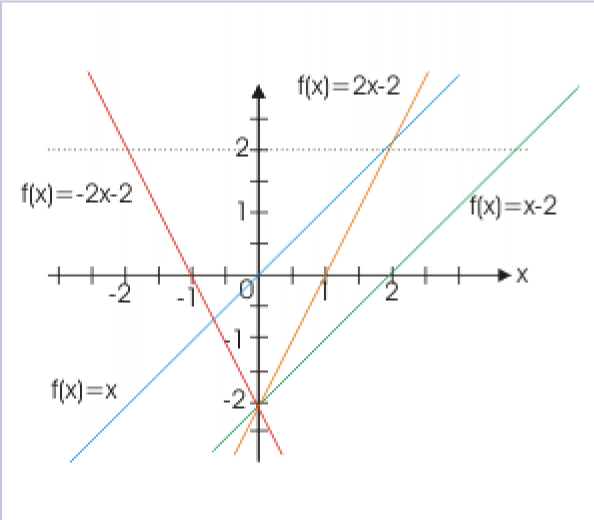
Keeping above two rules in mind, let us rework transformation steps :

(i) $f(x-2)$ i.e. subtract 2 from independent variable x i.e. shift the graph right by 2 units.

(ii) $f(2x-2)$ i.e. multiply independent variable x by 2 i.e. shrink the graph horizontally by half.

(iii) $f(-2x-2)$ i.e. negate independent variable i.e. flip the graph across y-axis.

This is the correct sequence as all transformations involved are as defined. The resulting graph is shown in the figure below :
Graph of transformed function



Operations are carried in sequence.

It is important the way graph is shrunk horizontally towards origin. Important thing is to ensure that y-intercept is not changed. It can be seen that function before being shrunk is :

$$f(x) = x - 2$$

Its y-intercept is 2. When the graph is shrunk by a factor by 2, the function is :

$$f(x) = 2x - 2$$

The y-intercept is again 2.The graph moves 1 unit half of x-intercept towards origin. Further, we can verify validity of critical points like x and y intercepts to ensure that transformation steps are indeed correct. Here,

$$x = 0, \quad y = -2X0 - 2 = -2$$

$$y = 0, \quad x = -\frac{(y+2)}{2} = -\frac{2}{2} = -1$$

We can decompose a given function in more than one ways so long transformations are valid as defined. Can we rewrite function as y = f{-2(x+1)}? Let us see :

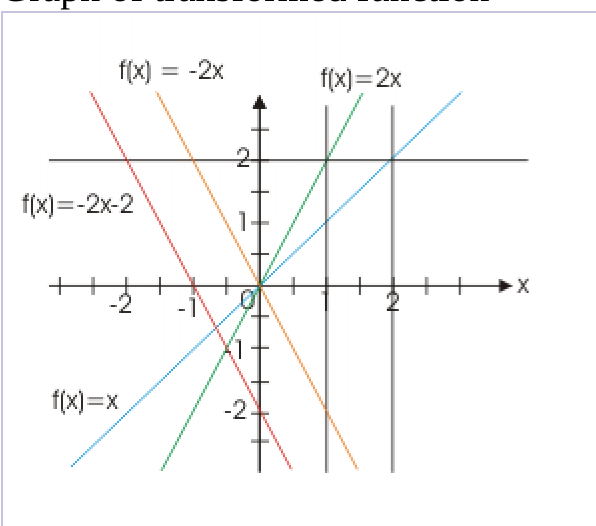
(i) f(2x) i.e. multiply independent variable x by 2 i.e. i.e. shrink the graph horizontally by half.

(ii) f(-2x) i.e. negate independent variable i.e. flip the graph across y-axis.

(iii) f{-2(x+1} i.e. add 1 to independent variable x x i.e. shift the graph left by 1 unit.

This decomposition is valid as transformation steps are consistent with the transformations allowed for arithmetic operations on independent variable.

Graph of transformed function



Operations are carried in sequence.

Horizontal shift

We have discussed transformation resulting in horizontal shift. In the simple case of operation with independent variable alone, the horizontal shift is “c”. In this case, transformation is represented by f(x+c). What is horizontal shift for more general case of transformation represented by f(bx+c)? Let us rearrange argument of the function,

$$f(bx+c) = f\left\{b\left(x+\frac{c}{b}\right)\right\}$$

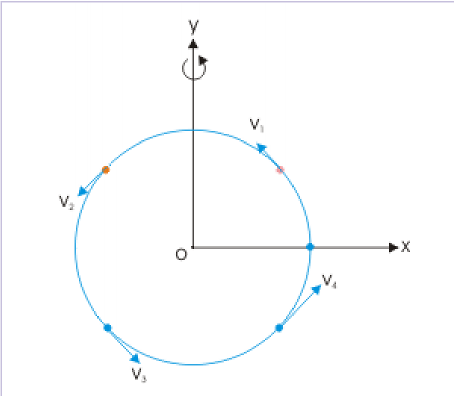
Comparing with f(x+c), horizontal shift is given by :

$$\text{Horizontal shift} = \frac{c}{b}$$

Non-uniform circular motion

What do we mean by non-uniform circular motion? The answer lies in the definition of uniform circular motion, which defines it be a circular motion with constant speed. It follows then that non-uniform circular motion denotes a change in the speed of the particle moving along the circular path as shown in the figure. Note specially the change in the velocity vector sizes, denoting change in the magnitude of velocity.

Circular motion



The speed of the particle changes with time in non-uniform circular motion.

A change in speed means that unequal length of arc (s) is covered in equal time intervals. It further means that the change in the velocity (**v**) of the particle is not limited to change in direction as in the case of uniform circular motion. In other words, the magnitude of the velocity (**v**) changes with time, in addition to continuous change in direction, which is inherent to the circular motion owing to the requirement of the particle to follow a non-linear circular path.

Radial (centripetal) acceleration

We have seen that change in direction is accounted by radial acceleration (centripetal acceleration), which is given by following relation,

$$a_R = \frac{v^2}{r}$$

The change in speed have implications on radial (centripetal) acceleration. There are two possibilities :

1: The radius of circle is constant (like in the motion along a circular rail or motor track)

A change in "v" shall change the magnitude of radial acceleration. This means that the centripetal acceleration is not constant as in the case of uniform circular motion. Greater the speed, greater is the radial acceleration. It can be easily visualized that a particle moving at higher speed will need a greater radial force to change direction and vice-versa, when radius of circular path is constant.

2: The radial (centripetal) force is constant (like a satellite rotating about the earth under the influence of constant force of gravity)

The circular motion adjusts its radius in response to change in speed. This means that the radius of the circular path is variable as against that in the case of uniform circular motion.

In any eventuality, the equation of centripetal acceleration in terms of “speed” and “radius” must be satisfied. The important thing to note here is that though change in speed of the particle affects radial acceleration, but the change in speed is not affected by radial or centripetal force. We need a tangential force to affect the change in the magnitude of a tangential velocity. The corresponding acceleration is called tangential acceleration.

Angular velocity

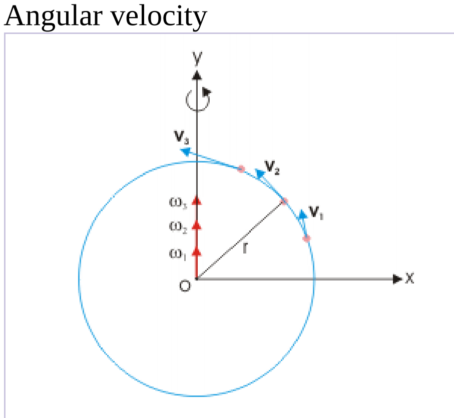
The angular velocity in non-uniform circular motion is not constant as ω = v/r and “v” is varying.

We construct a data set here to have an understanding of what is actually happening to angular speed with the passage of time. Let us consider a non-uniform circular motion of a particle in a centrifuge, whose linear speed, starting with zero, is incremented by 1 m/s at the end of every second. Let the radius of the circle be 10 m.

```
--------------------------- t v ω (s) (m/s) (rad/s) ----------------
------------ 0 0 0.0 1 1 0.1 2 2 0.2 3 3 0.3 ----------------------
-----
```

The above data set describes just a simplified situation for the purpose of highlighting variation in angular speed. We can visualize this change in terms of angular velocity vector with increasing magnitudes as shown in the figure here :

Angular velocity



Angular velocity changes with time in non-uniform circular motion.

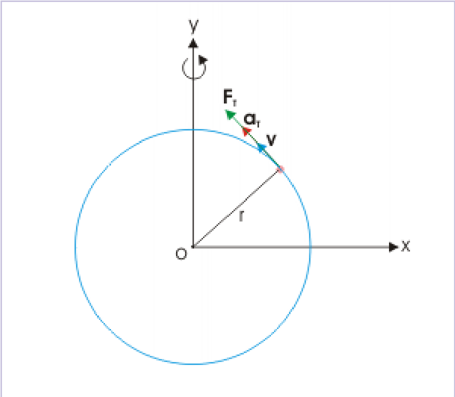
Note that magnitude of angular velocity i.e. speed changes, but not its direction in the illustrated case. However, it has been pointed out earlier that there are actually two directional possibilities i.e.

clockwise and anti-clockwise rotation. Thus, a circular motion may also involve change of direction besides a change in its magnitude.

Tangential acceleration

The non-uniform circular motion involves a change in speed. This change is accounted by the tangential acceleration, which results due to a tangential force and which acts along the direction of velocity circumferentially as shown in the figure. It is easy to realize that tangential velocity and acceleration are tangential to the path of motion and keeps changing their direction as motion progresses.

Tangential acceleration



Velocity, tangential acceleration and tangential force all act along the same direction.

We note that velocity, tangential acceleration and tangential force all act along the same direction. It must, however, be recognized that force (and hence acceleration) may also act in the opposite direction to the velocity. In that case, the speed of the particle will decrease with time.

The magnitude of tangential acceleration is equal to the time rate of change in the speed of the particle.

Equation:

$$a_T = \frac{đv}{đt}$$

Example:

Problem : A particle, starting from the position (5 m,0 m), is moving along a circular path about the origin in xy – plane. The angular position of the particle is a function of time as given here,

$$\theta = t^2 + 0.2t + 1$$

Find (i) centripetal acceleration and (ii) tangential acceleration and (iii)direction of motion at t =0 .

Solution : From the data on initial position of the particle, it is clear that the radius of the circle is 5 m.
(i) For determining centripetal acceleration, we need to know the linear speed or angular speed at a given time. Here, we differentiate angular position function to obtain angular speed as :

$$\omega = \frac{d\theta}{dt} = 2t + 0.2$$

Angular speed is varying as it is a function of time. For t = 0,

$$\omega = 0.2 \text{ rad}/s$$

Now, the centripetal acceleration is :

$$a_R = \omega^2 r = (0.2)^2 \text{ x } 5 = 0.04 \text{ x } 5 = 0.2\, m/s^2$$

(ii) For determining tangential acceleration, we need to have expression of linear speed in time.

$$v = \omega r = (2t + 0.2) \text{ x } 5 = 10t + 1$$

We obtain tangential acceleration by differentiating the above function :

$$a_T = \frac{dv}{dt} = 10\, m/s^2$$

Evidently, the tangential acceleration is constant and is independent of time.
(iii) Since, the angular speed is evaluated to be positive at t = 0, it means that angular velocity is positive. This, in turn, means that the particle is rotating anti-clockwise at t = 0.

Angular acceleration

The magnitude of angular acceleration is the ratio of angular speed and time interval.
Equation:

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

If the ratio is evaluated for finite time interval, then the ratio is called average angular acceleration and If the ratio is is evaluated for infinitesimally small period ($\Delta t \rightarrow 0$), then the ratio is called instantaneous angular acceleration. Mathematically, the instantaneous angular acceleration is :
Equation:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

The angular acceleration is measured in " rad/s^2 ". It is important to emphasize here that this angular acceleration is associated with the change in angular speed (ω) i.e. change in the linear speed of the particle (v = ωr) - not associated with the change in the direction of the linear velocity (**v**). In the case of uniform circular motion, ω = constant, hence angular acceleration is zero.

Relationship between linear and angular acceleration

We can relate angular acceleration (α) with tangential acceleration (a_T) in non – uniform circular motion as :

$$a_T = \frac{đv}{đt} = \frac{đ^2 s}{đt^2} = \frac{đ^2}{đt^2}\left(\mathrm{r}\ \theta\right) = \mathrm{r}\ \frac{đ^2\theta}{đt^2}$$
$$\Rightarrow a_T = \alpha\ \mathrm{r}$$

We see here that angular acceleration and tangential acceleration are representation of the same aspect of motion, which is related to the change in angular speed or the equivalent linear speed. It is only the difference in the manner in which change of the magnitude of motion is described.

The existence of angular or tangential acceleration indicates the presence of a tangential force on the particle.

Note : All relations between angular quantities and their linear counterparts involve multiplication of angular quantity by the radius of circular path "r" to yield to corresponding linear equivalents. Let us revisit the relations so far arrived to appreciate this aspect of relationship :
Equation:

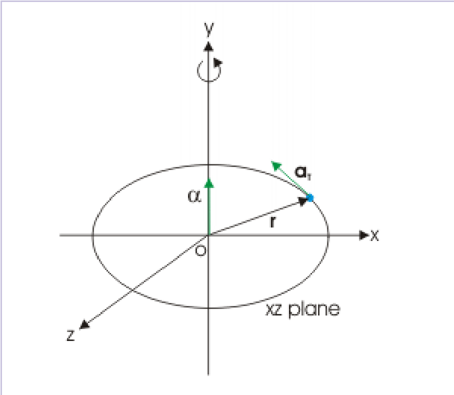
$$s = \theta r$$
$$v = \omega r$$
$$a_T = \alpha r$$

Linear and angular acceleration relation in vector form

We can represent the relation between angular acceleration and tangential acceleration in terms of vector cross product :

$$\mathbf{a_T} = \boldsymbol{\alpha}\mathbf{X}\mathbf{r}$$

Tangential and angular acceleration



Angular acceleration is an axial vector.

The order of quantities in vector product is important. A change in the order of cross product like ($\mathbf{rX\alpha}$) represents the product vector in opposite direction. The directional relationship between thee vector quantities are shown in the figure. The vectors " $\mathbf{a_T}$ " and "**r**" are in "xz" plane i.e. in the plane of motion, whereas angular acceleration (**α**) is in y-direction i.e. perpendicular to the plane of motion. We can know about tangential acceleration completely by analyzing the right hand side of vector equation. The spatial relationship among the vectors is automatically conveyed by the vector relation.

We can evaluate magnitude of tangential acceleration as :

$$\mathbf{a_T} = \alpha \mathbf{Xr}$$
$$\Rightarrow a_T = |\mathbf{a_T}| = \alpha r \sin\theta$$

where θ is the angle between two vectors **α** and **r**. In the case of circular motion, θ = 90°, Hence,

$$a_T = |\boldsymbol{\alpha}\mathbf{Xr}| = \alpha r$$

Uniform circular motion

In the case of the uniform circular motion, the speed (v) of the particle in uniform circular motion is constant (by definition). This implies that tangential acceleration, a_T, is zero. Consequently, angular acceleration ($\frac{a_T}{r}$) is also zero.

$$\Rightarrow a_T = 0$$
$$\Rightarrow \alpha = 0$$

Description of circular motion using vectors

In the light of new quantities and new relationships, we can attempt analysis of the general circular motion (including both uniform and non-uniform), using vector relations. We have seen that :

$$\mathbf{v} = \boldsymbol{\omega}\mathbf{Xr}$$

A close scrutiny of the quantities on the right hand of the expression of velocity indicate two possible changes :

- change in angular velocity (**ω**) and
- change in position vector (**r**)

The angular velocity (**ω**) can change either in its direction (clockwise or anti-clockwise) or can change in its magnitude. There is no change in the direction of axis of rotation, however, which is fixed. As far as position vector (**r**) is concerned, there is no change in its magnitude i.e. | **r** | or r is constant, but its direction keeps changing with time. So there is only change of direction involved with vector " **r**".

Now differentiating the vector equation, we have

$$\frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\omega}}{dt}\mathbf{X}\mathbf{r} + \boldsymbol{\omega}\mathbf{X}\frac{d\mathbf{r}}{dt}$$

We must understand the meaning of each of the acceleration defined by the differentials in the above equation :

- The term " $\frac{d\mathbf{v}}{dt}$ " represents total acceleration ($\mathbf{a}$) i.e. the resultant of radial ($\mathbf{a_R}$) and tangential acceleration($\mathbf{a_T}$).
- The term $\frac{d\omega}{dt}$ represents angular acceleration ($\boldsymbol{\alpha}$)
- The term $\frac{d\mathbf{r}}{dt}$ represents velocity of the particle ($\mathbf{v}$)

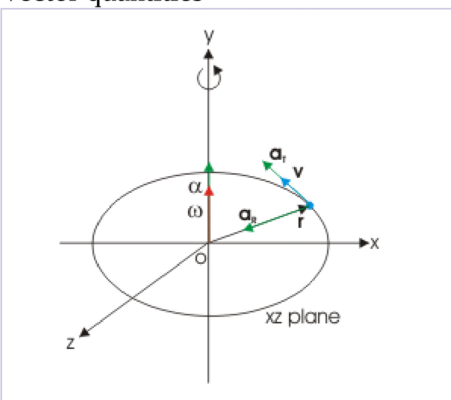
$$\Rightarrow \mathbf{a} = \boldsymbol{\alpha}\mathbf{X}\mathbf{r} + \boldsymbol{\omega}\mathbf{X}\mathbf{v}$$

$$\Rightarrow \mathbf{a} = \mathbf{a_T} + \mathbf{a_R}$$

where, $\mathbf{a_T} = \boldsymbol{\alpha}\mathbf{X}\mathbf{r}$ is tangential acceleration and is measure of the time rate change of the magnitude of the velocity of the particle in the tangential direction and $\mathbf{a_R} = \boldsymbol{\omega}\mathbf{X}\mathbf{v}$ is the radial acceleration also known as centripetal acceleration, which is measure of time rate change of the velocity of the particle in radial direction.

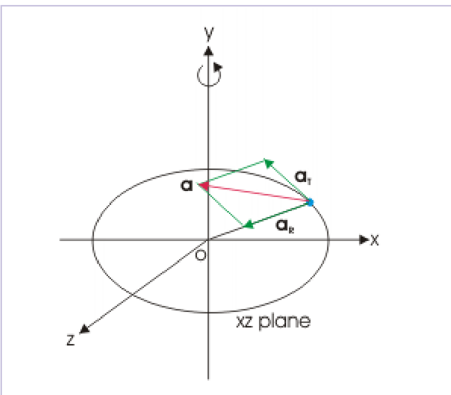
Various vector quantities involved in the equation are shown graphically with respect to the plane of motion (xz plane) :

Vector quantities



The magnitude of total acceleration in general circular motion is given by :

Resultant acceleration



$$a = \left|\mathbf{a}\right| = \sqrt{\left({a_T}^2 + {a_R}^2\right)}$$

Example:
Problem : At a particular instant, a particle is moving with a speed of 10 m/s on a circular path of radius 100 m. Its speed is increasing at the rate of 1 m/s^2 . What is the acceleration of the particle ?
Solution : The acceleration of the particle is the vector sum of mutually perpendicular radial and tangential accelerations. The magnitude of tangential acceleration is given here to be 1 m/s^2 . Now, the radial acceleration at the particular instant is :

$$a_R = \frac{v^2}{r} = \frac{10^2}{100} = 1\, m/s^2$$

Hence, the magnitude of the acceleration of the particle is :

$$a = \left|\mathbf{a}\right| = \sqrt{\left({a_T}^2 + {a_R}^2\right)} = \sqrt{\left(1^2 + 1^2\right)} = \sqrt{2}\, m/s^2$$

Exercises

Exercise:

Problem: A particle is moving along a circle in yz - plane with varying linear speed. Then

(a) acceleration of the particle is in x – direction

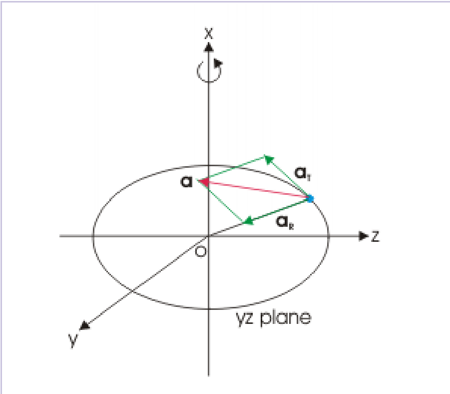
(b) acceleration of the particle lies in xy – plane

(c) acceleration of the particle lies in xz – plane

(d) acceleration of the particle lies in yz – plane

Solution:

The figure here shows the acceleration of the particle as the resultant of radial and tangential accelerations. The resultant acceleration lies in the plane of motion i.e yz – plane.
Circular motion



Hence, option (d) is correct.

Exercise:

Problem:

A particle is moving along a circle of radius “r”. The linear and angular velocities at an instant during the motion are “v” and “ω” respectively. Then, the product vω represents :

(a) centripetal acceleration

(b) tangential acceleration

(c) angular acceleration divided by radius

(d) None of the above

Solution:

The given product expands as :

$$v\omega = v \text{ x } \frac{v}{r} = \frac{v^2}{r}$$

This is the expression of centripetal acceleration.

Hence, option (a) is correct.

Exercise:

Problem:

Which of the following expression represents the magnitude of centripetal acceleration :

$$\text{(a) } \left|\frac{d^2\mathbf{r}}{dt^2}\right| \text{ (b) } \left|\frac{d\mathbf{v}}{dt}\right| \text{ (c) } r\frac{d\theta}{dt} \text{ (d) None of these}$$

Solution:

The expression $\left|\frac{d\mathbf{v}}{dt}\right|$ represents the magnitude of total or resultant acceleration. The differential $\frac{d\theta}{dt}$ represents the magnitude of angular velocity. The expression $\mathbf{r}\frac{d\theta}{dt}$ represents the magnitude of tangential velocity. The expression $\frac{d^2\mathbf{r}}{dt^2}$ is second order differentiation of position vector ($\mathbf{r}$). This is actually the expression of acceleration of a particle under motion. Hence, the expression $\left|\frac{d^2\mathbf{r}}{dt^2}\right|$ represents the magnitude of total or resultant acceleration.

Hence, option (d) is correct.

Exercise:

Problem:

A particle is circling about origin in xy-plane with an angular speed of 0.2 rad/s. What is the linear speed (in m/s) of the particle at a point specified by the coordinate (3m,4m) ?

(a)1 (b) 2 (c) 3 (d) 4

Solution:

The linear speed "v" is given by :

$$v = \omega r$$

Now radius of the circle is obtained from the position data. Here, x = 3 m and y = 4 m. Hence,

$$r = \sqrt{(3^2 + 4^2)} = 5\,m$$
$$\Rightarrow v = 0.2 \text{ x } 5 = 1\,m/s$$

Hence, option (a) is correct.

Exercise:

Problem:

A particle is executing circular motion. Man The velocity of the particle changes from zero to (0.3**i** + 0.4**j**) m/s in a period of 1 second. The magnitude of average tangential acceleration is :

$$\text{(a)}\quad 0.1\,m/s^2 \quad \text{(b)}\quad 0.2\,m/s^2 \quad \text{(c)}\quad 0.3\,m/s^2 \quad \text{(d)}\quad 0.5\,m/s^2$$

Solution:

The magnitude of average tangential acceleration is the ratio of the change in speed and time :

$$a_T = \frac{\Delta v}{\Delta t}$$

Now,

$$\Delta v = \sqrt{(0.3^2 + 0.4^2)} = \sqrt{0.25} = 0.5\,m/s$$
$$a_T = 0.5\,m/s^2$$

Hence, option (d) is correct.

Exercise:

Problem:

The radial and tangential accelerations of a particle in motions are a_T and a_R respectively. The motion can be circular if :

(a) $a_R \neq 0,\ a_T \neq 0$ (b) $a_R = 0,\ a_T = 0$ (c) $a_R = 0,\ a_T = \neq$ (d) $a_R = \neq,\ a_T = 0$

Solution:

Centripetal acceleration is a requirement for circular motion and as such it should be non-zero. On the other hand, tangential acceleration is zero for uniform circular acceleration and non-zero for non-uniform circular motion. Clearly, the motion can be circular motion if centripetal acceleration is non-zero.

Hence, options (a) and (d) are correct.

Exercise:

Problem:

Which of the following pair of vector quantities is/are parallel to each other in direction ?

(a) angular velocity and linear velocity

(b) angular acceleration and tangential acceleration

(c) centripetal acceleration and tangential acceleration

(d) angular velocity and angular acceleration

Solution:

The option (d) is correct.

Exercise:

Problem:

A particle is moving along a circle in a plane with axis of rotation passing through the origin of circle. Which of the following pairs of vector quantities are perpendicular to each other :

(a) tangential acceleration and angular velocity

(b) centripetal acceleration and angular velocity

(c) position vector and angular velocity

(d) angular velocity and linear velocity

Solution:

Clearly, vector attributes in each given pairs are perpendicular to each other.

Hence, options (a), (b), (c) and (d) are correct.

Exercise:

Problem:

A particle is executing circular motion along a circle of diameter 2 m, with a tangential speed given by v = 2t.

(a) Tangential acceleration directly varies with time.

(b) Tangential acceleration inversely varies with time.

(c) Centripetal acceleration directly varies with time.

(d) Centripetal acceleration directly varies with square of time.

Solution:

Tangential acceleration is found out by differentiating the expression of speed :

$$a_T = \frac{dv}{dt} \frac{d(2t)}{dt} = 2\, m/s$$

The tangential acceleration is a constant. Now, let us determine centripetal acceleration,

$$a_R = \frac{v_2}{r} = \frac{4t_2}{1} = 4t_2$$

The option (d) is correct.

Non – uniform circular motion(application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on problem solving

1: Calculation of acceleration as time rate of change of speed gives tangential acceleration.

2: Calculation of acceleration as time rate of change of velocity gives total acceleration.

3: Tangential acceleration is component of total acceleration along the direction of velocity. Centripetal acceleration is component of acceleration along the radial direction.

4: We exchange between linear and angular quantities by using radius of circle, "r", as multiplication factor. It is helpful to think that linear quantities are bigger than angular quantities. As such, we need to multiply angular quantity by “r” to get corresponding linear quantities and divide a linear quantity by “r” to get corresponding angular quantity.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to non-uniform circular motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Velocity
- Average total acceleration
- Total acceleration

Velocity

Example:
Problem : A particle, tied to a string, starts moving along a horizontal circle of diameter 2 m, with zero angular velocity and a tangential acceleration given by " $4t$ ". If the string breaks off at t = 5 s, then find the speed of the particle with which it flies off the circular path.
Solution : Here, tangential acceleration of the particle at time "t" is given as :

$$a_T = 4t$$

We note here that an expression of tangential acceleration (an higher attribute) is given and we are required to find lower order attribute i.e. linear speed. In order to find linear speed, we need to integrate the acceleration function :

$$a_T = \frac{đv}{đt} = 4t$$

$$\Rightarrow đv = 4tđt$$

Integrating with appropriate limits, we have :

$$\Rightarrow \int đv = \int 4tđt = 4 \int tđt$$

$$\Rightarrow v_f - v_i = 4\left[\frac{t^2}{2}\right]_0^5 = \frac{4X5^2}{2} = 50 \quad m \Big/ s$$

$$\Rightarrow v_f - 0 = 50$$

$$\Rightarrow v_f = 50 \quad m / s$$

Average total acceleration

Example:
Problem : A particle is executing circular motion. The velocity of the particle changes from (0.1**i** + 0.2**j**) m/s to (0.5**i** + 0.5**j**) m/s in a period of 1 second. Find the magnitude of average total acceleration.
Solution : The average total acceleration is :

$$\mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{(0.5\mathbf{i}+0.5\mathbf{j})-(0.1\mathbf{i}+0.2\mathbf{j})}{1}$$

$$\mathbf{a} = (0.4\mathbf{i} + 0.3\mathbf{j})$$

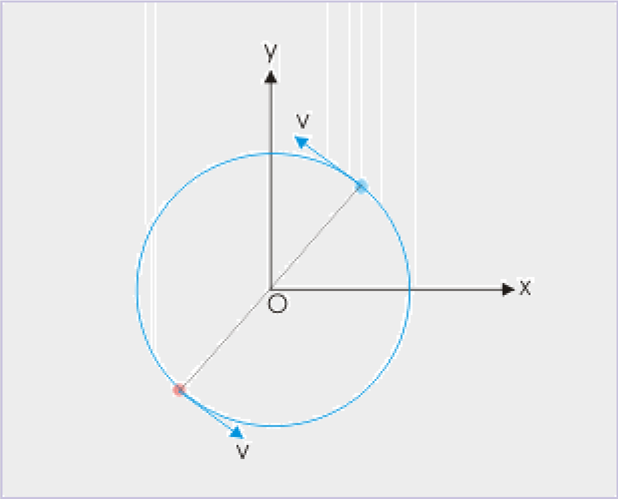
The magnitude of acceleration is :

$$a = \sqrt{(0.4^2 + 0.3^2)} = \sqrt{0.25} = 0.5\, m/s$$

Example:
Problem : A particle starting with a speed “v” completes half circle in time “t” such that its speed at the end is again “v”. Find the magnitude of average total acceleration.
Solution : Average total acceleration is equal to the ratio of change in velocity and time interval.

$$\mathbf{a}_{\text{avg}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{\Delta t}$$

From the figure and as given in the question, it is clear that the velocity of the particle has same magnitude but opposite directions.
Circular motion



The speeds of the particle are same at two positions.

$$v_1 = v$$

$$v_2 = -v$$

Putting in the expression of average total acceleration, we have :

$$\mathbf{a}_{\text{avg}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{\Delta t} = \frac{-\mathbf{v} - \mathbf{v}}{t}$$

$$\mathbf{a}_{\text{avg}} = \frac{-2\mathbf{v}}{t}$$

The magnitude of the average acceleration is :

$$\Rightarrow a_{\text{avg}} = \frac{2v}{t}$$

Total acceleration

Example:

Problem : The angular position of a particle (in radian), on circular path of radius 0.5 m, is given by :

$$\theta = -0.2t^2 - 0.04$$

At t = 1 s, find (i) angular velocity (ii) linear speed (iii) angular acceleration (iv) magnitude of tangential acceleration (v) magnitude of centripetal acceleration and (vi) magnitude of total acceleration.

Solution : Angular velocity is :

$$\omega = \frac{đ\theta}{đt} = \frac{đ}{đt}\left(-0.2t^2 - 0.04\right) = -0.2X2t = -0.4t$$

The angular speed, therefore, is dependent as it is a function in "t". At t = 1 s,

$$\omega = -0.4 \quad \text{rad}/s$$

The magnitude of linear velocity, at t = 1 s, is :

$$v = \omega r = 0.4X0.5 = 0.2 \quad m/s$$

Angular acceleration is :

$$\alpha = \frac{đ\omega}{đt} = \frac{đ}{đt}(-0.4t) = -0.4 \quad \text{rad}/s^2$$

Clearly, angular acceleration is constant and is independent of time.

The magnitude of tangential acceleration is :

$$\Rightarrow a_T = \alpha r = 0.4X0.5 = 0.2 \quad m/s^2$$

Tangential acceleration is also constant and is independent of time.

The magnitude of centripetal acceleration is :

$$\Rightarrow a_R = \omega v = 0.4tX0.2t = 0.08t^2$$

At t = 1 s,

$$\Rightarrow a_R = 0.08 \quad m/s^2$$

The magnitude of total acceleration, at t =1 s, is :

$$a = \sqrt{\left(a_T^2 + a_R^2\right)}$$

$$a = \sqrt{\left\{(0.2)^2 + (0.08)^2\right\}} = 0.215 \quad m/s^2$$

Example:
Problem : The speed (m/s) of a particle, along a circle of radius 4 m, is a function in time, "t" as :

$$v = t^2$$

Find the total acceleration of the particle at time, t = 2 s.
Solution : The tangential acceleration of the particle is obtained by differentiating the speed function with respect to time,

$$a_T = \frac{đv}{đt} = \frac{đ}{đt}\left(t^2\right) = 2t$$

The tangential acceleration at time, t = 2 s, therefore, is :

$$\Rightarrow a_T = 2X2 = 4 \quad m/s^2$$

The radial acceleration of the particle is given as :

$$a_R = \frac{v^2}{r}$$

In order to evaluate this expression, we need to know the velocity at the given time, t = 2 s :

$$\Rightarrow v = t^2 = 2^2 = 4 \quad m/s$$

Putting in the expression of radial acceleration, we have :

$$\Rightarrow a_R = \frac{v^2}{r} = \frac{4^2}{4} = 4 \quad m/s^2$$

The total acceleration of the particle is :

$$a = \sqrt{\left(a_T^2 + a_R^2\right)} = \sqrt{(4^2 + 4^2)} = 4\sqrt{2} \quad m \Big/ s^2$$

Circular motion with constant acceleration

The circular motion with constant acceleration allows constructing an equivalent linear system with the help of angular quantities.

The description of non-uniform circular motion of a particle is lot simplified, if the time rate of the change in angular velocity (angular acceleration) is constant. We have seen such simplification, while dealing with motion along a straight line. It is observed that the constant acceleration presents an additive frame work for motion along straight line. The velocity of the particle changes by a fixed value after every second (i.e. unit time). In translational motion along a straight line, we say, for example, that :

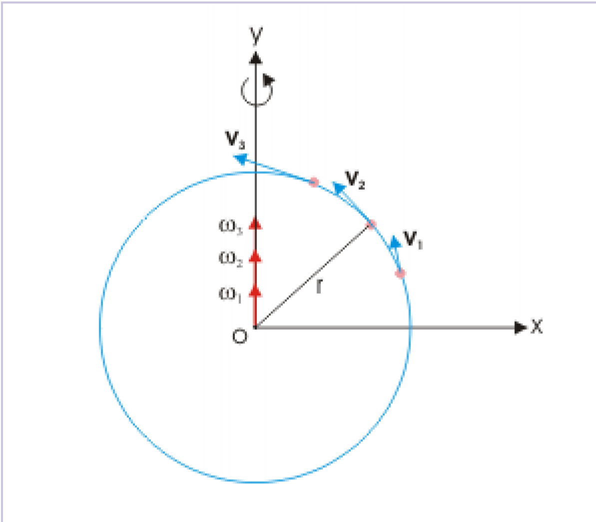
$$v = v_0 + at$$

where “v”, “ v_0 ” and “a” are attributes of motion along straight line with directional property. We can recall here that we are able to treat these vector quantities as signed scalars, because there are only two possible directions along a straight line. Now the moot question is : are these simplifications also possible with non-uniform circular motion, when angular acceleration is constant ?

The answer is yes. Let us see how does it work in the case of circular motion :

1: First, the multiplicity of directions associated with the velocity of the particle rotating about an axis is resolved to be equivalent to one directional angular velocity,"ω", acting along the axis of rotation. See the figure. If we stick to translational description of motion, then velocity vector is tangential to path. The direction of velocity vector changes as the particle moves along the circular path. On the other hand, angular velocity vector aligns in a single direction.

Angular velocity



Angular velocity acts along axis of rotation.

2: Second, the acceleration resulting from the change in the speed of the particle is represented by angular acceleration,"α". Like angular velocity, it is also one directional.

3: Third, there are only two possible directions as far as directional angular quantities are considered. The direction can be either clockwise (negative) or anti-clockwise (positive). This situation is similar to linear consideration where motion in the reference direction is considered positive and motion in opposite direction is considered negative.

4: Fourth, the change in either angular displacement or angular speed is basically translated to a change in the magnitude of one directional angular quantities. This allows us to treat the change in angular quantities as a consideration along straight line as in the case of motion along straight line.

The direction of angular quantities

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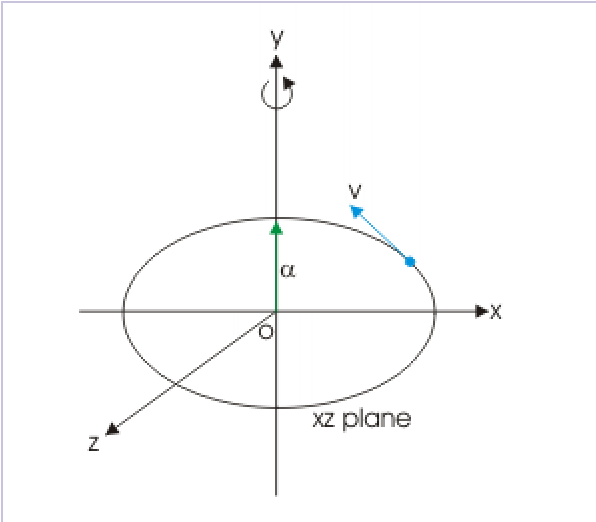


There are two possible directions with angular quantities.

Clearly, it is possible to covert a motion along a circular path into an equivalent system along straight line. As the angular quantities are represented along a straight line, it is then logical that relations available for linear motion for constant acceleration are also available for circular motion with appropriate substitution of linear quantities by angular quantities.

For example, the equation " $v = v_0 + at$ " has its corresponding relation in the circular motion as :

$$\omega = \omega_0 + \alpha t$$

As a matter of fact, there exists one to one correspondence between two types of equation sets. Importantly, we can treat angular vector quantities as signed scalars in the equations of motion, dispensing with the need to use vector notation. The similarity of situation suggests that we need not derive equations of motion again for the circular motion. We, therefore, proceed to simply write equation of angular motion with appropriate substitution.

Note: In this section of circular motion kinematics, our interest or domain of study is usually limited to the motion or acceleration in tangential direction. We may not refer to the requirement of motion in the radial direction in the form of centripetal acceleration, unless stated specifically.

What corresponds to what ?

The correspondence goes like this : the angular position is "θ" for linear "x"; the angular displacement is "Δθ" for linear "Δx"; the angular velocity is "ω" for linear "v" and the angular acceleration is "α" for linear "a".

Angular quantities

The different angular quantities corresponding to their linear counterparts are listed here fore ready reference :

Quantities	Linear variables	Angular variables
Initial position	x_0	θ_0
Final position	x	θ
Displacement	Δx	$\Delta\theta$
Initial velocity	v_0	ω_0
Final velocity	v	ω
Acceleration	a	α
Time interval	t	t

Basic equations

The corresponding equations for the two types of motion are :

S.N.	Linear equation	Angular equation
1:	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
2:	$v_{\text{avg}} = \frac{(v_0+v)}{2}$	$\theta\omega_{\text{avg}} = \frac{(\omega_0+\omega)}{2}$
3:	$\Delta x = x - x_0 = v_0 t + \frac{1}{2}at^2$	$\Delta\theta = \theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$

Derived equations

The derived equations for the two types of motion are :

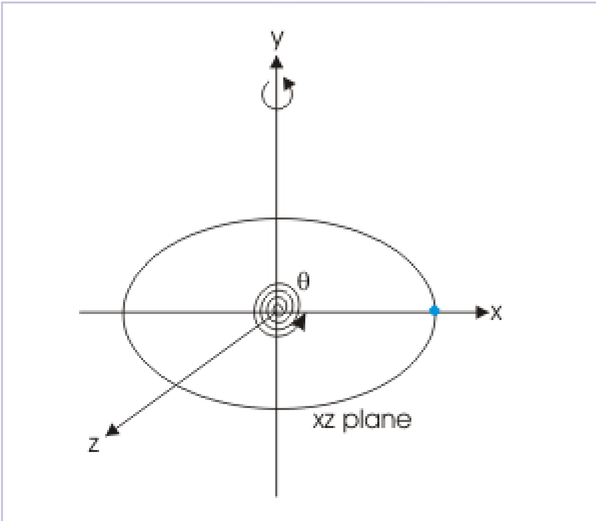
S.N.	Linear equation	Angular equation
1:	$v^2 = {v_0}^2 + 2a\left(x - x_0\right)$	$\omega^2 = {\omega_0}^2 + 2\alpha\left(\theta - \theta_0\right)$
2:	$\Delta x = \left(x - x_0\right) = \frac{(v_0+v)t}{2}$	$\Delta\theta = \left(\theta - \theta_0\right) = \frac{(\omega_0+\omega)t}{2}$
3:	$\Delta x = x - x_0 = vt - \frac{1}{2}at^2$	$\Delta\theta = \theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

Sign of angular quantities

The sign of angular quantities represents direction. A positive sign indicates anti-clockwise direction, whereas a negative sign indicates clockwise direction.

In the measurement of angle, a typical problem arises from the fact that circular motion may continue to rotate passing through the reference point again and again. The question arises, whether we keep adding angle or reset the measurement from the reference point ? The answer is that angle measurement is not reset in rotational kinematics. This means that we can have measurements like 540° and 20 rad etc.

Measurement of angle



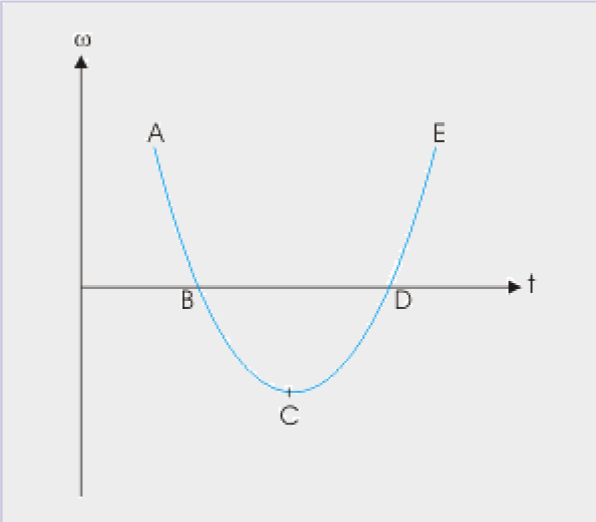
The measurement of angle is not reset.

This convention is not without reason. Equations of motion of circular motion with constant acceleration treats motion in an equivalent linear frame work, which considers only one reference position. If we reset the measurements, then equations of motion would not be valid.

Example:

Problem : The angular velocity – time plot of the circular motion is shown in the figure. (i) Determine the nature of angular velocity and acceleration at positions marked A, B, C, D and E. (ii) In which of the segments (AB, BC, CD and DE) of motion, the particle is decelerated and (iii) Is angular acceleration constant during the motion ?

Angular velocity – time plot



Solution :

(i) Angular velocity :

The angular velocities at A and E are positive (anti-clockwise). The angular velocities at B and D are each zero. The angular velocities at C is negative (clockwise).

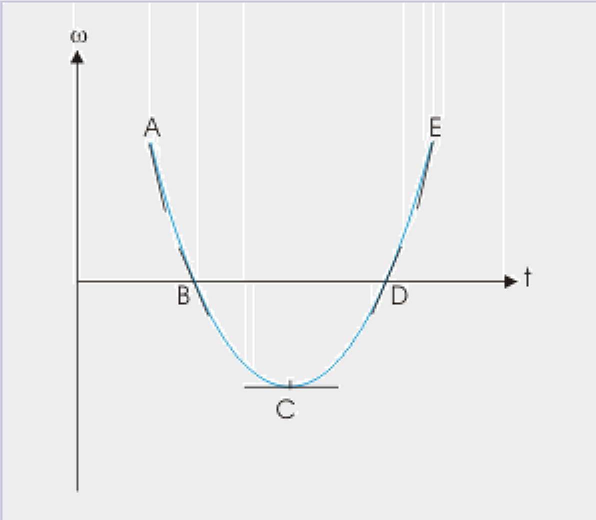
Angular acceleration :

The angular acceleration is equal to the first differential of angular velocity with respect to time.

$$\alpha = \frac{d\omega}{dt}$$

The sign of the angular acceleration is determined by the sign of the slope at different positions. The slopes at various points are as shown in the figure :

Slopes at different points



The angular accelerations at point A and B are negative (angular speed decreases with the passage of time). The angular accelerations at C is zero. The angular accelerations at point D and E are positive (angular speed increases with the passage of time).
(ii) Deceleration :
In the segments AB and CD, the magnitude of angular velocity i.e. angular speed decreases with the passage of time. Thus, circular motions in these two segments are decelerated. This is also confirmed by the fact that angular velocity and angular acceleration are in opposite directions in these segments.
(iii) The slopes on angular velocity - time plot are different at different points. Thus, angular accelerations are different at these points. Hence, angular acceleration of the motion is not constant.

Angular velocity

The angular velocity increases by a constant value at the end of every unit time interval, when angular velocity and acceleration act in the same direction. On the other hand, angular velocity decreases, when angular velocity and acceleration act in the opposite direction.

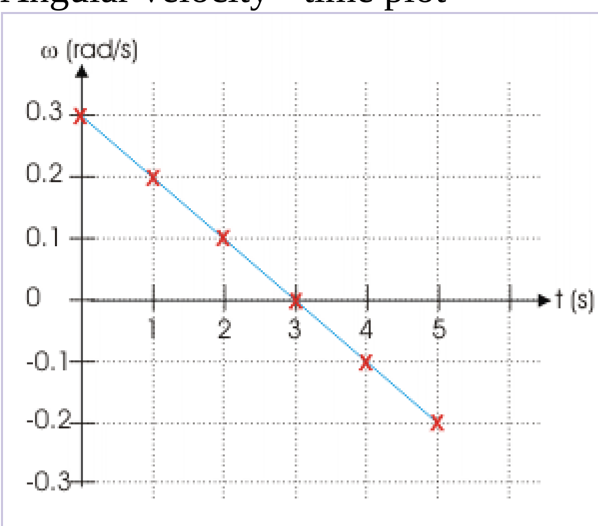
To appreciate this, we consider a circular motion of a particle whose initial angular velocity is 0.3 rad/s. The motion is subjected to an angular acceleration of magnitude 0.1 $\frac{\text{rad}}{s^2}$ in the opposite direction to the initial

velocity. In the table here, we calculate angular velocity of the particle, using relation, $\omega = \omega_0 + \alpha t$, at the end of every second and plot the data (for first 5 seconds) to understand the variation of angular velocity with time.

```
------------------------------------------------------
--- Time Angular acceleration Angular velocity (s)
(rad/s.s) (rad/s) ------------------------------------
--------------------- 0 -0.1 0.3 1 -0.1 0.3 - 0.1
x 1 = 0.2 2 -0.1 0.3 - 0.1 x 2 = 0.1 3 -0.1 0.3 -
0.1 x 3 = 0.0 4 -0.1 0.3 - 0.1 x 4 = -0.1 5 -0.1
0.3 - 0.1 x 5 = -0.2 ---------------------------------
------------------------
```

Here, the particle stops at the end of 3 seconds. The particle then reverses its direction (clockwise from anti-clockwise) and continues to move around the axis. The angular velocity – time plot is as shown here :

Angular velocity - time plot



We observe following aspects of the illustrated motion with constant acceleration :

1. The angular velocity decreases at uniform rate and the angular velocity – time plot is straight line. Positive angular velocity becomes less positive and negative angular velocity becomes more negative.
2. The slope of the plot is constant and negative.

Example:

Problem : A particle at the periphery of a disk at a radial distance 10 m from the axis of rotation, uniformly accelerates for a period of 5 seconds. The speed of the particle in the meantime increases from 5 m/s to 10 m/s. Find angular acceleration.

Solution : The initial and final angular velocities are :

$$\omega_0 = \frac{5}{10} = 0.5 \text{ rad}/s$$
$$\omega = \frac{10}{10} = 1 \text{ rad}/s$$
$$t = 5\ s$$

Using equation of motion, $\omega = \omega_0 + \alpha t$, we have :

$$\Rightarrow 1 = 0.5 + \alpha \text{ X } 5$$
$$\Rightarrow \alpha = \frac{0.5}{5} = 0.1 \text{ rad}/s^2$$

Angular displacement

The angular displacement is given as :

$$\Rightarrow \Delta\theta = \theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$$

If we start observation of motion as t = 0 and θ° = 0, then displacement (θ) is :

$$\Rightarrow \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

The particle covers greater angular displacement for every successive time interval, when angular velocity and acceleration act in the same direction and the particle covers smaller angular displacement, when angular velocity and acceleration act in the opposite direction.

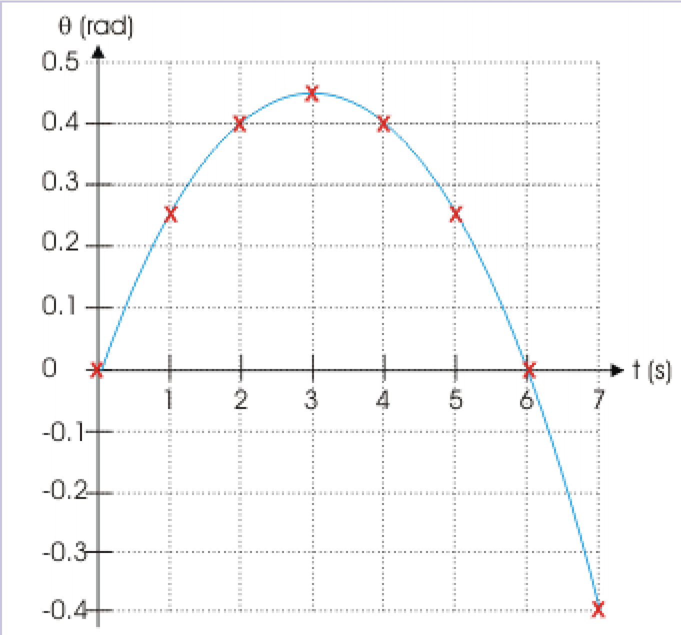
To appreciate this, we reconsider the earlier case of a circular motion of a particle whose initial angular velocity is 0.3 rad/s. The motion is subjected

to an angular acceleration of magnitude 0.1 $\mathrm{rad}\,/s^2$ in the opposite direction to the initial velocity. In the table here, we calculate angular displacement of the particle, using relation, $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, at the end of every second and plot the data (for first 7 seconds) to understand the variation of angular displacement with time.

```
--------------------------------------------------
---------------------------------- Time wo t
Angular displacement (θ) (s) (rad) (rad) ---------
--------------------------------------------------
------------------------- 0 0.0 0.00 1 0.3 x 1 0.3
- 0.5 x 0.1 x 1 = 0.3 - 0.05 = 0.25 2 0.3 x 2 0.6
- 0.5 x 0.1 x 4 = 0.6 - 0.20 = 0.40 3 0.3 x 3 0.9
- 0.5 x 0.1 x 9 = 0.9 - 0.45 = 0.45 4 0.3 x 4 1.2
- 0.5 x 0.1 x 16 = 1.2 - 0.80 = 0.40 5 0.3 x 5 1.5
- 0.5 x 0.1 x 25 = 1.5 - 1.25 = 0.25 6 0.3 x 6 1.8
- 0.5 x 0.1 x 36 = 1.8 - 1.80 = 0.00 7 0.3 x 7 2.1
- 0.5 x 0.1 x 49 = 2.1 - 2.50 = -0.40 ------------
--------------------------------------------------
----------------------
```

We see here that the particle moves in anti-clockwise direction for first 3 seconds as determined earlier and then turns back (clockwise) retracing the path till it reaches the initial position. Subsequently, the particle continues moving in the clockwise direction.

The angular displacement – time plot is as shown here :
Angular displacement – time plot



Example:

Problem : A particle at the periphery of a disk at a radial distance 10 m from the axis of rotation, uniformly accelerates for a period of 20 seconds. The speed of the particle in the meantime increases from 5 m/s to 20 m/s. Find the numbers of revolutions the particle completes around the axis.

Solution : We need to find angular displacement to know the numbers of revolutions made. Here,

$$\omega_0 = \frac{5}{10} = 0.5 \text{ rad}/s$$
$$\omega = \frac{20}{10} = 2 \text{ rad}/s$$
$$t = 20\ s$$

To calculate angular displacement, we need to know angular acceleration. Here, we can calculate angular acceleration as in the earlier exercise as :

$$\alpha = \frac{\omega - \omega_0}{\Delta t} = \frac{2.0 - 0.5}{20} = \frac{1.5}{20} = 0.075 \text{ rad} \Big/ s^2$$

Now, using equation of motion, $\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, we have :

$$\Rightarrow \Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\Delta\theta = 0.5 \ \text{X} \ 20 + \frac{1}{2} \ \text{X} \ 0.075 \ \text{X} \ 20^2 = 10 + 15 = 25 \ \text{rad}$$

The number of revolutions (nearest integer), n,

$$\Rightarrow n = \frac{\Delta\theta}{2\pi} = \frac{25}{6.28} = 3.97 = 3 \text{ (integer)}$$

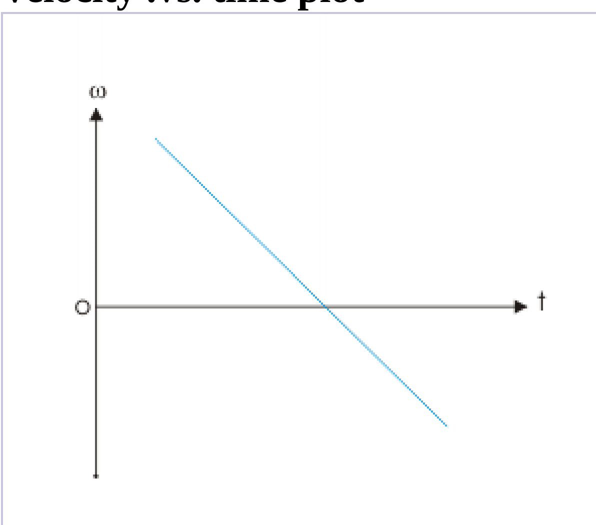
Exercises

Exercise:

Problem:

The angular velocity .vs. time plot of the motion of a rotating disk is shown in the figure. Then,

Velocity .vs. time plot



Velocity .vs. time plot

angular acceleration is constant.

angular acceleration is negative.

disk comes to a stop at a particular instant.

disk reverses direction during the motion.

Solution:

The slope of the straight line is a negative constant. Therefore, options (a) and (b) are correct. The line crosses time axis, when angular velocity is zero. Thus option (c) is correct. The angular velocities have opposite sign across time axis. It means that the disk reverses its direction. Thus option (d) is correct.

Hence, options (a),(b),(c) and (d) are correct.

Exercise:

Problem:

A point on a rotating disk, starting from rest, achieves an angular velocity of 40 rad/s at constant rate in 5 seconds. If the point is at a distance 0.1 meters from the center of the disk, then the distance covered (in meters) during the motion is :

$$\text{(a)}\ \ 5\ \ \text{(b)}\ \ 10\ \ \text{(c)}\ \ 20\ \ \text{(d)}\ \ 20$$

Solution:

We can find the distance covered, if we know the angular displacement. On the other hand, we can find angular displacement if we know the average angular speed as time is given.

$$\omega_{\text{avg}} = \frac{\omega_i + \omega_f}{2} = \frac{40+0}{2} = 20\,\text{rad}/s^2$$

We can consider this accelerated motion as uniform motion with average speed as calculated above. The angular displacement is :

$$\Rightarrow \theta = \omega_{\text{avg}} \text{ X } t = 20 \text{ X } 05 = 100\,\text{rad}$$

The distance covered is :

$$\Rightarrow s = \theta \text{ X } r = 100 \text{ X } 0.1 = 10\,\text{m}$$

Exercise:

Problem:

A point on a rotating disk completes two revolutions starting from rest and achieves an angular velocity of 8 rad/s. If the angular velocity of the disk is increasing at a constant rate, the angular acceleration (rad/s^2) is :

$$\text{(a)}\;\; 8 \;\;\text{(b)}\;\; 8\pi \;\;\text{(c)}\;\; \frac{8}{\pi} \;\;\text{(d)}\;\; \frac{16}{\pi}$$

Solution:

We can use the equation of motion for constant acceleration. Here,

$$\theta = 2\,\text{revolution} = 2 \text{ X } 2\pi = 4\pi$$

$$\omega_i = 0$$

$$\omega_f = 8\,\text{rad/s}$$

Looking at the data, it is easy to find that the following equation will serve the purpose,

$${\omega_f}^2 = {\omega_i}^2 = 2\alpha\theta$$

Solving for "α" and putting values,

$$\Rightarrow \alpha = \frac{{\omega_f}^2 - {\omega_i}^2}{2\theta} = \frac{8^2 - 0^2}{2\text{X}4\pi} = \frac{8}{\pi}\,\text{rad}/s_2$$

Hence, option (c) is correct.

Exercise:

Problem:

A point on a rotating disk is accelerating at a constant rate 1 rad/s^2 till it achieves an angular velocity of 10 rad/s. What is the angular displacement (radian) in last 2 seconds of the motion?

$$(\mathrm{a})\ \ 18\ \ (\mathrm{b})\ \ 24\ \ (\mathrm{c})\ \ 28\ \ (\mathrm{d})\ \ 32$$

Solution:

Here, final angular velocity, angular acceleration and time of motion are given. We can find the angular displacement using equation of motion for angular displacement that involves final angular velocity :

$$\theta = \omega_f t - \frac{1}{2} \alpha \frac{t}{2}$$

Putting values, we have :

$$\Rightarrow \theta = 10 \text{ X } 2 - \frac{1}{2} 1 \frac{2}{2} = 20 - 2 = 18 \, \mathrm{rad}$$

Hence, option (a) is correct.

Circular motion with constant acceleration (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Check understanding

An online examination, based on the basic subject matter, is available at the link given here. The examination session is designed only for the basic level so that we can ensure that our understanding of subject matter is satisfactory.

Circular motion with constant acceleration

Hints for problem solving

1: Visualize the circular motion as if we are dealing with straight line (pure translational) motion. Write down formula with substitution of linear quantities with angular quantities.

2: Use formula in scalar form. Stick to anticlockwise measurement as positive and clockwise measurement as negative.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to circular motion with constant acceleration. The questions are categorized in terms of the characterizing features of the subject matter :

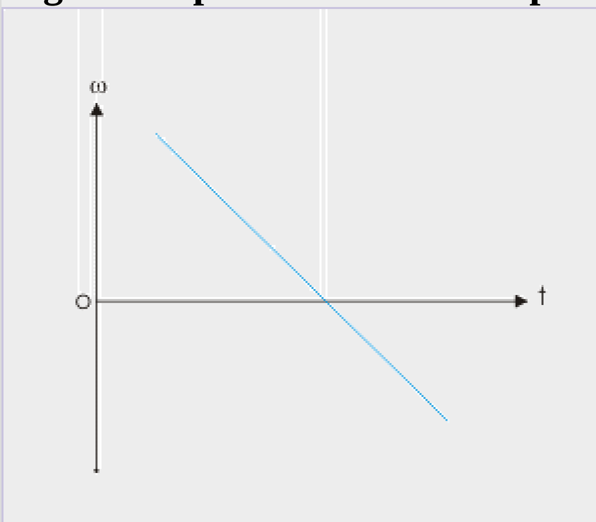
- Nature of angular motion
- Time interval
- Angular displacement

Nature of angular motion

Example:

Problem : The angular velocity .vs. time plot of the motion of a rotating disk is shown in the figure.

angular displacement .vs. time plot



Determine (i) nature of angular velocity (ii) nature of angular acceleration and (iii) whether the disk comes to a standstill during the motion?

Solution : The angular velocity is anticlockwise (positive) above time axis and clockwise (negative) below time axis.

The slope of angular velocity - time plot indicates nature of acceleration. Since the plot is a straight line, motion is accelerated/ decelerated at constant rate. Further, the slope of the straight line (angular velocity - time plot) is negative all through out.

Recall that it is easy to determine the sign of the straight line. Just move from left to right in the direction of increasing time along the time - axis. See whether the angular velocity increases of decreases. If increases, then slope is positive; otherwise negative. The angular velocity, here, becomes less positive above time axis and becomes more negative below time axis. Hence, slope is negative all through out.

It means that acceleration (negative) is opposite to angular velocity (positive) above time axis. Therefore, the disk is decelerated and the angular speed of disk decreases at constant rate. Below time - axis, the angular acceleration is still negative. However, angular velocity is also

negative below the time axis. As such, disk is accelerated and the angular speed increases at constant rate.
We see here that the plot intersects time - axis. It means that the disk comes to a stand still before changing direction from anticlockwise rotation to clockwise rotation.

Time interval

Example:
Problem : The angular position of a point on a flywheel is given by the relation :

$$\theta(\text{rad}) = -0.025t^2 + 0.01t$$

Find the time (in seconds) when flywheel comes to a stop.
Solution : The speed of the particle is :

$$\omega = \frac{d\theta}{dt} = -0.025 \text{ X } 2t + 0.1$$

When flywheel comes to a standstill, ω = 0,

$$\Rightarrow 0 = -0.025 \text{ X } 2t + 0.1$$

$$\Rightarrow t = \frac{0.1}{0.025} = \frac{100}{25} = 4\ s$$

Example:
Problem : The magnitude of deceleration of the motion of a point on a rotating disk is equal to the acceleration due to gravity (10 m/s^2). The point is at a linear distance 10 m from the center of the disk. If initial speed is 40 m/s in anti-clockwise direction, then find the time for the point to return to its position.

Solution : The disk first rotates in anti-clockwise direction till its speed becomes zero and then the disk turns back to move in clockwise direction. In order to analyze the motion, we first convert linear quantities to angular quantities as :

$$\omega_i = \frac{v_i}{r} = \frac{40}{10} = 4 \quad \text{rad}/s$$

$$\alpha = \frac{a_T}{r} = \frac{10}{10} = 1 \quad \text{rad}/s^2$$

When the point returns to its initial position, the total displacement is zero. Applying equation of motion for angular displacement, we have :

$$\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\Rightarrow 0 = 4 \text{ X } t - \frac{1}{2} \text{ X } 1 \text{ X } t^2$$

$$\Rightarrow t^2 - 8t = 0$$

$$\Rightarrow t(t-8) = 0$$

$$\Rightarrow t = 0 \quad or \quad 8s$$

The zero time corresponds to initial position. The time of return to initial position, therefore, is 8 seconds.

Angular displacement

Example:
Problem : A disk initially rotating at 80 rad/s is slowed down with a constant deceleration of magnitude 4 rad / s^2 . What angle (rad) does the disk rotate before coming to rest ?

Solution : Initial and final angular velocities and angular acceleration are given. We can use $\omega = \omega_0 + \alpha t$ to determine the time disk takes to come to stop. Here,

$$\omega_0 = 80 \text{ rad / s}, \ \alpha = 4 \text{ rad / s}^2$$

$$\Rightarrow 0 = 80 - 4 \text{ t}$$
$$\Rightarrow \text{t} = 20 \text{ s}$$

Using equation, $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, we have :

$$\Rightarrow \theta = 80 \text{ X } 20 - \frac{1}{2} \text{ X } 4 \text{ X } 20^2 = 1600 - 800 = 800 \text{ rad}$$

Example:

Problem : The initial angular velocity of a point (in radian) on a rotating disk is 0.5 rad/s. The disk is subjected to a constant acceleration of 0. 2 rad / s^2 . in the direction opposite to the angular velocity. Determine the angle (in radian) through which the point moves in third second.

Solution : Here, we need to be careful as the point reverses its direction in the third second ! For ω = 0,

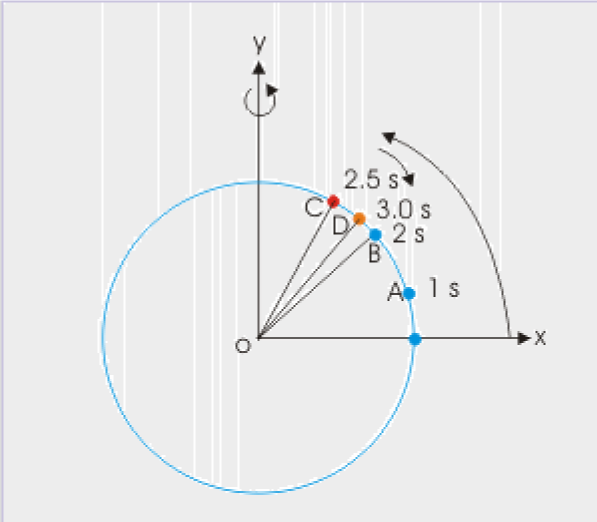
$$\omega = \omega_0 + \alpha t$$
$$\Rightarrow 0 = 0.5 - 0.2t$$
$$\Rightarrow t = \frac{0.5}{0.2} = 2.5 \ s$$

Thus the disk stops at t = 2.5 second. In this question, we are to find the angle (in rad) through which the point moves in third second – not the displacement. The figure here qualitatively depicts the situation. In the third second, the point moves from B to C and then from C to D. The displacement in third second is BOD, whereas the angle moved in the third second is |BOC| + |DOC|. Where,

∠BOC = displacement between 2 and 2.5 seconds.

∠DOC = displacement between 2.5 and 3 seconds.

Angle measurement



The angular velocity at the end of 2 seconds is :

$$\omega = 0.5 - 0.2 \text{ x } 2 = 0.1 \text{ rad}/s$$

$$\angle \mathrm{BOC} = \omega t - \frac{1}{2}\alpha t^2$$
$$\Rightarrow \angle \mathrm{BOC} = 0.1 \text{ X } 0.5 - \frac{1}{2} \text{ X } 0.2 \text{ X } 0.5^2$$
$$\Rightarrow \angle \mathrm{BOC} = 0.05 - 0.1 \text{ X } 0.25 = 0.05 - 0.025 = 0.025 \text{ rad}$$

The angular velocity at the end of 2.5 seconds is zero. Hence,

$$\Rightarrow \angle \mathrm{DOC} = 0 - \frac{1}{2} \text{ x } 0.2 \text{ x } 0.5^2$$
$$\Rightarrow \angle \mathrm{BOC} = -0.1 \text{ x } 0.25 = -0.025 \text{ rad}$$

It means that the point, at t = 3 s, actually returns to the position where it was at t = 2 s. The displacement is, thus, zero.
The total angle moved in third second = |0.025| + |-0.025| = 0.05 rad

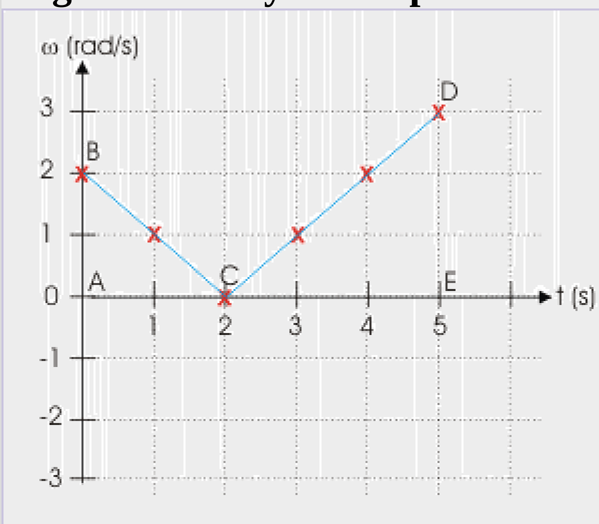
Example:
Problem : The angular velocity of a point (in radian) on a rotating disk is given by |t – 2|, where "t" is in seconds. If the point aligns with the reference direction at time t = 0, then find the quadrant in which the point falls after 5 seconds.

Problem : The area under the angular velocity – time plot and time axis is equal to angular displacement. As required, let us generate angular velocity data for first 5 seconds to enable us draw the requisite plot :

```
---------------------------------- Time (t) Angular velocity (θ) (s) (rad/s) -------------------------------- 0 2 1 1 2 0 3 1 4 2 5 3 ---------------------------------
```

The angular velocity – time plot is as shown in the figure :

Angular velocity – time plot



The displacement is equal to the area of two triangles :

$$\Rightarrow \theta = \frac{1}{2} \text{ X } 2 \text{ X } 2 + \frac{1}{2} \text{ X } 3 \text{ X } 3 = 6.5 \text{ rad}$$

Thus, the point moves 6.5 rad from the reference direction. Now, one revolution is equal to 2π = 2 x 3.14 = 6.28 rad. The particle is, therefore, in the fourth quadrant with respect to the reference direction.

Force

We all have extensive experience of force in its various forms. Newton’s laws of motion provide significant insight about the nature of force and what it does. The laws of motion emphasize that application of a single force (or resultant force) on a body changes its motion (velocity) either in direction or magnitude or both.

What is force ?

What is force? This basic important question about force remains unanswered even today. We do not know : what is force and how does it come into existence ? Our best guess today is that it arises from some short of change in the characterization of vacuum around us or according to quantum field theory, force is mediated by exchange of particles called "gravitons" (gravitational force), "photons" (electromagnetic force), "gluons" (strong nuclear force) and "bosons" (weak force). One thing common to all quantum models attempting to explain existence of force is that it propagates at the speed of light. For the time being, however, it would be better if we simply ignore this question and proceed ahead with what we know about force. We shall return to this topic once we familiarize ourselves with other aspects of force.

What we know today with definite authority about force is actually its effect on interaction with a particle or a body. Newton’s laws of motion precisely provide this information. The laws tell us about the change in the motion (velocity) of the body on which force is applied. An important description of force is that it is a "push” or “pull” on a body. Importantly, this is the nature of all types of force, irrespective of their class or genesis (gravitational, physical, mechanical, chemical, nuclear, electrical, etc.).

We can measure force in exact terms as ”acceleration produced in unit mass”. Using Newton’s second law,

For m = 1 kg,

F (Newton) = ma = 1 x a = a (m/s^2)

But, this measurement of force would essentially be in terms of what it does (acceleration) rather than in terms of what it is.

The question “what is force?” is, therefore, unanswered as we actually do not know.

Context of Newton’s laws of motion

Validity of Newton’s laws of motion is limited in important ways. Some of the limitations are basic. The relationship between force and acceleration as given by Newton's second law does not hold at great speeds (comparable to the speed of light) and is replaced by more general law as given by special theory of relativity. In this sense, Newton’s laws are a subset of more general laws as applicable to high speed motion. Further, Newton’s classical mechanics (so to speak) breaks down at atomic level and is substituted by quantum mechanics.

Even in the realm of classical mechanics, Newtonian mechanics is valid to motion as measured in certain type of reference system called inertial frame of reference. An inertial frame of reference is the one, in which Newton’s laws of motion are valid. The inertial frame of reference is a non-accelerated frame of reference.

Newton's second law of motion relates force, mass and acceleration. It is important that the measurements of the quantities involved in the relation be same in all inertial frames of reference. Thus, we should be careful while applying Newton’s second law. We must check whether the reference system is inertial or not ? For example, application of Newton’s second law in an accelerated lift will yield unexpected result. Fortunately, this limitation, on the part of reference system, is not insurmountable. We have developed techniques whereupon an accelerated non-inertial reference system can be converted into an equivalent inertial system by applying the concept of pseudo force. Hence, this limitation is not very serious and can be easily overcome.

Generally, we apply laws of motion to the terrestrial objects considering that Earth’s surface is an inertial frame. This assumption, as a matter of fact, is not far from the truth. The Earth rotates about its axis. As such, there is centripetal force working on each objects that we might choose to study. Clearly, the reference system attached to Earth is an accelerated frame - not an inertial frame of reference. But think about the motion of the terrestrial objects. They are confined to relatively smaller dimensions, where centripetal acceleration due to rotation may not cause appreciable or noticeable change in the velocity of the motion being observed. In other words, accounting of centripetal acceleration arising from rotation may not require to be taken care of (using pseudo force).

Force as a vector quantity

Force is a vector quantity. It acts in the direction of application. It is not always possible to identify direction of application in real time situation. As direction of acceleration is same as that of the direction of force, we may identify the direction of force as the direction in which "change of velocity (not velocity alone)" i.e. takes place.

Acceleration or rate of change in velocity takes place considering all forces acting on the body. Force, being vectors, are subject to superposition principle. The superposition principle states that a single vector can represent the effects of all vectors taken together. This single vector is known as net or resultant force.

It is possible that the resultant of the force system working on a body adds up to zero and as such there is no acceleration involved. Two forces may be equal, opposite and collinear and thus canceling each other. Mere existence of forces is not a guarantee of the acceleration od change in velocity. Alternatively, we can also state that absence of acceleration is not a guarantee of absence of force.

The fact, that force is a vector, has yet another important implication. The relation of force (cause) and acceleration (effect) can be studied in three mutually perpendicular directions. This feature of representing vector by

components is an experimentally verifiable mathematical construction. In simple words as applied to force vector, acceleration in a given direction can be studied by studying component of force (or forces) in that direction as if other components of force in other mutually perpendicular directions did not exist. This is a great simplification in studying mechanics and must be taken advantage, whenever a situation so presents itself.

System of forces

A body can be subjected to more than one force at a time. We must be careful while applying Newton's second law. The vector addition of forces should include all forces, which act on the body.

However, the list of force vectors, being added, must exclude the forces that the body applies on other bodies. Also, internal forces such as the intermolecular forces are not considered as internal forces can not change the velocity of the body. For example, we can not raise ourselves in air, using internal muscular force.

Effects of force

Generally we associate force with acceleration. This is, however, not always the case. When the object is a rigid body and free to move, then indeed acceleration is the only outcome of the application of force. However, if the object is not a rigid body, then a part of force may be used to deform the object and a part of it may be used to accelerate the object. If the body is constrained to move as well, then only deformation will result.

Force has no past or future

Force and its effect are current measurements at a particular position. What it means that if we apply a force of 2 N on a body of mass 2 kg for 1 second, then force will produce an acceleration of 1 m/s^2 (as given by

F=ma) exactly for 1 second. The moment force is removed, the acceleration of the body becomes zero. This is the case with a projectile. We need a force to impart an initial velocity. For the rest of flight when the projectile is airborne, there is only gravity (neglecting air resistance) that determines the acceleration of the projectile.

Exercise:

Problem:

A bob of mass 0.1 kg is hung from a mass-less string of 1 m length. The bob is set in oscillation. Then, the string is cut and the trajectory of the bob is observed. The bob takes different trajectory depending on the position of bob when the string is cut. It is found that the path is parabolic for the position(s) of bob :

(a) at mean position

(b) at extreme position

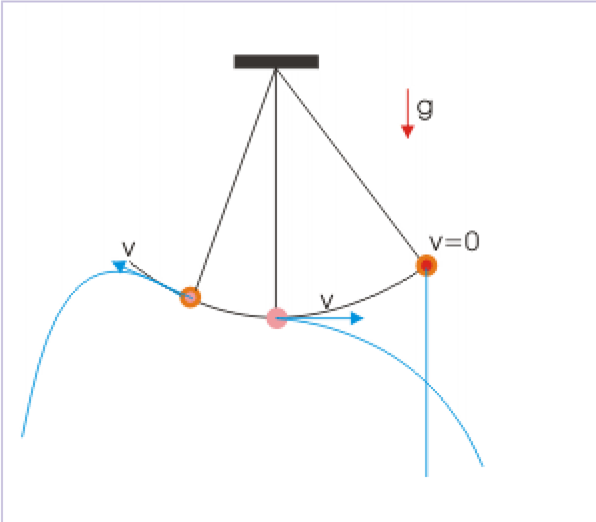
(c) while it is moving up towards extreme position

(d) while it is moving down towards mean position

Solution:

Force has no past or future. The moment the string is cut, there is no tension force on the bob. Only gravity acts on the bob. If the bob is at extreme position, then its velocity is zero and there is downward acceleration due to gravity. As a result, the bob falls vertically down.

Path of pendulum bob



Path of pendulum bob

In all other cases, the bob has certain velocity. Its direction is horizontal at mean position. It makes certain angle with horizontal when bob is moving either towards extreme position or mean position. In all these cases, velocity is in a direction which is not vertical. The bob is worked down by gravity acting downward. As such, there is downward acceleration due to gravity. This results in a parabolic trajectory.

Hence, options (b), (c) and (d) are correct.

Major classifications

Study of force and related phenomena is facilitated with certain classifications. One of the basic classifications is based on our experience with force. In our day to day life, we experience force when objects or particles are in contact. At the same time, we are aware that there are gravitational and electromagnetic forces, which come in to being without contact. These forces can change velocity of an object acting from a distance.

Contact forces : friction, tension, spring, muscle force etc.

“Action at a distance” forces : gravitational, Electromagnetic

We should understand that “contact” is a relative concept. What appears to be in contact is actually not in contact at atomic or sub-atomic levels. Further, this classification is also in the root of other important classification that divides force types between “fundamental” and “non-fundamental or other force types” :

Fundamental force : gravitational, Electromagnetic, weak force and strong force (nuclear force)

Other force types : friction, tension, spring, muscle force etc.

The important interpretation of this classification is that all other forces are actually macroscopic manifestation of fundamental forces. For example, we push an object with our hand using muscle. This muscle force is a macroscopic outcome of chemical force, which in turn is outcome of electromagnetic force. Also, we should keep in mind that the “effect of force” is same. It either pulls an object or pushes an object. In inertial frame of reference (non-accelerated frame of reference), the effect of force is determined by Newton’s laws of motion.

Newton’s laws of motion and motion types

Newton’s laws of motion form the basic framework in which interaction of force with a particle or a body is studied. As a matter of fact, these are the only laws other than conservation laws which govern whole of kinematics and dynamics i.e. study of motion. Newton’s laws of motion are historically postulated for pure translational motion of a particle and that of a body by extension. We, however, employ a similar set of Newton’s laws for studying rotational motion.

There are three corresponding Newton’s laws of motion for pure rotation. Newton’s first law for rotation, for example, states that a body in rotation remains in rotation unless there is an external torque is applied on it.

Similarly, Newton’s second law of motion for rotation relates torque, moment of inertia and angular acceleration as :

$$\boldsymbol{\tau} = I\boldsymbol{\alpha}$$

This relation is very much like the famous relation $\mathbf{F} = m\mathbf{a}$ for pure translational motion. Clearly, “force” is replaced by “torque”, “mass” by “moment of inertia” and “linear acceleration” by “angular acceleration”.

Generalization of Newton’s laws of motion

We have worked with Newton’s laws with fair degree of accuracy as far as “effect of force” i.e. “change in velocity” i.e. “acceleration" is concerned. It is true when we consider interaction at speeds, which are fraction of the speed of light. At higher speed, it is seen that we need a greater force to accelerate a body than that required to accelerate it by the same degree at a lower speed. The special theory of relativity quantifies (or accounts) this difference. The form of Newton’s law in momentum form remains unaltered even at higher speed and is given by :

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}$$

However, the definition of linear momentum is different at higher speed :

$$\mathbf{P} = \frac{m\mathbf{v}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

where “m” is the invariant mass, “v” is velocity and “c” is speed of light. Clearly, the new definition of momentum reduces to classical definition as product of mass and velocity when v<<c. For this reason, we can say that Newton’s laws are subset of more general “special theory of relativity”.

$$\mathbf{P} = m\mathbf{v}$$

Unifying theories

The realization that all forces are manifestation of few fundamental forces, gives rise to a temptation to conclude that all fundamental forces may have a unifying origin or model of description. This tempting idea to unify fundamental forces has led many different approaches and models to describe force. The foremost among these has been the successful unification of electrical and magnetic forces via electromagnetic theory. Both these forces are now considered to be manifested due to the presence of charge and its motion. However, no further progress towards unifying other fundamental forces could be made based on this model.

Two promising theories which attempt to unify fundamental forces are “quantum field theory” and “string theory”. The quantum field theory describes force in terms of the exchange of “momentum carrying” particles (photons, bosons etc). According to this theory, the very idea of force is redundant and is replaced by "momentum".

Newton's first law of motion

The first law of motion talks about the motion of a body for a particular situation. There is either "no force" or "zero net force" acting on the body. The first condition of "no force" is not common in our immediate surrounding. All bodies are acted by gravity i.e. force of attraction due to Earth. On the other hand, the second condition of "zero net force" is common in our immediate surrounding, where most bodies are stationary in Earth's reference as net force is zero.

First law of motion

> Unless acted upon by a net external force, a body, at rest, will remain at rest and a body, in motion, will remain in motion.

The state of motion of a given body, including the state of rest, is completely defined by its velocity. Stationary state is just one important case of constant velocity or uniform motion. If the object is stationary in a frame of reference, then

$\mathbf{v} = 0$

We can restate the first law of motion more concisely in velocity term as :

First law of motion

> If net external force on a body is zero, then its velocity remains constant.

Mathematically equivalent statements of the first law of motion are :

1: If $\sum \mathbf{F} = 0$, then $\mathbf{v}$ = a constant.

2: If $\sum \mathbf{F} = 0$, then $\mathbf{a} = 0$.

The substance of first law of motion is expressed in many ways. Here, we sum them all for ready reference (for the condition that the net force on a body is zero) :

- The body may either be at rest or may move with constant velocity.
- The body is not associated with any acceleration.

- If the body is moving, then the body moves along a straight line with a constant speed without any change of direction.
- If the body is moving, then the motion of the body is an uniform linear motion.
- If a body is moving with uniform linear motion, then we can be sure that the net force on the body is zero.

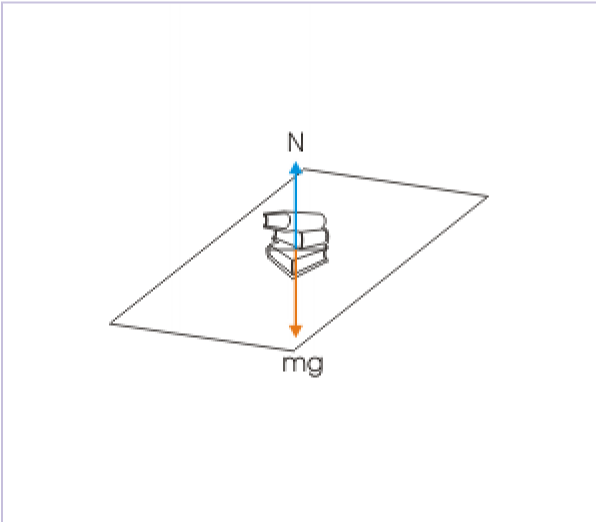
First law of motion and our experience

Let us now examine the interpretation of the law in a bit more detail as the statement may not be completely in agreement of what we see around.

A body at rest remains at rest

The part of the statement, which says that the body remains in stationary state, is a comprehensible argument, supported by our daily life experience. We actually experience that a body requires some external force to be moved around form its stationary state. As a matter of fact, this experience instills the notion that the state of rest is the natural state of matter. We, however, know that notion of rest is actually an experience or a perspective in specific reference. A body at rest in Earth's reference is in motion for other heavenly bodies.

Rest does not mean absence of force. The body, in question, may actually experience a system of force whose net force is zero. Consider a book lying on the study table. The book experiences two forces (i) its weight acting downwards and (ii) normal reaction of the table acting upwards. The two forces are equal and opposite and hence their resultant is zero.
Books lying on a table



The book is acted upon by a pair of balanced forces.

In the nutshell, this part of first law of motion provides additional possibility of rest other than when rest may result from “no force” being applied to the body. This part of the law, therefore, characterizes two important aspects of rest as :

- Rest may arise from “no force” being applied on the body.
- Rest may arise from “zero net force” being applied on the body

A body in motion keeps moving

The second part of the Newton’s first law is not directly supported from daily experience. Our general perception is actually contrary to what this law of motion says. We have seen that all bodies in motion, if left unattended, comes to rest.

We need to look a bit closer at the situation in hand, surrounding us. We live under the force of gravitation and almost always encounter force of friction. The two forces are generally the reason that an object apparently does not

follow this part of the law. Since the requirement of “no force” or “zero net force” on a "body in motion" is not fulfilled, we do not find real time example to support this part of the law.

We may get an insight into the basis of the law observing that an object like a base ball travels a longer distance on smoother plane. Lesser the friction longer the distance traveled. This fact is indicative that if there had been no friction, the ball would have kept moving and that would have been possible if the ball did not change its velocity.

In summary, this part of the law characterizes following important aspects of motion :

- The state of uniform linear motion results when “no force” is applied on the body.
- The state of uniform linear motion results, when “zero net force” is applied on the body.
- Uniform linear motion is the natural state of motion, rest being just one important case.

Newton’s first law and inertia

Newton’s first law of motion can be interpreted to mean that objects do not change its state of motion on its own. This property of an object is known as “inertia” i.e. sluggishness or inactivity.

The property of the object of maintaining state of motion unless being forced externally is characterized by first law. For this reason, Newton’s first law is also referred as “law of inertia”. Incidentally, “inertia” is quantized by Newton’s second law of motion, which measures the “unwillingness” on the part of an object to change. Every object resists change in velocity.

A body moves with acceleration only when it is acted by external net force. The amount of acceleration i.e. rate of change in velocity for a given net force is different for different mass of the bodies.

From Newton's second law of motion (we shall read about this law in next module) :

F = ma

For a given force, F, we have :

$$\Rightarrow a \propto \frac{1}{m}$$

In words, a smaller mass yields a greater acceleration and a greater mass yields a smaller acceleration. Thus, "mass" of a body is the measure of the inertia of the body in translational motion. Here, we have specified that mass is the measure of inertia in translational motion, because the inertia to rotational motion is measured by a corresponding rotational term called "moment of inertia".

Newton's first law and inertial frame of reference

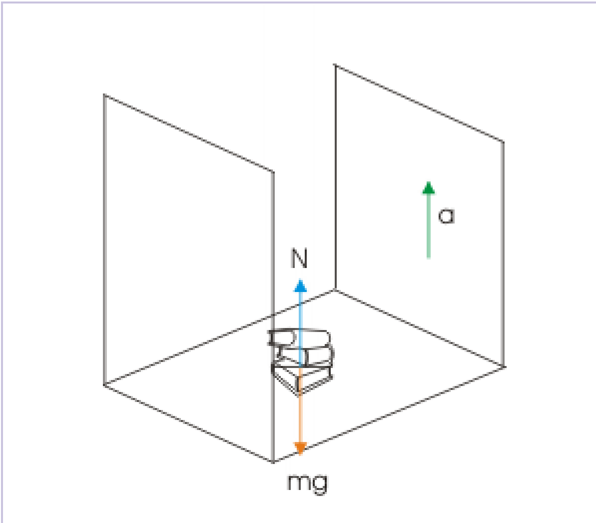
The relation between Newton's first law and inertial frame of reference is very close. The inertial frame of reference is actually defined in terms of Newton's first law.

Inertial frame of reference

A frame of reference in which Newton's laws of motion are valid is called inertial frame of reference.

An inertial frame of reference, where Newton's first law is valid, moves with constant velocity without acceleration. A frame of reference moving with constant relative velocity with respect to an inertial frame of reference is also inertial frame of reference. The context of inertial frame of reference is important for Newton's first law. Otherwise, we may encounter situations where a body may be found to be accelerated, even when net force acting on the body is zero. Consider the book lying on the floor of an accelerated lift moving upward.

Books lying on the floor of an accelerated lift

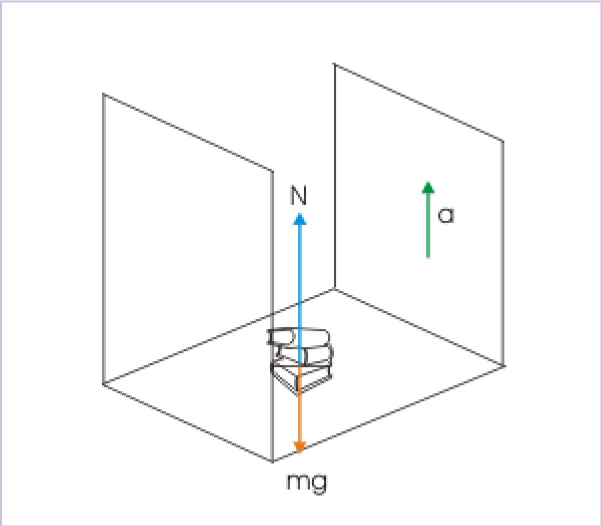


The book is acted upon by a
pair of balanced forces.

To an observer in the lift, the books are under a pair of balanced forces : the weight of the books acting downward and an equal normal force acting upward. The net force on the book is zero. An observer on the ground, however, finds that book is accelerated up. To support this observation in Earth's inertial frame, the observations in the lift has to be incorrect.

This apparent paradox is resolved by restraining ourselves to apply Newton's first law in the inertial frame of reference only. The observer on the ground determines that the book is moving with upward acceleration. He concludes that the normal force is actually greater than the weight of the books such that

Books lying on the floor of an accelerated lift



The book is acted upon by a net force.

N - mg = ma

Aletnatively, we may use the technique of psuedo force and convert the accelerated frame into an inertial frame and then apply Newton's first law. We shall discuss this technique subsquently.

Further, we can always convert an accelerated non-inertial frame of reference to an equivalent inertial frame of reference, using the concept of “pseudo force”. This topic will be dealt in detail separately.

We, now, sum up the discussion so far as :

- Inertial frame of reference is one in which Newton’s first law of motion is valid.
- Inertial frame of reference is one which moves at uniform velocity.
- Any reference system, which is moving with uniform velocity with respect to an inertial frame of reference is also an inertial frame of reference.
- Earth’s frame of reference approximates to inertial frame for motion, which is limited in dimension.

- We can convert an accelerated non-inertial frame of reference to an equivalent inertial frame of reference, using the concept of “pseudo force”.

Exercises

Exercise:

Problem: In which case(s), the net force is non-zero ?

(a) An air bubble moving up inside soda bottle at a speed 0.1 m/s

(b) A cork floating on water

(c) A car moving with 60 km/hr on a rough horizontal road

(d) None of above

Solution:

The objects in cases (a) and (c) are moving with constant velocity. Thus, there is no net force in these cases. On the other hand, floatation results when net force is zero.

Hence, option (d) is correct.

Newton's second law of motion
Second law of motion is the centerpiece of classical dynamics, providing exact connection between force (cause) and acceleration (effect).

The second law of motion determines the effect of net force on a body. The first law only defines the natural state of the motion of a body, when net force on the body is zero. It does not provide us with any tool to quantitatively relate force and acceleration (rate of change in velocity).

Second law of motion is the centerpiece of classical dynamics as it states the exact relation between force (cause) and acceleration (effect). This law has an explicit mathematical form and, therefore, has the advantage of quantitative measurement. As a matter of fact, the only available quantitative definition of force is given in terms of second law : “Force is equal to acceleration produced in unit mass.”

It must be clearly understood that the three laws of motion could well have been replaced by this single law of motion. However, the three laws are presented as they are, because first and third laws convey fundamental nature of "motion" and "force" which are needed to complete our understanding about them.

The second law of motion is stated in terms of linear momentum. It would, therefore, be appropriate that we first familiarize ourselves with this term.

Linear momentum

Linear momentum of a particle is defined as a vector quantity, having both magnitude and direction. It is the product of mass (a scalar quantity) and velocity (a vector quantity) of a particle at a given instant.

p = m**v**

The dimensional formula of linear momentum is $\left[MLT^{-1}\right]$ and its SI unit of measurement is " $kg - m / s$ ".

Few important aspects of linear momentum need our attention :

First, linear momentum is a product of positive scalar (mass) and a vector (velocity). It means that the linear momentum has the same direction as that of velocity.

Second, we have earlier referred that motion of a body is represented completely by velocity. But, the velocity alone does not convey anything about the inherent relation that “change in velocity” has with force. The product of mass and velocity in linear momentum provides this missing information.

In order to fully appreciate the connection between motion and force, we may consider two balls of different masses, moving at same velocity, which collide with a wall. It is our everyday common sense that tells us that the ball with greater mass exerts bigger force on the wall. We may, therefore, conclude that linear momentum i.e. the product of mass and velocity represents the “quantum of motion”, which can be connected to force.

It is this physical interpretation of linear momentum that explains why Newton’s second of motion is stated in terms of linear momentum as this quantity (not the velocity alone) connects motion with force.

Newton’s second law of motion

The second law of motion is stated differently. We have chosen to state the law as given here :

Newton’s second law of motion

> The time rate of change of momentum of a body is equal to the net (resultant) external force acting on the body.

In mathematical terms,

Equation:

$$\mathbf{F} = \frac{đ\mathbf{p}}{đt}$$

$$\Rightarrow \mathbf{F} = \frac{đ(m\mathbf{v})}{đt}$$

For invariant mass (the mass of the body under consideration does not change during application of force), we can take out mass "m" from the differential sign :
Equation:

$$\Rightarrow \mathbf{F} = m\frac{\partial \mathbf{v}}{\partial t}$$
$$\Rightarrow \mathbf{F} = m\mathbf{a}$$

As mass “m” is a positive scalar quantity, directions of force and acceleration are same. The dimensional formula of force is $\left[MLT^{-2}\right]$ and its SI unit of measurement is Newton, which is equal to " $kg - m / s^2$ ". One Newton (denoted by symbol "N") is defined as the force, which when applied on a point mass of 1 kg produces an acceleration of 1 m / s^2 .

Example

Problem 1 : A ball, weighing 10 gm, hits a hard surface in normal direction with a speed of 10 m/s and rebounds with the same speed. The ball remains in contact with the hard surface for 0.1 s. Find the magnitude of average force on the ball, applied by the surface.

Solution : We can find average force as the product of mass and average acceleration during the contact. Average acceleration is :

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{\Delta t}$$

If the rebound direction is considered positive, then v_1 = -10 m/s and v_2 = 10 m/s.

$$\Rightarrow a_{\text{avg}} = \frac{10-(-10)}{0.1} = 200\, m/s^2$$

The average force is :

$$\Rightarrow F_{\text{avg}} = ma_{\text{avg}} = 10 \;\text{ X }\; 200 = 2000\, N$$

Interpreting Newton's second law of motion

Few important aspects of Newton’s second law of motion are discussed in the following sections :

Deduction of first law of motion

Newton's first law of motion is a subset of second law in the sense that first law is just a specific description of motion, when net external force on the body is zero.

$$\mathbf{F} = m\mathbf{a} = 0$$

Equation:

$$\Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 0$$

This means that if there is no net external force on the body, then acceleration of the body is zero. Equivalently, we can state that if there is no net external force on the body, then velocity of the body can not change. Further, it is also easy to infer that if there is no net external force on the body, it will maintain its state of motion. These are exactly the statements in which first law of motion are stated.

Forces on the body

The body under investigation may be acted upon by a number of forces. We must use vector sum of all external forces, while applying second law of motion. A more general form of Second law of motion valid for a system of force is :

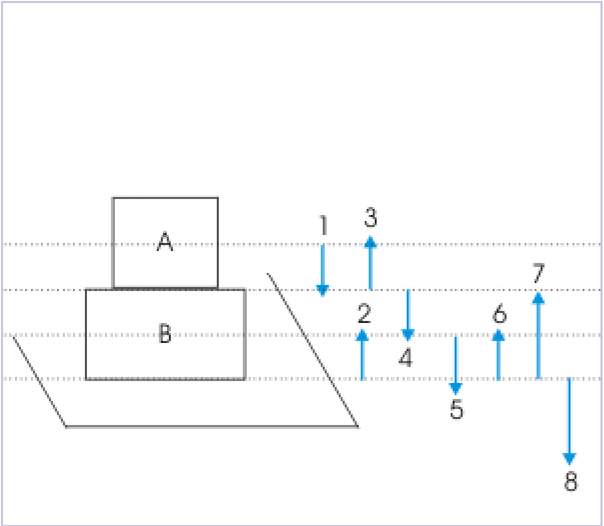
Equation:

$$\sum \mathbf{F} = m\mathbf{a}$$

The vector addition of forces as required in the left hand side of the equation excludes the forces that the body applies on other bodies. We shall know from Third law of motion that force always exists in the pair of “action” and “reaction”. Hence, if there are “n” numbers of forces acting on the body, then there are “n” numbers of forces that the body exerts on other bodies. We must carefully exclude all such forces that body as a reaction applies on other bodies.

Consider two blocks “A” and “B” lying on horizontal surface as shown in the figure. There are many forces in action here :

Forces acting on two blocks and Earth



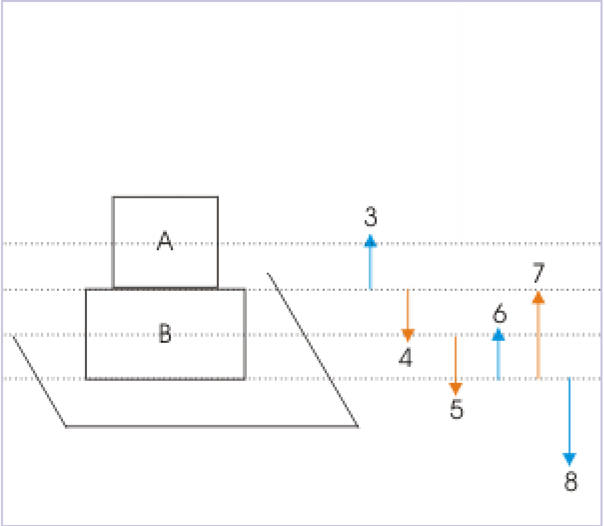
There are many forces acting on different bodies.

1. Earth pulls down “A” (force of gravitation)
2. “A” pulls up Earth (force of gravitation)
3. “B” pushes up “A” (normal force)
4. “A” pushes down “B” (weight of “A” : normal force)
5. Earth pulls down “B” (force of gravitation)
6. “B” pulls up Earth (force of gravitation)
7. Earth pushes up “B” (normal force) and

8. “B” pushes down the Earth (weight of “B” and "A" : normal force)

Let us now consider that we want to apply law of motion to block “B”. There are total of six forces related to “B” : 3, 4, 5, 6, 7 and 8. Of these, three forces 4, 5 and 7 act on “B”, while the remaining equal numbers of forces are applied by “B” on other bodies.

External forces acting on block B

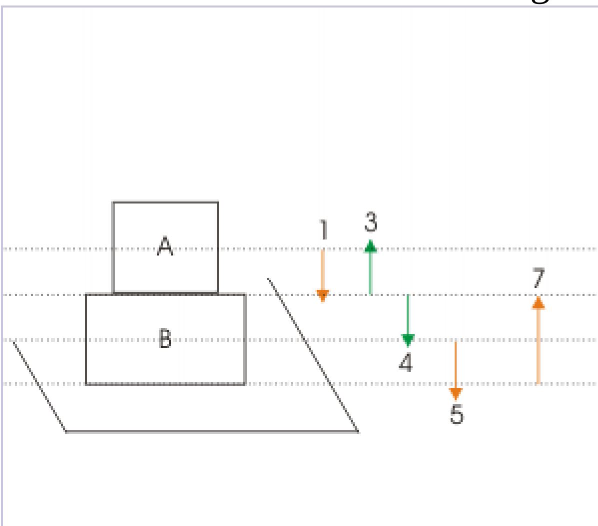


There are three external forces on B as shown with red arrow.

From the point of view of the law of motion, there are, thus, only three forces as shown with red arrow in the figure, which are external force. These are the forces that the surrounding applies on the block "B".

Also as pointed out before in the course, we must not include internal forces like intermolecular forces in the consideration. In order to understand the role and implication of internal forces, we consider interaction of two blocks together with Earth’s surface. In this case, the forces 3 (B pushes up A) and 4 (A pushes down B) are pair of normal forces acting at the interface between blocks A and B. They are internal to the combined system of two bodies and as such are not taken into account for using law of motion. Here, external forces are 1 (Earth pulls down A), 5(Earth pulls down B) and 7(Earth pushes up B).

External and internal forces acting on blocks A and B



External and internal forces acting on blocks A and B

In nutshell, we must exclude (i) forces applied by the body and (ii) internal forces. Clearly, we must only consider external forces applied on the body while using equation of motion as given by Newton's second law.

Equation of motion in component directions

As the force and acceleration are vector quantities, we can represent them with three components in mutually perpendicular directions. The consideration of dimension is decided by the force system as applied to the body.

If forces are collinear i.e. acting along a particular direction, then we use equation of motion in one direction. In such situation, it is possible to represent vector quantities with equivalent signed scalar quantities, in which sign indicates the direction. This is the simplest case.

If forces are coplanar i.e. acting in a plane, then we use component equations of motion in two directions. In such situation, we use component equations of motion in two directions.

Equation:

$$\sum F_x = ma_x$$
$$\sum F_y = ma_y$$

Since each coordinate direction is bi-directional, we treat component vectors by equivalent scalar representation whose sign indicates direction. The important aspect of component equation of motion is that acceleration in a particular component direction is caused by net of force components in that direction and is independent of other net of component force in perpendicular direction.

In case forces are distributed in three dimensional volume, then we must consider component equation of motion in third perpendicular direction also :

Equation:

$$\sum F_z = ma_z$$

Application of force

We implicitly consider that forces are applied at a single point object. The forces that act on a point object are concurrent by virtue of the fact that a point is dimensionless entity. This may appear to be confusing as we have actually used the word "body" – not "point" in the definition of second law. Here, we need to appreciate the intended meaning clearly.

Actually second law is defined in the context of translational motion, in which a three dimensional real body behaves like a point. We shall subsequently learn that application of a force system (forces) on a body in translation is equivalent to a point, where all mass of the body can be

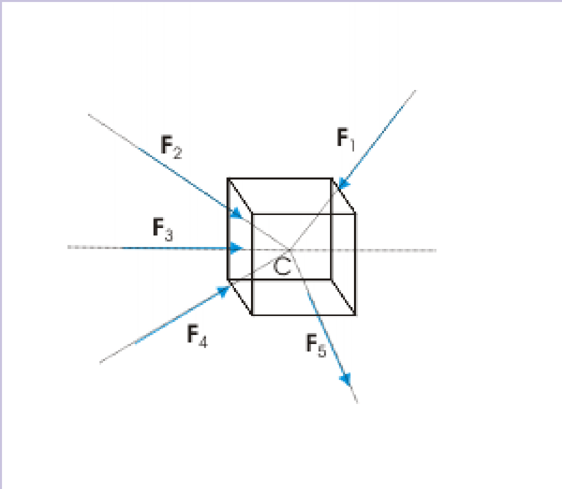
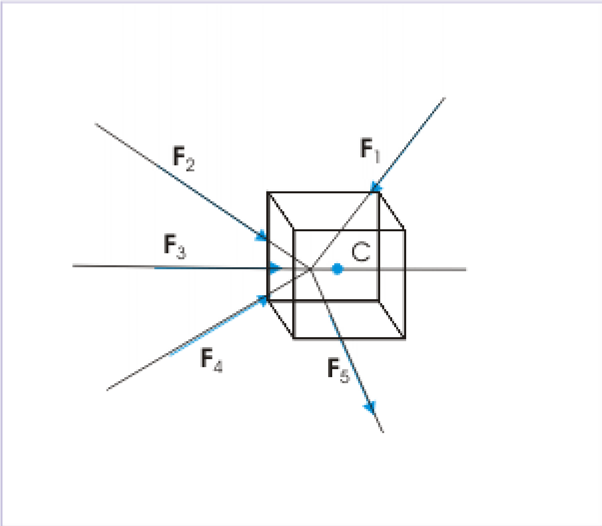
considered to be concentrated. In that case, the acceleration of the body is associated with that point, which is termed as “center of mass (C)”. The Newton's second law is suitably modified as :

Equation:

$$\sum \mathbf{F} = m\mathbf{a}_c$$

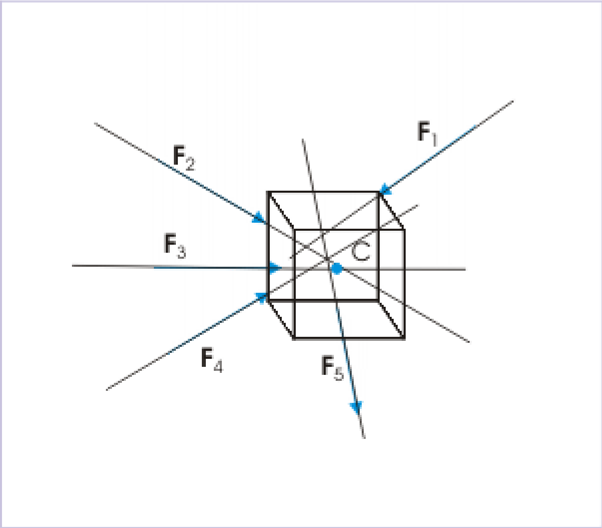
where $\mathbf{a}_c$ is the acceleration of the center of mass (we shall elaborate about the concept of center of mass in separate module). In general, application of a force system on a real body can involve both translational and rotational motion. In such situation, the concurrency of the system of forces with respect to points of application on the body assumes significance. If the forces are concurrent (meeting at a common point), then the force system can be equivalently represented by a single force, applied at the common point. Further, if the common point coincides with “center of mass (C)”, then body undergoes pure translation. Otherwise, there is a turning effect (angular/rotational effect) also involved.

Concurrent force system

Common point coincides with center of mass	Common point does not coincide with center of mass
	

What if the forces are not concurrent? In this case, there are both translational and rotational effects to be considered. The translational motion is measured in terms of center of mass as in pure translation, whereas the turning effect is studied in terms of “moment of force” or “torque”. This is defined as :

Non-concurrent force system



Forces as extended do not meet at a common point.

Equation:

$$\tau = |\mathbf{r} \times \mathbf{F}| = r_{\perp} F$$

where $r_{\perp}$ is perpendicular distance from the point of rotation.

Most importantly, same force or force system is responsible for both translational effect (force acting as "force" as defined by the second law) and angular/rotational effect (force manifesting as "torque" as defined by the angular form of Newton's second law in the module titled Second law of motion in angular form). We leave the details of these aspects of application of force as we will study it separately. But the point is made. Linear acceleration is not the only “effect” of the application of force (cause).

Also, force causes “effect” not necessarily as cause of acceleration – but can manifest in many ways : as torque to cause rotation; as pressure to change volume, as stress to deform a body etc. We should, therefore, always keep in mind that the study of translational effect of force is specific and not inclusive of other possible effects of force(s).

In the following listing, we intend to clarify the context of the study of the motional effect of force :

1: The body is negligibly small to approximate a point. We apply Newton’s second law for translation as defined without any consideration of turning effect.

2: The body is a real three dimensional entity. The force system is concurrent at a common point. This common point coincides with the center of mass. We apply Newton’s second law for translation as defined without any consideration of turning effect. Here, we implicitly refer the concurrent point as the center of mass.

3: The body is a real three dimensional entity. The force system is concurrent at a common point. But this common point does not coincide with the center of mass. The context of study in this case is also same as that for the case in which force system is not concurrent. We apply Newton’s second law for translation as defined for the center of mass and Newton’s second law for angular motion (we shall define this law at a later stage) for angular or rotational effect.

The discussion so far assumes that the body under consideration is free to translate and rotate. There are, however, real time situations in which rotational effect due to external force is counter-balanced by restoring torque. For example, consider the case of a sliding block on an incline. Application of an external force on the body along a line, not passing through center of mass, may not cause the body to overturn (rotate as it moves). The moment of external force i.e. applied torque may not be sufficient enough to overcome restoring torque due to gravity. As such, if it is stated that body is only translating under the given force system, then we assume that the body is a point mass and we apply Newton’s second law straight way as if the body were a point mass.

Unless otherwise stated or specified, we shall assume that the body is a point mass and forces are concurrent. We shall, therefore, apply Newton's second law, considering forces to be concurrent, even if they are not. Similarly, we will consider that the body is a point mass, even if it is not. For example, we may consider a block, which is sliding on an incline. Here, "friction force" is along the interface, whereas the "normal force" and "weight" of the block act through center of mass (C). Obviously these forces are not concurrent. We, however, apply Newton's second law for translation, as if forces were concurrent.

Exercises

Exercise:

Problem:

A ball of mass 0.1 kg is thrown vertically. In which of the following case(s) the net force on the ball is zero? (ignore air resistance)

(a) Just after the ball leaves the hand

(b) During upward motion

(c) At the highest point

(d) None of above cases

Solution:

The only force acting on the ball is gravity (gravitational force due to Earth). It remains constant as acceleration due to gravity is constant in the vicinity of Earth. Thus, net force on the ball during its flight remains constant, which is equal to the weight of the ball i.e.

$$F = \mathrm{mg} = 0.1 \text{ X } 10 = 1\ N$$

Hence, option (d) is correct.

Exercise:

Problem:

A pebble of 0.1 kg is subjected to different sets of forces in different conditions on a train which can move on a horizontal linear direction. Determine the case when magnitude of net force on the pebble is greatest. Consider g = 10 m/s^2 .

(a) The pebble is stationary on the floor of the train, which is accelerating at 10 m/s^2 .

(b) The pebble is dropped from the window of train, which is moving with uniform velocity of 10 m/s.

(c) The pebble is dropped from the window of train, which is accelerating at 10 m/s^2 .

(d) Magnitude of force is equal in all the above cases.

Solution:

In the case (a), the pebble is moving with horizontal acceleration of 10 m/s^2 as seen from the ground reference (inertial frame of reference). The net external force in horizontal direction is :

$$\Rightarrow F_{\text{net}} = \text{ma} = 0.01 \text{ X } 10 = 1\,N\ (\text{acting horizontally})$$

There is no vertical acceleration. As such, there is no net external force in the vertical direction. The magnitude of net force on the pebble is, therefore, 1 N in horizontal direction.

In the case (b), the pebble is moving with uniform velocity of 10 m/s in horizontal direction as seen from the ground reference (inertial frame of reference). There is no net force in horizontal direction. When dropped, only force acting on it is due to gravity. The magnitude of net force is, thus, equal to its weight :

$$\Rightarrow F_{\text{net}} = \text{mg} = 0.01 \text{ X } 10 = 1\,N\ (\text{acting downward})$$

In the case (c), the pebble is moving with acceleration of 10 m/s^2 as seen from the ground reference (inertial frame of reference). When the pebble is dropped, the pebble is disconnected with the accelerating train. As force has no past or future, there is no net horizontal force on the pebble. Only force acting on pebble is its weight. The magnitude of force on the pebble is :

$$\Rightarrow F_{\text{net}} = \text{mg} = 0.01 \text{ X } 10 = 1\, N \text{ (acting downward)}$$

Hence, option (d) correct.

Exercise:

Problem:

A rocket weighing 10000 kg is blasted upwards with an initial acceleration 10 m/s^2 . The thrust of the blast is :

$$\text{(a) } 1 \text{ X } 10^2 \text{ (b) } 2 \text{ X } 10^5 \text{ (c) } 3 \text{ X } 10^4 \text{ (d) } 4 \text{ X } 10^3$$

Solution:

The net force on the rocket is difference of the thrust and weight of the rocket (thrust being greater). Let thrust be F, then applying Newton’s second law of motion :

$$F_{\text{net}} = F - \text{mg} = \text{ma}$$

$$\Rightarrow F = m\left(g + a\right) = 10000 \text{ X } \left(10 + 10\right) = 200000 = 2 \text{ X } 10^5\, N$$

Hence, option (b) is correct.

Exercise:

Problem:

The motion of an object of mass 1 kg along x-axis is given by equation,

$$x = 0.1t + 5t^2$$

where “x” is in meters and “t” is in seconds. The force on the object is :

$$\text{(a) } 0.1\,N \text{ (b) } 0.5\,N \text{ (c) } 2.5\,N \text{ (d) } 10\,N$$

Solution:

The given equation of displacement is a quadratic equation in time. This means that the object is moving with a constant acceleration. Comparing given equation with the standard equation of motion along a straight line :

$$x = \text{ut} + \frac{1}{2}at^2$$

We have acceleration of the object as :

$$a = 5 \text{ X } 2 = 10\,m/t^2$$

Applying Newton’s second law of motion :

$$F = \text{ma} = 1 \text{ X } 10 = 10\,N$$

Hence, option (d) is correct.

Exercise:

Problem:

A bullet of mass 0.01 kg enters a wooden plank with a velocity 100 m/s. The bullet is stopped at a distance of 50 cm. The average resistance by the plank to the bullet is :

$$\text{(a) } 10\,N \text{ (b) } 100\,N \text{ (c) } 1000\,N \text{ (d) } 10000\,N$$

Solution:

The plank applies resistance as force. This force decelerates the bullet and is in opposite direction to the motion of bullet. Since we seek to know average acceleration, we shall consider this force as constant force assuming that wooden has uniform constitution. Let the corresponding deceleration be “a”. Then, according to equation of motion,

$$v^2 = u^2 + 2\,\text{as}$$

$$\Rightarrow a = -\frac{v^2 - u^2}{2s}$$

Putting values,

$$\Rightarrow a = -\frac{10^4 - 0}{2 \text{ X } 0.5} = -10^4\, m/t^2$$

Applying Newton’s second law of motion :

$$\Rightarrow F = \text{ma} = 0.01 \text{ X } 10^4 = 100\, N$$

Hence, option (b) is correct.

Newton's third law of motion

Third law of motion is different to other two laws of motion in what it describes. This law states about an important characteristic of force rather than the relation between force and motion as described by the first two laws.

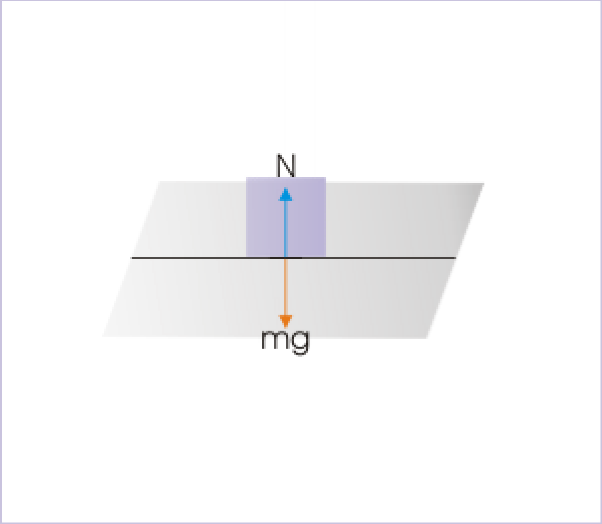
Newton's third law of motion

> One body interacts with other body exerting force on each other, which are equal in magnitude, but opposite in direction.

The action and reaction pair acts along the same line. Their points of application are different as they act on different bodies. This is a distinguishing aspect of third law with respect first two laws, which consider application of force on a single entity.

The law underlines the basic manner in which force comes into existence. Force results from interaction of two bodies, always appearing in pair. In other words, the existence of single force is impossible. In the figure below, we consider a block at rest on a table. The block presses the table down with a force equal to its weight (mg). The horizontal table surface, in turn, pushes the block up with an equal normal force (N), acting upwards.

Newton's third law of motion



Two bodies exert equal but opposite force on each other.

$$N = \mathrm{mg}$$

In this case, the net force on the block and table is zero. The force applied by the table on the block is equal and opposite to the force due to gravity acting on it. As such, there is no change in the state of block. Similarly, net force on the table is zero as ground applies upward reaction force on table to counterbalance the force applied by the block. We should, however, be very clear that these action and reaction force arising from the contact are capable to change the state of motion of individual bodies, provided they are free to move. Consider collision of two billiard balls. The action and reaction forces during collision change the course of motion (acceleration of each ball).

The "action" and "reaction" forces are external forces on individual bodies. Depending on the state of a body (i.e. the state of other forces on the body), the individual "action" or "reaction" will cause acceleration in the particular body. For this reason, book and table do not move on contact, but balls after collision actually moves with certain acceleration.

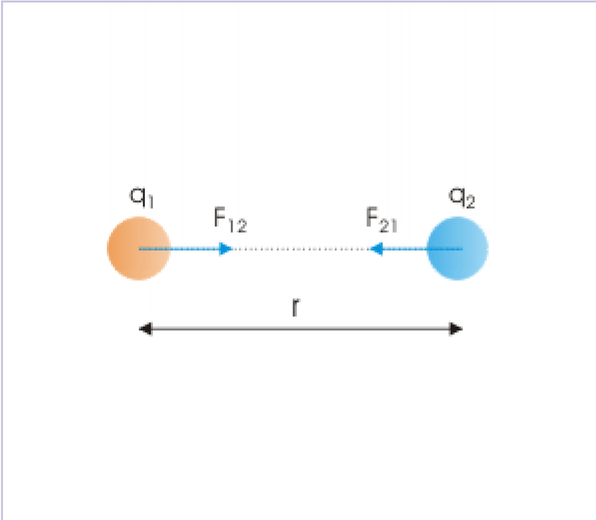
The scope of this force is not limited to interactions involving physical contact. This law appears to apply only when two bodies come in contact. But, in reality, the characterization of force by third law is applicable to all force types. This requirement of pair existence is equally applicable to forces like electrostatic or gravitational force, which act at a distance without coming in contact.

Let us consider the force between two charges q_1 and $-q_2$ placed at a distance "r" apart. The magnitude of electrostatic force is given by :

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

The charge q_1 applies a force F_{21} on the charge q_2 and charge q_2 applies a force F_{12} on q_1 . The two forces are equal in magnitude, but opposite in direction such that :

Electrostatic force



Force appears in pair.

Equation:

$$F_{12} = -F_{21}$$
$$\Rightarrow F_{12} + F_{21} = 0$$

Note: Here, we read the subscripted symbol like this : F_{12} means that it is a force, which is applied on body 1 by body 2.

It should be emphasized that though vector sum of two forces is zero, but this condition does not indicate a state of equilibrium. This is so because two forces, often called as action and reaction pair, are acting on different bodies. Equilibrium of a body, on the other hand, involves consideration of external forces on the particular body.

Deduction of Third law from Newton’s Second law

We have pointed out that “action” and “reaction” forces are external forces on the individual bodies. However, if we consider two bodies forming a “system of two bodies”, then action and reaction pairs are internal to the system of two bodies. The forces on the system of bodies are :

Equation:

$$F = \quad F_{\text{int}} + \quad F_{\text{ext}}$$

If no external forces act on the system of bodies, then :

$$F_{\text{ext}} = 0$$

From second law of motion, we know that only external force causes acceleration to the body system under consideration. As such, acceleration of the “system of two bodies” due to net internal forces should be zero. Hence,

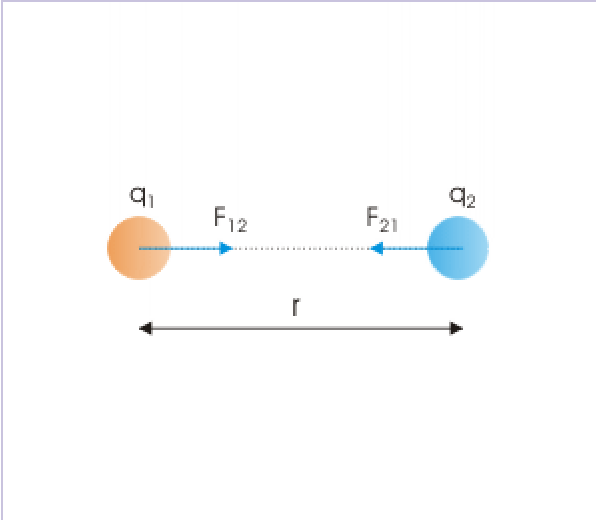
$$F_{\text{int}} = 0$$

This is possible when internal forces are pair forces of equal magnitude, which are directed in opposite directions.

The internal forces are incapable to produce acceleration of the system of bodies. The term “system of bodies” is important (we shall discuss the concept of system of bodies and their motion in separate module). The acceleration of the system of bodies is identified with a point known as center of mass. When we say that no acceleration is caused by the pair of third law forces, we mean that the “center of mass” has no acceleration. Even though individual body of the system is accelerated, but “center of mass” is not accelerated and hence, we say that "system of bodies" is not accelerated.

Referring to two charged body system that we referred earlier, we can consider forces F_{12} and F_{21} being the internal forces with respect to the system of two charged bodies. As such, applying Newton’s second law,

Electrostatic force



Force appears in pair.

$$F_{int} = F_{12} + F_{21} = 0$$

$$\Rightarrow F_{12} = -F_{21}$$

The “action” and “reaction” forces are, therefore, equal in magnitude, but opposite in direction. Clearly, third law is deducible from second law of motion.

The important point to realize here is that “action” and “reaction” forces are external forces, when considered in relation to individual bodies. Each of the two forces is capable to produce acceleration in individual bodies. The same forces constitute a pair of equal and opposite internal forces, when considered in relation to the system of bodies. In this consideration, the center of mass of the body system has no acceleration and net internal force is zero.

Summary

Here, we summarize the interpretation of third law of motion as discussed above :

- This law does not address the issue of force and acceleration as applied to a body like first two laws.
- The law characterizes the nature of force irrespective of its class and genesis that they exist in pair of two equal but opposite forces. The existence of a single force is impossible.
- The pair of forces acts on two different bodies.
- The two forces do not neutralize each other, because they operate on different bodies. Each of the bodies will accelerate, if free to do so.

Fundamental force types

There are different types of forces that may operate on a body. The forces are different in origin and characterization. However, there are only four fundamental forces. Other force types are simply manifestation of these fundamental forces.

In this module, we shall describe the four fundamental force types. The other force types, which are required to be considered in mechanics, will be discussed in a separate module. In this sense, this module is preparatory to the study of dynamics. The treatment of the force types, however, will be preliminary and limited in scope to the extent which fulfills the requirement of dynamics.

The four fundamental force types are :

1. Gravitational force
2. Electromagnetic force
3. Weak force
4. Strong force (nuclear force)

Gravitational force

The force of gravitation is a long distance force, arising due to the very presence of matter. Netwon’s gravitation law provides the empirical expression of gravitational force between two point like masses m_1 and m_2 separated by a distance “r” as :

Equation:

$$F_G = \frac{Gm_1m_2}{r^2}$$

where “G” is the universal constant. $G = 6.7 \text{ x } 10^{-11}\, N - m^2/\,\text{kg}^2$.

Gravitational force is a pair of pull on the two bodies directed towards each other. It is always a force of attraction. Gravitation is said to follow inverse

square law as the force is inversely proportional to the square of the distance between the bodies.

Since the force of gravitation follows inverse square law, the force can be depicted as a conservative force field, in which work done in moving a mass from one point to another is independent of the path followed. The gravitation force is the weakest of all fundamental forces but can assume great magnitude as there are truly massive bodies present in the universe.

In the case of Earth (mass “M”) and a body (mass “m”), the expression for the gravitational force is :

Equation:

$$F_G = \frac{GMm}{r^2}$$

Equation:

$$\Rightarrow F_G = mg$$

where “g” is the acceleration due to gravity.

Equation:

$$\Rightarrow g = \frac{GM}{r^2}$$

The most important aspect of acceleration due to gravity here is that it is independent of the mass of the body “m”, which is being subjected to acceleration. Its value is taken as 9.81 m/s^2 .

Gravitation force has a typical relation with the mass of the body on which its effect is studied. We know that mass (“m”) is part of the Newton’s second law that relates force with acceleration. Incidentally, the same mass of the body (“m”) is also a part of the equation og gravitation that determines force. Because of this special condition, bodies of different masses are subjected to same acceleration. Such is not the case with other forces and the resulting acceleration is not independent of mass.

Consider for example a body of mass "m'" instead of "m". Then,

$$F_G = \frac{GM\text{m'}}{r^2} = \text{m'}g$$

$$\Rightarrow g = \frac{GM}{r^2}$$

We see here that gravitational force on the body is proportional to the mass of the body itself. As such, the acceleration, which is equal to the force divided by mass, remains same.

Knowing that acceleration due to gravitational force in the Earth's vicinity is a constant, we can calculate gravitational pull on a body of mass "m", using relation second law of motion :

F = mg

Further, gravitational force is typically a force which operates at a distance. The force is said to be communicated to an object at a distance through a field known as gravitational field. For this reason, gravitational force is classified as "force at a distance".

Electromagnetic force

Electromagnetic force is an intermediate range force that plays central role in the constitution of matter in its various forms. Coulomb's law provides the empirical expression of electromagnetic force between two point like charges q_1 and q_2 separated by a distance "r" as :
Equation:

$$F_E = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

where ε_0 is the permittivity of vacuum. $\varepsilon_0 = 8.854$ x $10^{-12}\ A^2/Nm^2s^2$

Electromagnetic force is a pair of pulls on the two charged bodies. The force is said to follow inverse square law as the force is inversely proportional to the square of the distance between the bodies.

The charge is a signed scalar quantity, meaning thereby that the nature of electromagnetic force may be attractive or negative, depending on the polarity of two charges. Two similar charges (positive or negative) repel each other, whereas two dissimilar charges (positive and negative) attract each other.

Since the electromagnetic force follows inverse square law, the force can be depicted as a conservative force field, in which work done in moving a mass from one point to another is independent of the path followed. The electromagnetic force is second strongest force after nuclear force.

The acceleration of a charged body of mass “m”, carrying a charge “q” is given as :
Equation:

$$\Rightarrow a = \frac{F_E}{m}$$
$$\Rightarrow a = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2 m}$$

Here, acceleration of the body is not independent of “mass” of the body as in the case of gravitational force.

Like gravitational force, however, the electromagnetic force operates at a distance. The electromagnetic force is said to be communicated to an object at a distance through a field known as electromagnetic field. For this reason, electromagnetic force is also classified as "force at a distance" and "field force".

Strong (nuclear) force

Nuclear force is strongest of all forces and is a short range ($<10^{-14}$ m) force that is effective within the dimension of a nucleus. This force is responsible for holding together charged nucleons i.e. protons, which other wise would have been repelled away due to electromagnetic force operating between charged particles of same polarity.

When inter-particle distance increases ($>10^{-14}$ m), nuclear force sharply decreases and becomes smaller than the electromagnetic force. Further, nuclear force is not a charge based force like electromagnetic force. It operates between a pair of nucleons whether charged or not charged and is largely attractive in nature. Three possible nucleon force pairs are :

- neutron (n) – neutron (n)
- proton (p) – neutron (n)
- proton (p) – proton (p)

When the inter-particle distance decreases less than 0.5 fermi ($<10^{-15}$ m), the nuclear force becomes repulsive.

No quantitative expression for nuclear force is yet known.

Weak force

Like nuclear force, weak force is a short range force and in addition also a short duration force. Existence of this force came into consideration to explain variation in energy levels in β-decay during radioactive disintegration. According to classical model, the energy of all β- particles (electrons) should have been same. However, it is found that they possess energy from zero to a certain maximum value.

This apparent contradiction was resolved by postulating existence of “weak force”, which interacts through elementary particles like neutrino and anti-neutrino to manifest variation in energy distribution in radioactive decay. Subsequently, postulation of these elementary particles has been supported with their actual existence.

No quantitative expression for weak force is yet known.

Contrary to its name, it is not the weakest force. It is a “weak” force with respect to other nucleus – based nuclear force, but is stronger than gravitational force. The relative strength of fundamental forces is as given here :

Gravitational force (F_G) < Weak force (F_W) < Electromagnetic force (F_E) < Nuclear force (F_N)

Other force types

There are scores of other forces with which we come across in our daily life. Some of these are contact, tension, spring, muscular, chemical force etc. These forces result from fundamental forces. In most of the cases, electromagnetic force is the basic force, which ultimately manifests in a particular force type.

Operation of force

Forces differ in the manner they operate on a body. Some need physical contact, whereas others act from a distance. Besides, there are forces, which apply along the direction of attachments to the body. We can classify forces on the basis of how do they operate on a body. According to this consideration, the broad classification is as given here :

- Field force
- Contact force
- Force applied through an attachment

Forces like electromagnetic and gravitational forces act through a field and apply on a body from a distance without coming into contact. These forces are said to transmit through the field at the speed of light. These forces are, therefore, known as field forces.

Forces such as normal and friction forces arise, when two bodies come into contact. These forces are known as contact forces. They operate at the interface between two bodies and act along a line, which forms specific angle to the tangent drawn at the interface.

Besides, some of the forces are applied through flexible entities like string and spring. For example, we raise bucket from a well by applying force on the bucket through the flexible medium of a rope. The direction of the taut rope (string) determines the direction of force applied. Application of force through string provides tremendous flexibility as we can change string direction by virtue of some mechanical arrangement like pulley.

The spring force is a mechanism through which variable force (F = kx) can be applied to a body. Like string, the spring also provides for changing direction of force by changing the orientation of the spring.

Other force types relevant to the study of dynamics

We have already discussed field forces like gravitational and electromagnetic forces in the module named "Fundamental forces". Here, we embark to study other force types, which may form the part of the study of dynamics. These are :

- Contact force
- String tension
- Spring force

Contact force

When an object is placed on another object, two surfaces of the bodies in contact apply normal force on each other. Similarly, when we push a body over a surface, the force of friction, arising from electromagnetic attraction among molecules at the contact surface, opposes relative motion of two bodies.

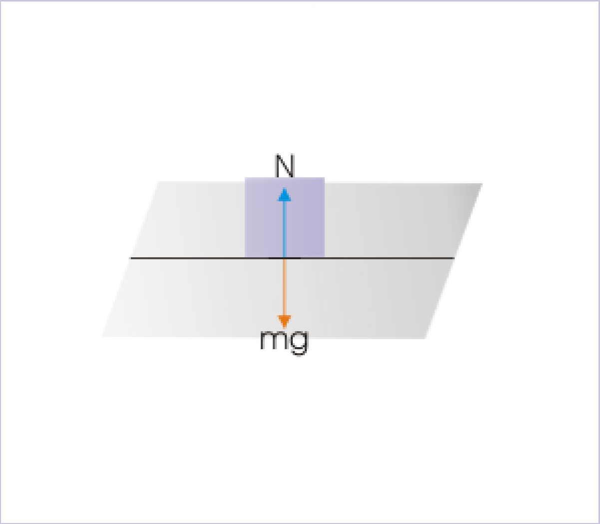
In dynamics, we come across these two particular contact forces as described here.

Normal force

When two bodies come in contact, they apply equal and opposite forces on each other in accordance with Newton’s third law of motion. As the name suggest, this "normal force" is normal to the common tangent drawn between two surfaces.

We consider here a block of mass “m” placed on the table. The weight of the block (mg) acts downward. This force tries to deform the surface of the table. The material of the table responds by applying force to counteract the force that tends to deform it.

Normal force



The surface applies a normal force on the body.

The rigid bodies counteract any deformation in equal measure. A rigid table, therefore, applies a force, which is equal in magnitude and opposite in direction. As there is no relative motion at the interface, this contact force has no component along the interface and as such it is normal to the interface.

Equation:

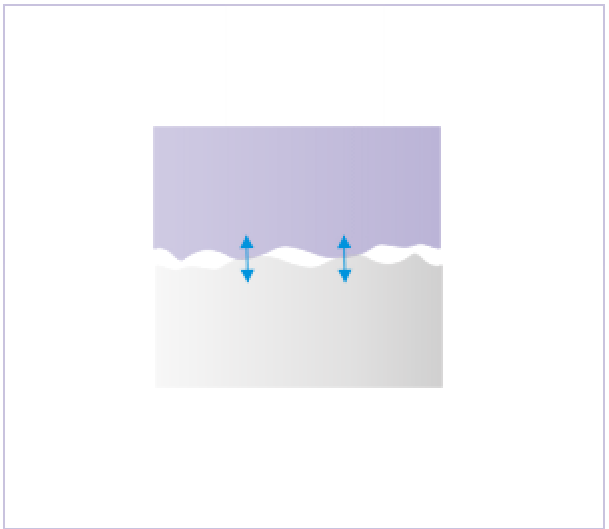
$$N = mg$$

Friction force

Force of friction comes into play whenever two bodies in contact either tends to move or actually moves. The surfaces in contact are not a plane

surface as they appear to be. Their microscopic view reveals that they are actually uneven with small hills and valleys. The bodies are not in contact at all points, but limited to elevated points. The atoms/molecules constituting the surfaces attract each other with electrostatic force at the contact points and oppose any lateral displacement between the surfaces.

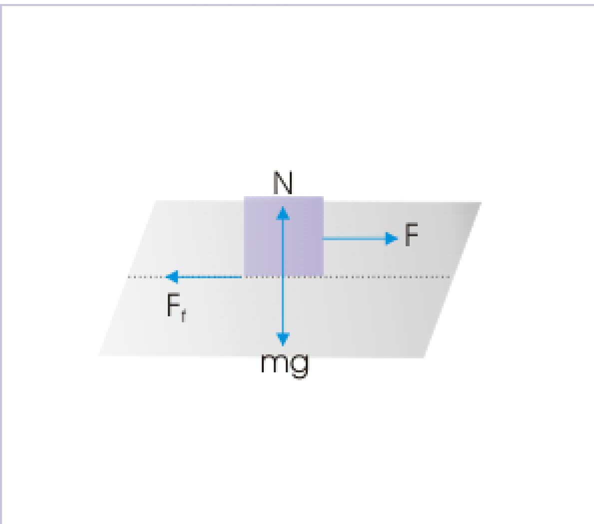
Friction force



The surfaces are uneven.

For this reason, we need to apply certain external force to initiate motion. As we increase force to push an object on horizontal surface, the force of friction also grows to counteract the push that tries to initiate motion. But, beyond a point when applied force exceeds maximum friction force, the body starts moving.

Friction force



The friction force acts tangential to the interface between two surfaces.

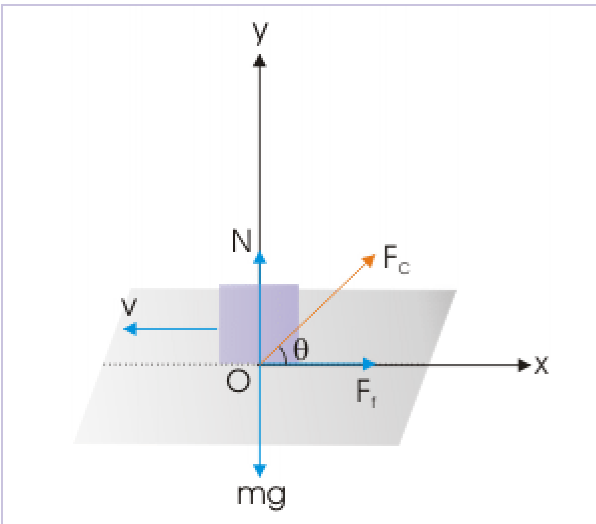
Incidentally, the maximum force of friction, also known as limiting friction, is related to the normal force at the surface.

$$F_f = \mu N$$

where "μ" is the coefficient of friction between two particular surfaces in question. Its value is dependent on the nature of surfaces in contact and the state of motion. When the body starts moving, then also force of friction applies in opposite direction to the direction of relative motion between two surfaces. In general, we use term “smooth” to refer to a frictionless interface and the term “rough” to refer interface with certain friction. We shall study more about friction with detail in a separate module.

The two contact forces i.e. normal and friction forces are perpendicular to each other. The magnitude of net contact force, therefore, is given by the magnitude of vector sum of two contact forces :

Contact force



The contact force is equal to the vector sum of normal and friction forces.

Equation:

$$F_C = \sqrt{N^2 + F_f^2}$$

The tangent of the angle formed by the net contact force to the interface is given as :

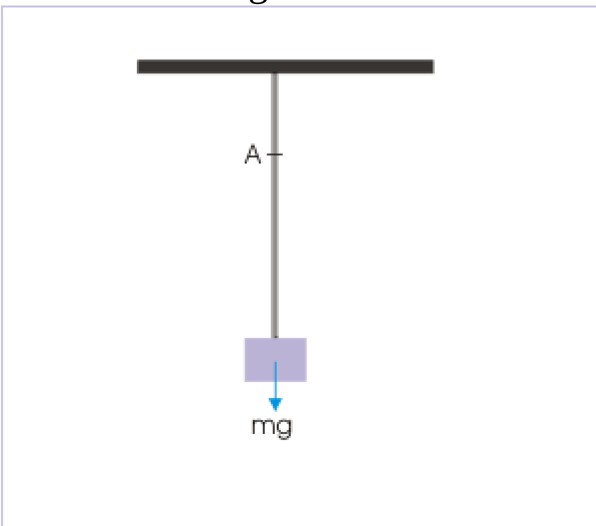
Equation:

$$\tan\theta = \frac{N}{F_f}$$

String Tension

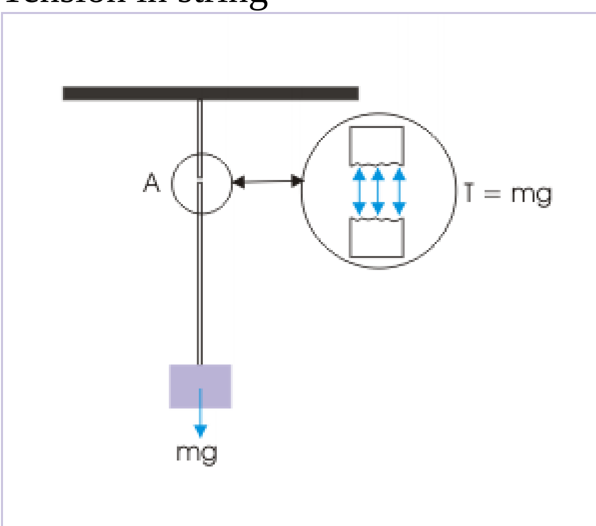
String is an efficient medium to transfer force. We pull objects with the help of string from a convenient position. The string in taut condition transfers force as tension.

Tension in string



Let us consider a block hanging from the ceiling with the help of a string. In order to understand the transmission of force through the string, we consider a cross section at a point A as shown in the figure. The molecules across "A" attract each other to hold the string as a single piece.

Tension in string

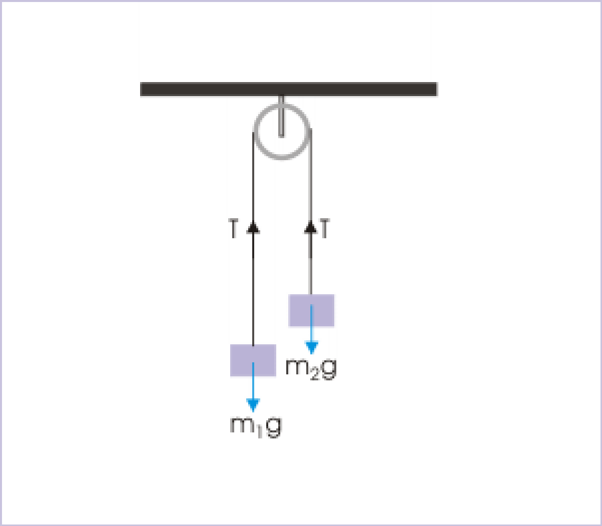


If we consider that the mass of the string is negligible, then the total downward pull is equal to the weight of the block (mg). The electromagnetic force at “A”, therefore, should be equal to the weight acting in downward direction. This is the situation at all points in the string and thus, the weight of the block is transmitted through out the length of the string without any change in magnitude.

However, we must note that force is transmitted undiminished under three important conditions : the string is (i) taut (ii) inextensible and (ii) mass-less. Unless otherwise stated, these conditions are implied when we refer string in the study of dynamics.

We must also appreciate that string is used with various combination of pulleys to effectively change the direction of force without changing the magnitude. There, usually, is a doubt in mind about the direction of tension in the string. We see that directions of tension in the same piece of string are shown in opposite directions.

Tension in string

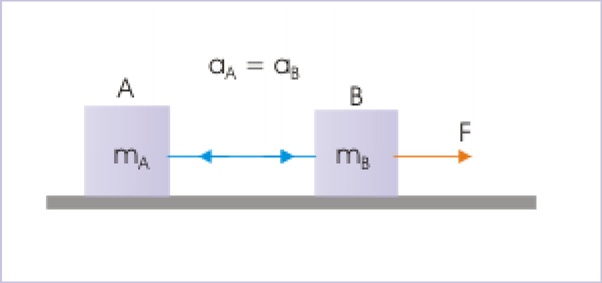


However, this is not a concern that should be overemphasized. After all, this is the purpose of using string that force is communicated undiminished (T) but with change in direction. We choose the appropriate direction of tension in relation to the body, which is under focus for the study of motion. For example, the tension (T) is acting upward on the block, when we consider forces on the block. On the other hand, the tension is acting downward on the pulley, when we consider forces on the pulley. This inversion of direction of tension in a string is perfectly fine as tension work in opposite directions at any given intersection.

While considering string as element in the dynamic analysis, we should keep following aspects in mind :

1: If the string is taught and inextensible, then the velocity and acceleration of each point of the string (also of the objects attached to it) are same.

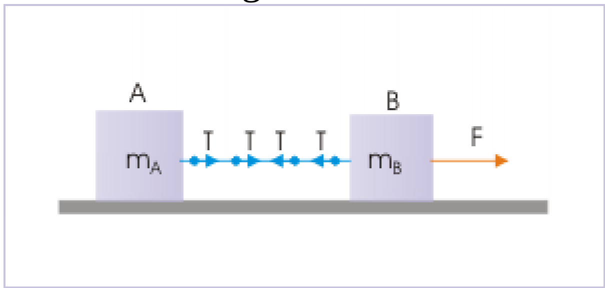
Inextensible string



The velocity and acceleration are same at each point on the string.

2: If the string is "mass-less", then the tension in the string is same at all points on the string.

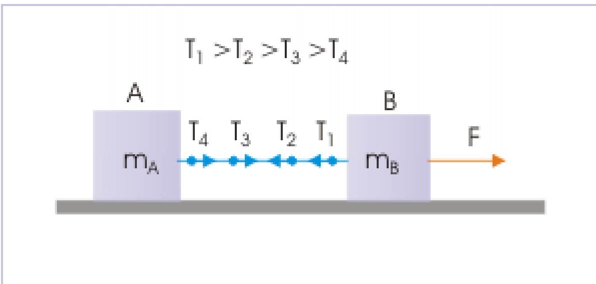
Mass-less string



The tension is same at each point on the string.

3: If the string has certain mass, then the tension in the string is different at different points. If the distribution of mass is uniform, we account mass of the string in terms of "mass per unit length (λ)" - also called as linear mass density of the string.

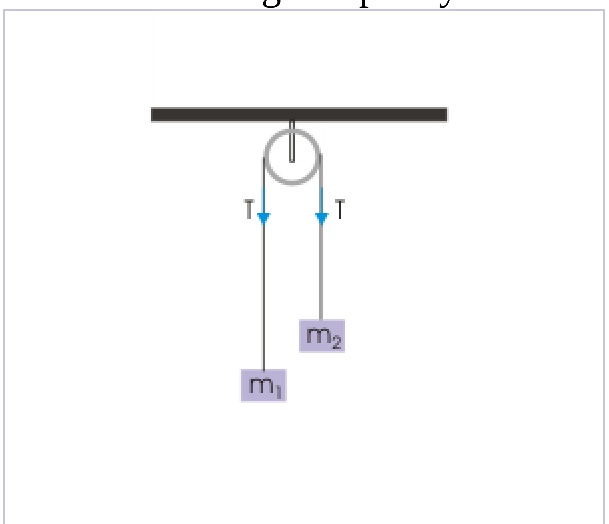
String with certain mass



The tension is different at different points on the string.

4: If "mass-less" string passes over a "mass-less" pulley, then the tension in the string is same on two sides of the pulley.

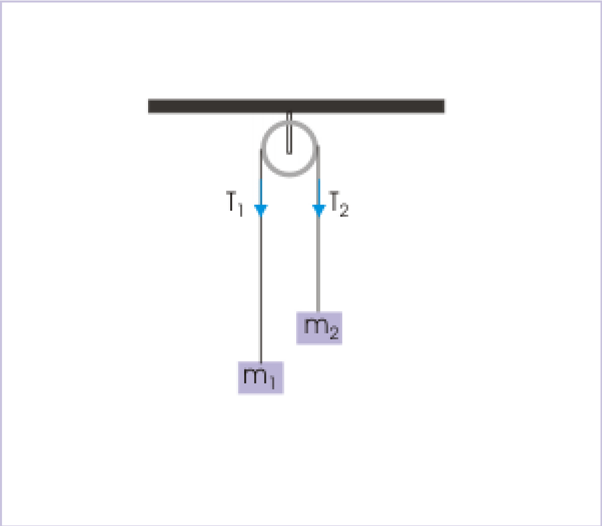
Mass - less string and pulley



The tensions in the string are same on two sides of the pulley.

5: If string having certain mass passes over a mass-less pulley, then the tension in the string is different on two sides of the pulley.

String and pulley



The tensions in the string are different on two sides of the pulley.

6: If "mass-less" string passes over a pulley with certain mass and there is no slipping between string and pulley, then the tension in the string is different on two sides of the pulley. Tensions of different magnitudes form the requisite torque required to rotate pulley of certain mass.

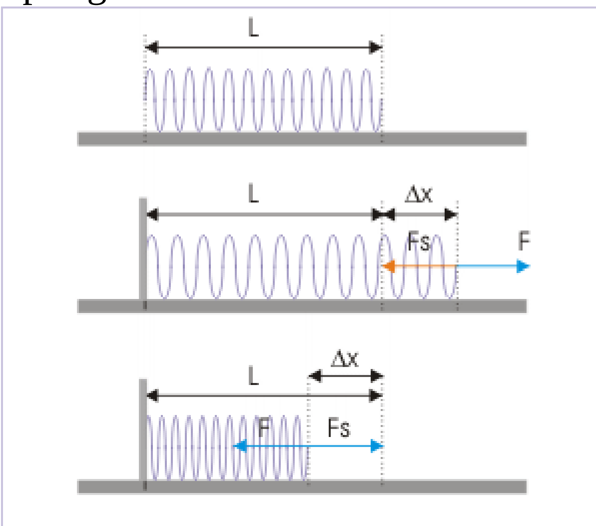
We have enlisted above scenarios involving string which are usually considered during dynamic analysis. Notably, we have not elaborated the reasons for each of the observations. We intend, however, to supplement these observations in appropriate context during the course.

Spring

Spring is a metallic coil, which can be stretched or compressed. Every spring has a “natural length” that can be measured, while the spring is lying on a horizontal surface (shown in the figure at the top).

If we keep one end fixed and apply a force at the other end to extend it as shown in the figure, then the spring stretches by a certain amount say Δx. In response to this, the spring applies an equal force in opposite direction to resist deformation (shown in the figure at the middle).

Spring force



Expansion and compression of spring

For a "mass-less" spring, it is experimentally found that :

Equation:

$$F_S \propto \Delta x$$
$$\Rightarrow F_S = -k\Delta x$$

where “k” is called spring constant, specific to a given spring. The negative sign is inserted to accommodate the fact that force exerted by spring is opposite to the direction of change in the length of spring. This relation is known as Hook's law and a spring, which follows Hook's law, is said to be perfectly elastic.

Similarly, when an external force compresses the spring, it opposes compression in the same manner (shown in the figure at the bottom).

Analysis framework for laws of motion
A planned approach to problems is essential to effectively apply laws of motion.

The analysis of motion involves writing Newton's second law of motion in mutually perpendicular coordinate directions. For this, we carry out force analysis following certain simple steps. These steps may appear too many, but we become experienced while working with them. Some of the steps are actually merged. Nevertheless, we need to recognize the importance of each of the steps as these are critical steps to ensure correct analysis of the motion/ situation. The steps are :

1. Identifying body system
2. Identifying external forces
3. Identifying a suitable coordinate system
4. Constructing a free-body diagram
5. Resolving force along the coordinate axes
6. Applying second law of motion

The identification of body system is based on the description of the problem. The body system decides the nature of force - whether it is internal or external. The selection of body system should suit the requirement of the analysis i.e. end objective of analysis. This will be clear as we discuss this aspect subsequently in the module.

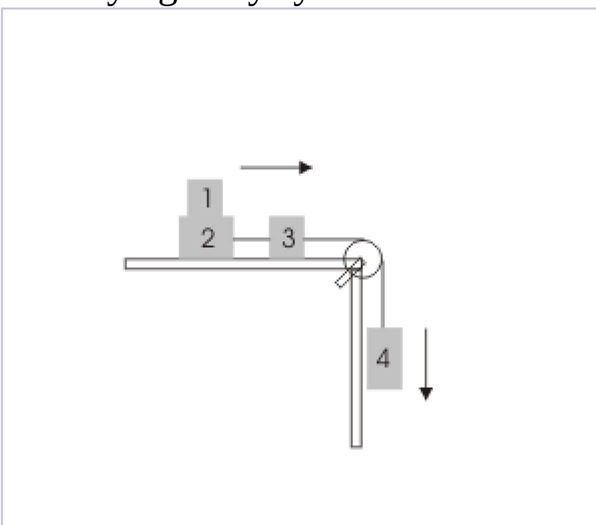
The identification of external forces means to identify forces, which are applied on the object by other objects/sources and exclude forces, which the object applies on other objects. The important aspect of identification of external force is that external characterization depends on the definition of body system. The same force, which is external to a body, may become internal force for a different body system involving that body.

In this module, we shall not discuss steps 2 and 6. As far as identification of external forces (step 2) is concerned, we have discussed the same with an appropriate example in module titled Newton’s second law of motion . On the other hand, we shall apply second law of motion in mutually perpendicular directions (step 6) in the analysis of individual cases and as such we shall not discuss this step separately.

Identifying body system

If we are studying a single or two body system, then there is no issue. However, let us consider an illustration here as shown in the figure. There are possibilities of having different combination of bodies constituting different body systems.

Identifying body system



in selection of a body system, the guiding principle is to treat bodies as a single body system, when we know that bodies have common acceleration.

In the figure, the objects numbered 2 and 3 can be considered a single body system as two bodies have same acceleration. The objects 3 and 4 can not be combined as a single body system as they have acceleration of same magnitude, but with different direction. The objects 1 and 2 can be treated as a single body system, if two bodies do not have relative motion with respect to each other. Otherwise, we would need to treat them as separate body systems.

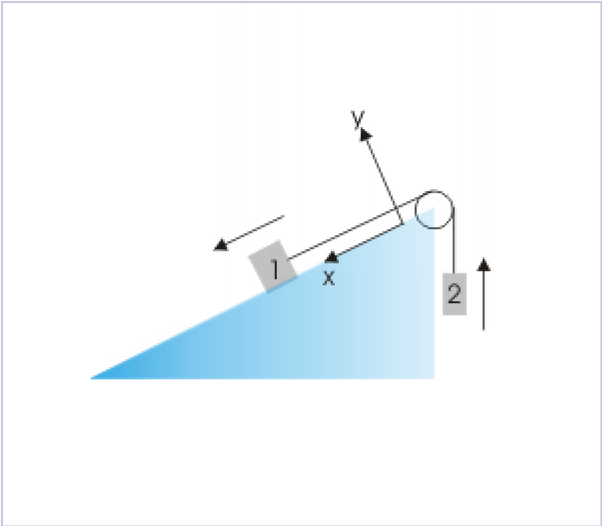
It is not always required that we must know the direction of acceleration before hand. We can assume a direction with respect to one body system and then proceed to find the directions of the accelerations of the other body systems in the arrangement. Even if our assumption about the direction of acceleration is incorrect, the solution of the problem automatically corrects the direction. We shall see this aspect while working with specially designed example, higlighting the issue.

But the basic question is to know : why should be look for more than one body system in the first place? It is because such considerations will yield different sets of equations involving laws of motion and thus facilitate solution of the problem in hand.

Identifying a suitable coordinate system

There is no rule in this regard. However, a suitable coordinate system facilitates easier analysis of problem in hand. The guiding principle in this case is to select a coordinate system such that one of the axes aligns itself with the direction of acceleration and other is perpendicular to the direction of acceleration.

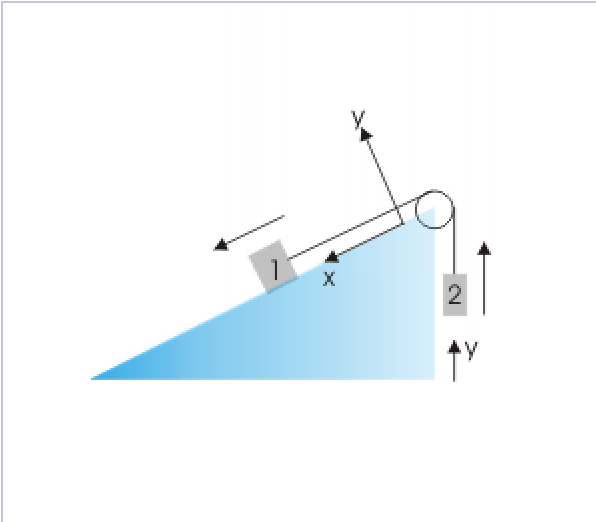
Identifying a suitable coordinate system



For the case as shown in the figure, the coordinate system for body 1 has its x-axis aligned with the incline, whereas its y-axis is perpendicular to the incline. In case, bodies are in equilibrium, then we may align axes such that they minimize the requirement of taking components.

It must be realized here that we are free to utilize more than one coordinate system for a single problem. However, we would be required to appropriately choose the signs of vector quantities involved.

Identifying a suitable coordinate system

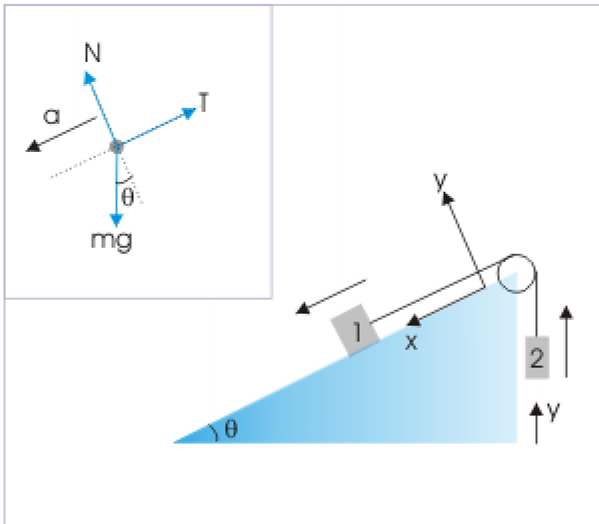


In the figure above, we have selected two coordinate systems; one for body 1 and other for body 2. Note that y-axis for body 2 is vertical, whereas y – axis of body 1 is perpendicular to incline. Such considerations are all valid so long we maintain the sign requirements of the individual coordinate systems. In general, we combine "scalar" results from two different coordinate systems. For example, the analysis of force on body 2 yields the magnitude of tension in the string. This value can then be used for analysis of force on body 1, provided string is mass-less and tension in it transmits undiminished.

Constructing a free-body diagram

A free body diagram is a symbolic diagram that represents the body system with a “point” and shows the external forces in both magnitude and direction. This is the basic free-body diagram, which can be supplemented with an appropriate coordinate system and some symbolic representation of acceleration. The free body diagram of body 1 is shown in the upper left corner of the figure :

Constructing a free-body diagram



We may have as many free body diagrams as required for each of the body systems. If the direction of acceleration is known before hand, then we may show its magnitude and direction or we may assign an assumed direction of acceleration. In the later case, the solution finally lets us decide whether the chosen direction was correct or not?

What force system do we study?

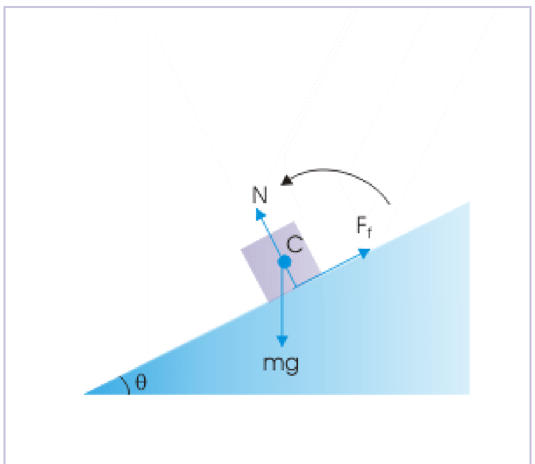
We have discussed in the module titled "Newton's second law of motion" that we implicitly consider force system on a body as concurrent forces. It is so because, Newton's second law of motion connects force (cause) with "linear" acceleration (effect). The words "linear" is implicit as there is no reference to angular quantities in the statement of second law for translation.

We, however, know that such con-currency of forces can only be ensured if we consider point objects. What if we consider real three dimensional bodies? A three dimensional real body involves only translation, if external forces are concurrent (meeting at a common point) and coincides with the "center of mass" of the body. In this situation, the body can be said to be in pure translation. We need only one modification that we assign acceleration of the body to a specific point called "center of mass" (we shall elaborate this aspect in separate module).

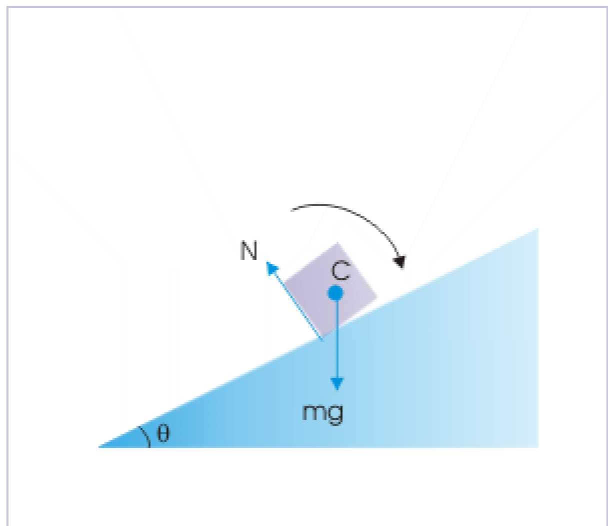
The real time situation presents real three dimensional bodies. We encounter situation in which forces are not concurrent. Still, we consider them to be concurrent (by shifting force in parallel direction). It is because physical set up constrains motion to be translational. For example, consider the motion of a block on an incline as shown in the figure.

A block on an incline

Friction induces a tendency to overturn.



The weight counterbalances the tendency to overturn.



The forces weight (mg), normal force (N) and friction force (Ff) are not concurrent. The friction force, as a matter of fact, does not pass through "center of mass" and as such induces "turning" tendency. However, block is not turned over as restoring force due to the weight of the block is greater and inhibits turning of the block. In the nutshell, block is constrained to translate without rotation. Here, translation is enforced not because forces are concurrent, but because physical situation constrains the motion to be translational.

Resolving force along the coordinate axes

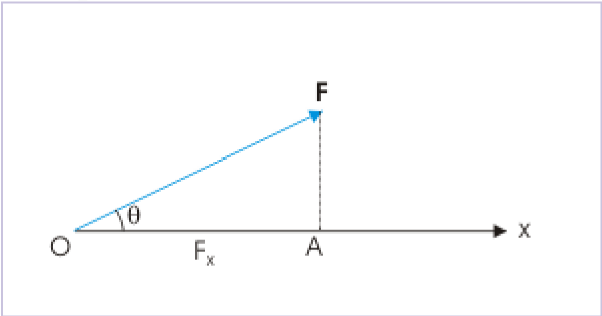
The application of a force and resulting motion, in general, is three dimensional. It becomes convenient to analyze force (cause) and

acceleration (effect) analysis along coordinate axes. This approach has the benefit that we get as many equations as many there are axes involved. In turn, we are able to solve equations for as many unknowns.

We must know that the consideration in each direction is an independent consideration – not depending on the motion in other perpendicular directions. A force is represented by an equivalent system of components in mutually perpendicular coordinate directions.

The component of force vector along a direction (say x-axis) is obtained by :

Component of force



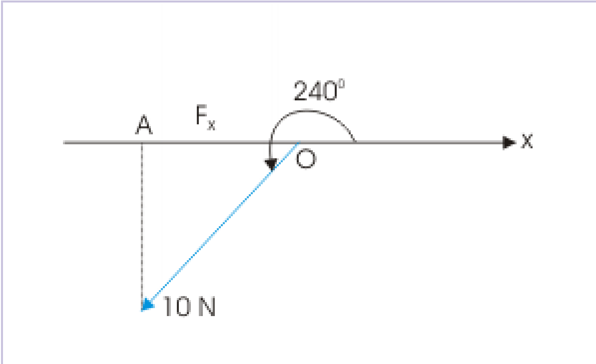
The angle is measured from the positive reference direction.

Equation:

$$F_x = F\cos\theta$$

where "θ" is the angle that force vector makes with the positive direction of x-axis. For example, consider the force as shown in the figure. We need to find the component of a force along x-axis. Here,

Component of force

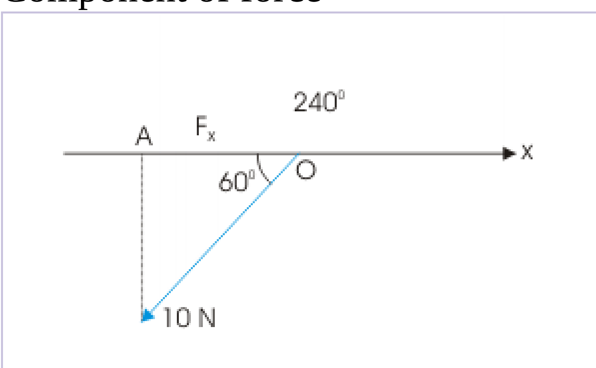


Acute angle is considered.

$$F_x = F\cos\theta = 10\cos 240^0 = 10X - \frac{1}{2} = -5N$$

Alternatively, we consider only the acute angle that the force makes with x-axis. We need not measure angle from the The angle here is 60°. As such, the component of the force along x-axis is :

Component of force



Acute angle is considered.

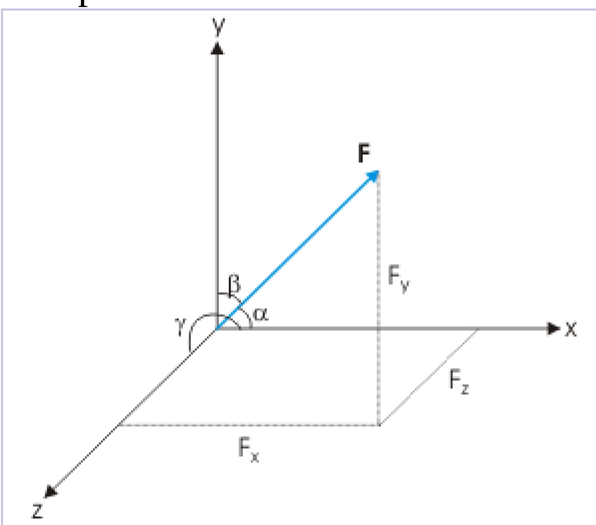
$$F_x = F\cos\theta = 10\cos 60^0 = 10X\frac{1}{2} = 5N$$

We decide the sign of the component by observing whether the projection of the force is in the direction of x-axis or opposite to it. In this case, it is in the opposite direction. Hence, we apply negative sign as,

$$F_x = -F\cos\theta = -10\cos 60^0 = -10X\frac{1}{2} = -5N$$

Clearly, if a force "**F**" makes angles α, β and γ with the three mutually perpendicular axes, then :

Components of force



The angle is measured from respective directions of axes.

$$F_x = F\cos\alpha$$

$$F_y = F\cos\beta$$

$$F_z = F\cos\gamma$$

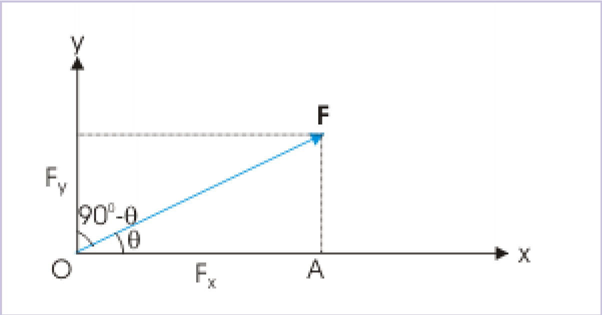
We can write vector force, using components, as :

Equation:

$$\Rightarrow \quad = F\cos\alpha \; + F\cos\beta \; + F\cos\gamma$$

In the case of coplanar force, we may consider only one angle i.e. the angle that force makes with one of the coordinate direction. The other angle is compliment of this angle. If “θ” be the angle that force makes with the positive direction of x-axis, then

Components of coplanar force



There are two components of a coplanar force.

$$F_x = F\cos\theta$$

$$F_y = F\cos\ 90^0 - \theta\ \ = F\sin\theta$$

Example:

Problem : Find the magnitude of components of the weight of the block of 1 kg in the directions, which are parallel and perpendicular to the incline of angle 30°.

Solution :

The weight of the block acts in vertically downward direction. Its magnitude is given as :

Components of force

Components parallel and perpendicular to incline

$$W = mg = 1X10 = 10 \quad N$$

The direction of weight is at an angle 30° with the perpendicular to the incline. Hence, components are :

$$W_{\perp} = W\cos\theta = 10\cos 30^0 = 10X\frac{\sqrt{3}}{2} = 5\sqrt{3} \quad N$$

$$W_{||} = W\sin\theta = 10\sin 30^0 = 10X\frac{1}{2} = 5 \quad N$$

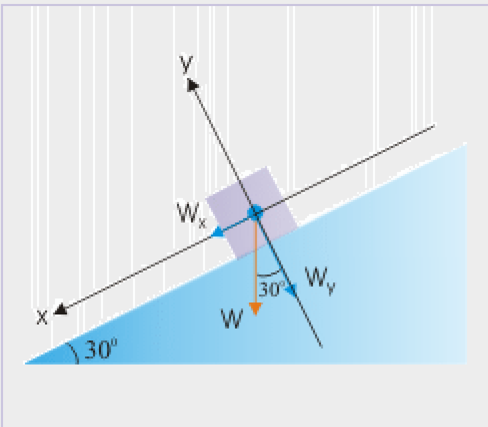
Note : We should realize here that these are the magnitudes of components in the referred directions. The components have directions as well. The values here, therefore, represent the scalar components of weight. The vector components should also involve directions. Since components are considered along a given direction, it is possible to indicate direction by assigning appropriate sign before the values.

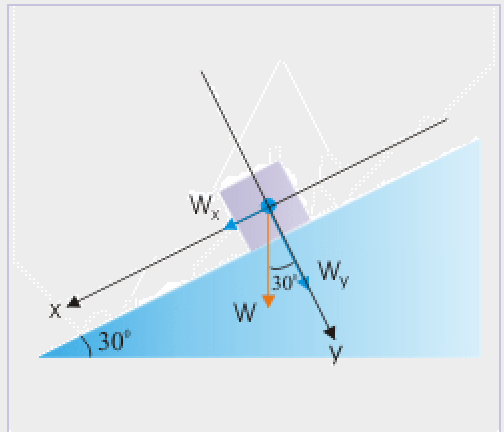
The sign of the components depend on the reference directions i.e. coordinate system. Let us consider the given components in two coordinate systems as shown. In the first coordinate system,

Components of weight

Upward perpendicular direction is positive.

Downward perpendicular direction is positive.





$$W_y = -W\cos\theta = -10\cos 30^0 = -10X\frac{\sqrt{3}}{2} = -5\sqrt{3} \quad N$$

$$W_x = W\sin\theta = 10\sin 30^0 = 10X\frac{1}{2} = 5 \quad N$$

In the second coordinate system,

$$W_y = W\cos\theta = 10\cos 30^0 = 10X\frac{\sqrt{3}}{2} = 5\sqrt{3} \quad N$$

$$W_x = W\sin\theta = 10\sin 30^0 = 10X\frac{1}{2} = 5 \quad N$$

Free body diagram (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

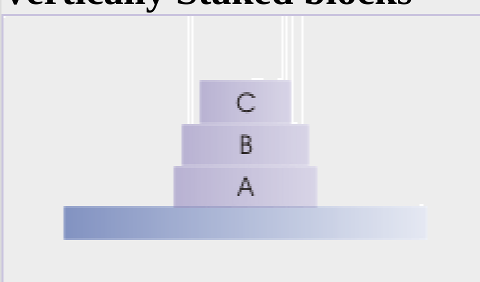
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the Free body diagram (FBD). The questions are categorized in terms of the characterizing features of the subject matter :

- Vertically Staked blocks
- Block, string and pulley
- Block, spring and incline
- Hinged rod
- Rod and spherical shell

Vertically Staked blocks

Example:
Problem : Draw free body diagram of three blocks placed one over other as shown in the figure.
Vertically Staked blocks

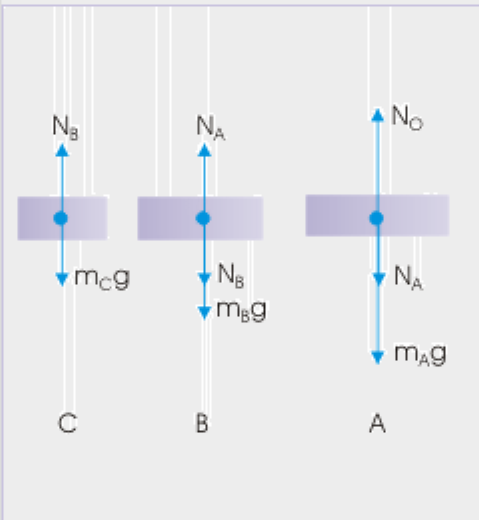


Three blocks of different

masses are placed one over other on a horizontal surface.

Solution : We start drawing FBD of the topmost block as we do no need to consider normal force due to an overlying body as the case with other blocks.

Forces on the blocks



The forces on the individual elements of the system are shown.

The forces on the block "C" are :

1. $W_C = m_C g =$ its weight, acting downward
2. $N_B =$ normal reaction on "C" due to the upper surface of block B, acting upward

The forces on the block "B" are :

1. $W_B = m_B g =$ its weight, acting downward
2. $N_B =$ normal reaction on "B" due to the lower surface of block C, acting downward

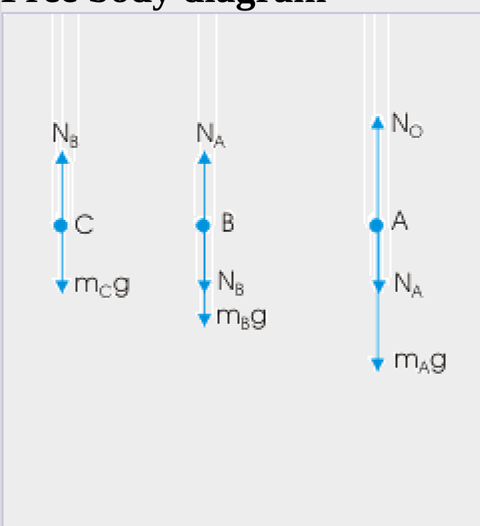
3. N_A = normal reaction on "B" due to the upper surface of block A, acting upward

The forces on the block "A" are :

1. $W_A = m_A g$ = its weight, acting downward
2. N_A = normal reaction on "A" due to the lower surface of block B, acting downward
3. N_O = normal reaction on "A" due to horizontal surface, acting upward

The FBD of the blocks as points with external forces are shown here. Note that motion is not involved. Hence, no information about acceleration is given in the drawing.

Free body diagram



The elements are shown as point with forces.

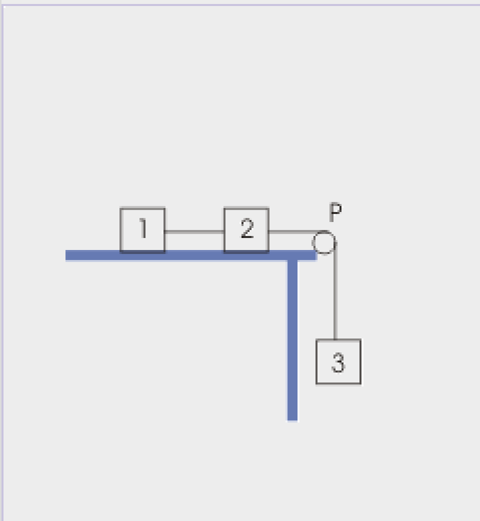
We have deliberately not shown the coordinate system which may be selected, keeping in mind the inputs available. In this instant case, however, a vertical axis is clearly the only choice.

Block, string and pulley

Example:

Problem : Three blocks are connected with the help of two “mass-less” strings and a “mass-less” pulley as shown in the figure. If there is no friction involved and strings are taught, then draw free body diagram of each of the blocks.

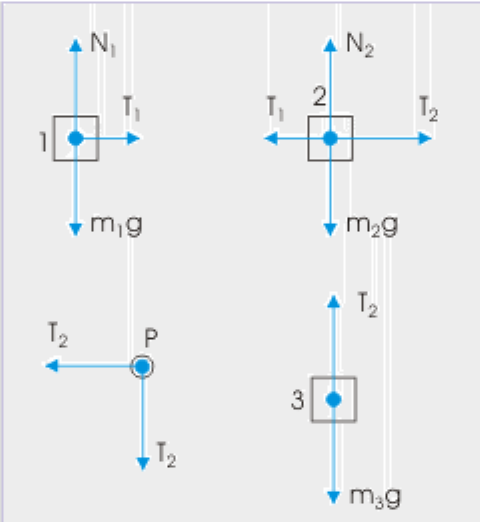
Three blocks connected with two strings



Three blocks are connected through two "mass-less" strings and one "mass-less" pulley.

Solution : Since there are two separate strings. Tensions in two strings are different. We see here that block -1 has no attachment to its left. Hence, it would involve minimum numbers of forces. Thus, we start from block -1.

Forces on the blocks



The forces on the individual elements of the system are shown.

The forces on the block – 1 are :

1. $W_1 = m_1 g =$ its weight, acting downward
2. $N_1 =$ normal force on block – 1 due to the surface of table, acting upward
3. $T_1 =$ tension in the string, towards right

The forces on the block – 2 are :

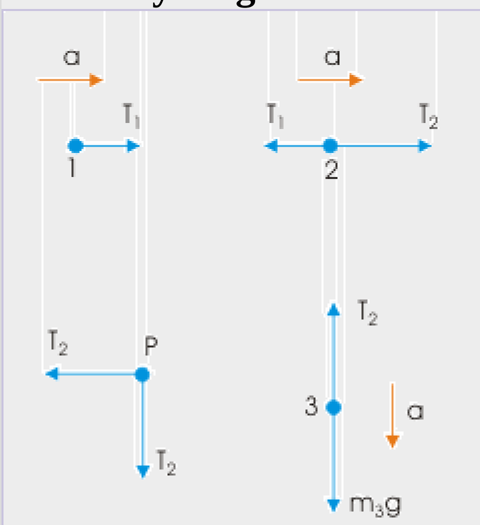
1. $W_2 = m_2 g =$ its weight, acting downward
2. $N_2 =$ normal force on block – 2 due to the surface of table, acting upward
3. $T_1 =$ tension in the string, towards left
4. $T_2 =$ tension in the string, towards right

We note here that “mass-less” string passes over a “mass-less” pulley and no friction is involved. As such, the tensions in the string on either side of the pulley are equal.
The forces on the block – 3 are :

1. $W_3 = m_3 g =$ its weight, acting downward
2. $T_2 =$ tension in the string, acting upward

Since strings are taught, it is evident that the acceleration of the blocks and string are same. Also, we note that motion of the blocks on the table is in horizontal direction only. There is no motion in vertical direction. The forces in the vertical direction, therefore, constitute a balanced force system. Thus, for the analysis of motion, the consideration of forces in vertical directions for blocks of masses " m_1 " and " m_2" is redundant and can be simply ignored. Now, taking these two considerations in account, the FBD of the blocks are as shown here :

Free body diagram



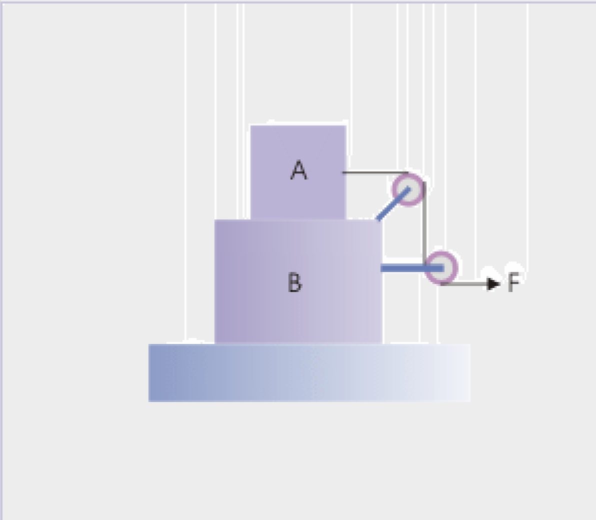
The elements are shown as point with forces.

We should note that the FBD of the blocks show acceleration. Idea here is that we should supplement FBD with as much information as is available. However, we have deliberately not shown the coordinate system which may be selected, keeping in mind the inputs available.

Example:

Problem : Draw free body diagrams of two blocks “A” and “B” in the arrangement shown in the figure, where Block “B” is lying on a smooth horizontal plane.

Two blocks

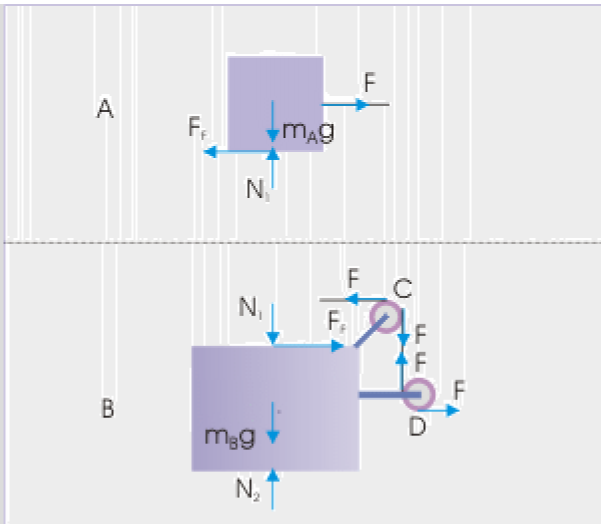


A force "F" is applied with string and pulley arrangement.

Solution : Drawing FBD is a methodological process. However, its efficient use is intuitive and sometimes experience based.

This problem highlights these aspects of drawing FBD. The figure below gives the sketch of various forces on each of the blocks. Note specially that there are indeed large numbers of forces on block "B".

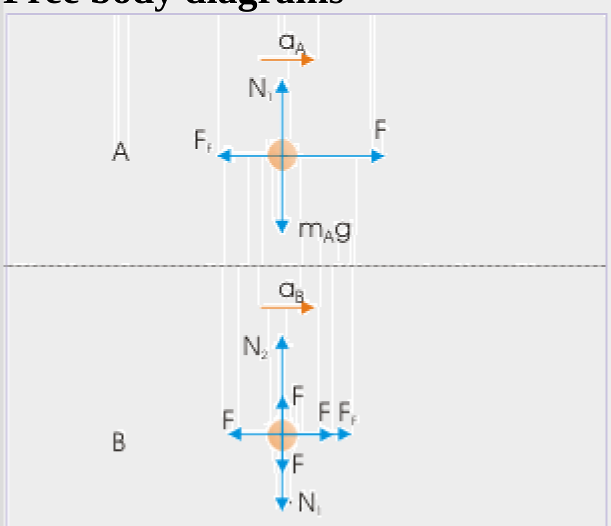
Two blocks



Forces on the two blocks.

The FBD of each of the blocks are shown assuming that only translation is involved. The forces are, therefore, shown concurrent at a single point in each case.

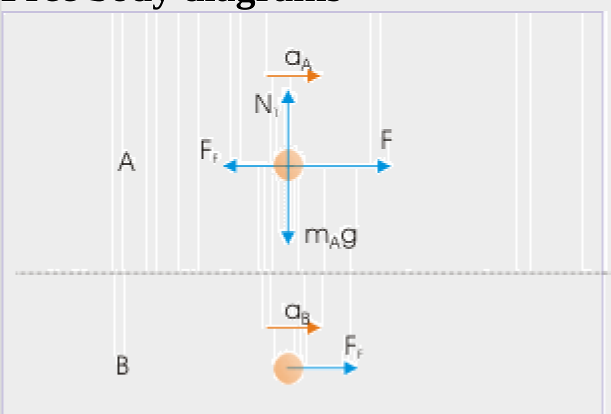
Free body diagrams



Forces on the two blocks are shown concurrent.

If we look closely at the forces on the block "B", then we realize that tension in the string is equal to external force "F". These tensions act on the block "B" through two attached pulleys. Knowing that tension in the string is same everywhere, we could have neglected all the four tensions as far as block "B" is concerned. They form two pairs of equal and opposite forces and, therefore, tensions form a balanced force system for block "B".
We could have further simplified drawing of FBD in the first attempt. We are required to consider only translation in horizontal direction. The forces in vertical direction on block "B", as a matter of fact, may not be required for the force analysis in horizontal direction. We will learn subsequently that we can determine friction in this case by considering vertical normal force only on block "A". Thus, we can simplify drawing FBD a lot, if we have lots of experience in analyzing forces.

Free body diagrams



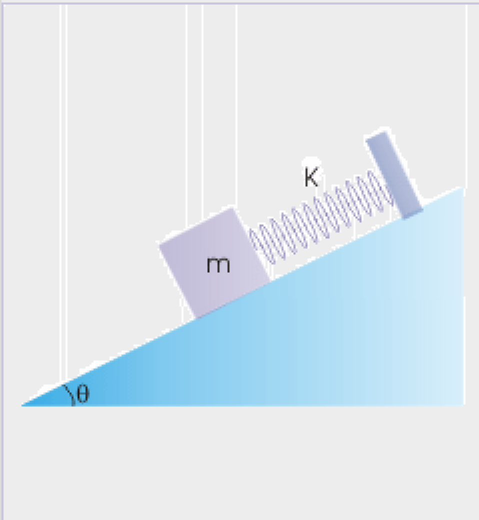
Abridged version of free body diagram.

Block, spring and incline

Example:

Problem : A block of mass “m” is held with the support of a spring of constant “k” on a rough incline of angle “θ”. Draw the free body diagram (FBD) of the block.

A block on an incline



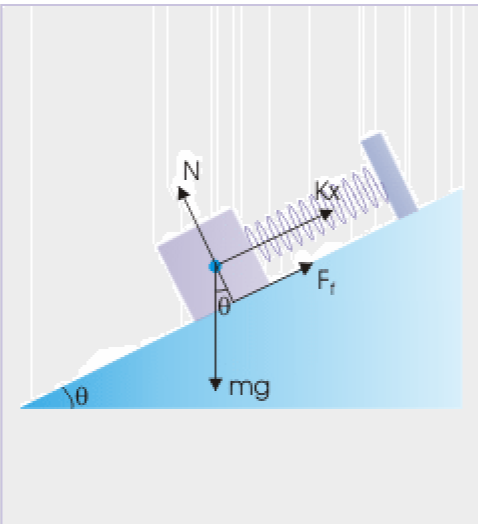
A block of mass “m” is held with the support of a spring of constant “k” on a rough incline of angle “θ”.

Solution : The external forces on the block are :

1. Weight of the block
2. Normal force due to incline
3. Friction force
4. Spring force

The forces are drawn from the points of respective applications. Note specially that forces are not concurrent.

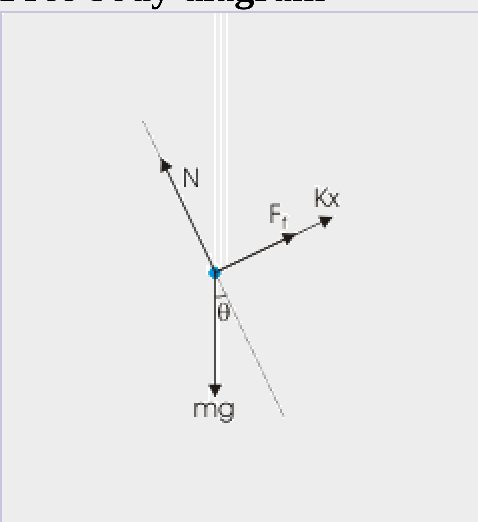
Forces on the blocks



The forces on the block are shown.

As there is no rotation involved, we consider forces to be concurrent and represent them as such with a common point.
The FBD of the block as point object is shown here :

Free body diagram



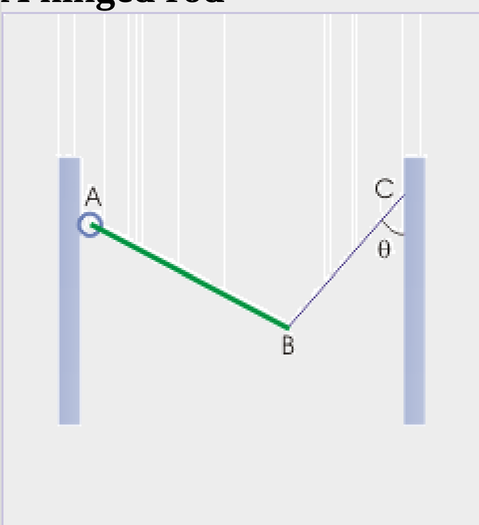
The block is shown as point with external forces on it.

We have indicated the angle that normal force makes with the direction of perpendicular to the incline. Idea here is that we should supplement FBD with as much information as is available. We have deliberately not shown the coordinate system which may be selected, keeping in mind the inputs available.

Hinged rod

Example:
Problem : A rod “AB” is hinged at “A” from a wall and is held with the help of a string as shown in the figure. Draw the free body diagram (FBD) of the rod.

A hinged rod



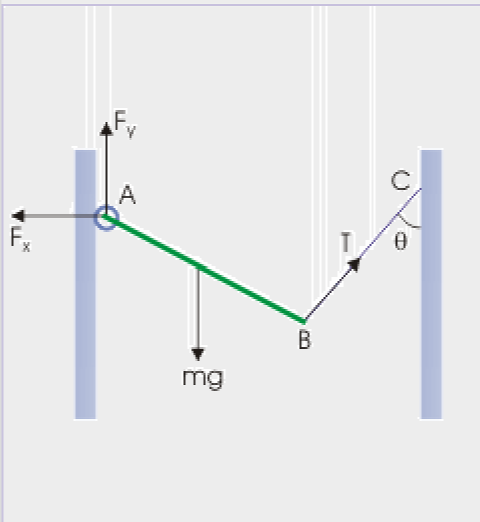
The rod is hinged from a wall and is held with the help of a string .

Solution : This example is designed to highlight the characteristics of a hinge. A hinge changes the nature of contact force at the contact between

two objects. The direction of contact forces are not predefined like in the normal case, but can assume any direction depending on the other forces acting on the body under consideration.

As a consequence, the contact force is represented by a unknown force “F”. In the case of coplanar force system, this unknown force can, in turn, be represented by a pair of components in “x” and “y” directions. The figure below shows the forces acting on the rod “AB”.

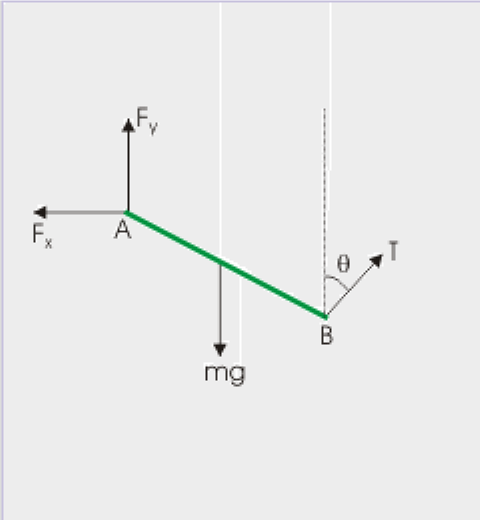
Forces on the rod



The forces on the rod are shown.

The FBD of the rod after removing other elements of the system is shown here :

Free body diagram

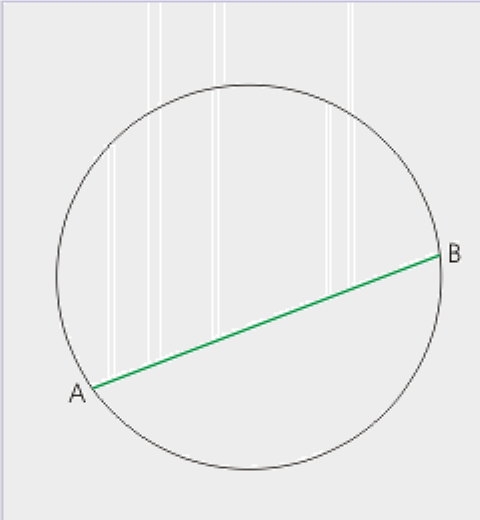


The rod is shown with external forces on it.

We should note that the FBD of the rod shows the angle that the tension force makes with the vertical. Idea here is that we should supplement FBD of the rod with as much information as is available. Also, we should note that we have not reduced the rod to a point as earlier to emphasize the lateral placements of forces on the rod. As a result, the rod may involve tendencies for both translation and rotation. If only translational is involved, we can treat rod as point with its center of mass.
We have considered horizontal and vertical directions as "x" and "y" directions for denoting unknown force components " F_x " and " F_y ".

Rod and spherical shell

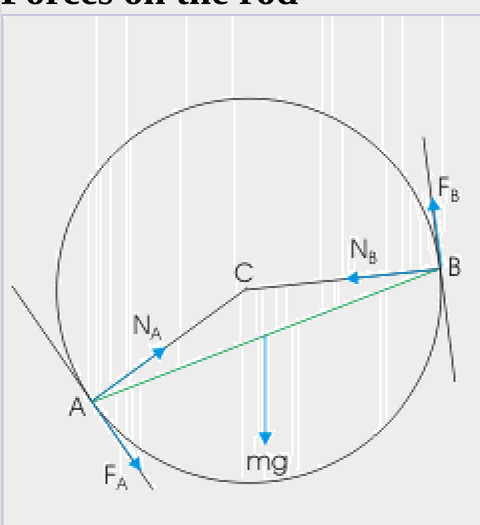
Example:
Problem : A rod AB is placed inside a spherical shell, whose inside surface is rough. Draw the free body diagram (FBD) of the rod.
A rod placed inside a spherical shell



The rod is placed inside a spherical shell, whose inside surface is rough.

Solution : We note that inside surface of the spherical shell is rough. It means that there will be friction between spherical shall and the rod. Thus, there are three forces operating on the rod : (i) weight of rod (ii) normal force between rod and spherical shell and (iii) friction force between rod and spherical shell. Since the rod is in contact at two end points, contact forces operate at both these end points.

Forces on the rod

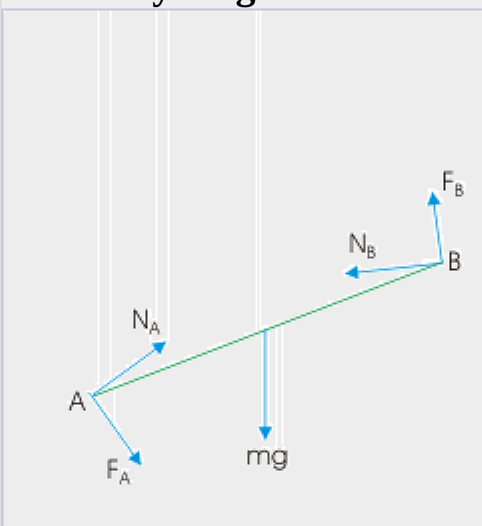


The forces on the rod are shown.

The normal forces are perpendicular to the tangents drawn at “A” and “B”. As such, normal forces at these points, when extended meet at the center of the spherical shell. The friction force at the contact surface is along the tangent drawn.

Here we see that weight is shifted laterally towards “B”. Considering rod has a downward tendency at “B”, the friction is shown in the upward direction at “B” and downward direction at “A”. The FBD of the rod after removing other elements of the system is shown here.

Free body diagram



The rod is shown with external forces on it.

We have deliberately not shown the coordinate system which may be selected, keeping in mind the inputs available. Also, we have not reduced the rod as point as the rod may undergo both translational and rotational motion. As such, lateral placements of forces along the rod are shown with FBD.

Resolution of forces (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints for solving problems

We resolve a force along the axes of a coordinate system (see Components of a vector) in following manner :

1 : Select coordinate system such that maximum numbers of forces are along the axes of chosen coordinate system.

2 : Determine "x" and "y" components of force by considering acute angle between the direction of axis and force.

3 : Use cosine of the acute angle for the component of axis with which angle is measured. Use sine of the angle for the axis perpendicular to the other axis. If "θ" be the angle that the force vector makes with x – axis, then components along "x" and "y" axes are :

$$F_x = F\cos\theta$$

$$F_y = F\sin\theta$$

4 : If the component i.e. projection of force is in the opposite direction of the reference axis, then we prefix the component with a negative sign.

Representative problems and their solutions

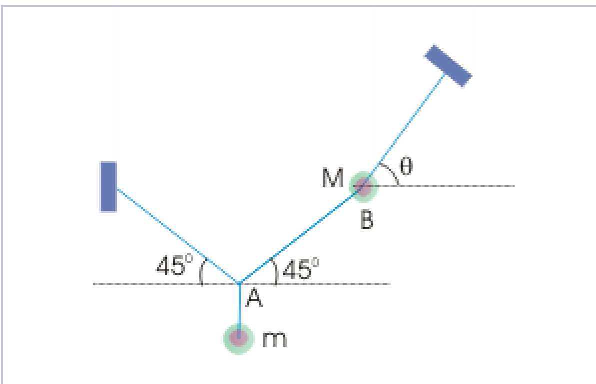
We discuss problems, which highlight certain aspects of the study leading to the analysis framework for law of motion. The questions are categorized in terms of the characterizing features of the subject matter :

- Balanced forces
- Unbalanced forces
- Acceleration due to gravity
- Forces on an incline

Balanced forces

Problem 1 : Two small spherical objects of mass “m” and “M” are attached with strings as shown in the figure. Find the angle “θ” such that the given system is in equilibrium.

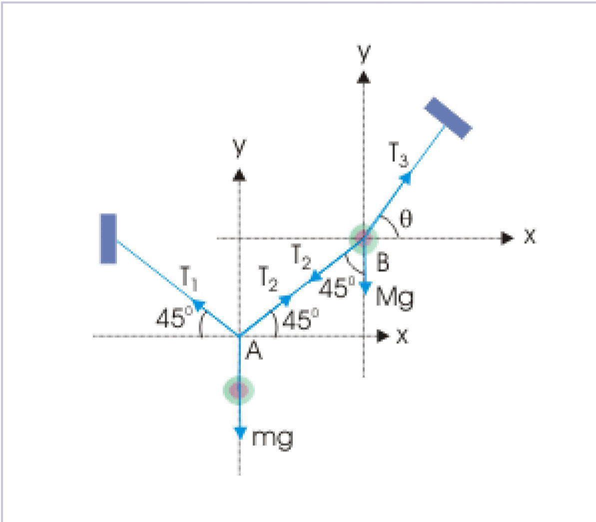
Balanced forces



Two blocks are hanging with the help of three strings.

Solution : Every point of the system is under action of balanced force system. We shall work through the points "A" and "B", where masses are attached.

Forces on the objects



Forces on the spherical objects are weights and tensions as shown.

Let " T_1 " and " T_2 " be the tensions in the two strings, meeting at point "A". The forces at "A" are shown in the figure above. Considering balancing of forces in "x" and "y" directions :

$$\sum F_x = T_1 \cos 45^0 - T_2 \cos 45^0 = 0$$

$$\Rightarrow T_1 = T_2 = T(\text{say})$$

$$\sum F_y = T_1 \sin 45^0 + T_1 \sin 45^0 = mg$$

$$\Rightarrow T \sin 45^0 + T \sin 45^0 = mg$$

$$\Rightarrow \frac{2T}{\sqrt{2}} = mg$$

$$\Rightarrow T = \frac{mg}{\sqrt{2}}$$

Let “ T_3 ” be the tension in the upper section. The forces on the mass “M” is shown in the figure above. Considering balancing of forces in "x" and "y" directions :

$$\sum F_x = T_3 \cos\theta - T \sin 45^0 = 0$$

Substituting for “T” and evaluating, we have :

$$\Rightarrow T_3 \cos\theta = \frac{mg}{\sqrt{2}} X \frac{1}{\sqrt{2}} = \frac{mg}{2}$$

$$\sum F_y \Rightarrow T_3 \sin\theta = T \cos 45^0 + Mg$$

Substituting for "T",

$$\Rightarrow T_3 \sin\theta = \frac{T}{\sqrt{2}} + Mg = \frac{mg}{\sqrt{2}} X \frac{1}{\sqrt{2}} + Mg$$

$$\Rightarrow T_3 \sin\theta = \frac{\text{mg}}{2} + Mg$$

Taking ratio of the resulting equations in the analysis of forces on "M",

$$\tan\theta = \frac{\frac{mg}{2} + Mg}{\frac{mg}{2}}$$

$$\Rightarrow \tan\theta = \left(1 + \frac{2M}{m}\right)$$

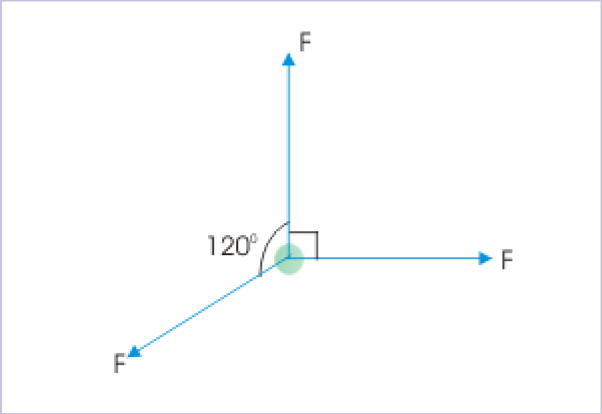
$$\Rightarrow \theta = \tan^{-1}\left(1 + \frac{2M}{m}\right)$$

Unbalanced forces

Problem 2 : A force “F” produces an acceleration “a” when applied to a body of mass “m”. Three coplanar forces of the same magnitudes are

applied on the same body simultaneously as shown in the figure. Find the acceleration of the body.

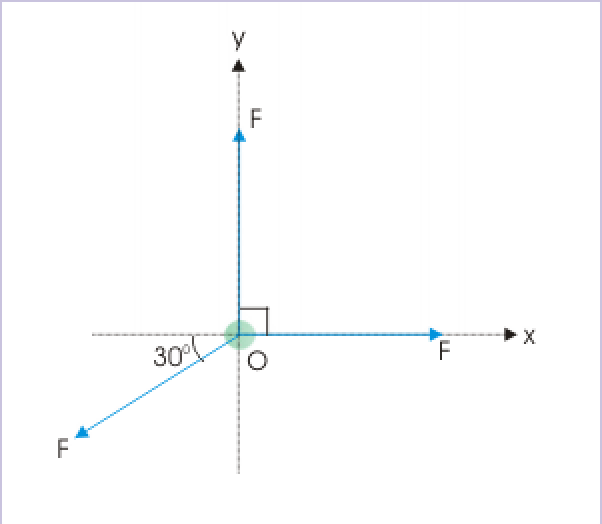
Forces on a body



Three forces act on a body as shown.

Solution : We select two axes of coordinate system so that they align with two mutually perpendicular forces as shown in the figure. We keep in mind that three forces are coplanar.

Coordinate system



Coordinate system is selected to align with force.

Taking components of forces in “x” and “y” directions,

$$\sum F_x = F - F\cos 30^0 = F\left(1 - \frac{\sqrt{3}}{2}\right) = F\left(\frac{2-\sqrt{3}}{2}\right)$$

$$\sum Fy = F - F\sin 30^0 = F\left(1 - \frac{1}{2}\right) = \frac{F}{2}$$

The net force is given by :

$$F_{net} = \sqrt{\left(F_x^2 + F_y^2\right)} = \frac{F}{2}\sqrt{\left\{1 + \left(2-\sqrt{3}\right)^2\right\}}$$

$$\Rightarrow F_{net} = \frac{F}{2}\sqrt{\left(1+4+3-4\sqrt{3}\right)}$$

$$\Rightarrow F_{\text{net}} = \frac{F}{2}\sqrt{\left(8-4\sqrt{3}\right)}$$

$$\Rightarrow \sum F_{\text{net}} = F\sqrt{\left(2-\sqrt{3}\right)}$$

But, it is given that F = ma. Substituting for "F" in the equation, we have :

$$\Rightarrow F_{\text{net}} = ma\sqrt{\left(2-\sqrt{3}\right)}$$

Let acceleration of the body under three coplanar forces be a’. Then,

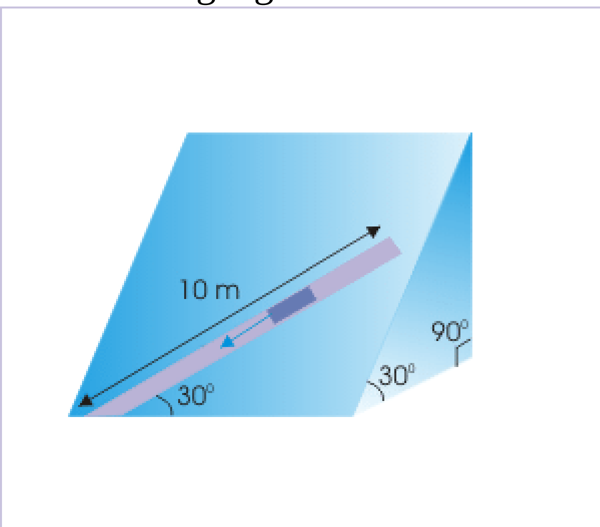
$$\Rightarrow ma\prime = ma\sqrt{\left(2-\sqrt{3}\right)}$$

$$\Rightarrow a\prime = a\sqrt{\left(2-\sqrt{3}\right)}$$

Acceleration due to gravity

Problem 3 : A small cylindrical object slides down the smooth groove of 10 m on the surface of an incline plane as shown in the figure. If the object is released from the top end of the groove, then find the time taken to travel down the length.

Motion along a groove

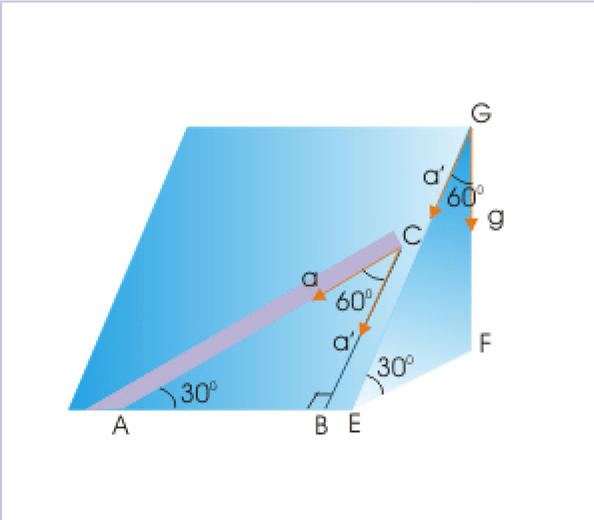


The cylindrical object slides down the smooth groove.

Solution : In order to find the time taken to travel down the incline, we need to know acceleration along the groove. The object travels under the influence of gravity. The component of acceleration due to gravity along the incline (GE), as shown in the figure below, is :

$$a\prime = g\cos 60^{0}$$

Component of acceleration due to gravity



Acceleration due to gravity is resolved along the groove in two stages.

This component of acceleration makes an angle 60° with the groove. Hence, component of acceleration along the groove (CA) is given as :

$$a = a\prime \cos 60^0$$

Combining two equations, the acceleration along the groove, "a", is :

$$a = g \cos 60^0 \cos 60^0 = gX\frac{1}{2}X\frac{1}{2} = \frac{g}{4}$$

The acceleration along the groove is constant. As such, we can apply equation of motion for constant acceleration :

$$x = ut + \frac{1}{2}at^2$$

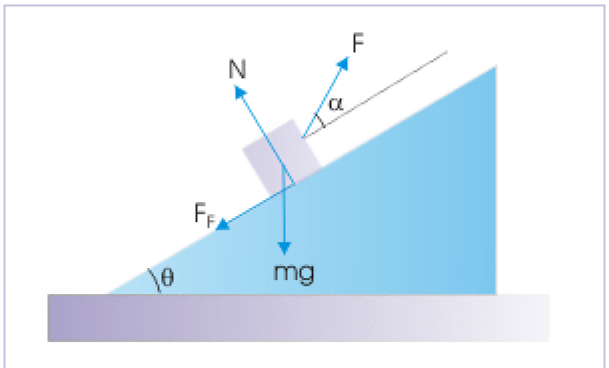
$$\Rightarrow 10 = 0 + \frac{1}{2}X\frac{g}{4}Xt^2$$

$$\Rightarrow t = \sqrt{\left(\frac{80}{g}\right)} = \sqrt{8} = 2\sqrt{2} \quad s$$

Forces on an incline

Problem 4 : Four forces act on a block of mass “m”, placed on an incline as shown in the figure. Then :

A block on an incline

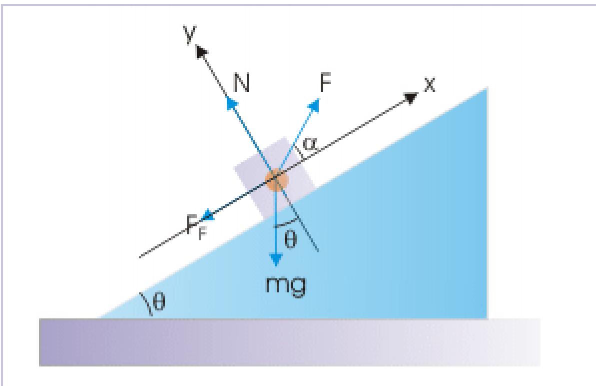


A force "F" acts as shown.

1. Resolve forces along parallel and perpendicular to incline and find net component forces in two directions.
2. Resolve forces along horizontal and vertical directions and find net component forces in two directions.

Solution : The free body diagram with four forces with coordinates are as shown in the figure below.

A block on an incline



Forces are resolved in directions parallel and perpendicular to incline.

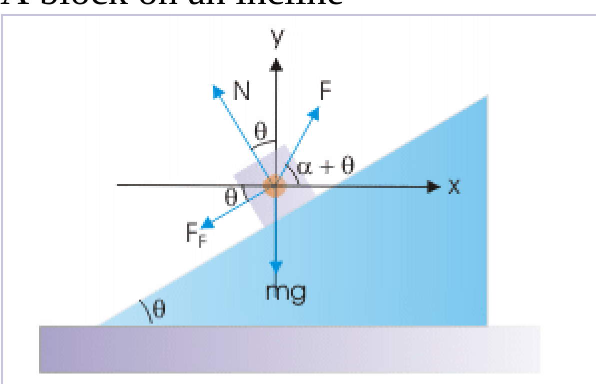
The net component of forces along two axes are :

$$\sum F_x = F\cos\alpha - F_F - mg\sin\theta$$

$$\sum Fy = N + F\sin\alpha - mg\cos\theta$$

Now, we select “x” and “y” axes along horizontal and vertical directions as shown. Note the angles that different forces make with the axes.

A block on an incline



Forces are resolved in horizontal and vertical directions.

The net component of forces along two axes are :

$$\sum Fx = F\cos(\alpha+\theta) - N\sin\theta - F_F\cos\theta$$

$$\sum F_y = N\cos\theta + F\sin(\alpha+\theta) - mg - F_F\sin\theta$$

Balanced force system
Resultant of a balanced force system is zero.

The motion under balanced force system is the subject matter of Newton's first law of motion. The condition of balanced force system simply implies that the net external force on the body is zero. This is exactly the situation for which Newton’s first law predicts that the body maintains its state of translational motion.

Mathematically, the net external force is zero for a balanced force system :

$$\sum \mathbf{F} = 0$$

This implies that linear acceleration of the point mass is zero,

$$\mathbf{a} = 0$$

For three dimensional rigid body, linear acceleration of the center of mass is zero (we shall discuss this aspect in detail subsequently in the course) ,

$$\mathbf{a}_C = 0$$

In component form,

$$\sum F_x = ma_x = 0$$

$$\sum F_y = ma_y = 0$$

$$\sum F_z = ma_z = 0$$

It is to be noted here that the condition of zero net force does not mean necessarily “rest” as may be inferred from most of the examples in the immediate surrounding. A balanced force system simply indicates that velocity of the body does not change i.e.

$$\mathbf{v} = \text{constant}$$

Equilibrium

Equilibrium denotes a state of motion, which does not change due to external force or force system. Inversely, a force system, which does not change the state of motion of a body is said to be in equilibrium.

The state of motion of a body consists of both its translational and rotational states. A balanced force system, having zero net external force, does not guarantee equilibrium. A general consideration of equilibrium requires that force system should meet two conditions :

1. The net force is zero. In other words, force system is a balanced force system.
2. The net moment of force (torque) about any axis should be equal to zero.

Mathematically,

$$\sum \mathbf{F} = 0$$

$$\sum \boldsymbol{\tau} = 0$$

Equilibrium and balanced force system

It is clear that a balanced force system is a necessary condition for equilibrium, but it is not the sufficient condition to ensure equilibrium. However, a close look, at the body systems that we generally consider in dynamics, reveals that many of the systems eventually are restricted to translational motion.

There are following three situations under which a body is restricted to have only translational motion :

1: The body is an infinitesimally small entity to be approximated as a point. In this case, forces are concurrent, meeting at a common point.

2: The body is a real three dimensional entity. The forces on the body are concurrent at a common point. The common point, in turn, coincides with "center of mass" of the body.

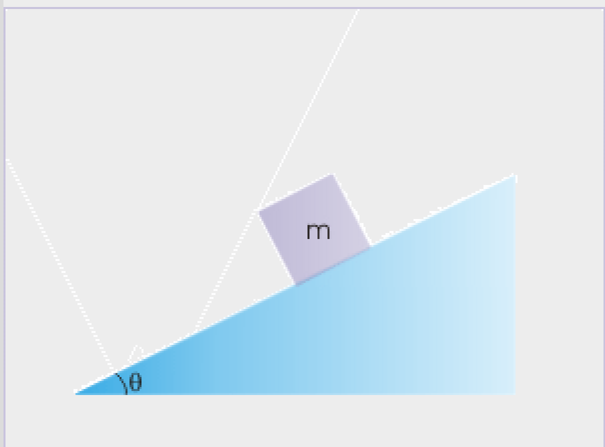
3: The body is a real three dimensional entity. The forces on the body are not concurrent at a common point. However, the body is restrained to have only translational motion. The forces are though not concurrent, but we consider them concurrent at the "center of mass" of the body, as no rotational motion is involved.

In the nutshell, if we can treat the motion as translational motion without any angular or rotational motion involved, then a balanced force system with zero net force alone is sufficient to meet the requirement of equilibrium of the body.

Example:

Problem : A block of mass “m” is at rest on a rough incline of angle “θ”. Find the contact forces and the net contact force on the block.

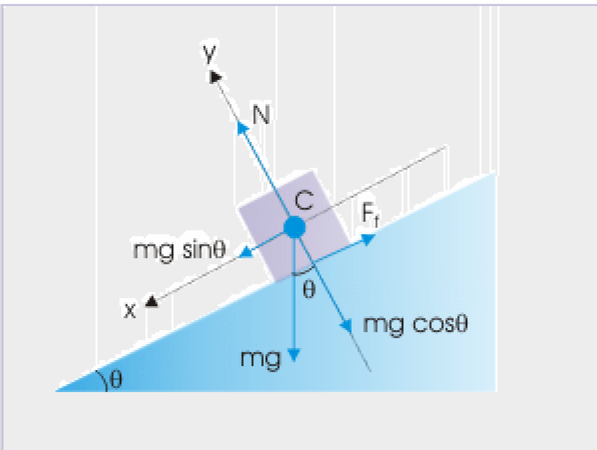
A block on an incline



The block rests on the incline.

Solution : The forces on the block are (i) its weight (ii) normal force and (iii) Friction force. These forces are not concurrent (see figure).

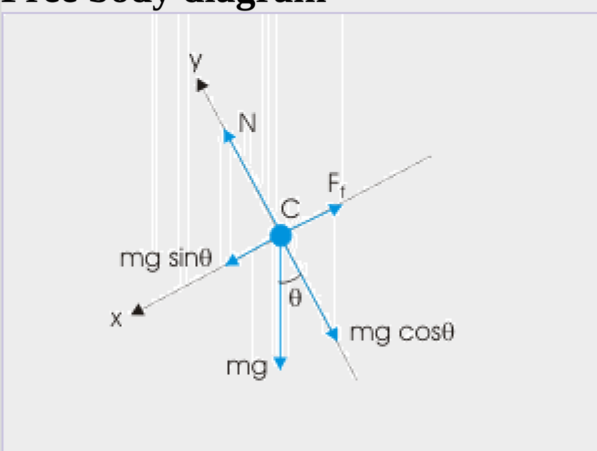
Forces on the block



Forces are not concurrent.

However, no turning effect is involved. We can, therefore, treat force system concurrent with the "center of mass" of the block. In order to analyze the forces, we consider a coordinate system as shown in the figure. Since the block is at rest, the forces on the block are balanced :

Free body diagram



Forces are shown with block as a point with concurrent forces .

$$\sum F_x = mg\sin\theta - F_f = 0$$

$$\sum F_y = N - mg\cos\theta = 0$$

There are two contact forces (i) normal force and (ii) friction. The friction is given by the first equation :

$$F_f = mg\sin\theta$$

The normal force is given by the second equation :

$$N = mg\cos\theta$$

The net contact force is vector sum of two contact forces, which are at right angle to each other. Hence, magnitude of net contact force is :

$$F_C = \sqrt{\left(N^2 + F_f^2\right)}$$

$$F_C = \sqrt{\left\{(mg)^2\left(\cos^2\theta + \sin^2\theta\right)\right\}}$$

$$\Rightarrow F_C = mg$$

Coplanar force system

Though it has not been explicitly mentioned at any point in our study, but a scrutiny of the illustrations so far discussed indicates that most of the situations involve coplanar force systems. This simplifies force analysis greatly as we need to analyze force in two directions only :

$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

If the force system is a balanced force system, then :

$$\sum F_x = ma_x = 0$$

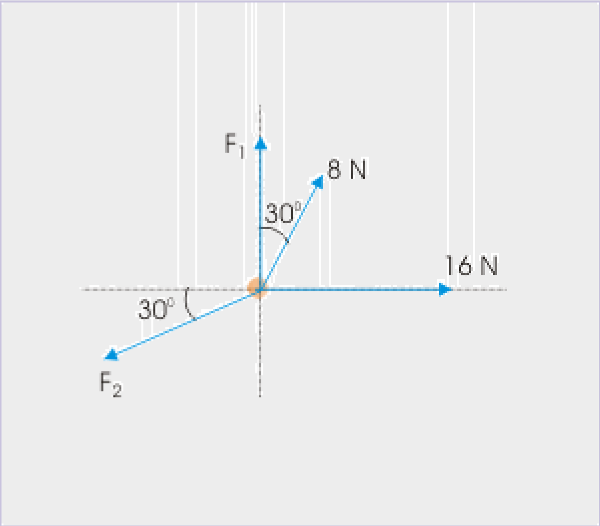
$$\sum F_y = ma_y = 0$$

In case only translation is involved, then above two equations fulfills the requirement of the equilibrium as well.

Example:

Problem : : A body is subjected to four coplanar concurrent forces as shown in the figure. The body is under equilibrium. Find the forces " F_1 " and " F_2 ".

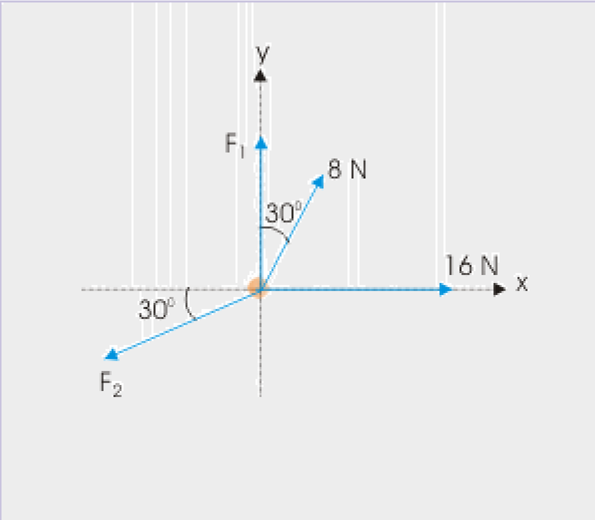
Four forces on a body



The forces are concurrent.

Solution : The body is in equilibrium under the action of coplanar concurrent forces. It means that forces form a balanced force system. We can treat the body as a point on the free body diagram. Let "x" and "y" be the axes of the coplanar coordinate system as shown in the figure. Then,

Free body diagram



The forces are concurrent.

$$\sum F_x = 16 + 8\sin 30^0 - F_2 \cos 30^0 = 0$$

$$\sum F_y = F_1 + 8\cos 30^0 - F_2 \sin 30^0 = 0$$

We can find " F_2 " from the first equation,

$$\Rightarrow F_2 X \frac{\sqrt{3}}{2} = 16 + 8X\frac{1}{2} = 20$$

$$\Rightarrow F_2 = 20X\frac{2}{\sqrt{3}} = \frac{40}{\sqrt{3}} \quad N$$

Substituting in second equation and solving for " F_1 ", we have :

$$\Rightarrow F_1 = F_2 \sin 30^0 - 8\cos 30^0 = \frac{40}{\sqrt{3}} X\frac{1}{2} - 8X\frac{\sqrt{3}}{2}$$

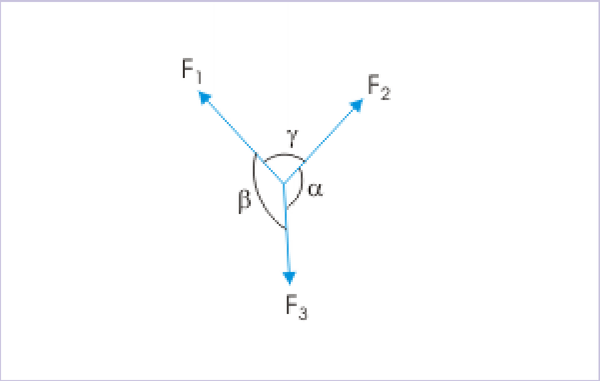
$$\Rightarrow F_1 = \frac{20}{\sqrt{3}} - 4X\sqrt{3}$$

$$\Rightarrow F_1 = \frac{20-12}{\sqrt{3}} = \frac{8}{\sqrt{3}} \quad N$$

Lami’s theorem

We have studied Lami’s theorem for three concurrent vectors (non-collinear vectors). If three concurrent forces are balanced, then

Lami's theorem



Three concurrent balanced forces.

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

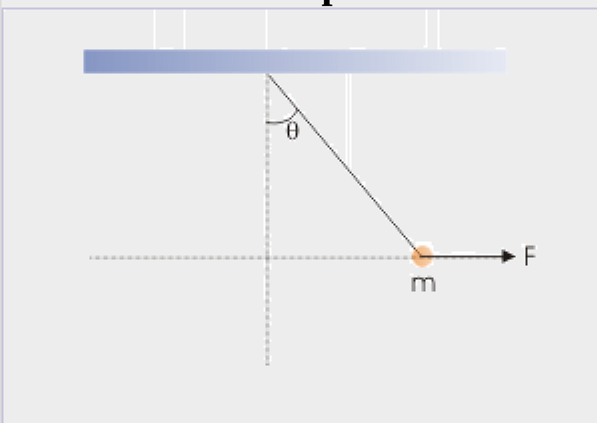
where α,β,γ are the angles between remaining two force not involved in the particular ratio. We can refer to the figure to understand the relation between forces and the angle involved. As net force is zero, forces represent three sides of a closed triangle. Clearly, three balanced concurrent forces are coplanar.

This theorem helps us to analyze balanced force system, comprising of exactly three concurrent coplanar forces.

Example:

Problem : A spherical mass “m” hanging from ceiling is displaced by applying a horizontal force,F. The string makes an angle “θ” with vertical as shown in the figure. Find force F.

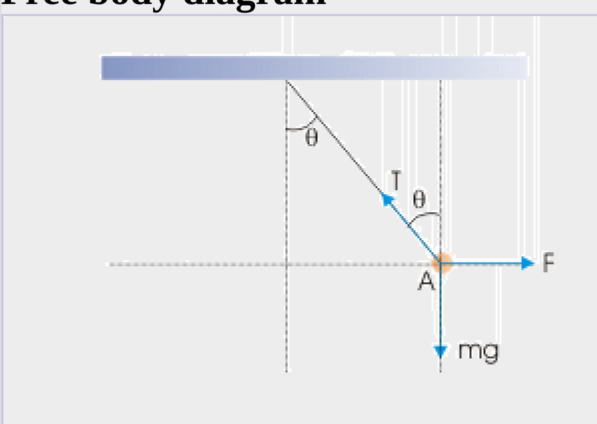
Three forces on a spherical mass



The forces are concurrent.

Solution : Here, we shall employ Lami’s theorem as there are exactly three forces acting on the spherical mass. The FBD of the spherical mass is as shown in the figure.

Free body diagram



The forces are concurrent.

Applying Lami’s theorem, we have :

$$\frac{mg}{\sin\left(90^0+\theta\right)}=\frac{T}{\sin 90^0}=\frac{F}{\sin\left(180^0-\theta\right)}$$

$$\Rightarrow \frac{mg}{\cos\theta}=\frac{T}{1}=\frac{F}{\sin\theta}$$

From the last two ratios, we have :

$$\Rightarrow F=T\sin\theta$$

From the first two ratios, we have :

$$\Rightarrow T=\frac{mg}{\cos\theta}$$

Combining two equations,

$$\Rightarrow F=\frac{mg\sin\theta}{\cos\theta}=mg\tan\theta$$

Lami's theroem provides an alternative to normal component analysis - may be with some computational advantage. Nevertheless, we should keep in mind that it is just an alternative method. We could have solved this problem, using component analysis as :

$$\sum F_x=F-T\sin\theta=0$$
$$\Rightarrow F=T\sin\theta$$

and

$$\sum F_y=\mathrm{mg}-T\cos\theta=0$$
$$\Rightarrow \mathrm{mg}=T\cos\theta$$

Taking ratio to eliminate “T”,

$$\Rightarrow \frac{F}{\mathrm{mg}}=\tan\theta$$
$$\Rightarrow F=\mathrm{mg}\tan\theta$$

Balanced force system (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

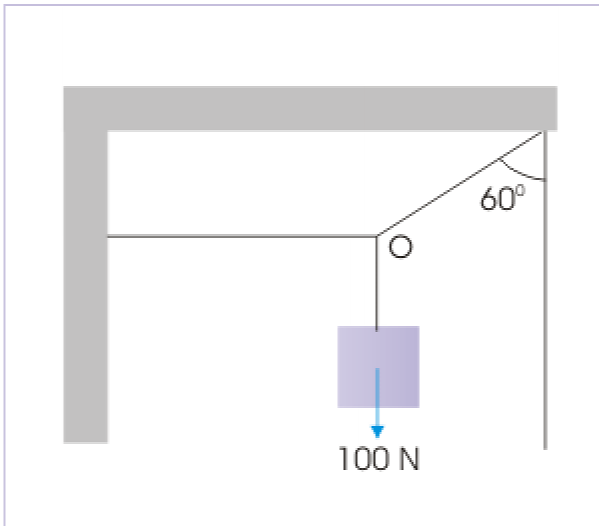
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the balanced force system. The questions are categorized in terms of the characterizing features of the subject matter :

- String and block system
- Incline and block system
- Pulley, string and block system
- Two blocks system
- Pulley, string and multiple blocks system

String and block system

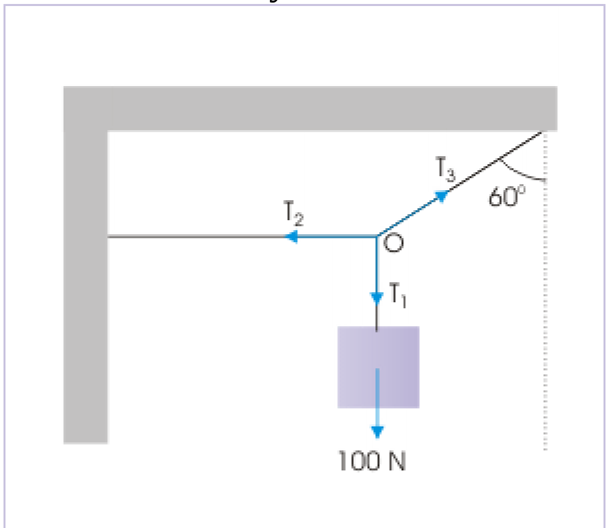
Problem 1 : A block weighing 100 N is suspended with the help of three strings as shown in the figure. Find the tension in each of the strings.
Balanced force system



Solution : This example illustrates one important aspect of force diagram. We can even draw force diagram of a point on the system like “O”, where three strings meet. The point does not represent a body, but force diagram is valid so long we display the forces acting through the point, O.

Let T_1 , T_2 and T_3 be the tensions in the string as shown in the figure here.

Balanced force system

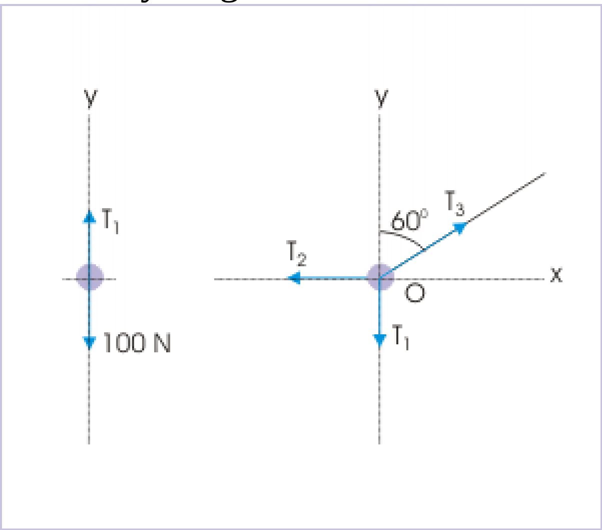


A preliminary assessment of forces suggests that analysis of forces on block will provide value for the unknown, T_1. Hence, we first analyze force on the block.

Free body diagram of block

$$T_1 = 100\ N$$

Free body diagram



Free body diagram of “O”

The external forces at point “O” are (i) Tension, T_1 (ii) Tension, T_2 and (iii) Tension, T_3

$$\sum F_x = T_3 \sin 60^0 - T_2 = 0$$
$$\Rightarrow T_2 = T_3 \sin 60^0$$

and

$$\sum F_y = T_3 \cos 60^0 - T_1 = 0$$
$$\Rightarrow T_3 = \frac{T_1}{\cos 60^0} = 200\ N$$

Putting this in the equation for T_2 , we have :

$$\Rightarrow T_2 = T_3 \sin 60^0 = 200 \text{ x } \frac{\sqrt{3}}{2} = 100\sqrt{3}\ N$$

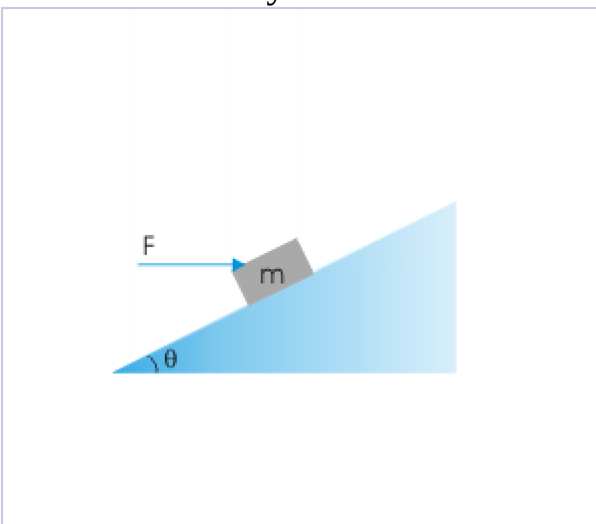
We should note that direction of tension " T_1 " acts up with respect to the body, whereas " T_1 " acts down with respect to point "O". We need not be overly concerned and just try to figure out, what a taut string does to the body or point in consideration. The tension pulls down the point "O" and

pulls up the body. For this reason, it has different directions with respect to them.

Incline and block system

Problem 3 : Find the force, F, required to keep the block stationary on an incline of angle "θ" having friction-less surface as shown in the figure.

Balanced force system

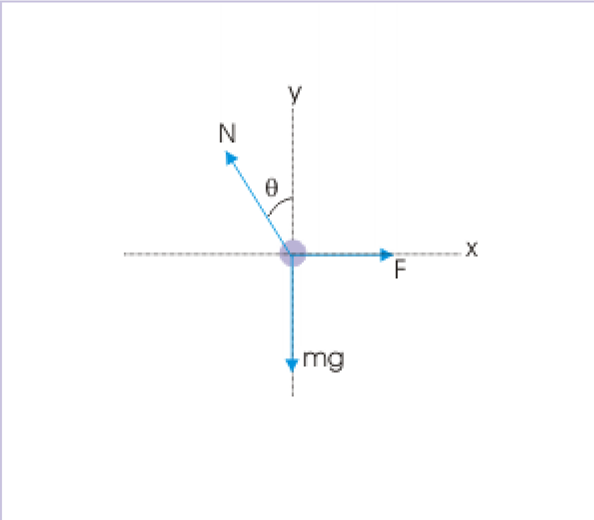


Solution : We can either have (a) axes in horizontal and vertical directions or (b) parallel to incline and perpendicular to it. Which of the two is better suited here ? In this case, one force (mg) is along vertical direction, whereas other external force (F) is along horizontal direction. As such, it is advantageous to have a horizontal and vertical axes as two of three forces are along the coordinate axes.

Free body diagram of the block

The external forces on the block are (i) Force, F (ii) Weight, mg and (iii) Normal force, N.

Free body diagram



$$\sum F_x = F - N\sin\theta = 0$$
$$\Rightarrow F = N\sin\theta$$

and

$$\sum F_y = N\cos\theta - \text{mg} = 0$$
$$\Rightarrow \text{mg} = N\cos\theta$$

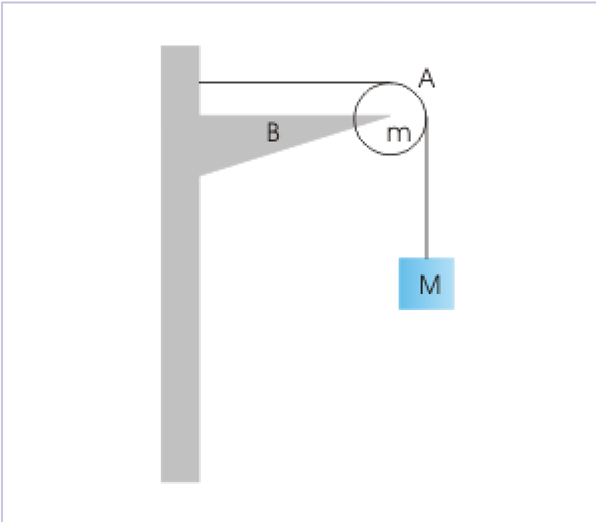
Taking ratio, we have :

$$\Rightarrow \frac{F}{\text{mg}} = \tan\theta$$
$$\Rightarrow F = \text{mg}\tan\theta$$

Pulley, string and block system

Problem 4 : A string going over a pulley "A" of mass "m" supports a mass "M" as shown in the figure. Find the magnitude of force exerted by the clamp "B" on pulley "A".

Balanced force system

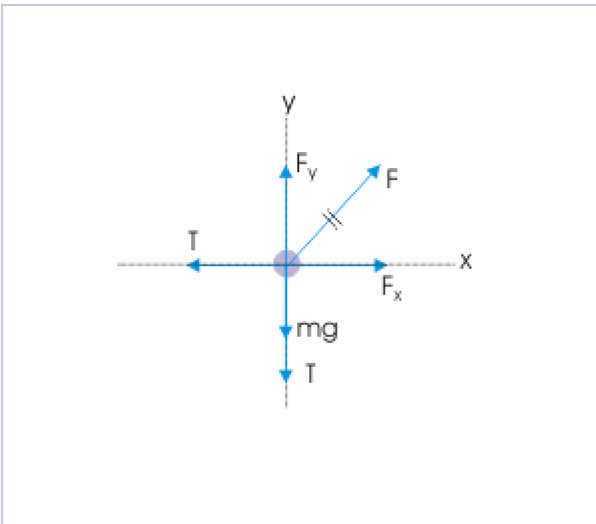


Solution : Here, we consider pulley as the body system. Let us also consider that clamp “B” exerts a force “F” in an arbitrary direction, making an angle with the horizontal. We should note that pulley, unless otherwise specified, is considered to be of negligible mass and friction-less. In this case, however, pulley has finite mass “m” and its weight should be considered to be an external force on the pulley.

Free body diagram of pulley

The string is single piece and mass-less, whereas pulley is not mass-less. However, pulley is static. As such, there is no torque involved. Hence, tension in the string all through out is same. From the consideration of block, we see that tension in the string is equal to the weight of the block i.e. Mg.

Now, the external forces on pulley are (i) Horizontal tension "T" (ii) Weight, mg, of the pulley (iii) Vertical Tension, T, and (iv) force, F applied by clamp “B”.
Free body diagram



$$\sum F_x = F_x - T = F_x - \text{Mg} = 0$$
$$\Rightarrow F_x = \text{Mg}$$

and

$$\sum F_y = F_y - \text{mg} - T = F_y - \text{mg} - \text{Mg} = 0$$
$$\Rightarrow F_y = (M+m)g$$

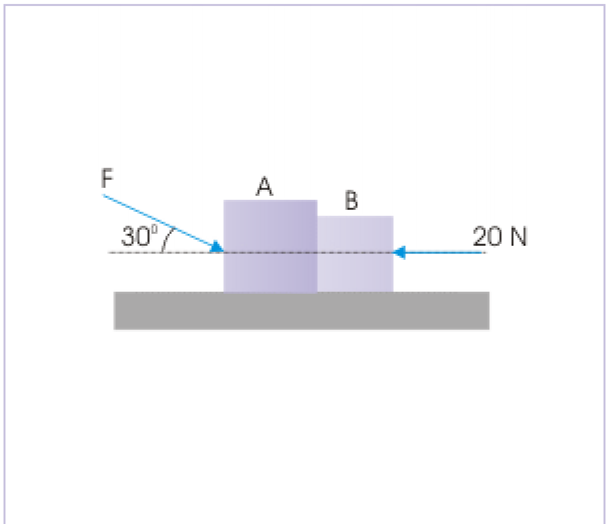
The force exerted by the clamp,F, is :

$$F = \sqrt{({F_x}^2 + {F_y}^2)} = \sqrt{\left\{(\text{Mg})^2 + (M+m)^2 g^2\right\}}$$
$$\Rightarrow F = g\sqrt{\left\{M^2 + (M+m)^2\right\}}$$

Two blocks system

Problem 5 : Two blocks "A" and "B", weighing 20 N and 10 N respectively are in contact with each other. If the blocks are at rest, then find the force "F" and the normal reactions between all contact surfaces.

Balanced force system



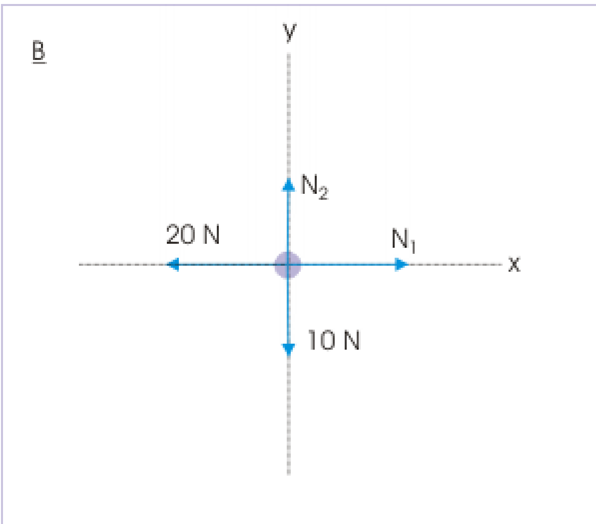
Solution : The question demands that we draw free body diagram of each of the block separately as we are required to know normal reactions at all surfaces. Here, there are three contact surfaces between (i) A and horizontal surface (ii) B and horizontal surface and (iii) A and B.

A preliminary assessment of forces on the blocks suggests that analysis of forces on B will provide values of unknown force(s). It is so because the forces on B are mutually perpendicular (thus, they would not need to be resolved), if appropriate coordinate system is chosen. Hence, we first analyze force on block B.

Free body diagram of B

The external forces are (i) weight of B = 10 N (ii) Normal force applied by A i.e. N_1 (say) (iii) Normal force applied by surface i.e. N_2 and (iv) external force of 20 N

Free body diagram



$$\sum F_x = N_1 - 20 = 0$$
$$\Rightarrow N_1 = 20\ N$$

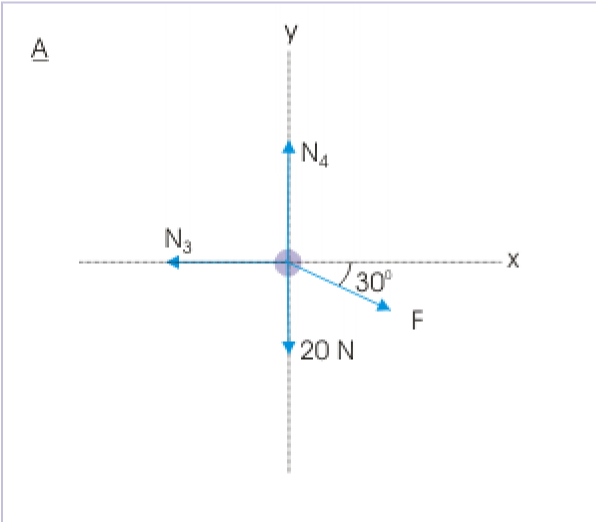
and

$$\sum F_y = N_2 - 10 = 0$$
$$\Rightarrow N_2 = 10\ N$$

Free body diagram of B

The external forces are (i) weight of A, 20 N (ii) Normal force applied by B i.e. N_3 (say) = N_1 = 20 N (iii) Normal force applied by surface i.e. N_4 and (iv) force, F = ?

Free body diagram



$$\sum F_x = F\cos 30^0 - N_3 = F\cos 30^0 - N_1 = F\cos 30^0 - 20 = 0$$
$$\Rightarrow 20 = F\cos 30^0$$
$$\Rightarrow F = \frac{20}{\cos 30^0} = \frac{20}{\frac{\sqrt{3}}{2}} = \frac{40}{\sqrt{3}}\ N$$

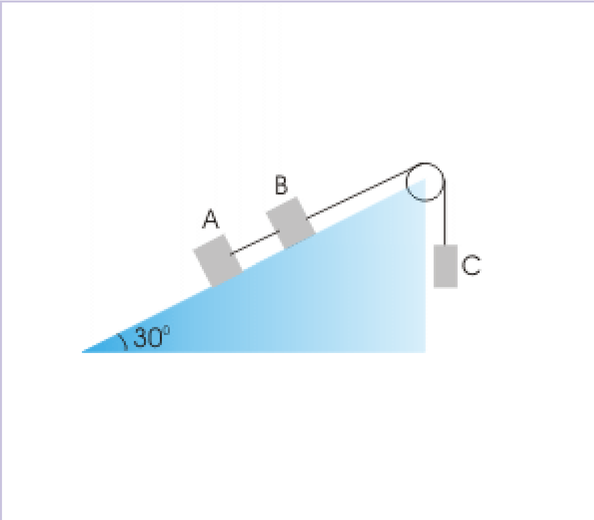
and

$$\sum F_y = N_4 - F\sin 30^0 - 20 = 0$$
$$\Rightarrow N_4 = F\sin 30^0 + 20 = \frac{40}{\sqrt{3}}\ \text{X}\ \frac{1}{2} + 20 = \frac{20}{\sqrt{3}} + 20 = 20\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)\ N$$

Pulley, string and multiple blocks system

Problem 6 : The blocks A and B weighing 10 N and 20 N are connected by a string. The block B, in turn, is connected to block C with another string passing over a pulley. Friction forces at all interfaces is negligible. If the block system is in equilibrium, find the weight of C and tensions in the two strings.

Balanced force system



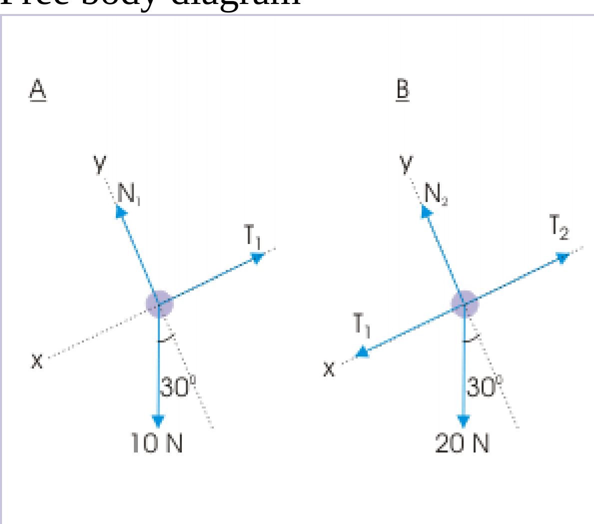
Solution : There are two strings. Hence, the tensions in the strings will be different. Let T_1 and T_2 be the tension in strings AB and BC respectively.

Looking at the various body systems, we guess that the simplest force system is the one associated with block C. However, force analysis of block C will not yield anything as we do not know its weight or the tension T_2 .

Thus, we begin with block A.

Free body diagram of A

Free body diagram



The external forces are (i) weight of A = 10 N (ii) Normal force applied by incline i.e. N_1 (say) (iii) tension in AB, T_1 .

$$\sum F_x = 10 \sin 30^0 - T_1 = 0$$
$$\Rightarrow T_1 = 10 \sin 30^0 = 5\ N$$

We need not analyze forces in y – direction as we are not required to determine normal force N_1 and it is not expected to be used for analyzing force on block B.

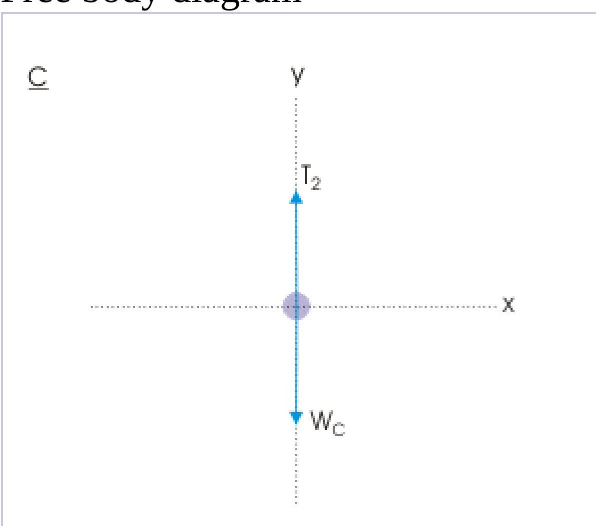
Free body diagram of B

The external forces are (i) weight of B, W_C = 20 N (ii) Normal force applied by incline i.e. N_2 (say) (iii) tension in AB, T_1 and (iv) tension in BC, T_2 .

$$\sum F_x = 20 \sin 30^0 + T_1 - T_2 = 0$$
$$\Rightarrow T_2 = 20 \text{ x } \frac{1}{2} + 5 = 15\ N$$

Free body diagram of C

Free body diagram



The external forces are (i) weight of c = ? and (ii) tension in BC, T_2 .

$$\sum F_y = T_2 - W_C = 0$$
$$\Rightarrow W_C = T_2 = 15\ N$$

Acknowledgment

Author wishes to thank Scott Kravitz, Programming Assistant, Connexions for making suggestion to remove error in the module.

Unbalanced force system

We are now appropriately equipped to study motion under a system of forces, which is not a balanced force system. The resulting motion, in this situation, is an accelerated motion. The approach and methodology for studying motion with unbalanced force system are same as employed for the study of balanced force system.

There are large numbers of natural motions that fall under this category. One such motion involves motional mechanism that allows animals to move around – surprisingly using internal muscular force. We shall find out that this is not an exception to the laws of motion, rather an intelligent endeavor on the part of living beings that distinguish them from non-living counterparts.

There are many more such real time examples. A large numbers of real time examples of accelerated motion belong to the category of circular motion. We shall, however, save these circular motion cases for the study at a later stage.

Basic framework

The laws of motion are again the basic frame work for the study of body system under unbalanced force system. The treatment of the laws differs in only one respect to the one used for balanced force system. The net force, now, does not equate to zero, but to the product of mass and acceleration.

$\sum \mathbf{F} = m\mathbf{a}$

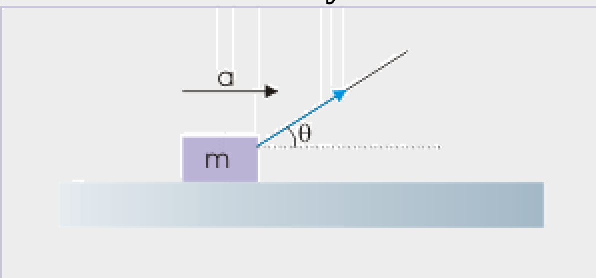
In component form,

$$\sum F_x = ma_x; \;\; \sum F_y == ma_y; \sum F_z == ma_z$$

Example:
Problem : A block of mass “m” is pulled by a string on a smooth horizontal surface with acceleration “a”. The string maintains an angle “θ”

with horizontal. Find (i) tension in the string (ii) force applied by the surface on the block and (iii) force applied by the block on the surface.

Unbalanced force system

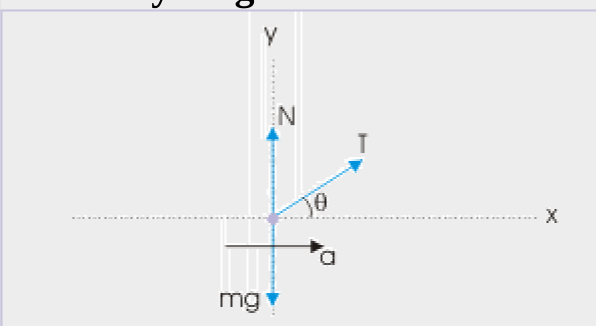


Solution : Here, we consider block as the body system.

Free body diagram of the block

The external forces on the block are (i) Tension in the string,T, (ii) weight of the block, mg, and (ii) Normal force applied by the surface,N.

Free body diagram



The block moves in x-direction with acceleration “a’. On the other hand, there is no motion involved in vertical i.e. y-direction as implied by the question. The forces in y-direction, therefore, constitute a balanced force system. Now,

$$\sum F_x = T\cos\theta = ma_x$$
$$\Rightarrow T\cos\theta = ma$$
$$\Rightarrow T = \frac{ma}{\cos\theta}$$

and

$$\sum F_y = N + T\sin\theta - mg = 0$$
$$\Rightarrow N = mg - T\sin\theta$$

Putting values of tension, T, in the above equation,

$$\Rightarrow N = mg - \frac{ma\sin\theta}{\cos\theta}$$
$$\Rightarrow N = mg - ma\tan\theta$$

According to Newton's third law, the surface is acted upon by the same magnitude of force that the surface applies on the block. Thus, force on surface is also N, acting downward.

Note in the example above that vertical component of tension has reduced the normal force from its magnitude, when external force (T) is not applied. This highlights the important aspect of contact forces that they need not be equal to the weight of the body in contact. Equality of contact forces, however, is always maintained in accordance with Newton's third law. This means that surface is also acted by a reduced normal force.

Composite mass system

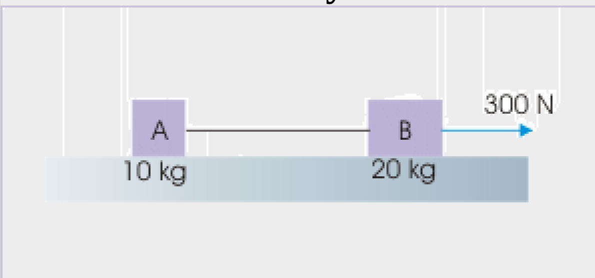
A composite body system can be selected with elements having same acceleration. We must understand here that when we consider a composite body system then all internal forces should be neglected from our consideration. The external force causes a common acceleration to the composite body system. Clearly, this technique of combining bodies into a single composite body system is possible only when accelerations of the constituents are same.

Let us work out a problem here to illustrate this point about the composite mass system.

Example:
Problem : Two blocks A and B of masses 10 kg and 20 kg respectively, connected by a string, are placed on a smooth surface. The blocks are pulled by a horizontal force of 300 N as shown in the figure. Find (i) acceleration of the blocks (ii) tension in the string and determine (iii)

whether magnitudes of acceleration and tension change when force is applied on other mass? Consider, $g = 10\, m/s^2$.

Unbalanced force system

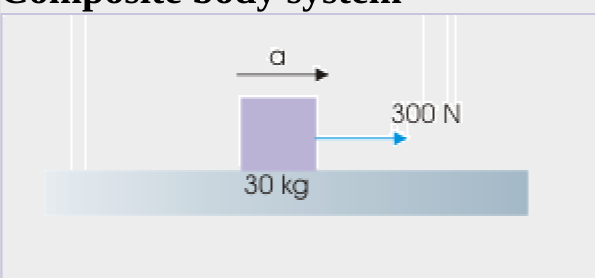


Solution : Let us first investigate acceleration and tension for the given configuration in the figure above. Here, we see that two masses have same acceleration. We can use this opportunity to apply the concept of composite body system (comprising of both blocks). We neglect tension in the string as it constitutes internal force for the whole composite body system.

Let "a" be the acceleration in the direction of force. The mass of the composite body system, m_c , is :

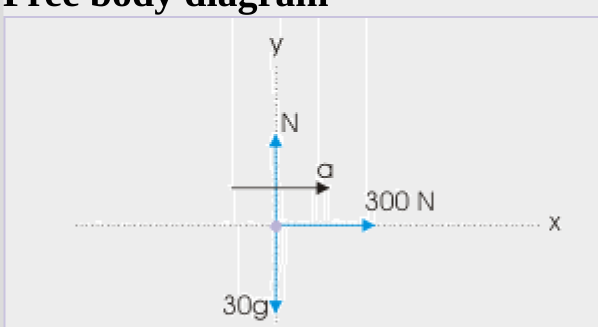
$$m_c = 10 + 20 = 30 \text{ kg}$$

Composite body system



Free body diagram of the composite body system

Free body diagram



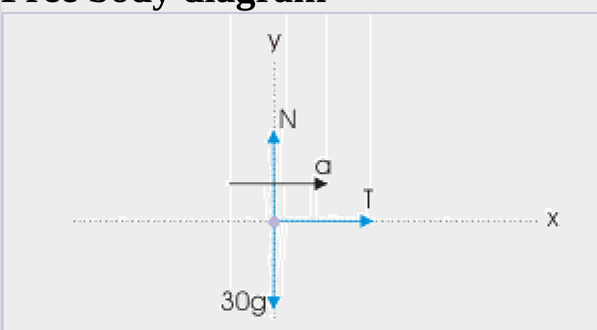
$$\sum F_x = F = ma_x$$
$$\Rightarrow 300 = 30a$$
$$\Rightarrow a = 10\, m/s^2$$

Note that this force equation does not depend on whether we apply force on A or B. In either case, the force equation remains same – only direction of acceleration changes with the change in the direction of applied force. It means that magnitude of acceleration of the two body system will remain same in either case.

In order to find the tension in the string, however, we need to consider individual block. Here, we consider block "A" for the simple reason that it is acted upon by a single force (T) in x-direction and analysis of force on block "A" will be simpler than that of block "B", which is acted by two forces (T and 300 N force).

Free body diagram of the body A

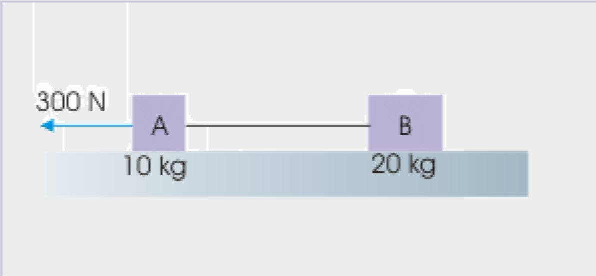
Free body diagram



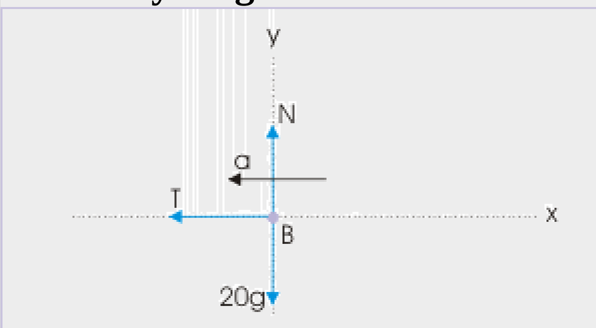
$$\sum F_x = T = ma_x$$
$$\Rightarrow T = 10a$$
$$\Rightarrow T = 10 \text{ x } 10 = 100\ N$$

Now, we answer the third part of the question : whether magnitudes of acceleration and tension change when force is applied on other mass? Let us consider the case when force is applied on body “A” as shown in the figure.

Free body diagram



Now, we consider force and acceleration of block "B" as it is now acted by only one force i.e. tension in the string in x-direction.
Free body diagram of the body B
Free body diagram



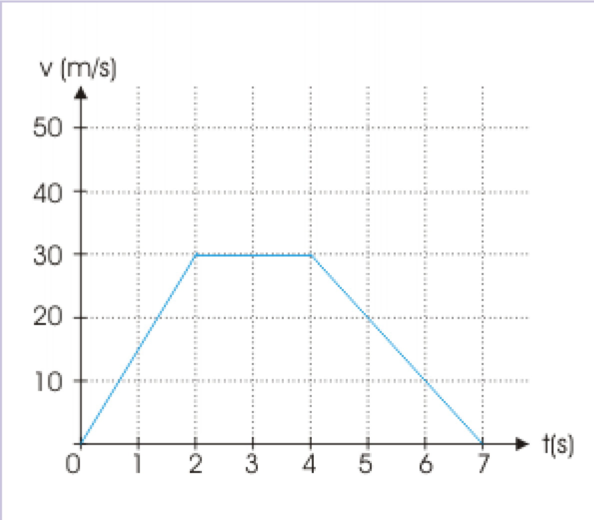
$$\sum F_x = T = ma_x$$
$$\Rightarrow T = 20a$$
$$\Rightarrow T = 20 \text{ x } 10 = 200\ N$$

Thus, we see that acceleration (a) is independent, but tension (T) in the string is dependent on the point of application of the external force.

Time dependent force

The force as applied to a body may change with time. The change may occur in any combination of magnitude and direction of the force. The resulting acceleration will accordingly vary with time as given by force equation. In turn, the rate of change of velocity will change.

In a simplified scheme of thing, the velocity profile of a motion due to a particular variable force may look like the one shown here.
Velocity – time plot

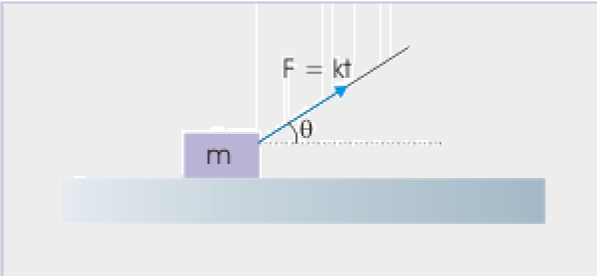


The velocity in first two seconds is increasing at constant rate, meaning that the body is under constant acceleration. The body is, thus, acted upon by a constant net force during this period. Subsequently, velocity is constant for time between 2s and 4s. Acceleration and hence force on the body , therefore, is zero in this interval. Finally, body is brought to rest with a constant force acting in opposite direction to that of velocity.

As a matter of fact, this velocity profile is typical of any vehicle, which is first accelerated, then run with minimum acceleration or even zero acceleration and finally brought to rest with constant deceleration.

We work out an example for studying motion under variable force, which is similar to the earlier example except that force is now dependent on time.

Example:
Problem : A block of mass "m" is pulled by a string on a smooth horizontal surface with force F = kt, where "k" is a constant. The string maintains an angle "θ" with horizontal. Find the time when block breaks off from the surface.
Unbalanced force system

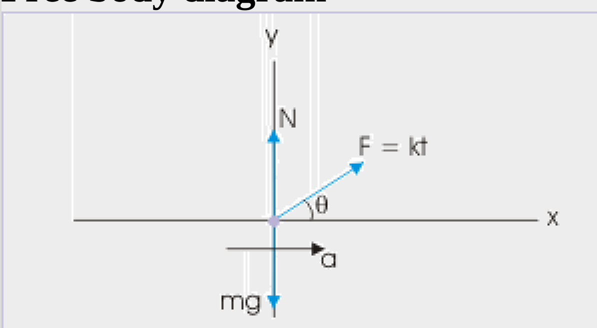


Solution : Here, important thing is to know the meaning of "breaking off". It means that physical contact between two surfaces is broken off. In that condition, the normal force should disappear as there is no contact between block and surface.

Since normal force is directed vertically up, we need to analyze forces in y-direction only so that we could apply the condition corresponding to "breaking off".

Free body diagram of the block

Free body diagram



$$\sum F_y = N + kt\sin\theta - mg = 0$$
$$\Rightarrow N = mg - kt\sin\theta$$

Now, let $t = t_B$ (break off time) when N = 0 (breaking off condition)

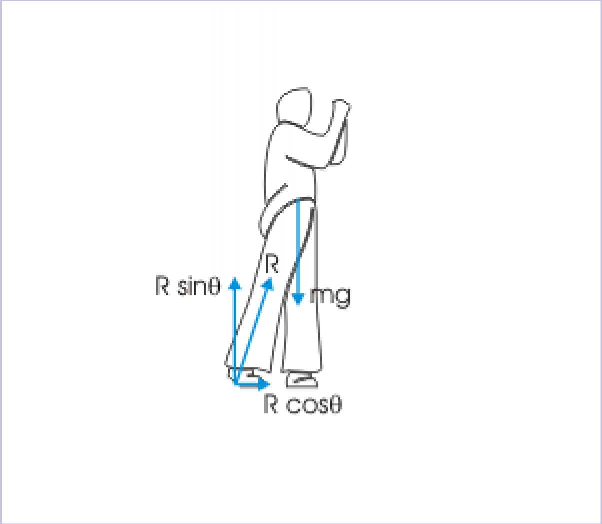
$$\Rightarrow 0 = mg - kt_B\sin\theta$$
$$\Rightarrow kt_B\sin\theta = mg$$
$$\Rightarrow t_B = \frac{mg}{k\sin\theta}$$

Motional mechanism of animals

According to laws of motion, motion of a body is not possible by the application of internal force. On the other hand, animals move around using internal muscular force. This is not a contradiction, but an intelligent maneuvering on the part of animals, which use internal muscular force to generate external force on them.

Let us take the case of our own movement. So long we stand upright, applying weight on the ground in vertical direction; there is no motion.

Motion of a human being



To move forward (say), we need to press back the surface at an angle. The ground applies an equal and opposite force (say, reaction of the ground). The reaction of the surface is an external force for our body. The horizontal component of the reaction force moves our body forward, whereas the vertical force balances our weight.

Elements of body system

The underlying frame work of the analysis of force systems in inertial frame of reference is now almost complete. There is nothing new as far as application of laws of motion is concerned. But, there is a big "but" with respect to details of various elements of body systems that we consider during study of motion. These elements typically are block, string, incline, pulleys and spring.

The whole gamut of analysis in dynamics requires systemic approach to answer following questions :

1. what are the forces ?
2. which of them are external forces ?
3. does friction is part of the external force system ?
4. whether forces are collinear, coplanar or three-dimensional ?
5. are forces balanced or unbalanced ?
6. are the forces time dependent ?
7. is the motion taking place in inertial frame or accelerated frame ?
8. what would be the appropriate coordinate system for analysis ?
9. what are the characteristics of system elements ?

It is not very difficult to realize that our job is half done if we are able to classify the system in hand based on the answers to above questions. Though, we have listed the system elements at the end of the list, we shall soon realize that a great deal of our effort in getting answers to questions 1, 2, 4 and 8 are largely determined by the elements involved, while the rest are situation specific.

We have briefly described system elements like block, string, pulley etc. In subsequent modules, we shall emphasize details of these and other elements.

Unbalanced force system (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to unbalanced force system. The questions are categorized in terms of the characterizing features of the subject matter :

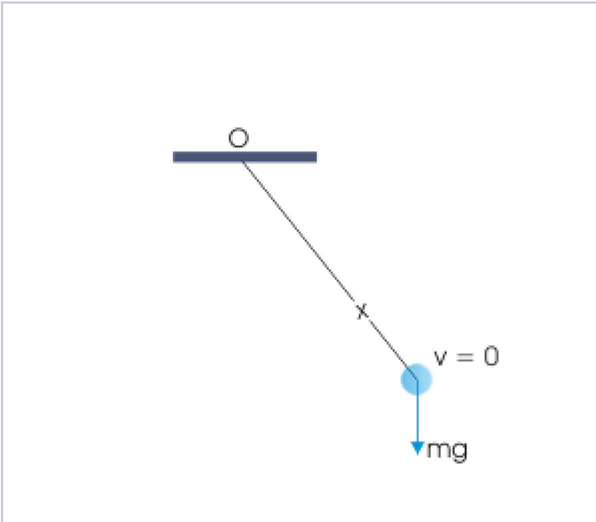
- Nature of force
- Motion under gravity
- Combined motion
- Constrained motion

Nature of force

Problem 1 : A pendulum bob oscillates between extreme positions in a vertical plane. The string connected to the bob is cut, when it is at one of the extreme points. What would be the trajectory of the bob after getting disconnected?

Solution : This question emphasizes two important aspects of force. First, force has no past or future. It means that when force is removed, its effect is over. Second, if the body is at rest, then motion (velocity) of the body is in the direction of applied force.

Pendulum



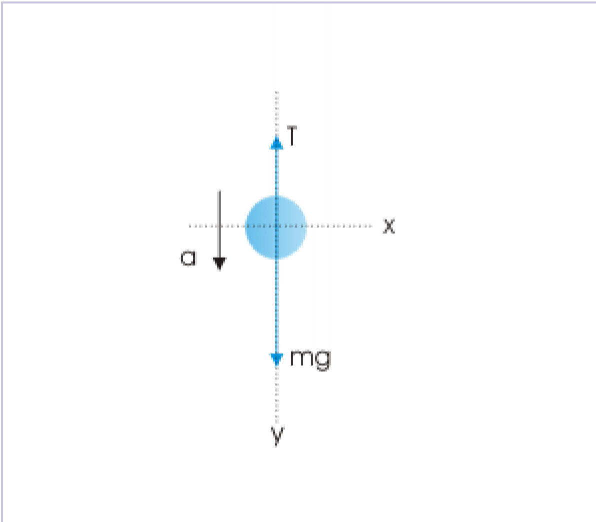
When the string is cut, pendulum bob is at the extreme position and its velocity is zero. The tension in the string immediately disappears as the string is cut. At that instant, the only force on the pendulum bob is force due to gravity i.e. "mg", which acts in vertically downward direction. Thus, pendulum bob has straight trajectory in the downward direction.

Motion under gravity

Problem 2 : With what minimum acceleration a person should slide down a hanging rope, whose breaking strength is $2/3^{\text{rd}}$ of his weight?

Solution : The breaking strength of rope is less than the weight of the person. As such, the rope will break, if the person simply hangs on the rope. It is, therefore, required that the person should climb down with acceleration greater than a minimum value so that tension in the rope is less than its breaking strength.

Let the person is sliding down with an acceleration, "a". The forces on the person are (i) tension in upward direction and (ii) weight of the person in downward direction. The free body diagram of the person sliding down the rope is shown here.
Free body diagram



$$\sum F_y = mg - T = ma$$
$$\Rightarrow T = mg - ma$$

According to question, the limiting value of tension in the rope is :

$$T = \frac{2mg}{3}$$

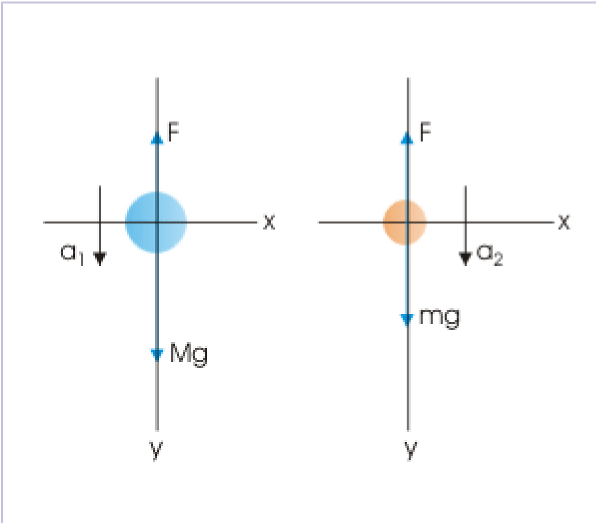
Putting the value of limiting tension in the equation of analysis, we have :

$$\Rightarrow \frac{2mg}{3} = mg - ma$$
$$\Rightarrow a = g - \frac{2g}{3} = \frac{g}{3}$$

Problem 3 : Two small spheres of same dimensions “A” and “B” of mass “M” and “m” (M > m) respectively are in free fall from the same height against air resistance “F”. Then, which of the spheres has greater acceleration?

Solution : The free body diagrams of the two bodies are shown in the figure. Let a_1 and a_2 be the accelerations of “A” and “B” blocks respectively. The force analysis in y-direction is as given here :

Free body diagram



For “A”

$$a_1 = \frac{Mg - F}{M}$$
$$\Rightarrow a_1 = g - \frac{F}{M}$$

Similarly,

$$\Rightarrow a_2 = g - \frac{F}{m}$$

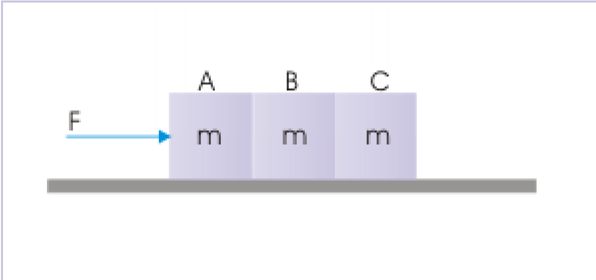
As M > m,

$$\Rightarrow a_1 > a_2$$

Motion in horizontal direction

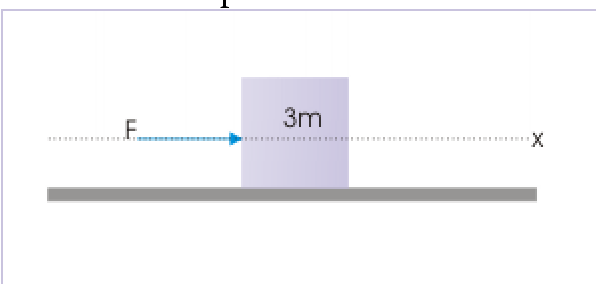
Problem 4 : Three blocks “A”, “B” and “C” of identical mass “m” are in contact with each other. The blocks are pushed with a horizontal force "F". If N_1 and N_2 be the normal forces in horizontal direction at the contact surface between the pairs of blocks “A and B" and "B and C" respectively, then find (i) the net force on each of the blocks and (ii) normal forces, N_1 and N_2.

Three identical blocks being pushed by external force



Solution : Net force means resultant force on individual block. Here, the three blocks in contact under the given situation have same acceleration as block on left pushes the block ahead. As such, we can treat the motion of three blocks as that of a single composite block of mass “3m”. Let the common acceleration of the composite mass be “a”. Then,

Force on composite mass



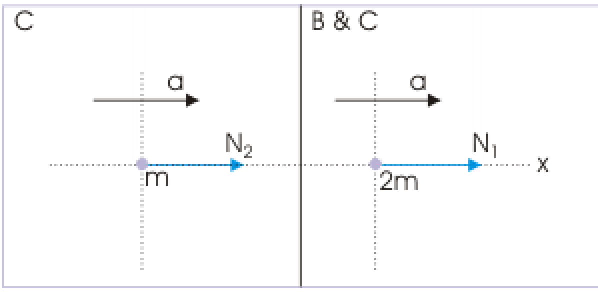
$$a = \frac{F}{3m}$$

Note that we consider only horizontal forces as net force in vertical direction is zero. Since acceleration of each block of identical mass,"m", is same, the net force on each block must also be same. This is given by :

$$F_{\text{net}} = ma = \frac{F}{3}$$

In order to analyze forces for normal forces at the interface in horizontal direction, we need to carry out force analysis of the blocks appropriately. We start force analysis with the third block (C), as there is only one external force on the block in the horizontal direction.

Free body diagram



Free body diagram of block “C”

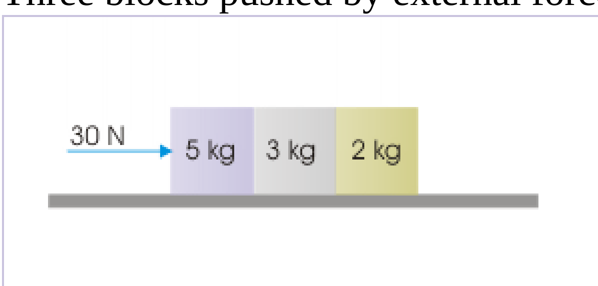
$$\sum F_x = N_2 = ma = m \text{ x } \frac{F}{3m} = \frac{F}{3}$$

We can now consider block “B” and “C” as the composite block of mass “2m”. Considering Free body diagram of block “B” and “C” together, we have :

$$\sum F_x = N_1 = 2ma$$
$$\Rightarrow N_1 = 2ma = 2m \text{ x } \frac{F}{3m} = \frac{2F}{3}$$

Problem 5 : Three blocks of mass 5 kg, 3 kg and 2 kg are placed side by side on a smooth horizontal surface as shown in the figure. An external force of 30 N acts on the block of 5 kg. Find the net external force on the block of mass 3 kg.

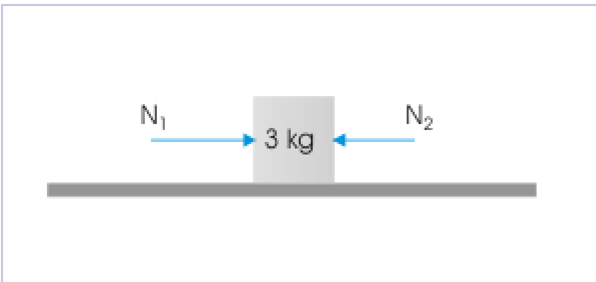
Three blocks pushed by external force



The blocks move together with
a common acceleration

Solution : Here, we know that there is no motion in vertical direction. It means that forces in vertical directions are balanced force system and consideration in vertical direction can be conveniently excluded from the analysis. We can find net external force on the middle block by determining normal forces due to blocks on each of its sides. Evidently, the net external force will be the difference of normal forces in horizontal direction. This approach, however, will require us to analyze force on each of the blocks on the sides.

Net force on the block



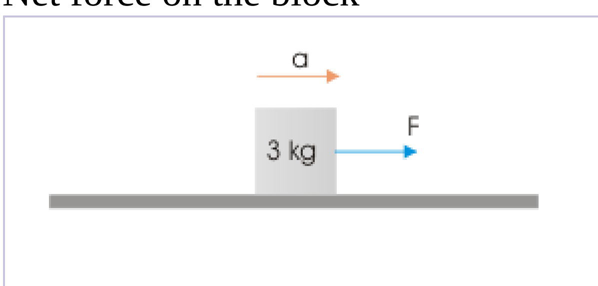
Net force is equal to the resultant of normal forces due to blocks on the side.

Alternatively, we find common acceleration, "a", of three blocks as :

$$a = \frac{F}{M} = \frac{30}{5+3+2} = \frac{30}{10} = 3 \quad m/s^2$$

According to second law of motion, the net force on the block is equal to the product of its mass and acceleration. Let the net force on the block be "F". Then,

Net force on the block



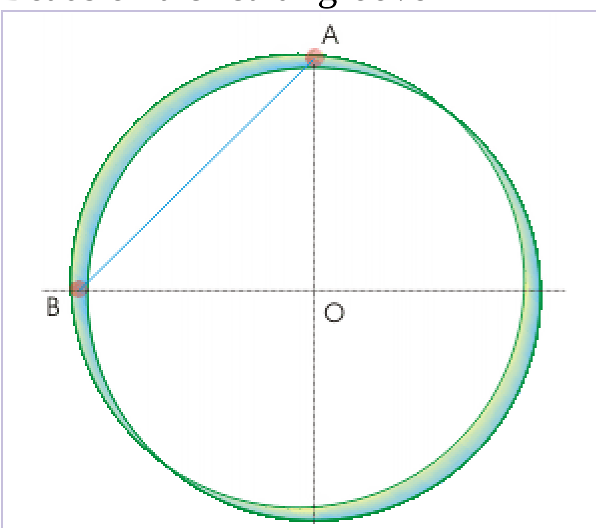
Net force is equal to the product of mass and acceleration.

$$\Rightarrow F = ma = 3X3 = 9 \quad N$$

Constrained motion

Problem 6 : At a given instant, two beads, “A” and “B” of equal mass “m” are connected by a string. They are held on a circular grove of radius “r” as shown in the figure. Find the accelerations of beads and the tension in the string, just after they are released from their positions.

Beads on a circular groove



The beads are released from the positions shown.

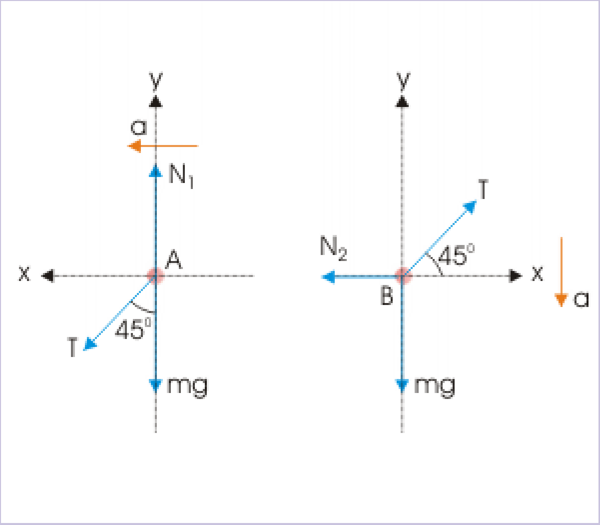
Solution : Since motion is initiated, velocity of the beads are zero. No centripetal acceleration is associated at this instant. At this instant, the beads “A” and “B” have tangential accelerations in horizontal and vertical direction respectively. As the beads are connected with a string, magnitude

of accelerations of the beads are equal in magnitude. As such, magnitude of tangential accelerations are equal when beads are released.

We can find acceleration and tension by considering free body diagram of each of the beads. The initial positions of the beads are at the end of a quarter of circle. It means that triangle OAB is an isosceles triangle and the string makes an angle 45° with the radius at these positions. Let N_1 and N_2 be the normal force at these positions on the beads. Let also consider that the magnitude of acceleration of either bead is “a”.

The free body diagrams of two beads are shown in the figure.

Free body diagrams



The FBD of beads are shown.

Considering FBD of “A” and forces in horizontal direction, we have :

$$T \cos 45^0 = ma$$

$$\Rightarrow T = \sqrt{2}ma$$

Considering FBD of “B” and considering forces in vertical, we have :

$$mg - T \sin 45^0 = ma$$

Substituting for “T”,

$$\Rightarrow mg - \sqrt{2}maX\frac{1}{\sqrt{2}} = ma$$

$$\Rightarrow mg - ma = ma$$

$$\Rightarrow a = \frac{g}{2}$$

We find tension in the string by putting value of acceleration in its expression as :

$$T = \frac{\sqrt{2}mg}{2} = \frac{mg}{\sqrt{2}}$$

String and spring

String and spring are very close in their roles and functionality in a mechanical arrangement. Main features of two elements in their ideal forms are :

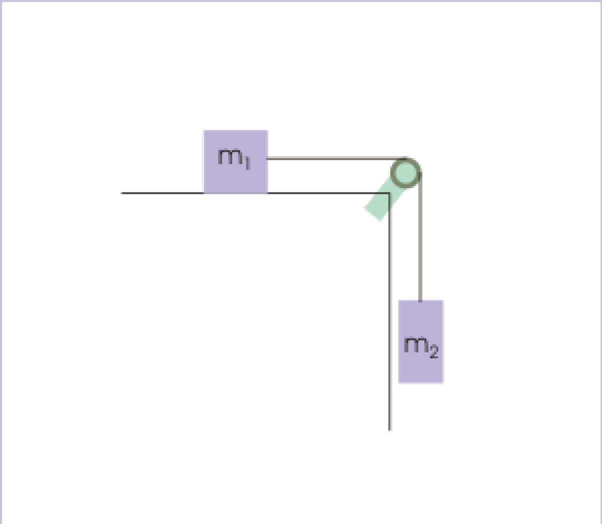
- **String** : Inextensible, mass-less : Transmits force
- **Spring**: Extensible and compressible, mass – less : Transmits and stores force

We have already discussed the basics of these two elements in the module titled " Analysis framework for laws of motion ". Here, we shall examine their behaviour as part of a body system and shall also discuss similarities and differences between two elements.

String

The elements of a body system connected by a string have common acceleration. Consider the body system comprising of two blocks. As the system comprises of inextensible elements, the body system and each element, constituting it, have same magnitude of acceleration (not acceleration as there is change of direction), given by :

Connected system involving string



The elements of a body system connected by a string have

common acceleration.

$$a = \frac{F}{m_1 + m_2}$$

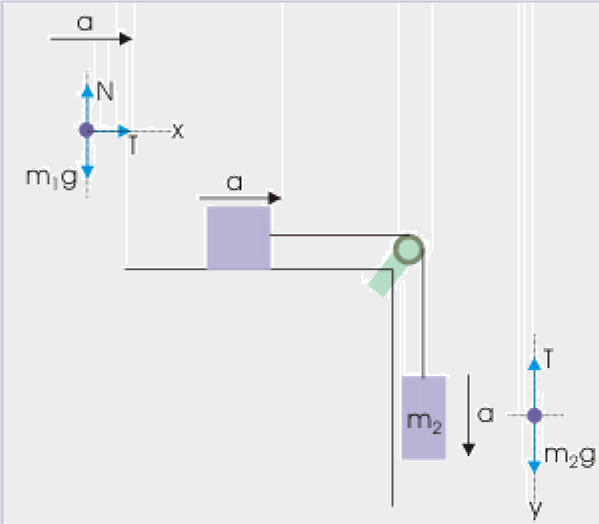
In the process, string transmits force undiminished for the ideal condition of being “inextensible” and “mass-less”. This is the basic character of string. We must however be careful in stating what we have stated here. If the ideal condition changes, then behavior of string will change. If string has certain “mass”, then force will not be communicated “undiminished”. If string is extensible, then it behaves like an extensible spring and accelerations of the string will not be same everywhere.

Inextensible and mass-less string

Determination of common acceleration and tension in the string is based on application of force analysis in component forms. In general, force analysis for each of the block gives us an independent algebraic relation. It means that we shall be able to find as many unknowns with the help of as many equations as there are blocks in the body system.

Individual string has single magnitude of tension through out its length. On the other hand, if the systems have different pieces of strings (when there are more than two blocks), then each piece will have different tensions.

Example:
Problem : Two blocks of masses “ m_1 ” and “ m_2 ” are connected by a string that passes over a mass-less pulley as shown in the figure. Neglecting friction between surfaces, find acceleration of the block and tension in the string.
Connected system involving string



The elements of a body system connected by a string have common acceleration.

Solution : Since there is no friction, the block of mass " m_1 " will be pulled to right by the tension in the string (T). On the other hand, block of mass " m_2 " will move down under net force in vertical direction.
Two blocks are connected by a taut string. Hence, magnitude of accelerations of the bodies are same. Let us assume that the system moves with magnitude of acceleration “a” in the directions as shown. It should be understood here that a string has same magnitude of acceleration - not the acceleration as string in conjunction with pulley changes the direction of motion and hence that of acceleration.
The external forces on " m_1 " are (i) weight of block, m_1g , (ii) tension, T, in the string and (iii) Normal force on the block, N. We need not consider analysis of forces on m_1g in y-direction as forces are balanced in that direction. In x-direction,

$$\sum F_x = T = m_1a$$

On the other hand, the external forces on " m_2 " are (i) weight of block, m_2g and (ii) tension, T, in the string.

$$\sum F_y = m_2g - T = m_2a$$

Combining two equations, we have :

$$a = \frac{m_2 g}{m_1 + m_2}$$

and

$$\Rightarrow T = m_1 a = \frac{m_1 m_2 g}{m_1 + m_2}$$

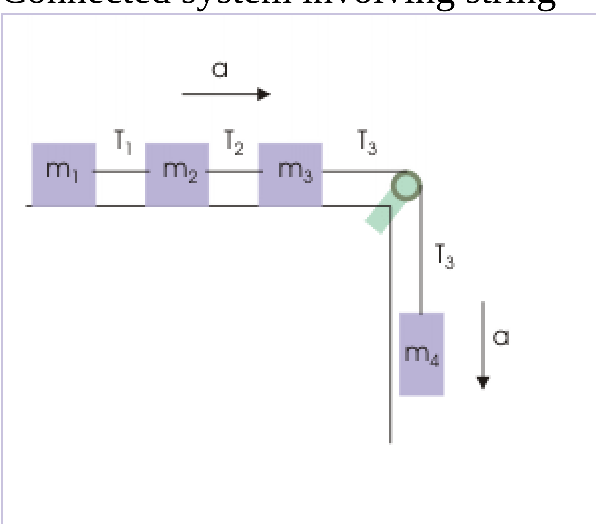
Useful deduction

The result for acceleration as obtained above is significant for this type of arrangement, where blocks are pulled down by a string passing over a pulley. The acceleration as derive is equivalent to :

$$\Rightarrow a = \frac{F}{m_1 + m_2}$$

where $F = m_2 g$. It suggests that we can treat the arrangement as composite mass of ($m_1 + m_2$) subjected to an external force “F” of magnitude “ $m_2 g$ ”. This analysis helps in situations where more than one block is pulled by a hanging block. Take the example as shown in the figure :

Connected system involving string



Three blocks are being pulled on a smooth surface.

In this case,

$$\Rightarrow a = \frac{F}{m_1+m_2+m_3+m_4} = \frac{m_4 g}{m_1+m_2+m_3+m_4}$$

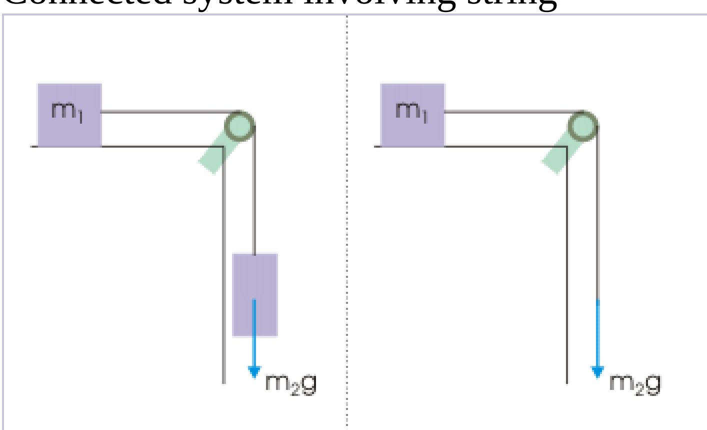
$$\Rightarrow T_1 = m_1 a;\ T_2 = (m_1 + m_2)a;\ T_3 = (m_1 + m_2 + m_3)a;$$

Important to note here is that the separate strings have different tensions.

Mode of force application

Consider the two cases when a block is pulled down by a force in two different manners. In one case, string is pulled by another mass m_2 , hanging from it. In the second case, the string is pulled by a force equal to hanging weight ($m_2 g$). Now the question is : whether two situations are equivalent or different ?

Connected system involving string



The block on the table is pulled in two different ways.

As worked out earlier, the acceleration in the first case, a_1 , is :

$$\Rightarrow a_1 = \frac{m_2 g}{m_1 + m_2}$$

In the second case, let the acceleration of the block be “ a_2 ”. Then force analysis of the block on the table is :

$$\Rightarrow m_1 a_2 = T$$

But,

$$T = m_2 g$$

Now, combining two equations, we have :

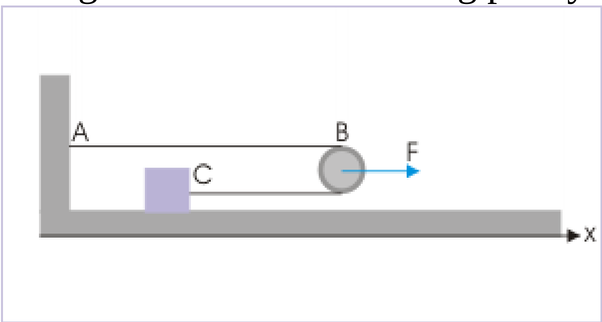
$$\Rightarrow a_2 = \frac{m_2 g}{m_1}$$

Evidently, $a_2 > a_1$. This result is expected because applying force as weight and as force are two different things. In the first case, mass of the body applying force is also involved as against the later case, when no "body" of any mass is involved as far as application of force is concerned. The bottom line is : “do not hang a weight simply to apply force”.

Variable acceleration of a single string

Further, the behavior of string is true to ideal string so long string is not intervened by other element, which is itself accelerating. Consider a pulley – string system, in which the pulley is accelerating as shown in the figure.

String in contact with moving pulley



Different parts of the string
have different accelerations.

In such case, the accelerations of different parts of the string are not same. Note in the example shown above that one end of the string is fixed, but the other end of string is moving. As a matter of fact, the acceleration of block (hence that of the string at that end) is twice that of the pulley. The acceleration of the other end of the string, which is fixed, is zero. This aspect is explained in a separate module on movable pulley in the course.

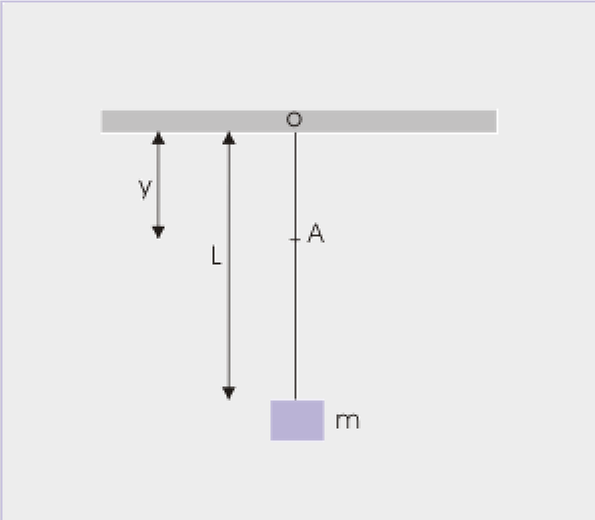
String with certain mass

The tension in the string having certain mass is not same everywhere. A part of the force is required to accelerate string as well. Usually,mass is distributed uniformly along the length of the string. We describe mass distribution in terms of "mass per unit length" and denotes the same as "λ" such that the mass of the string,"m" of length "L" is given as :

$m = \lambda L$

If we take cross section at any point in the string, then the part of the string on each side forms an mass element. having mass proportional to the length of that part of the string. This aspect is brought out in the example given here.

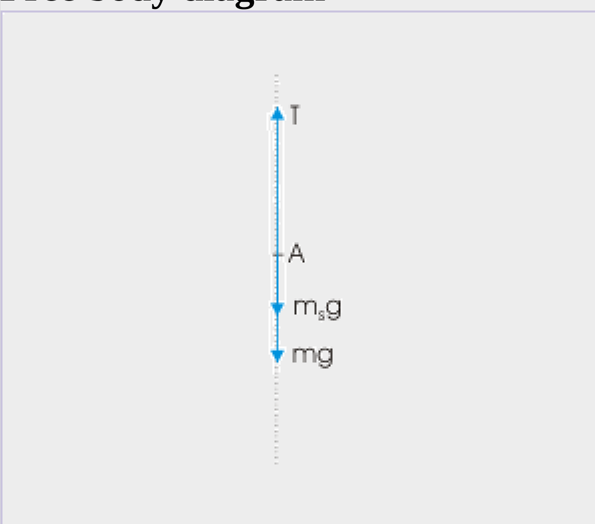
Example:
Problem : A body of mass “m” is hanging with a string having linear mass density “λ”. What is the tension at point “A” as shown in the figure.
Balanced force system



Solution : The tension in the string is not same. The tension above “A” balances the weight of the block and the weight of the string below point “A”.

Free body diagram of point “A”

Free body diagram



Mass of the string below point A, " m_s ", is :

$$m_s = \lambda(L - y)$$

The external forces at point "A" are (i) Tension, T (ii) weight of block, mg, and (iii) weight of string below A," $m_s g$ ".

$$\sum F_y = T - \text{mg} - m_s g = 0$$
$$\Rightarrow T = (m + m_s)g$$
$$\Rightarrow T = \{m + \lambda(L - y)\}g$$

Spring

Spring is extensible and compressible. What it means that spring involves change in its length, when force is applied to it. It must be understood that the role of spring in mechanical arrangement is more significant with respect to “storage” of energy (force) rather than transmission of force. This aspect is dealt in modules related to energy. For the time being, we describe only those aspects of spring, which involve transmission of force.

As a matter of fact, spring is exactly like string – though as an element it generally gives impression of complexity. We shall, however, find that such is not the case except for minor adjustment in our perception.

Constant acceleration

When a block (body) hangs from a spring, the body system is stationary. The spring is in extended condition for the given force system working on it. The forces at the two ends of the spring are equal. Net force on the spring is zero in the extended condition. This is same as string. There is no difference. As a matter of fact, ideal spring not only transmits force as truly as string, but also provides the magnitude of force in terms of force constant as :

$$F = kx_0$$

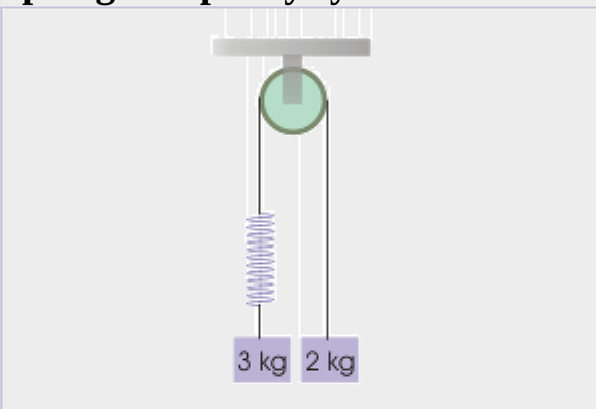
where x_0 is the extension for a given force as applied at either end. When a block is suspended from a spring hanging from the ceiling, the spring is acted by equal forces at both ends. We calculate extension in terms of either of the two forces and not for the net force, which is zero in any case.

Example:

Problem : Consider the arrangement consisting of spring in the figure. If extension in the spring is 0.01 m for the given blocks, find the spring constant.

Solution : We treat spring exactly like string. The tension in the connecting string is equal to spring force,

Spring and pulley system

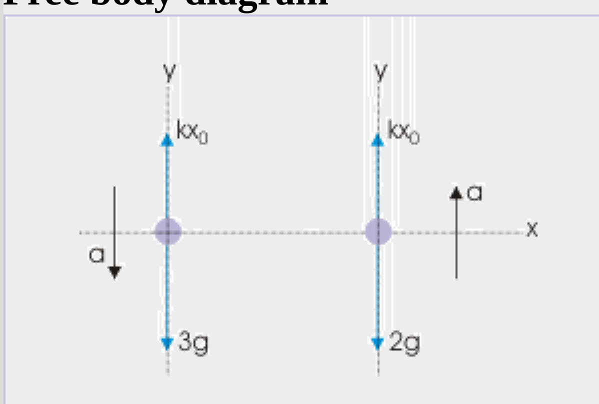


The tension in the string is equal to spring force.

$$T = kx_0$$

The free body diagrams of 3 kg and 2 kg blocks with assumed common acceleration "a" are shown in the figure. Here all forces are along y-direction only. Now,

Free body diagram



$$3g - kx_0 = 3a$$

and

$$kx_0 - 2g = 2a$$

Combining two equations, we have :

$$\Rightarrow a = \frac{g}{5}$$

Putting in first force equation, we have :

$$3g - kx_0 = 3a = 3 \text{ x } \frac{g}{5}$$
$$kx_0 = 3g - \frac{3g}{5} = \frac{12g}{5}$$

For x_0 = 0.01 m,

$$\Rightarrow k = \frac{12g}{5x_0} = 12 \text{ x } \frac{10}{5} \text{ x } 0.01 = 2400\ N/m$$

Variable acceleration

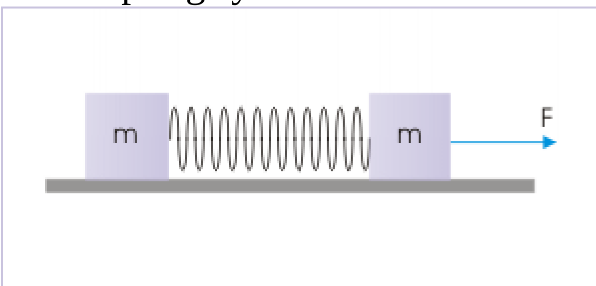
Spring, however, behaves differently for a certain period in the beginning. When we apply force, spring begins to stretch or compress. During this period, the spring force is not constant but its magnitude depends on the extension as :

F = -kx

where "x" is the displacement. Clearly, spring force is opposite in direction to the displacement. As force is variable here, acceleration of the body attached to spring is also variable. We can not handle variable acceleration, using equation of motion for constant acceleration. The detail of spring motion under variable acceleration, therefore, is handled or analyzed using energy concepts.

We can, therefore, analyze spring motion only in limited manner during which length of spring changes. For illustration, consider the case of two blocks, which are connected by a spring and pulled by an external force.

Block spring system

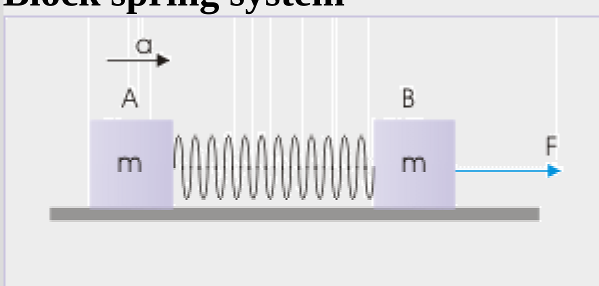


The accelerations of the blocks are not same in this period. It must be understood that extension of the spring takes finite time to extend or compress during which blocks will have different accelerations and spring will be subjected to different forces at the two ends. Eventually, however, extension is stabilized for the given external force and the blocks move with same accelerations.

Example:

Problem : Consider the configuration as given in the figure. If "a" be the acceleration of block "A" at the instant shown, then find the acceleration of block "B".

Block spring system

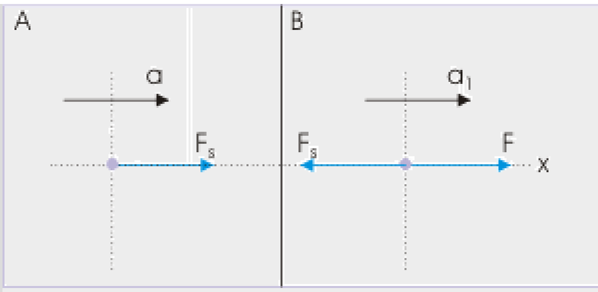


Solution : It is important to note the wording of the question. It assumes accelerations to be different. It means that the spring is getting extended during the motion. We do not know the spring force.

Let us consider that spring force be F_s .

Free body diagram of "A"

Free body diagram



$$\sum F_x = F_s = ma$$
$$\Rightarrow a = \frac{F_s}{m}$$

Free body diagram of "B"

$$\sum F_x = F - F_s = ma_1$$
$$\Rightarrow ma_1 = F - F_s$$

Combining two equations and eliminating, F_s , we have:

$$\Rightarrow a_1 = \frac{F}{m} - a$$

Spring (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

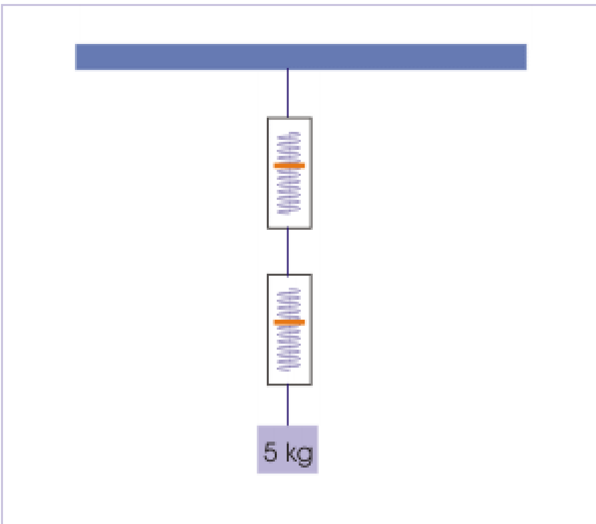
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to spring. The questions are categorized in terms of the characterizing features of the subject matter :

- Spring balance
- Spring and block
- Spring and pulley
- Extension in spring
- Spring force as contact force

Spring and block

Problem 1 : A mass of 5 kg is suspended with two “mass-less” spring balances as shown in the figure. What would be the reading in each of the spring balances.
Spring balance



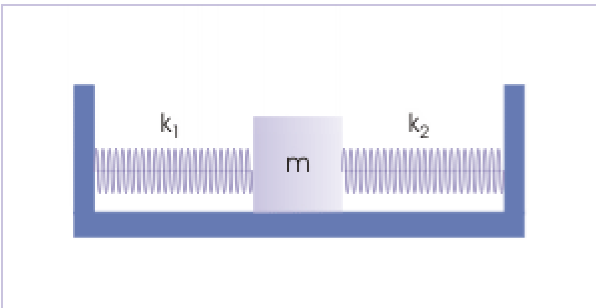
A mass of 5 kg is suspended from a combination of two springs

Solution : The spring balance is calibrated to measure the mass of the object suspended from it. In the stabilized condition, the spring force in the lower spring is equal to the weight of the object suspended from it. Since spring balances are "mass-less", the lower spring reads 5 kg.

The lower "spring balance" and the "object" together are attached to the lower end of the upper spring balance. Again, a total of 5 kg is suspended from the upper spring balance (note that springs have no weight). As such, the spring force in the upper spring is equal to the weight of the object suspended from it. The upper spring, therefore, also reads 5 kg.

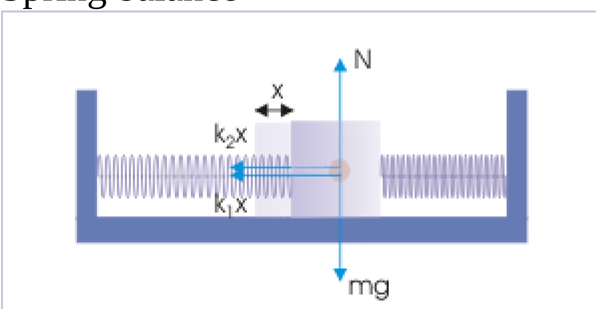
Spring and block

Problem 2 : In the arrangement shown in the figure, the block is displaced by a distance “x” towards right and released. Find the acceleration of the block immediately after it is released.
Spring balance



The block is displaced towards right.

Solution : When block is displaced towards right, the spring on left is stretched. The spring applies a force on the block towards left as spring tries to contract. The force is equal to the spring force and is given by :
Spring balance



Both forces acts towards left.

$$F_1 = -k_1 x$$

The force is opposite to the assumed positive reference direction of displacement (towards right). On the other hand, the spring on right is compressed. As such, it also applies a force towards left as spring tries to expand.

The force is equal to the spring force and is given by :

$$F_2 = -k_2 x$$

Thus, we see that both springs apply force in the direction opposite to the reference direction. The net force on the block, at the time of extension “x” in the spring is given by :

$$\Rightarrow F_1 + F_2 = -(k_1 + k_2)x$$

The acceleration of the block, immediately after it is released, is :

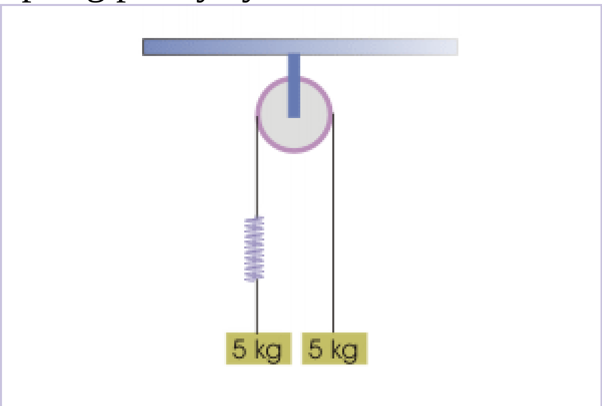
$$\Rightarrow a = -\frac{(k_1 + k_2)x}{m}$$

The acceleration of the block is negative as all other quantities in the above expression are positive. It means that acceleration is in the opposite direction of the displacement. The magnitude of acceleration is :

$$\Rightarrow |a| = \frac{(k_1 + k_2)x}{m}$$

Spring and pulley

Problem 3 : A spring of spring constant 200 N/m is attached to a “block – pulley” system as shown in the figure. Find the extension in the spring.
Spring pulley system

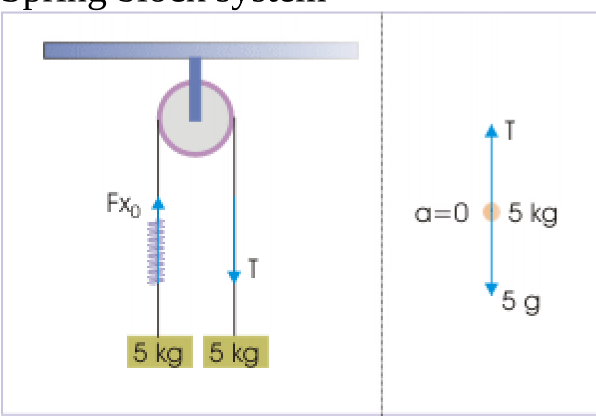


The blocks have same mass.

Solution : We have studied that there is two phases of extension/compression of a spring. One is a brief period in which spring dynamically changes its length depending on the net axial force on it. This means that forces at the two ends of the spring are not same. Quickly, however, spring acquires the stabilized extension in the second phase, usually denoted by symbol " x_0 ". In this stabilized condition, the forces at the two ends of an ideal (without mass) spring are equal.

This problem deals the situation in which spring has acquired the stabilized extension. Thus forces across the spring are equal at two ends and are also equal to the tensions in the attached string at those ends. In the force analysis, we shall treat spring just like string with the exception that the spring force equals tension in the string. Hence,

Spring block system



Forces on the block

$$T = kx_0$$

As the masses of the blocks are equal, the net pulling force on the blocks in the downward direction is zero.

$$F = 5g - 5g = 0$$

It means that blocks have zero acceleration. Now, free body diagram (as shown on the right hand side of the figure above) of any of the given blocks yields :

$$\Rightarrow T - 5g = m \text{ X } a = 0$$

$$\Rightarrow T = 5g = kx_0$$

$$\Rightarrow x_0 = \frac{5g}{k} = \frac{5X10}{200} = 0.25 \quad m$$

Extension in spring

Problem 4 : A spring of spring constant "k", has extended lengths " x_1 " and " x_2 " corresponding to spring forces 4 N and 5 N. Find the spring length when spring force is 9 N.

Solution : Let the natural length of spring be “x”. According to Hooke’s law, the magnitude of spring force,

$$F = k\Delta x$$

where " Δx " is the extension. Let the spring length for spring force 9N be " $x3$ ". Then,

$$9 = k(x_3 - x) = kx_3 - kx$$

$$\Rightarrow x_3 = \frac{9 + kx}{k}$$

We need to find expressions of spring constant and natural length in terms of given values. For the given two extensions, we have :

$$4 = k(x_1 - x)$$

$$5 = k(x_2 - x)$$

Subtracting first equation from second and solving for “k”,

$$\Rightarrow 1 = k(x_2 - x_1)$$

$$k = \frac{1}{(x_2 - x_1)}$$

Substituting for “k” in the first equation,

$$\Rightarrow 4 = \frac{1}{(x_2 - x_1)} X(x_1 - x)$$

$$\Rightarrow 4(x_2 - x_1) = (x_1 - x)$$

$$\Rightarrow 4x_2 - 4x_1 = x_1 - x$$

$$\Rightarrow x = 5x_1 - 4x_2$$

We can, now, evaluate the required expression of “ x_3

$$\Rightarrow x_3 = \frac{9 + \frac{1}{(x_2 - x_1)} X(5x_1 - 4x_2)}{\frac{1}{(x_2 - x_1)}}$$

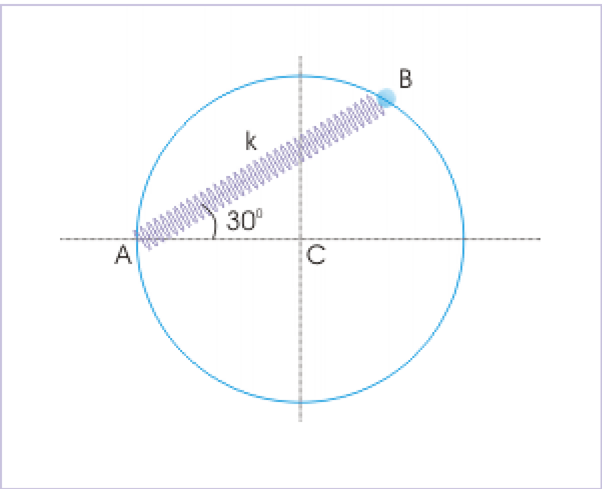
$$\Rightarrow x_3 = 9(x_2 - x_1) + (5x_1 - 4x_2)$$

$$x_3 = 5x_2 - 4x_1$$

Spring force as contact force

Problem 5 : A bead of mass, “m”, is placed on a circular rim of radius “r” and is attached to a fixed point “A” through a spring as shown in the figure. The natural length of spring is equal to the radius “r” of the circular rim and spring constant is “k”. Find the normal force on the circular rim at the moment spring is released from the position “B”. The spring makes 30° with the horizontal in this position.

A bead attached to a spring

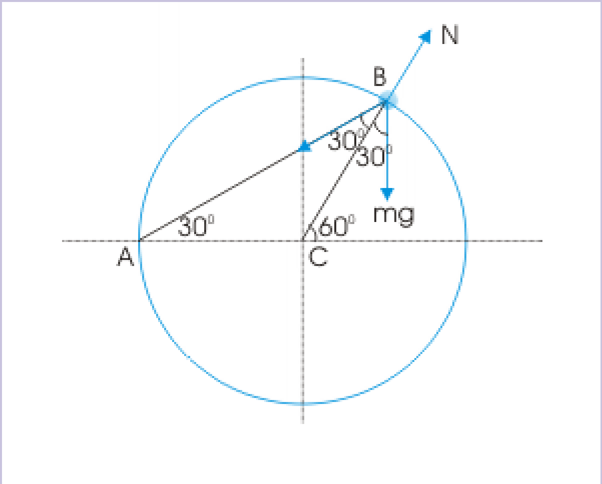


The bid applies a force on the circular ring.

Solution : The forces on the bead are (i) its weight (ii) spring force and (iii) normal force.

In order to carry force analysis, we need to determine magnitudes and directions of forces. Here, we know the directions of all three forces. Spring force acts along spring length, gravity acts vertically downward and normal force acts radially. In triangle ABC, CA and CB are the radii. Hence, triangle ABC is an isosceles triangle.

A bead attached to a spring



Forces on the bead.

$$\angle CAB = \angle CBA = 30^0$$

Let "F" be the spring force. Normal force is equal to sum of the components of spring force and gravity in radial direction. Carrying force analysis in CB direction, we have :

$$N = mg\cos 30^0 + F\cos 30^0$$

The magnitude of spring force, “F”, is given by :

$$F = kx$$

where “x” is the extension in the spring and is given by :

$$x = AB - r = 2r\cos 30^0 - r = \left(\sqrt{3} - 1\right)r$$

Substituting in the expression of normal force, we have :

$$N = mgX\frac{\sqrt{3}}{2} + k\left(\sqrt{3} - 1\right)r\sqrt{\frac{3}{2}}$$

$$N = \frac{\sqrt{3}}{2}\left\{mg + k\left(\sqrt{3} - 1\right)r\right\}$$

Incline plane

Incline is a plane whose two ends are at different elevation with respect to a base line. This is a useful device to reduce the requirement of force in exchange of extended motion. When we push an object over an incline to raise the object at higher level, we need to apply lesser force (mg sinθ) in comparison to lifting the same vertically against its weight (mg).

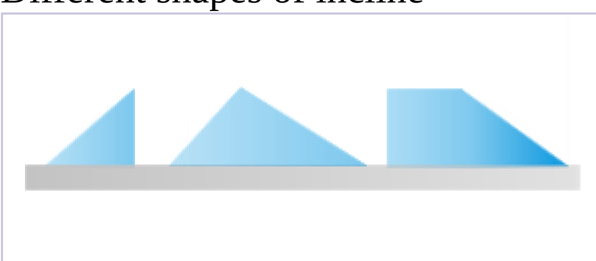
What we gain through reduction in force we have to work longer to cover a longer length. Moreover, no plane, however smooth, is friction-less. Such devices aiming to reduce the magnitude of effort are ultimately less energy efficient.

Incline

Shape

An incline can have different shapes. Three general shapes, including single and double inclines, are shown here for illustration.

Different shapes of incline

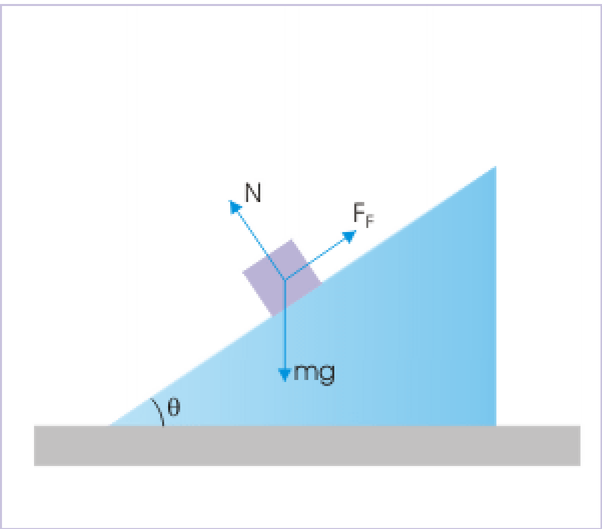


Forces

The force analysis with respect to motion of the block on a rough incline plane involves various forces. A minimum of three forces operate on the block placed on the incline : (i) weight of the block, mg, (ii) normal force (N) and (iii) friction (f_s , F_s or F_k). Any other external force is additional to these three named forces. On the other hand, named forces reduce to two

(weight and normal force), if the friction between incline and block is negligible to be considered zero.

Forces on the incline



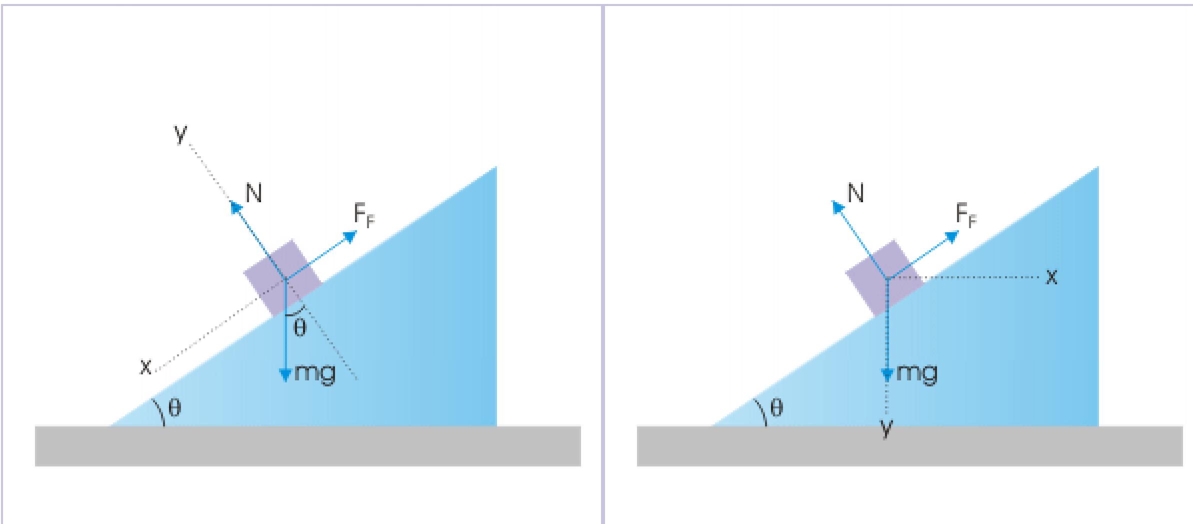
There are three external forces on the block.

Choice of planar coordinates

There are two useful coordinate orientations to analyze the forces and consequently the motion of the block.

We can either keep x-axis along incline and y-axis perpendicular to incline. In this setup, the weight of the block lying on the incline surface is resolved into components along these directions. The angle between vertical and perpendicular to incline is equal to the angle of incline “θ” as shown in the figure. The normal force on the block due to incline surface is in y-direction, whereas friction is along x-direction.

Choice of coordinate system



In the nutshell, contact forces are along the chosen coordinates, but weight of the body makes an angle with the axis, which needs to be resolved along the axes of the coordinate system for the analysis.

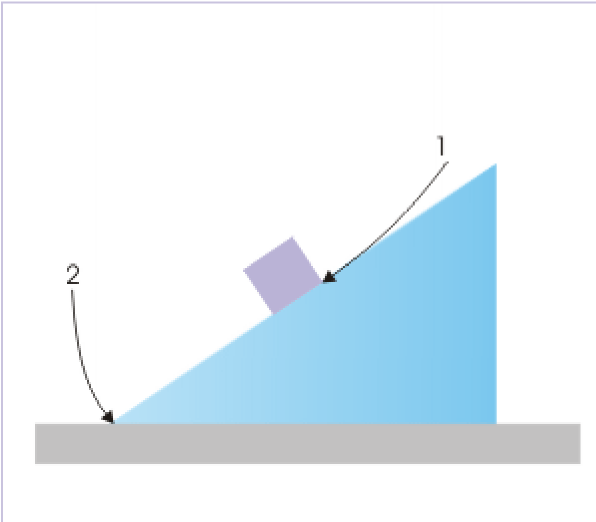
Alternatively, we can align x and y – axes along horizontal and vertical directions. In that case, weight of the block is along y-axis and as such need not be resolved. However, the two contact forces (normal and friction) on the block are now at an angle with the axes and are required to be resolved for force analysis.

It is evident, therefore, that the first choice is relatively better in most of the cases as there is only one force to be resolved into components against resolution of two forces as required in the second case.

Friction

The incline and block interface may be either termed as “smooth” or “rough”. The smooth surface indicates that we can neglect friction force. We should be aware that there are actually two contact interfaces as shown in the figure.

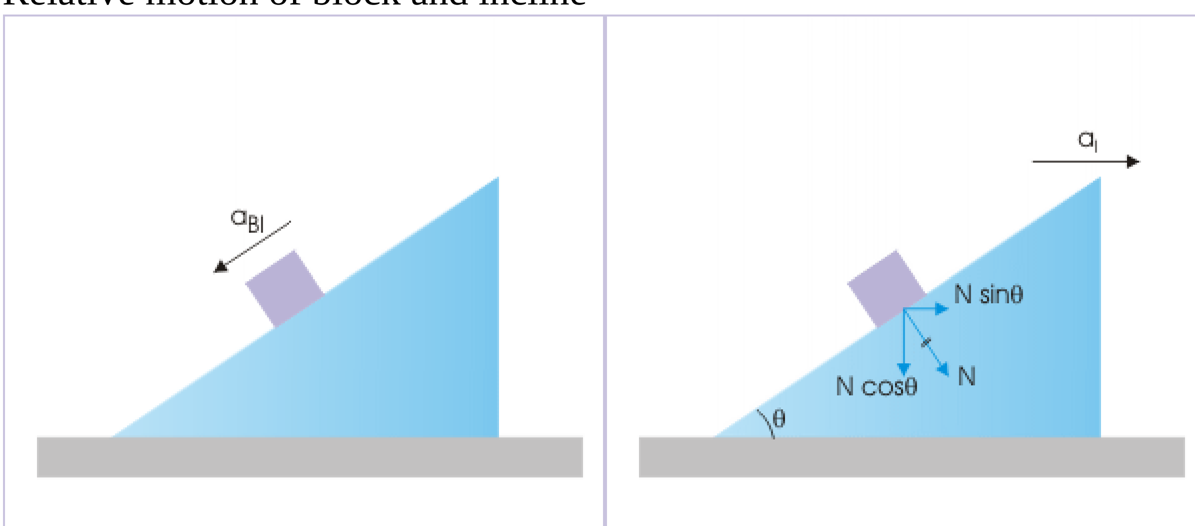
Friction at the interface between surfaces in contact



There are two contact
interfaces.

Generally, the incline is considered to be fixed to the ground. If it is not fixed, then it is important to know the nature of friction between the incline and horizontal surface on which the incline is placed. If their interface is smooth, then any force on incline will accelerate the incline itself. Even just placing a block on the incline will move the incline. The horizontal component of normal force (N sinθ) applied by the block will accelerate the incline to the right with acceleration, $\mathbf{a}_I$.

Relative motion of block and incline

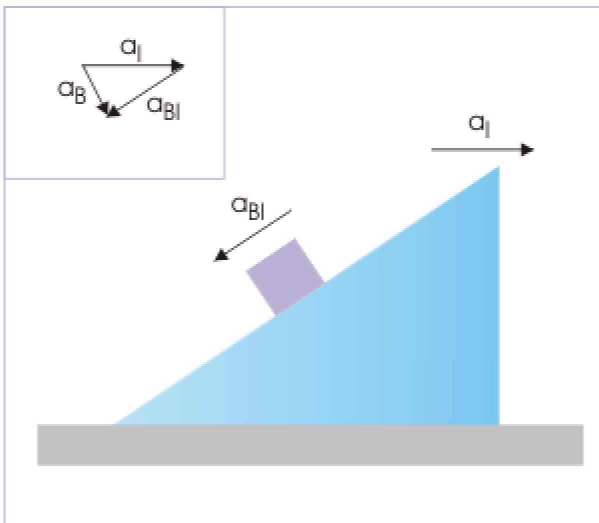


Block moves down the incline, whereas incline moves horizontally.

In that situation, the motion of block is taking place in the accelerated frame of the incline – not in the inertial frame of the ground. The acceleration of the block with respect to incline ($\mathbf{a}_{\text{BI}}$) is along the incline.

The acceleration of the block with respect to ground ($\mathbf{a}_B$) is obtained from the relation for relative acceleration given as :

Motion in the accelerated frame of incline



Acceleration of block with respect to ground.

Equation:

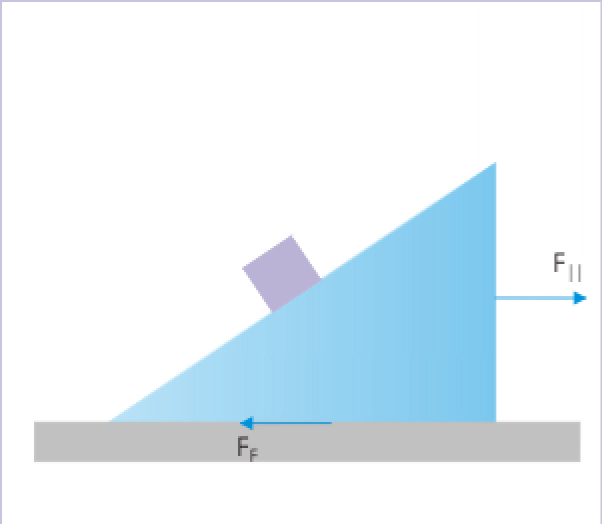
$$\mathbf{a}_{\text{BI}} = \mathbf{a}_B - \mathbf{a}_I$$
$$\Rightarrow \mathbf{a}_B = \mathbf{a}_{\text{BI}} + \mathbf{a}_I$$

where $\mathbf{a}_I$ is the acceleration of incline with respect to ground and $\mathbf{a}_{\text{BI}}$ is the acceleration of block with respect to incline. Now, evaluating the vector sum of the accelerations of the accelerations on the right hand side, the

direction and magnitude of acceleration of the block with respect to ground is obtained.

However, if the contact between the incline and horizontal surface is rough, then the motion of incline will depend on whether force on the incline in horizontal direction (parallel to contact surface) is greater than the maximum static friction or not?

Motion of incline



Motion of incline is determined by force parallel to contact surface compared to friction.

We need to be careful while applying Newton's force laws for the case, where incline itself is accelerated. We should evaluate the situation as required either using ground reference or the accelerated reference of the incline – whichever is suitable for the situation in hand. We shall discuss this aspect of incline motion in details after studying motion in accelerated reference.

In this module, we shall, therefore, restrict our discussion to motion of a body on an incline plane, which is (a) stationary with respect to ground and (b) offers negligible friction to the motion of the body.

Motion on a smooth fixed incline

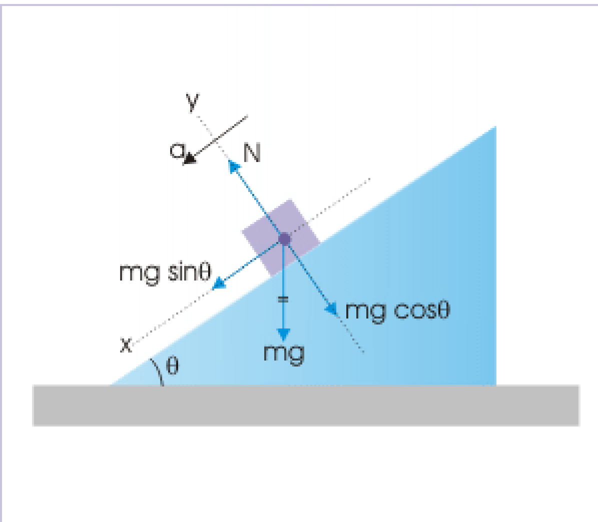
There are only two external forces on the block moving on a smooth incline plane. They are (a) normal force and (b) weight of the block.

There is no motion in the direction perpendicular to the incline. As such, forces in that direction form a balance force system. The normal force equals component of weight in opposite direction.

Equation:

$$\sum F_y = N - mg\cos\theta = 0$$
$$\Rightarrow N = mg\cos\theta$$

Forces on a fixed incline



Weight and normal force act on the block.

On the other hand, there is only component of weight along the incline in downward direction that accelerates the block in that direction. It must clearly be understood that the block will not be at rest on a smooth incline, unless some additional force stops the motion. The acceleration along the incline is obtained as :

Equation:

$$\sum F_y = mg\sin\theta = ma$$
$$\Rightarrow a = \frac{mg\sin\theta}{m} = g\sin\theta$$

In the following sections of the module, we shall consider three specific cases of motion of a block on smooth incline :

- Motion on an incline
- Motion of two blocks on an incline
- Motion on double incline

Motion on an incline

Problem 1 : With what speed a block be projected up an incline of length 10 m and angle 30° so that it just reaches the upper end (consider g = 10 m/s^2).

Solution : The motion of the block is a linear motion. The component of gravity acts in the opposite direction to the motion. Let initial velocity be "u". Here,

u = ?, x = 10 m, a = - g sin30°, v = 0 (final velocity). Using equation of motion,

$$v^2 = u^2 + 2ax$$
$$\Rightarrow 0 = u^2 - 2g\sin 30^0 \text{ x } x = u^2 - 2 \text{ X } 10 \text{ X } \frac{1}{2} \text{ X } 10$$

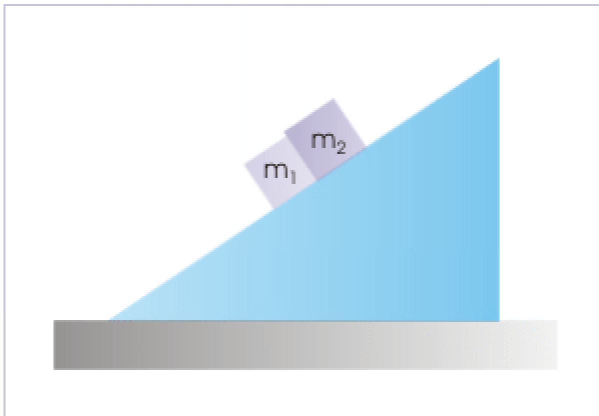
$$\Rightarrow u^2 = 100$$
$$\Rightarrow u = 10\, m/s$$

Motion of two blocks on an incline

Problem 1 : Two blocks of mass "m1" and "m2" are placed together on a smooth incline of angle "θ" as shown in the figure. If they are released

simultaneously, then find the normal force between blocks.

Motion of two blocks on a double incline



Two blocks placed on a smooth incline of angle “θ” are in contact initially.

The incline is smooth. Hence, there is no friction. The blocks, therefore, move down due to the component of gravitational force parallel to incline. In other words, acceleration of each of the block is “gsinθ”. As such, the blocks move with equal acceleration and hence there is no normal force between the surfaces of the two blocks. We can confirm the conclusion drawn by considering free body diagram of each of the blocks.

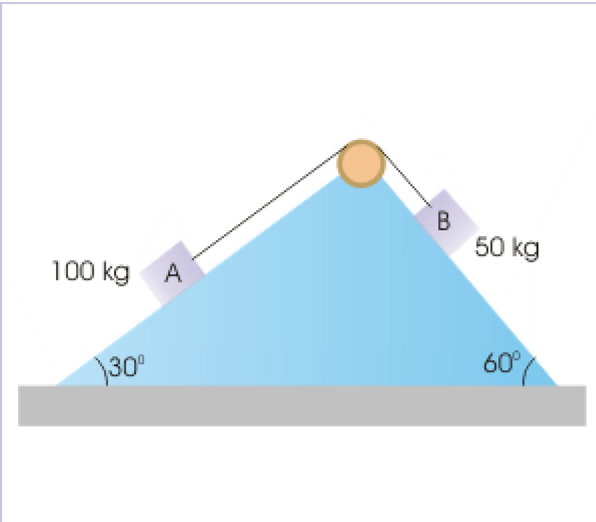
It is important to understand that this is a different situation than when two blocks are pushed together by an external force, “F”. The external force can produce different accelerations in the blocks individually and as such normal force appears between the surfaces when blocks move collectively with same acceleration.

In this case, however, the gravitational force produces same acceleration in each of the blocks - independent of their masses; and as such, normal force does not appear between blocks. Since blocks are together initially, they move together with same velocity in downward direction.

Motion on double incline

Problem 3 : Two blocks "A" and "B" connected by a string passing over a pulley are placed on a fixed double incline as shown in the figure and let free to move. Neglecting friction at all surfaces and mass of pulley, determine acceleration of blocks. Take g = 10 m/s^2 .

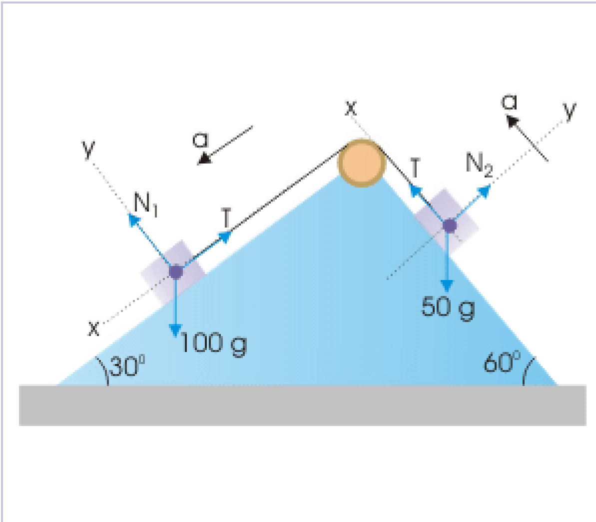
Motion on double incline



Solution : Here, we assume a direction for acceleration. If we are correct, then we should get positive value for acceleration; otherwise a negative value will mean that direction of motion is opposite to the one assumed. We also observe here that each of the block is, now, acted by additional force of tension as applied by the string.

Let us consider that block "A" moves down and block "B" moves up the plane. The figure shows the free body diagram as superimposed on the body system.

Motion on double incline



Free body diagram of "A"

The forces on the blocks are (i) weight of the block "A" (ii) Normal force applied by incline, " N_1 ", and (iii) Tension in the string, T. Analyzing force in x - direction :

$$\sum F_x = 100g \sin 30^0 - T = 100a$$
$$\Rightarrow 100 \ \text{X} \ g \ \text{X} \ \frac{1}{2} - T = 100a$$

Free body diagram of "B"

The forces on the blocks are (i) weight of the block "B" (ii) Normal force applied by incline, " N_2 ", and (iii) Tension in the string, T. Analyzing force in x - direction :

$$\sum F_x = T - 50g \sin 60^0 = 50a$$
$$\Rightarrow T - 50 \ \text{X} \ g \ \text{X} \ \frac{\sqrt{3}}{2} = 50a$$

Adding two equations, we have :

$$\Rightarrow 50 \ \text{X} \ g \ \ 1 - \frac{\sqrt{3}}{2} \ \ = 150a$$
$$\Rightarrow a = \frac{50 \text{ X } 10\text{X}(2-1.73)}{150\text{X}2} = 0.45 \, m/s^2$$

Positive value indicates that the chosen direction of acceleration is correct. Thus, the block “A” moves down and the block “B” moves up as considered.

Pulleys

A pulley is a part of a convenient arrangement that makes it possible to transfer force with change of direction. Unless otherwise stated, a pulley is considered to have negligible mass and friction. This is a relative approximation with respect to mass and friction involved with other elements. Pulleys are used in different combination with other elements – almost always with strings and blocks.

It is relatively difficult to fetch a bucket of water from a well with a “string” as compared to a “pulley and string” system. The basic consideration in making useful mechanical arrangements are two folds (i) improve the convenience of applying force and (ii) reduce the magnitude of force. The example of fetching of a bucket of water with “pulley and string” achieves the goal of improving convenience as we find it easier applying force in the level of arms horizontally rather than applying force vertically.

Had it been possible to reduce force for doing a mechanical activity, then that would have been wonder and of course against the well founded tenets of physical laws. What is meant here by reducing force is that we can fulfill a task (which comprises of force and motion) by reducing force at the expense of extending motion.

The important characterizing aspects of pulley are discussed in the sections named :

- Static or fixed pulley
- Moving pulley
- Combination or Multiple pulley system

Static pulley

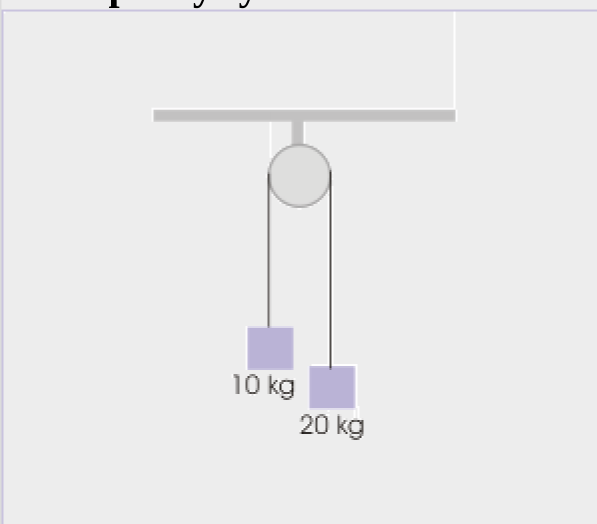
The pulley is fixed to a frame. In this situation, we are only concerned with the accelerations of the bodies connected to the string that passes over the pulley. Since string is a single piece inextensible element, the accelerations of the bodies attached to it are same.

We are at liberty to choose the direction of acceleration of the blocks attached to the pulley. A wrong choice will be revealed by the sign of acceleration that we get after solving equations. However, it is a straight forward choice here as it is obvious that the bigger mass will pull the blocks - string system down.

Example:

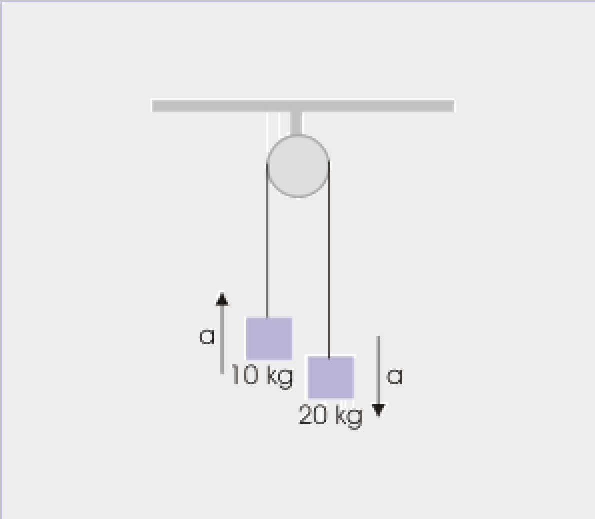
Problem : Two blocks of masses 10 kg and 20 kg are connected by a string that passes over a pulley as shown in the figure. Neglecting friction between surfaces, find acceleration of the blocks and tension in the string (consider $g = 10\, m/s^2$).

Static pulley system



Solution : The blocks are connected by a taut string. Hence, their accelerations are same. Let us assume directions of accelerations as shown in the figure. Also, let the magnitude of accelerations be “a”.

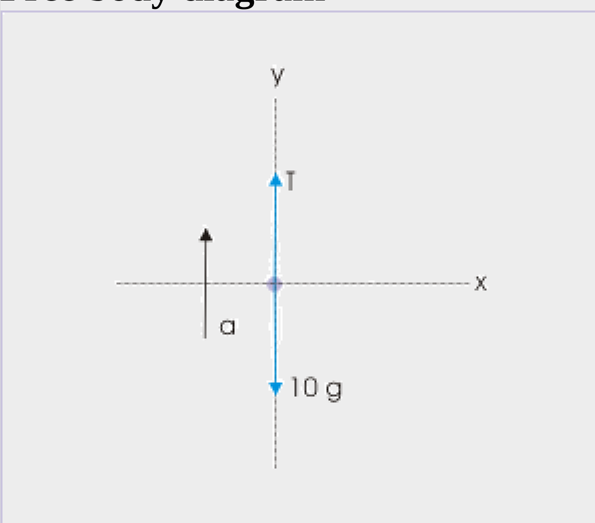
Static pulley system



Free body diagram of body of mass 10 kg

The external forces are (i) weight of block, 10g, and (ii) tension, T, in the string.

Free body diagram

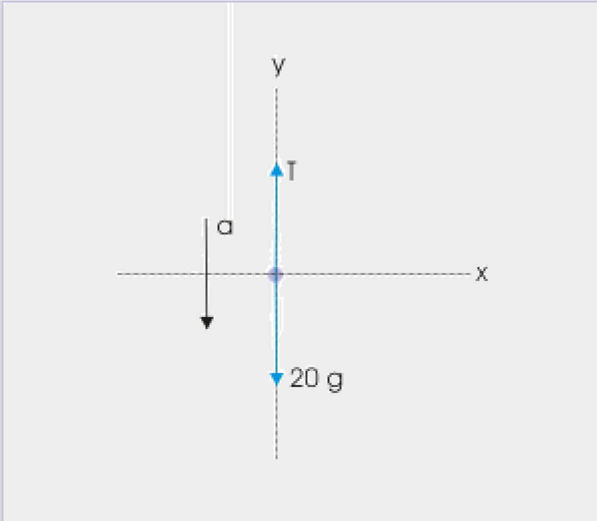


$$\sum F_y = T - 10g = 10a$$
$$\Rightarrow T - 10 \text{ x } 10 = 10a$$
$$\Rightarrow T = 100 + 10a$$

Free body diagram of body of mass 20 kg

The external forces are (i) weight of block, 20g, and (ii) tension, T, in the string.

Free body diagram



$$\sum F_y = 20g - T = 20a$$
$$\Rightarrow 20 \text{ x } 10 - T = 20a$$
$$\Rightarrow T = 200 - 20a$$

From two equations, we have :

$$\Rightarrow 200 - 20a = 100 + 10a$$
$$\Rightarrow a = \frac{100}{30} = 3.33\, m/s^2$$

Putting value of "a" in either of above two equations, we determine tension as :

$$\Rightarrow T = 100 + 10 \text{ X } 3.3 = 133\; N$$

There is a very useful technique to simplify the solution involving "mass-less" string and pulley. As string has no mass, the motion of the block-string system can be considered to be the motion of a system comprising of two blocks, which are pulled down by a net force in the direction of acceleration.

Let us consider two blocks of mass " m_1 " and " m_2 " connected by a string as in the previous example. Let us also consider that $m_2 > m_1$ so that block

of mass " m_2 " is pulled down and block of mass " m_1 " is pulled up. Let "a" be the acceleration of the two block system.

Now the force pulling the system in the direction of acceleration is :

$$F = m_2 g - m_1 g = (m_2 - m_1) g$$

The total mass of the system is :

$$m = m_1 + m_2$$

Applying law of motion, the acceleration of the system is :
Equation:

$$a = \frac{F}{m} = \frac{(m_2 - m_1) g}{(m_1 + m_2)}$$

Clearly, this method to find acceleration is valid when the block - string system can be combined i.e. accelerations of the constituents of the system are same. We can check the efficacy of this technique, using the data of previous example. Here,

$$m_1 = 10 \quad kg; \quad m_2 = 20 \quad kg$$

The acceleration of the block is :

$$a = \frac{(m_2 - m_1) g}{(m_1 + m_2)} = \frac{(20 - 10) X 10}{(10 + 20)}$$

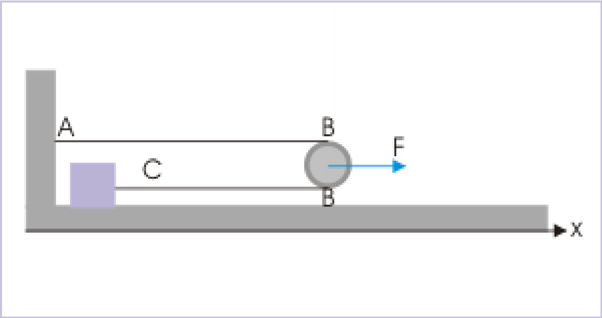
$$a = \frac{10}{3} = 3.33 \quad m/s^2$$

Moving pulley

Moving pulley differs to static pulley in one important respect. The displacements of pulley and block, which is attached to the string passing over it, may not be same. As such, we need to verify this aspect while

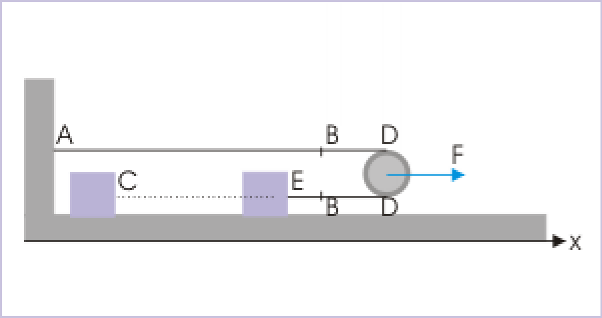
applying force law. The point is brought out with clarity in the illustration explained here. Here, we consider a block attached to a string, which passes over a mass-less pulley. The string is fixed at one end and the Pulley is pulled by a force in horizontal direction as shown in the figure.

Moving pulley system



In order to understand the relation of displacements, we visualize that pulley has moved a distance “x” to its right. The new positions of pulley and block are as shown in the figure. To analyze the situation, we use the fact that the length of string remains same in two situations. Now,

Moving pulley system



Length of the string, L, in two situations are given as :

$$L = \mathrm{AB} + \mathrm{BC} = \mathrm{AB} + \mathrm{BD} + \mathrm{ED}$$

$$\Rightarrow \mathrm{BC} = \mathrm{BD} + \mathrm{ED}$$

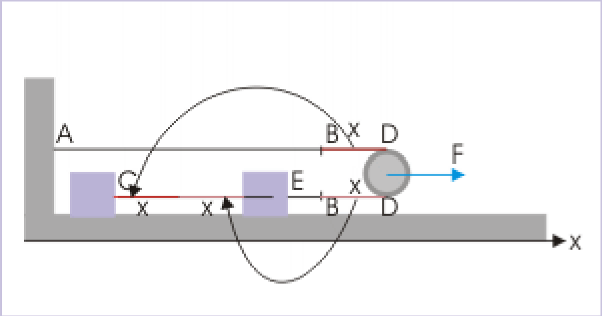
$$\Rightarrow \mathrm{ED} = \mathrm{BC} - \mathrm{BD}$$

Now, displacement of the block is :

$$\Rightarrow \mathrm{CE} = \mathrm{CD} - \mathrm{ED} = \mathrm{CD} - \mathrm{BC} + \mathrm{BD} = \mathrm{BD} + \mathrm{BD} = 2\mathrm{BD} = 2x$$

Note that for every displacement “x” of pulley, the displacement of block is 2x. We can appreciate this fact pictorially as shown in the figure below :

Moving pulley system



Further, as acceleration is second derivative of displacement with respect to time, the relation between acceleration of the block (a_B) and pulley (a_P) is :

Equation:

$$\Rightarrow a_B = 2a_P$$

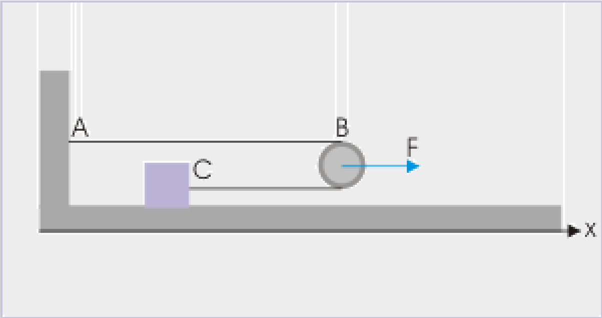
This is an important result that needs some explanation. It had always been emphasized that the acceleration of a taut string is always same through out its body. Each point of a string is expected to have same velocity and acceleration! What happened here ? One end is fixed, while other end is moving with acceleration. There is, in fact, no anomaly. Simply, the acceleration of the pulley is also reflected in the motion of the loose end of the string as they are in contact and that the motion of the string is affected by the motion of the pulley.

But the point is made. The accelerations of two ends of a string need not be same, when in contact with a moving body with a free end.

Example:

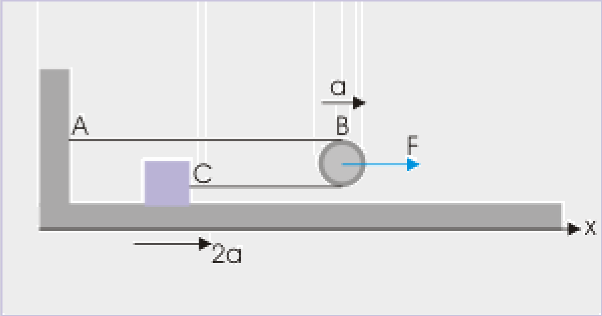
Problem : A block of mass, “m” is connected to a string, which passes over a smooth pulley as shown in the figure. If a force “F” acts in horizontal direction, find the accelerations of the pulley and block.

Moving pulley system



Solution : Let us consider that the acceleration of pulley is “a” in the direction of applied force. Now as analyzed before, acceleration of the block is “2a” and is in the same direction as that of pulley.

Moving pulley system

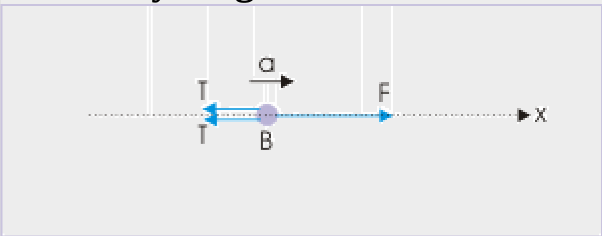


As motion is confined to x-direction, we draw free body diagram considering forces in x-direction only.

Free body diagram of pulley

The external forces are (i) force, F, (ii) tension, T, in the string and (ii) tension, T, in the string.

Free body diagram



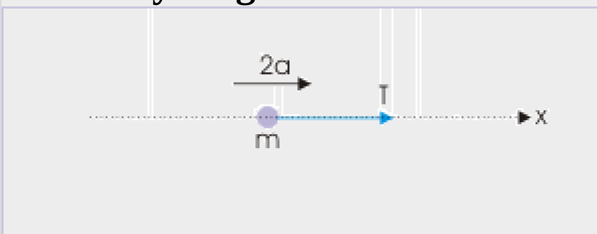
$$\sum F_x = F - 2T = m_p a_p$$

As mass of the pulley is zero,

$$\Rightarrow F = 2\mathrm{T}$$

$$\Rightarrow T = \frac{F}{2}$$

This is an unexpected result. The pulley is actually accelerated, but the forces on it form a balanced force system. This apparent contradiction of force law is due to our approximation that pulley is “mass-less”.
Free body diagram of block
The external force is (i) tension, T, in the string.
Free body diagram



$$\sum F_x = T = ma_x = 2ma$$

Combining two equations, we have :

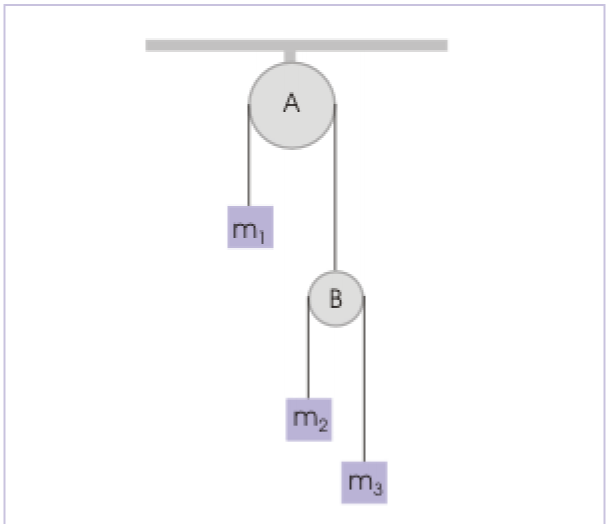
$$\Rightarrow a = \frac{F}{4m}$$

Thus, acceleration of the pulley is F/4m and that of block is F/2m.

Combination or Multiple pulley system

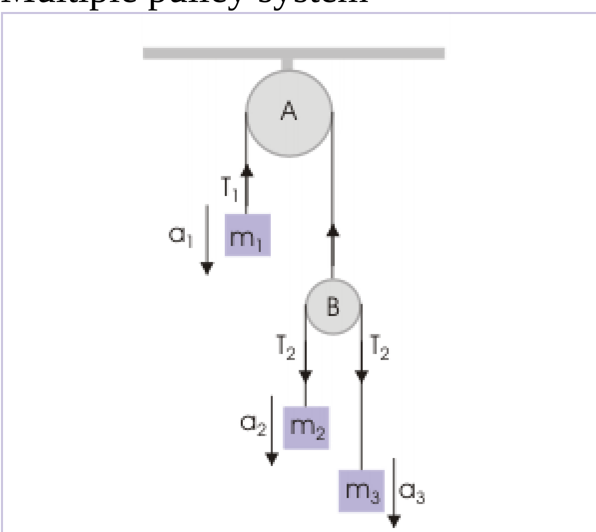
Multiple pulleys may involve combination of both static and moving pulleys. This may involve combining characterizing aspects of two systems.

Let us consider one such system as shown in the figure.
Multiple pulley system



Let us consider that accelerations of the blocks are as shown in the figure. It is important to note that we have the freedom to designate direction of acceleration without referring to any other consideration. Here, we consider all accelerations in downward direction.

Multiple pulley system



We observe that for given masses, there are five unknowns $a_1,\ a_2,\ a_3,\ T_1$ and T_2 . Whereas the free body diagram corresponding to three blocks provides only three independent relations. Thus, all five unknowns can not be evaluated using three equations.

However, we can add two additional equations; one that relates three accelerations and the one that relates tensions in the two strings. Ultimately,

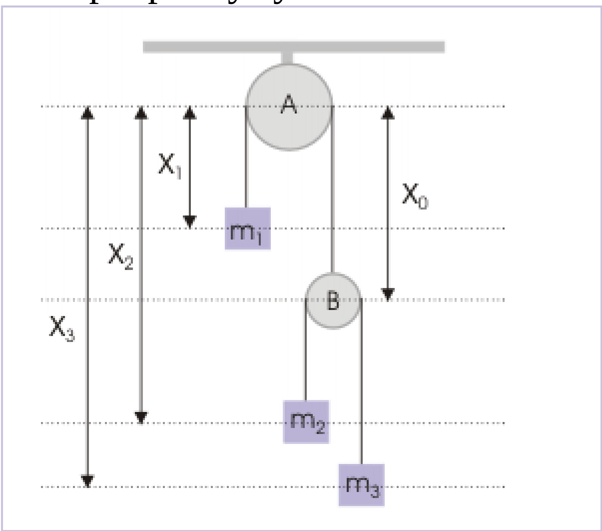
we get five equations for five unknown quantities.

1: Accelerations

The pulley “A” is static. The accelerations of block 1 and pulley “B” are, therefore, same. The pulley “B”, however, is moving. Therefore, the accelerations of blocks 2 and 3 may not be same as discussed for the case of static pulley. The accelerations of blocks 2 and 3 with respect to moving pulley are different than their accelerations with respect to ground reference.

We need to know the relation among accelerations of block 1, 2 and 3 with respect ground. In order to obtain this relation, we first establish the relation among the positions of moving blocks and pulley with respect to some fixed reference. Then we can obtain relation among accelerations by taking second differentiation of position with respect to time. Important to understand here is that these positions are measured with respect to a fixed reference. As such, their differentiation will yield accelerations of the blocks with respect to fixed reference i.e. ground reference. Let the positions be determined from a horizontal datum drawn through the static pulley as shown in the figure.

Multiple pulley system



Now, the lengths of the two strings are constant. Let they be L_1 and L_2 .

$$L_1 = x_0 + x_1$$
$$L_2 = (x_2 - x_0) + (x_3 - x_0) = x_2 + x_3 - 2x_0$$

Eliminating x_0 , we have :

$$\Rightarrow L_2 = x_2 + x_3 - 2L_1 + 2x_1$$
$$\Rightarrow x_2 + x_3 + 2x_1 = 2L_1 + L_2 = \text{constant}$$

This is the needed relation for the positions. We know that acceleration is second derivative of position with respect to time. Hence,

Equation:

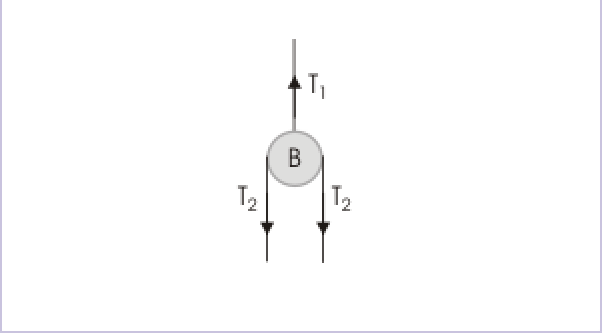
$$\Rightarrow a_2 + a_3 + 2a_1 = 0$$

This gives the relation of accelerations involved in the pulley system.

2: Tensions in the strings

We can, now, find the relation between tensions in two strings by considering the free body diagram of pulley "B" as shown in the figure.

Free body diagram



Here,

Equation:

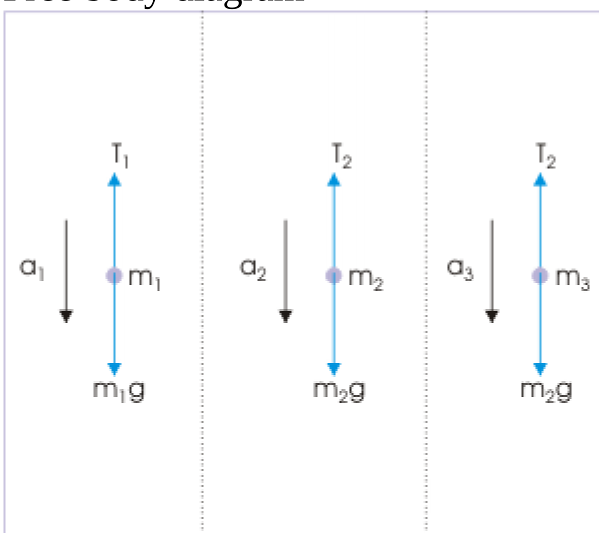
$$T_1 = 2T_2$$

Note: This result appears to be simple and on expected line. But it is not so. Note that pulley "B" itself is accelerated. The result, on the other hand, is exactly same as for a balanced force system. In fact this equality of forces in opposite direction is possible, because we have considered that pulley has negligible mass. This aspect has been demonstrated in the force analysis of the example given earlier (you may go through the example again if you have missed the point).

3: Free body diagrams of the blocks

The free body diagrams of the blocks are as shown in the figure.

Free body diagram



Equation:

$$m_1 g - T_1 = m_1 a_1$$
$$m_2 g - T_2 = m_2 a_2$$
$$m_3 g - T_2 = m_3 a_3$$

Thus, we have altogether 5 equations for 5 unknowns.

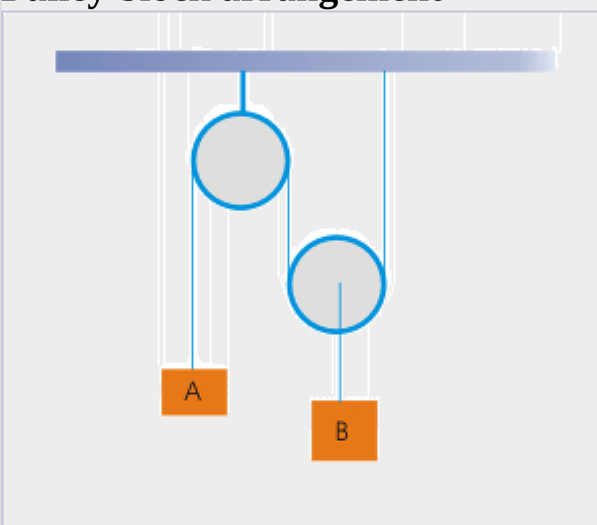
There is one important aspect of the motions of blocks of mass " m_2 " and " m_3 " with respect to moving pulley "B". The motion of blocks take place with respect to an accelerating pulley. Thus, interpretation of the

acceleration must be specific about the reference (ground or moving pulley). We should ensure that all measurements are in the same frame. In the methods, described above we have considered accelerations with respect to ground. Thus, if acceleration is given with respect to the moving pulley in an analysis, then we must first change value with respect to the ground.

Example:

Problem : In the arrangement shown in the figure, mass of A = 5 kg and mass of B = 10 kg. Consider string and pulley to be mass-less and friction absent everywhere in the arrangement. Find the accelerations of blocks.

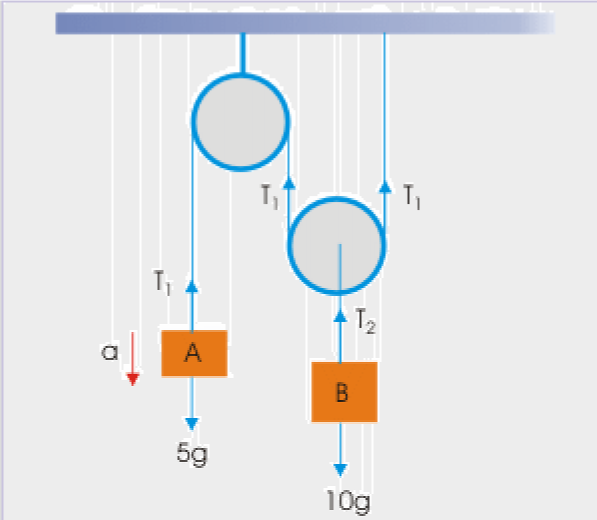
Pulley block arrangement



Pulley block arrangement

Solution : Let the acceleration of block "A" be "a" in the downward direction. Let the tensions in the string be "T_1" and "T_2". See figure below showing forces acting on the blocks and moving pulley. The force analysis of the block "A" yields :

Pulley block arrangement



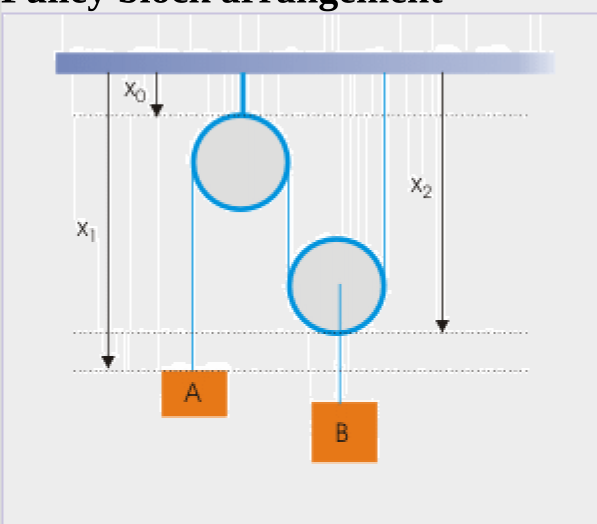
Pulley block arrangement

$$\Rightarrow 5g - T_1 = 5a$$

$$\Rightarrow 50 - T_1 = 5a$$

From constraint of string length, we see that :

Pulley block arrangement



Pulley block arrangement

$$x_1 - x_0 + x_2 - x_0 + x_2 = L$$

$$\Rightarrow x_1 + 2x_2 = L + 2x_0 = \text{Constant}$$

Differentiating twice with respect to time, we get relation between accelerations of two blocks as :

$$\Rightarrow a_1 = -2a_2$$

$$\Rightarrow a_2 = -\frac{a_1}{2} = -\frac{a}{2}$$

Force analysis of the moving mass-less pulley yields :

$$\Rightarrow T_2 = 2T_1$$

Force analysis of the block B results (refer to the force diagram shown in the beginning) :

$$\Rightarrow 2T_1 - 10X10 = 10X\frac{a}{2} = 5a$$

$$\Rightarrow 2T_1 - 100 = 5a$$

Simultaneous equations of forces on blocks A and B are :

$$\Rightarrow 50 - T_1 = 5a$$

$$\Rightarrow 2T_1 - 100 = 5a$$

Solving for "a", we have :

$$\Rightarrow a = 0$$

Thus, accelerations of two blocks are zero. Note, however, that tensions in the strings are not zero.

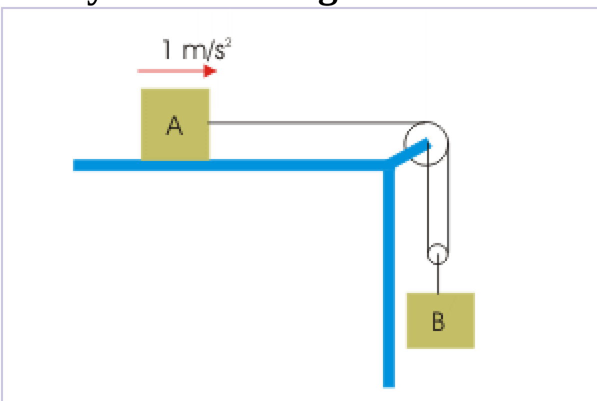
Exercises

Exercise:

Problem:

In the arrangement shown, the acceleration of block “A” is 1 m/s^2 . Consider string and pulley to be mass-less and friction absent everywhere in the arrangement. Then,

Pulley block arrangement



Pulley block arrangement

(a) The acceleration of block “B” is 1 m/s^2 .

(b) The acceleration of block “B” is 2 m/s^2 .

(c) The acceleration of block “B” is 0.5 m/s^2 .

(d) The acceleration of block “B” is 0.75 m/s^2 .

Solution:

We are required to determine acceleration of block "B" to answer this question. By inspection, we observe that if block “A” moves “x” distance towards right, then the string length is distributed in two equal halves about hanging pulley. Clearly, if “A” moves by “x”, then “B” moves by x/2. As such, acceleration of “A” is twice that of “B”. As it

is given that acceleration of “A” is 1 m/s^2, the acceleration of “B” is 2 m/s^2.

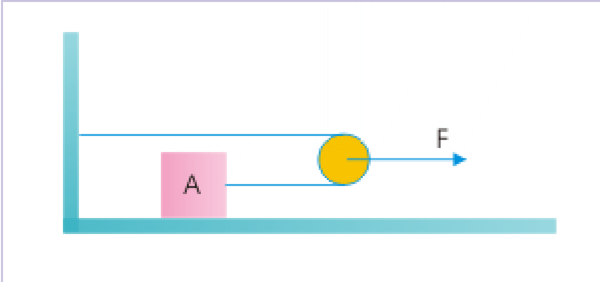
Hence, option (b) is correct.

Exercise:

Problem:

Given : mass of block “A” = 0.5 kg and acceleration of pulley = 1 m/s^2 . Consider string and pulley to be mass-less and friction absent everywhere in the arrangement. Then,

Pulley block arrangement



Pulley block arrangement

(a) The force “F” is 1 N.

(b) The force “F” is 2 N.

(c) The tension in the string is 1 N.

(d) The tension in the string is 0.5 N.

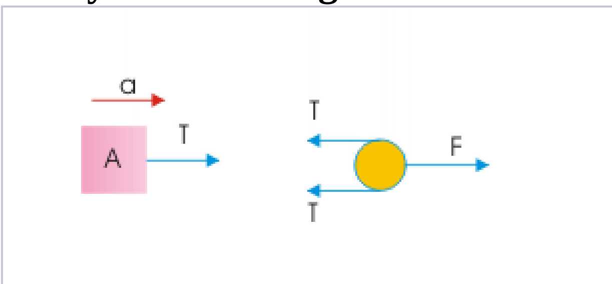
Solution:

Let the acceleration of block “A” is “a”. We know that the acceleration of block “A” is two times that of pulley. Hence,

$$a = 1 \text{ X } 2 = 2 \quad m/s^2$$

Let the tension in the string be “T”. Considering forces on block “A”, we have :

Pulley block arrangement



Pulley block arrangement

$$T = ma = 0.5X2 = 1 \quad N$$

Considering forces on the mass-less pulley, we have :

$$\Rightarrow F = 2T = 2X1 = 2 \quad N$$

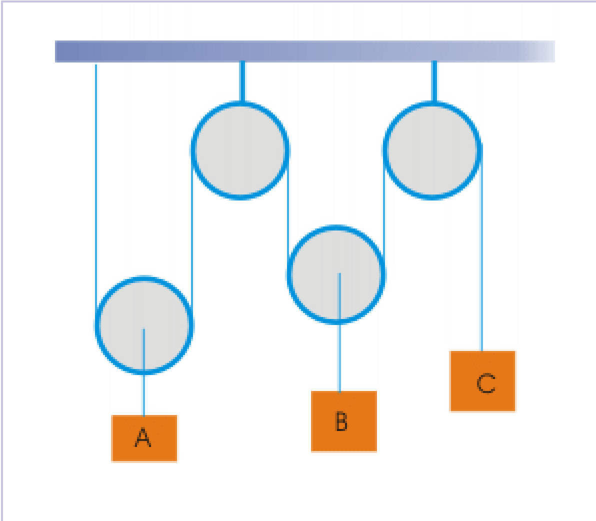
Hence, options (b) and (c) are correct.

Exercise:

Problem:

In the arrangement shown in the figure, accelerations of blocks A,B and C are a_A , a_B and a_C respectively. Consider string and pulley to be mass-less and friction absent everywhere in the arrangement. Then

Blocks and pulleys arrangement



Blocks and pulleys arrangement

$$(a)\ 2a_A + 2a_B + a_C = 0$$

$$(b)\ a_A + a_B + 2a_C = 0$$

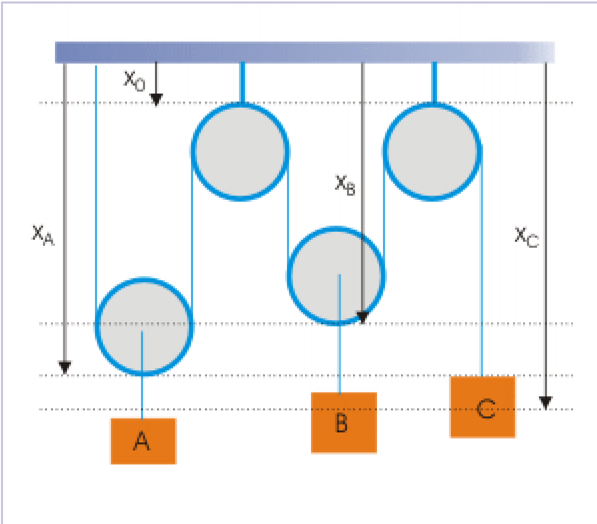
$$(c)\ a_A + a_B + a_C = 0$$

$$(d)\ 2a_A + a_B + a_C = 0$$

Solution:

Here, we need to determine relation of accelerations of blocks using constraint of string length.

Blocks and pulleys arrangement



Blocks and pulleys arrangement

$$x_A + x_A - x_0 + x_B - x_0 + x_B - x_0 + x_C - x_0 = L$$

$$\Rightarrow 2x_A + 2x_B + x_C = L + 3x_0 = \text{constant}$$

Differentiating two times, we have :

$$\Rightarrow 2a_A + 2a_B + a_C = 0$$

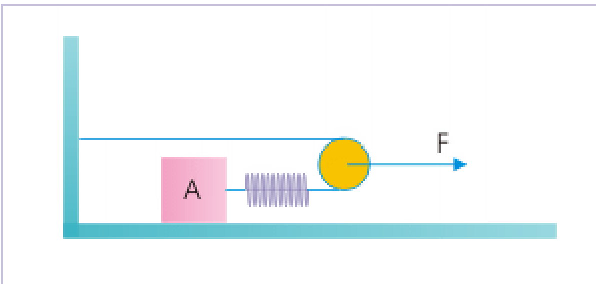
Hence, option (a) is correct.

Exercise:

Problem:

Given : mass of block “A” = 10 kg, spring constant = 200 N/m, extension in the string = 0.1 m. Consider string and pulley to be mass-less and friction absent everywhere in the arrangement. Then,

Pulley block arrangement



Pulley block arrangement

(a) The force “F” is 40 N.

(b) The acceleration of block A is 4 m/s^2 .

(c) The acceleration of pulley is 1 m/s^2

(d)The tension in the string is 10 N.

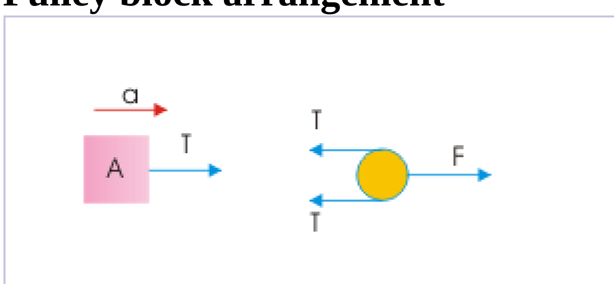
Solution:

Let “T” be the tension in the string. Here, spring force is equal to tension in the string.

$$T = kx = 200X0.1 = 20 \quad N$$

Let acceleration of block “A” is “a”. Considering forces on block “A”, we have :

Pulley block arrangement



Pulley block arrangement

$$\Rightarrow T = ma = 10Xa$$

$$\Rightarrow a = \frac{T}{10} = \frac{20}{10} = 2 \quad m/s^2$$

Considering forces on the mass-less pulley, we have :

$$\Rightarrow F = 2T = 2X20 = 40 \quad N$$

Now, acceleration of pulley is half of the acceleration of block. Hence, acceleration of pulley is 1 m/s^2 .

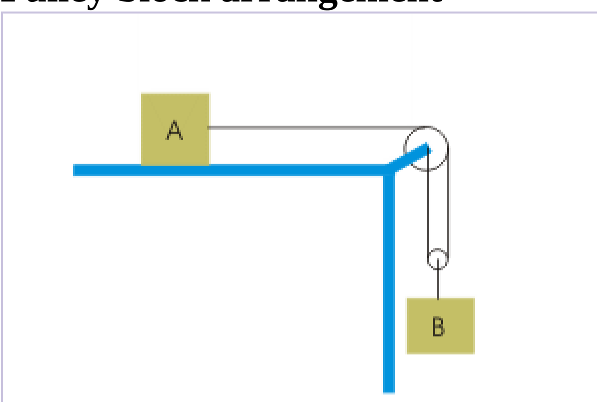
Hence, options (a) and (c) are correct.

Exercise:

Problem:

In the arrangement shown, the mass of “A” and “B” are 1 kg and 2 kg respectively. Consider string and pulley to be mass-less and friction absent everywhere in the arrangement. Then,

Pulley block arrangement



Pulley block arrangement

(a) The tension in the string connected to “A” is 1 N.

(b) The tension in the string connected to “A” is 20/3 N.

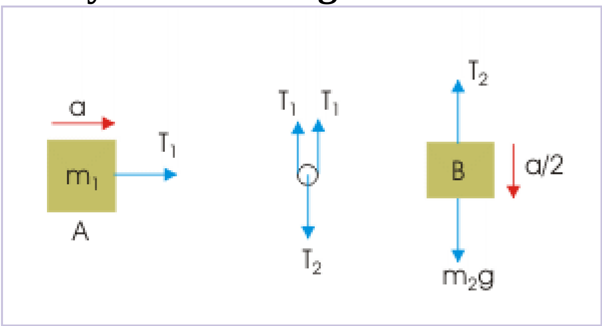
(c) The tension in the string connected to “B” is 2 N.

(d) The tension in the string connected to “B” is 20/3 N.

Solution:

Let the tensions in the strings connected to blocks “A” and “B” be “ T_1 ” and “ T_2 ” respectively. We see here that only force making block “A” to move on the table is tension in the string connected to it. Let acceleration of block “A” is “a”. From constraint relation, we know that the acceleration of “A” is twice the acceleration of “B”. Thus, acceleration of block “B” is “a/2”. Applying law of motion for block “A”, we have :

Pulley block arrangement



Pulley block arrangement

$$\Rightarrow T_1 = m_1 \text{ X } a = 1 \text{ X } a = a$$

Consideration of mass-less pulley,

$$\Rightarrow T_2 = 2T_1 = 2 \text{ X } a = 2a$$

Applying law of motion for block “A”, we have :

$$\Rightarrow m_2 g - T_2 = m_2 \text{ X } \frac{a}{2}$$

$$\Rightarrow 2 \text{ X } 10 - 2a = 2 \text{ X } \frac{a}{2} = a$$

$$\Rightarrow a = \frac{20}{3} \; m/s^2$$

Putting this value in the expressions of tensions, we have :

$$\Rightarrow T_1 = a = \frac{20}{3} \; N$$

$$\Rightarrow T_2 = 2a = \frac{40}{3} \; N$$

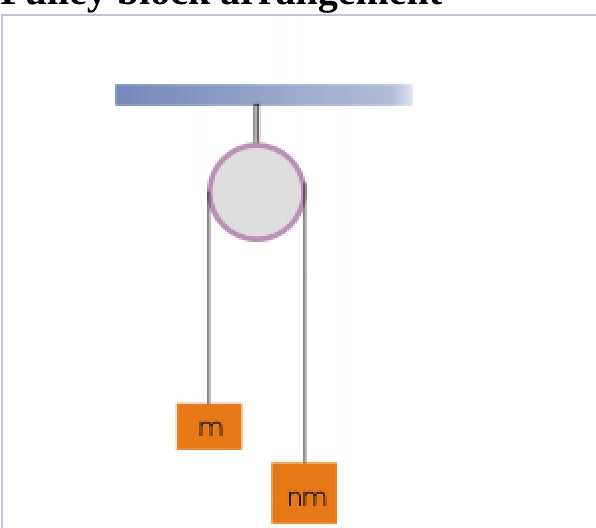
Hence, options (b) is correct.

Exercise:

Problem:

Two blocks of mass "m" and "nm" are hanging over a pulley as shown in the figure. Consider string and pulley to be mass-less and friction absent everywhere in the arrangement. If n>1 and acceleration of the blocks is g/3, then value of "n" is :

Pulley block arrangement



Pulley block arrangement

$$(a)\frac{3}{2} \quad (b)\frac{5}{3} \quad (c)3 \quad (d)2$$

Solution:

We consider blocks and string as one system. The net external force on the system is

$$F = nmg - mg = (n-1)mg$$

Total mass of the system, M, is :

$$M = nm + m = (n+1)m$$

The acceleration of the system i.e. the acceleration of either block is :

$$\Rightarrow a = \frac{(n-1)mg}{(n+1)m} = \frac{(n-1)g}{(n+1)}$$

According to question,

$$\Rightarrow a = \frac{(n-1)g}{(n+1)} = \frac{g}{3}$$

$$\Rightarrow 3n - 3 = n + 1$$

$$\Rightarrow 2n = 4$$

$$\Rightarrow n = 2$$

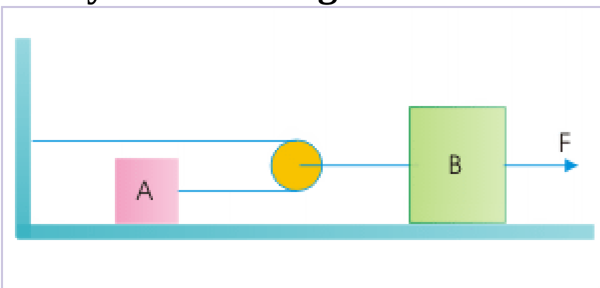
Hence, option (d) is correct.

Exercise:

Problem:

Given : mass of block “A” = 0.5 kg, mass of block “B”=1 kg and F = 2.5 N. Consider string and pulley to be mass-less and friction absent everywhere in the arrangement. Then,

Pulley block arrangement



Pulley block arrangement

(a) The acceleration of block “B” is 1 m/s^2 .

(b) The acceleration of block “A” is 1 m/s^2 .

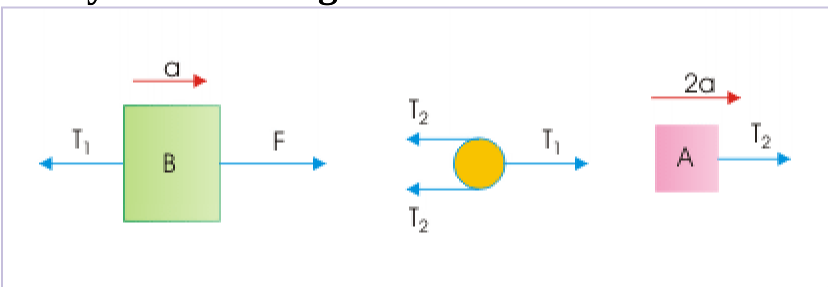
(c) The tension in the string connecting block “B” and pulley is 1 N.

(d) The tension in the string connected to block “A” is 2 N.

Solution:

Let the acceleration of block “B” is “a”. Considering forces on block of mass “B”, we have :

Pulley block arrangement



Pulley block arrangement

$$F - T_1 = Ma$$

$$\Rightarrow 2.5 - T_1 = 1Xa = a$$

Considering free body diagram of mass-less pulley,

$$\Rightarrow T_1 = 2T_2$$

The pulley and the block are connected by inextensible string. As such, their accelerations are same. Further, we know by constraint analysis that acceleration of block is twice that of pulley. Hence, acceleration of block “B” is “2a”. Considering forces on block “A”, we have :

$$\Rightarrow T_2 = mX2a = 0.5X2a = a$$

Thus,

$$\Rightarrow T_1 = 2Xa = 2a$$

Putting this value in the force equation of block “A”,

$$2.5 - 2a = 0.5a$$

$$\Rightarrow a = 1 \quad m/s^2$$

Clearly,

$$\Rightarrow T_1 = 2a = 2X1 = 2 \quad N$$

$$\Rightarrow T_2 = T_1/2 = 1 \quad N$$

Hence, option (a) is correct.

Pulleys(application-I)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints for solving problems

In some questions, we are required to find relation for the accelerations of different elements in the arrangement. We, therefore, need a framework or a plan to establish this relation. This relation is basically obtained by differentiating the relation of positions of the movable elements in the arrangement.

1. Draw reference, which is fixed. This reference can be the level of fixed pulley or the ground.
2. Identify all movable elements like blocks and pulleys (excluding static ones).
3. Assign variables for the positions of movable elements from the chosen reference.
4. Write down constraint relations for the positions of the elements. Usually, total length of the string is the “constraint” that we need to make use for writing relation for the positions.
5. Differentiate the relation for positions once to get relation of velocities and twice to get relation of accelerations.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the pulleys. The questions are categorized in terms of the characterizing features of the subject matter :

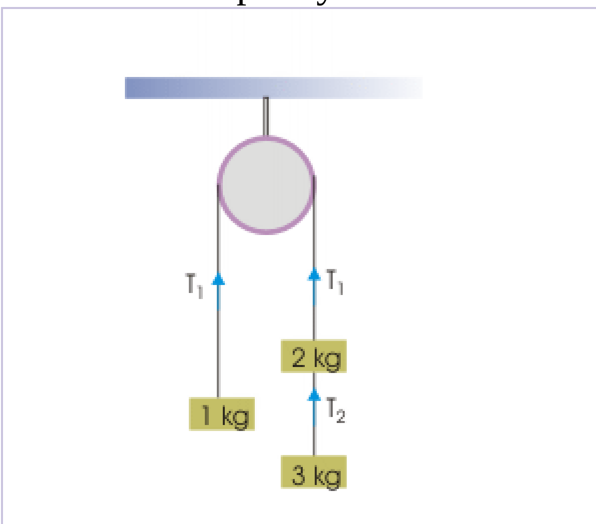
- Static or fixed pulley
- Moving pulley

- Combination or Multiple pulley system

Static or fixed pulley

Problem 1 : Pulley and strings are "mass-less" and there is no friction involved in the arrangement. Find acceleration of the block of mass 3 kg and tensions " T_1 " and " T_2 " as shown in the figure.

Static or fixed pulley



Solution 1 : We observe here that two strings are taught. Therefore, acceleration of the all constituents of the system is same. This situation can be used to treat the system as composed of three blocks only (we neglect string as it has no mass). The whole system is pulled down by a net force as given by :

$$F = 2g + 3g - g = 4g$$

Total mass of the system is :

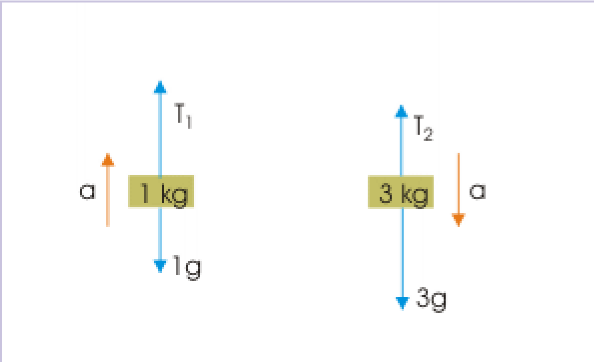
$$m = 1 + 2 + 3 = 6 \quad kg$$

Applying law of motion, the acceleration of the system and hence acceleration of the block of mass 3 kg is :

$$a = \frac{F}{m} = \frac{4g}{6} = \frac{40}{6} = \frac{20}{3} \quad m/s^2$$

In order to find tension " T_1 ", we consider FBD of mass of 1 kg. Here,

Free body diagrams



$$\sum F_y = T_1 - 1Xg = ma_y = 1X\frac{20}{3}$$

$$\Rightarrow T_1 = \frac{20}{3} + 10 = \frac{50}{3} \quad N$$

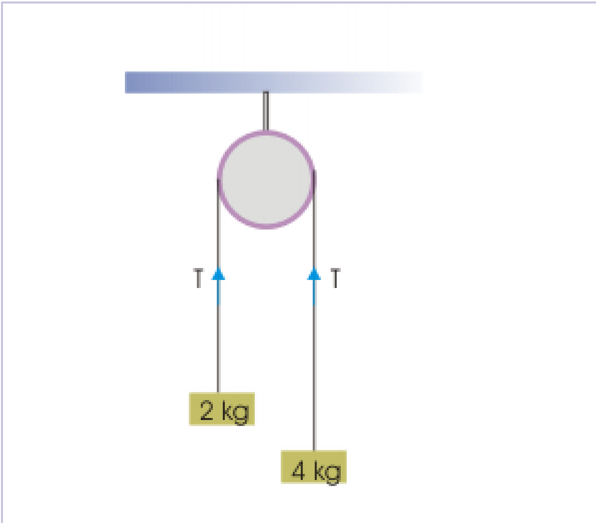
In order to find tension " T_2 ", we consider FBD of mass of 3 kg. Here,

$$\sum F_y = 3Xg - T_2 = ma_y = 3X\frac{20}{3} = 20$$

$$\Rightarrow T_2 = 3X10 - 20 = 10 \quad N$$

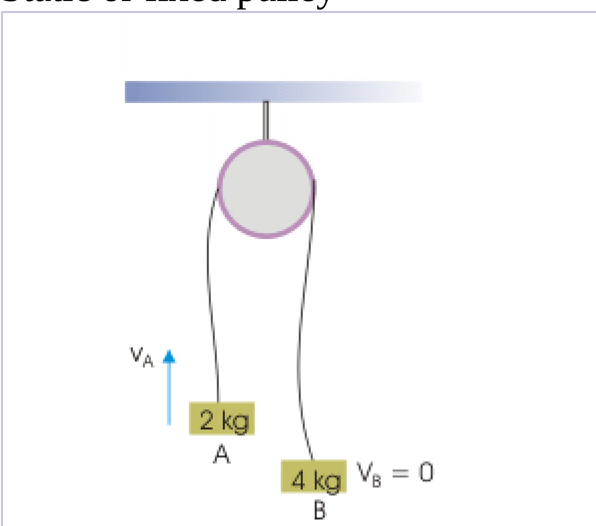
Problem 2 : Pulley and string are "mass-less" and there is no friction involved in the arrangement. The blocks are released from rest. After 1 second from the start, the block of mass 4 kg is stopped momentarily. Find the time after which string becomes tight again.

Static or fixed pulley



Solution 1 : We need to understand the implication of stopping the block. As soon as block of 4 kg (say block “B”) is stopped, its velocity becomes zero. Tension in the string disappears. For the other block of 2 kg (say block “A”) also, the tension disappears.

Static or fixed pulley



The block “A”, however, has certain velocity with which it is moving up. It will keep moving up against the force of gravity. On the other hand, block “B” will resume its motion of a free fall under gravity.

The string will become taught again, when the displacements of two blocks equal after the moment the block “B” was stopped momentarily. Let “t” be that time, then :

$$v_A t - \frac{1}{2}gt^2 = 0 + \frac{1}{2}gt^2$$

$$\Rightarrow v_A - \frac{1}{2}gt = \frac{1}{2}gt$$

$$\Rightarrow t = \frac{v_A}{g}$$

Clearly, we need to know the velocity of the block “A”, when block “B” was stopped. It is given in the question that the system of blocks has been in motion for 1 s. Now, the acceleration of the system (hence that of constituent blocks) is :

$$a = \frac{4g - 2g}{6} = \frac{20}{6} = \frac{10}{3} \quad m/s^2$$

Applying equation of motion for block “A”, we have :

$$v_A = u_A + at = 0 + \frac{10}{3}X1 = \frac{10}{3} \quad m/s$$

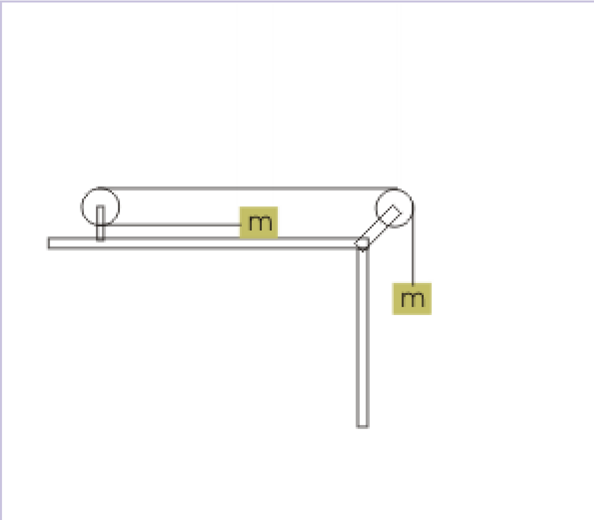
Now putting values in the equation for time, we have :

$$\Rightarrow t = \frac{v_A}{g} = \frac{10}{3X10} = \frac{1}{3} \quad s$$

Moving pulley

Problem 3 : Pulleys and string are “mass-less” and there is no friction involved in the arrangement. The two blocks have equal mass “m”. Find acceleration of the blocks and tension in the string.

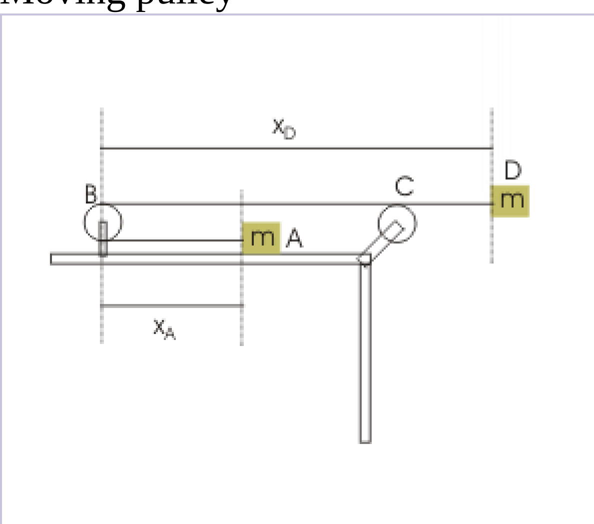
Moving pulley



Solution 1 : In order to determine accelerations, we shall write constraint equation to find the relation for the accelerations of two blocks. Going by the hints given in the module, we see that pulley on the table is fixed and can be used as reference. However, one of the blocks is hanging and is not in the level of other movable block on the table.

To facilitate writing of constraint relation, we imagine here that hanging block "D" is raised on the same level as block "A". Now, we assign variables to denote positions of two movable elements as shown in the figure.

Moving pulley



$$x_A + x_D = L$$

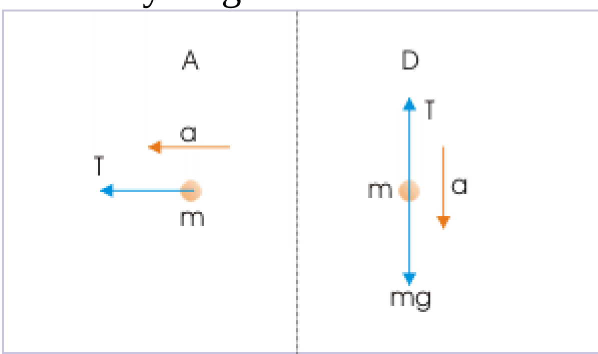
Differentiating twice,

$$\Rightarrow a_A + a_D = 0$$

$$\Rightarrow a_A = -a_D$$

This means that the accelerations of two blocks are same. Since string is single piece, tension in the string is same. Considering free body diagrams, we have :

Free body diagrams



Forces on blocks "A" and "D"

For A :

$$T = ma$$

For D :

$$mg - T = ma$$

Combining, we have :

$$mg - ma = ma$$

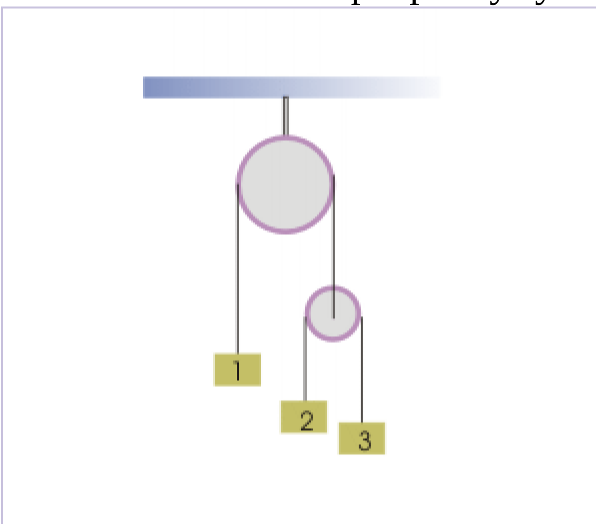
$$\Rightarrow a = \frac{g}{2}$$

$$\Rightarrow T = ma = \frac{mg}{2}$$

Combination or Multiple pulley system

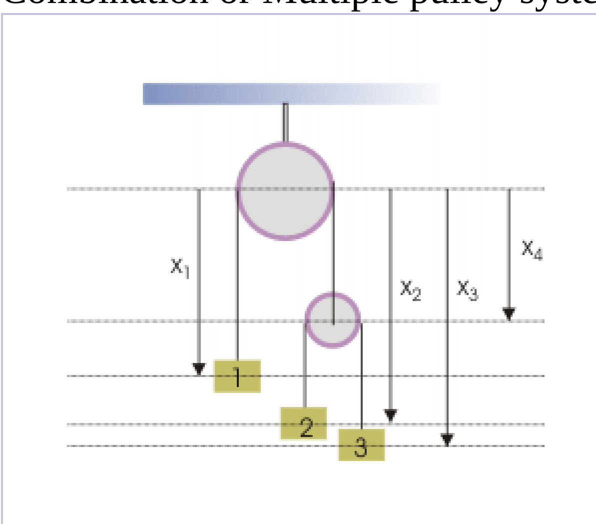
Problem 4 : Pulleys and string are "mass-less" and there is no friction involved in the arrangement. If at a given time velocities of block "1" and "2" are 2 m/s upward. Find the velocity of block "3" at that instant.

Combination or Multiple pulley system



Solution 1 : We see here that velocities of block "1" and "2" are given and we have to find the velocity of remaining block "3". It is apparent that if we have the relation of velocities of movable blocks, then we can get the result. We draw the reference as a horizontal line through the center of the static pulley and denote variables for all the movable elements of the system. Let " L_1 " and " L_1 " be the lengths of two strings, then :

Combination or Multiple pulley system



$$x_1 + x_4 = L_1$$

$$x_2 - x_4 + x_3 - x_4 = L_2$$

Differentiating once, we get the relations for velocities of the movable elements,

$$\Rightarrow v_1 + v_4 = 0$$

$$\Rightarrow v_2 - v_4 + v_3 - v_4 = 0$$

We need to eliminate " v_4 " to get the required relation of velocities of the blocks. Rearranging second equation, we have :

$$\Rightarrow v_2 + v_3 - 2v_4 = 0$$

Substituting for " v_4 " from the first equation,

$$\Rightarrow v_2 + v_3 - 2(-v_1) = 0$$

$$\Rightarrow v_2 + v_3 + 2v_1 = 0$$

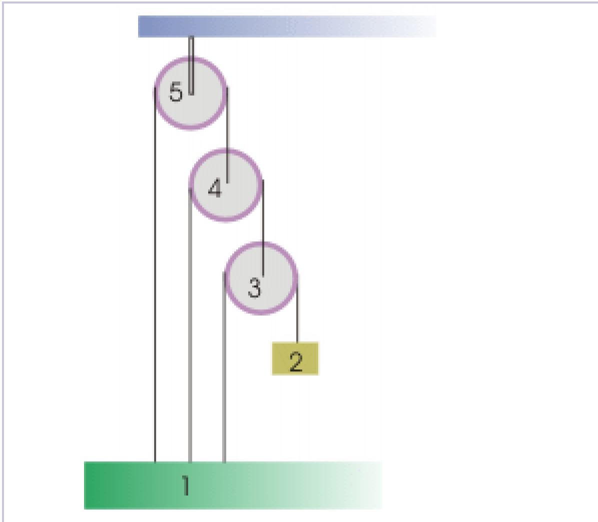
Now, considering upward direction as positive, $v_2 = v_3 = 2 \quad m/s$. Putting these values in the relation for velocities,

$$\Rightarrow 2 + v_3 + 2X2 = 0$$

$$\Rightarrow v_3 = -6 \quad m/s$$

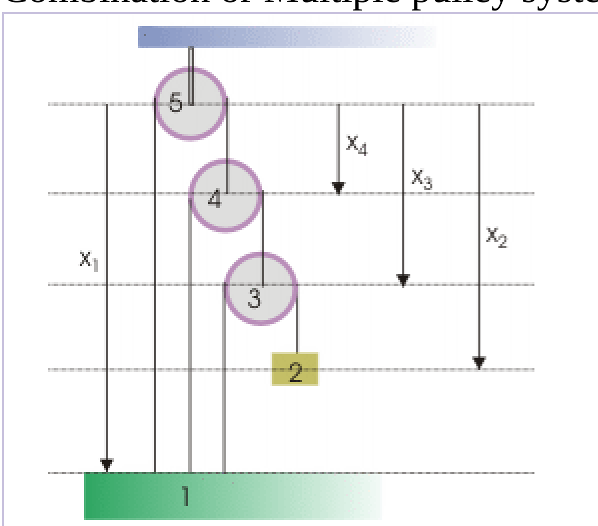
Negative sign means that velocity of the block “3” is vertically downward.

Problem 5 : Pulleys and string are “mass-less” and there is no friction involved in the arrangement. Find the relation for the accelerations between the hanging plank (marked “1”) and the block (marked “2”).
Combination or Multiple pulley system



Solution 1 : In order to obtain the relation for the accelerations of given elements, we first need to develop constraint relations for the three string having fixed lengths. For this, we choose a horizontal reference through the center of topmost fixed pulley as shown in the figure.

Combination or Multiple pulley system



With reference to positions as shown in the figure, the constraint relations are :

$$x_1 + x_4 = L_1$$

$$x_1 - x_4 + x_3 - x_4 = L_2$$

$$x_1 - x_3 + x_2 - x_3 = L_3$$

Differentiating above relations twice with respect to time, we have three equations :

$$\Rightarrow a_1 + a_4 = 0$$

$$\Rightarrow a_1 - a_4 + a_3 - a_4 = 0$$

$$\Rightarrow a_1 - a_3 + a_2 - a_3 = 0$$

Rearranging second and third equation,

$$\Rightarrow a_1 + a_2 - 2a_3 = 0$$

$$\Rightarrow a_1 + a_3 - 2a_4 = 0$$

Substituting for " a_3 " from second equation.

$$\Rightarrow a_1 + a_2 - 2(2a_4 - a_1) = 0$$

Substituting for " a_4 " from first equation,

$$\Rightarrow a_1 + a_2 - 2(-2a_1 - a_1) = 0$$

$$a_1 + a_2 + 6a_1 = 0$$

$$\Rightarrow a_2 + 7a_1 = 0$$

$$\Rightarrow a_2 = -7a_1$$

Thus, we need to pull the block a lot to raise the plank a bit. This is a mechanical arrangement that allows to pull a heavy plank with smaller force.

Pulleys (application - II)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

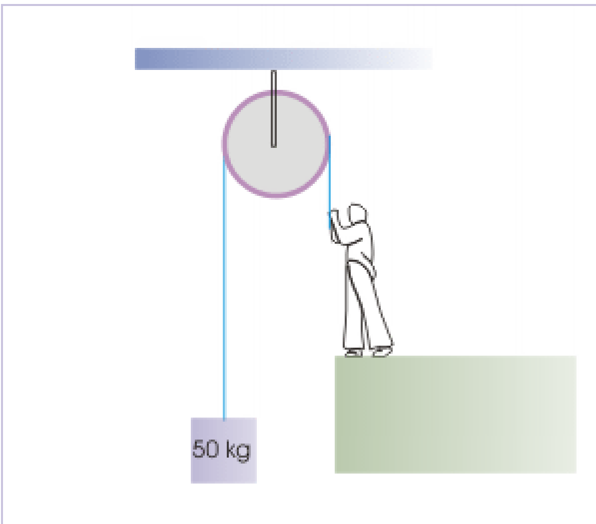
We discuss problems, which highlight certain aspects of the study leading to pulleys. The questions are categorized in terms of the characterizing features of the subject matter :

- Change of force direction
- External force on pulley system
- Horizontal pulley system
- Vertical pulley system
- Incline and pulley system

Change of force direction

Problem 1 : A person is pulling a block of 50 kg at a uniform speed with the help of a pulley and rope arrangement as shown in the figure. If the person weighs 60 kg, find the normal force that the person exerts on the surface underneath.

Change of force direction

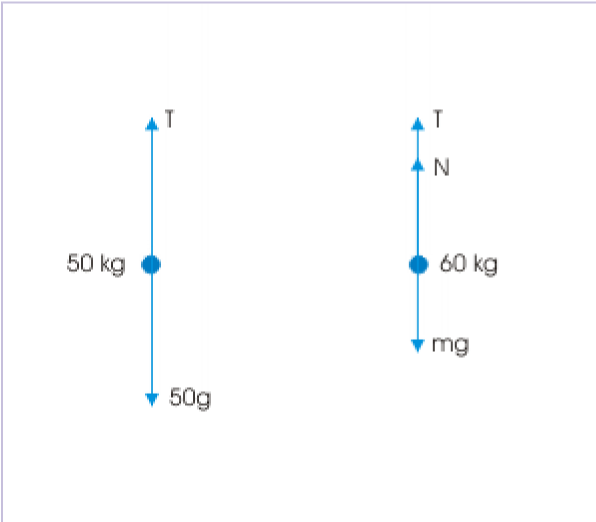


Applied force is opposite to the direction of force lifting the block.

Solution : In order to find normal force applied by the person on the surface underneath, we need to carry out force analysis of the external forces on the person. The surface and the person apply normal force on each other, which is equal in magnitude but opposite in direction. The forces on the person are (i) weight, “mg”, acting downward (ii) normal force, “N”, acting upward and (iii) tension, “T”, acting upward.

The free body diagram of the person is shown on the right side of the figure shown below.

Free body diagrams



Free body diagrams of block and boy are shown.

$$\sum F_y \Rightarrow T + N = mg$$

$$\Rightarrow N = 60 \text{ X } 10 - T = 600 - T$$

It is clear that we would need the value of tension, "T", to solve the equation for the normal force. We can find the value of "T" by considering the FBD of the block being raised. The FBD of the block is shown on the left side of the figure above. Note that we consider that person is raising the block without acceleration (at uniform velocity as given in the question).

$$\sum F_y = T - 50g = 0$$

$$\Rightarrow T = 50g = 50 \text{ X } 10 = 500 \quad N$$

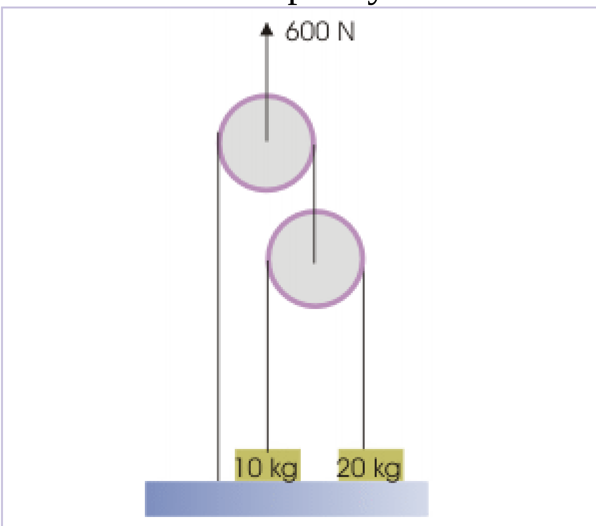
Putting this value in the equation for normal force, we have :

$$\Rightarrow N = 600 - T = 600 - 500 = 100 \quad N$$

External force on pulley system

Problem 2 : In the arrangement shown in the figure, the strings are taught initially. An external force of 600 N is applied in the vertically upward direction. Considering strings and pulleys to be "mass – less" and all contacts without friction, determine the accelerations of the blocks.

External force on pulley



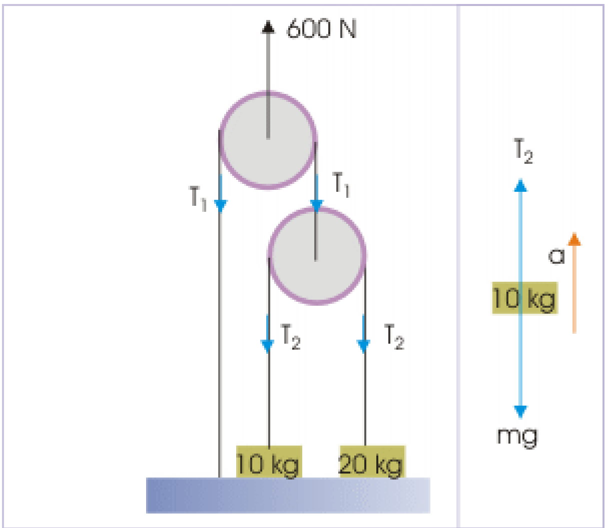
External force of 600 N acts upwards on the topmost pulley.

Solution : The forces on each of the blocks are (i) weight of the block (ii) tension in the string and (iii) normal force.

When a block is lifted, the normal force becomes zero as the contact between surfaces is broken. The external forces on the block are, then, only two, comprising of its weight and tension in the string. It is, therefore, clear that we need to know the tension in the string to know the accelerations of the blocks.

Let " T_1 " and " T_2 " be the tensions in the two strings as shown in the figure.

External force on pulley



Forces on the elements of the system are shown.

As the pulleys are “mass – less”, we have following force relations,

$$\Rightarrow T_1 = \frac{600}{2} = 300N$$

$$\Rightarrow T_2 = \frac{T_1}{2} = \frac{300}{2} = 150N$$

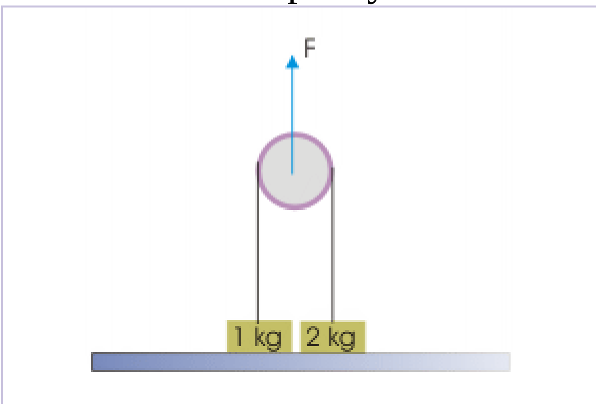
On the other hand, weights of the blocks are 100 N and 200 N. Since tension in the string is 150 N, only block of 10 kg with weight 100 N will move up. The other block will remain stationary. Force analysis on the mass of 10 kg in shown on the right hand side of the figure shown above. The acceleration of the block of mass 10 kg is :

$$\Rightarrow a = \frac{T_2 - mg}{m} = \frac{150 - 10X10}{10} = 5 \quad m/s^2$$

Problem 3 : Two blocks of mass 1 kg and 2 kg are connected with a string that passes over a “mass – less” pulley. A time dependent force "F = 20 t"

acts on the pulley as shown. Find the time when the block of mass 1 kg looses contact from the floor.

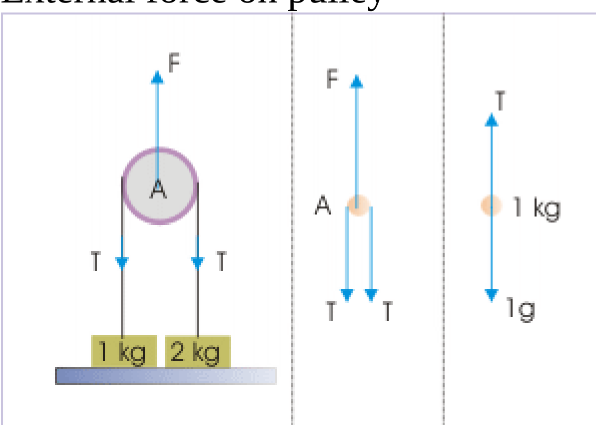
External force on pulley



A variable force acts upwards on the pulley.

Solution : The block of mass 1 kg is about to be lifted when tension in the string is equal to the weight of the block such that normal force between the surfaces becomes zero. This means that :

External force on pulley



A variable force acts upwards on the pulley.

$$\Rightarrow T = mg = 1X10 = 10N$$

On the other hand, the analysis of force on the pulley, "A", yields,

$$\Rightarrow F = 2T$$

$$\Rightarrow T = \frac{F}{2} = \frac{20t}{2} = 10t$$

Equating two equations, we have :

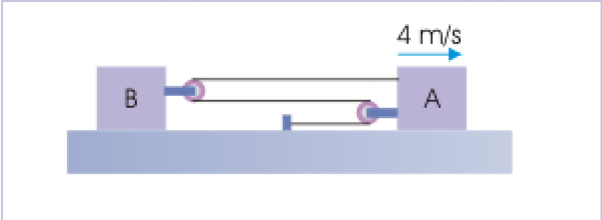
$$\Rightarrow 10t = 10$$

$$t = 1 \quad s$$

Horizontal pulley system

Problem 4 : In the arrangement shown in the figure, the block “A” moves with a velocity 4 m/s towards right. The string and the pulleys are “mass-less” and friction is absent everywhere. What is relative velocity of block “B” with respect to “A”?

Horizontal pulley system



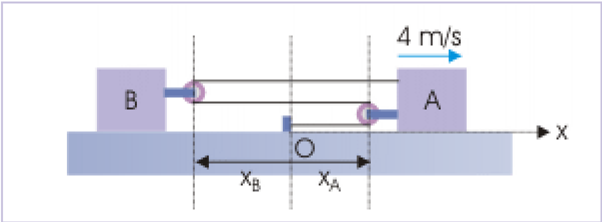
Pulley and block system.

Solution : The blocks are connected by a single string, which is intervened by two pulleys. We can connect velocities of blocks, if we have constraint relation for the length of string. First differentiation of the constraint relation will yield the required relation between velocities.

Since the pulleys are connected to blocks as their part (their relative positions do not change due to motion), we treat block and the attached

pulley as a single entity, whose positions are same. Following the procedure for writing constraint relation, we select the fixed end of the string as the reference point.

Constraint relation



Positions of moving parts from a fixed point.

From the figure,

$$x_A + (x_A + x_B) + (x_A + x_B) = L$$

$$\Rightarrow 3x_A + 2x_B = 0$$

Differentiating w.r.t time, we have :

$$\Rightarrow 3\frac{đx_A}{đt} + 2\frac{đx_B}{đt} = 0$$

There is an important subtle point here. The positions of blocks are on either side of the reference point (not on the same side as usually is the case). If positive direction of reference x-direction is towards right as shown in the figure, then velocities of two blocks are :

$$v_A = \frac{đx_A}{đt}$$

$$v_B = -\frac{đx_B}{đt}$$

Substituting in the equation, we have :

$$\Rightarrow 3v_A - 2v_B = 0$$

$$\Rightarrow v_B = \frac{3v_A}{2}$$

The sign of this relation (positive) denotes that velocities of blocks "A" and "B" are both in the reference direction of x-axis.

Relative velocity of “B” with respect to “A” is :

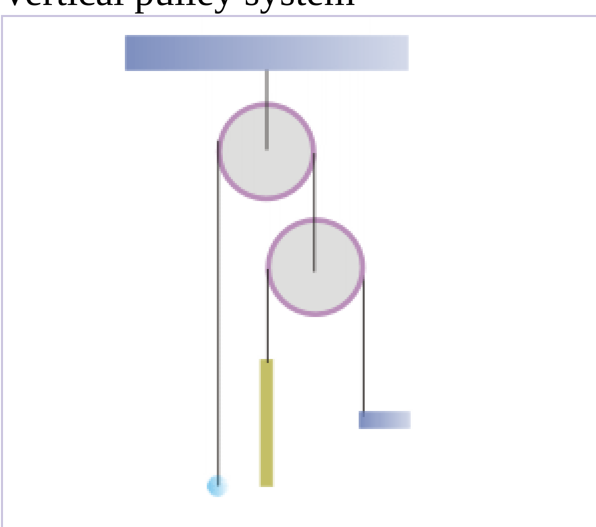
$$v_{BA} = v_B - v_A = \frac{3v_A}{2} - v_A$$

$$\Rightarrow v_{BA} = \frac{v_A}{2} = \frac{4}{2} = 2 \quad m/s \quad \text{towards right}$$

Vertical pulley system

Problem 5 : A ball and rod of mass “m” and “nm” respectively are held in positions as shown in the figure. The friction is negligible everywhere and mass of the pulleys and the strings are also negligible. At a certain time, the ball and rod are released simultaneously. If the length of the rod is “L”, what time would the ball take to reach the other end of the rod. Consider "n > 1".

Vertical pulley system



A ball travels past a vertical rod.

Solution : The ball and the moving pulley are connected through the same string that goes over a fixed pulley. As such, their accelerations are same. Now, the rod is hanging from a string that passes over the moving pulley and its other end is attached to a fixed point. From the constrain relation as worked out in the subject module Pulleys, we know that acceleration of the rod is twice that of the moving pulley. Hence, we can conclude that acceleration of the rod is twice as that of the ball. Let us, then, consider that ball and rod move with accelerations “a” and “2a” respectively.

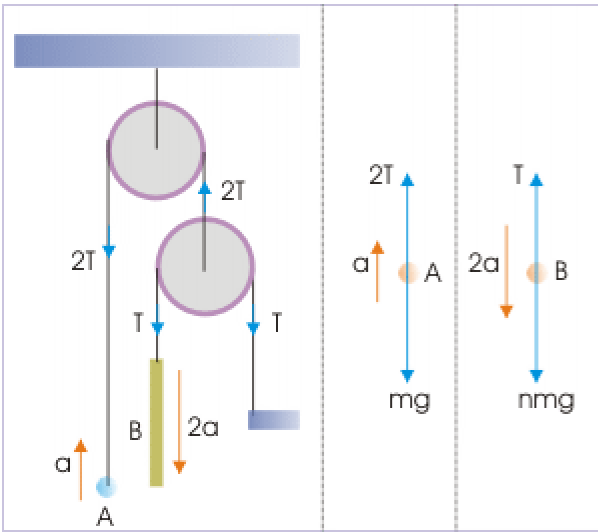
As the rod moves down (n >1), the ball moves up. The motions are in opposite direction. Since two elements are moving in opposite directions, the relative acceleration is arithmetic sum “a + 2a = 3a”. Now, using equation of motion for constant acceleration, we have :

$$L = \frac{1}{2}(3a)t^2$$

$$\Rightarrow T = \sqrt{\left(\frac{2L}{3a}\right)}$$

In order to evaluate this expression, we need to know acceleration “a”. From the free body diagram of the moving pulley, we note that the tension in the string attached to ball is twice the tension in the string attached to the rod.

Vertical pulley system



Forces on the elements of the system.

The free body diagram of the ball and the rod are shown on the right side of the above figure. The force analysis in the vertical directions yields following relations :

$$nmg - T = nmX2a$$

$$2T - mg = ma$$

Multiplying first equation by “2” and eliminating “T”, we have :

$$\Rightarrow 2nmg - mg = 4nma + ma = (4n + 1)ma$$

$$a = \frac{(2n - 1)g}{(4n + 1)}$$

Putting this value in the expression of time, we have :

$$t = \sqrt{\left(\frac{2L(4n + 1)}{3(2n - 1)g}\right)}$$

Working with moving pulleys

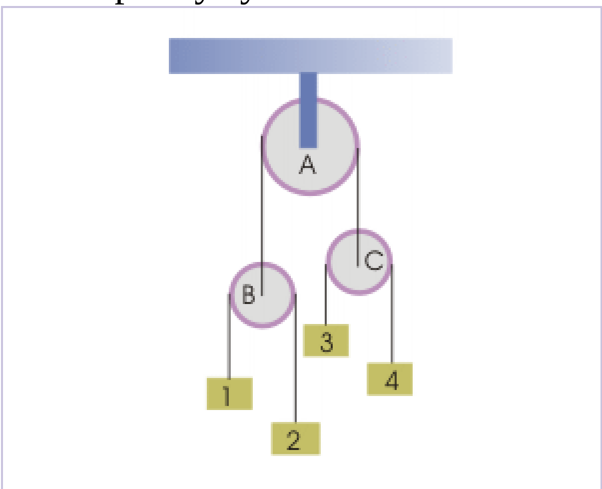
Analysis of motion with moving pulley relies on constraint relation and force analysis for moving entities like pulleys and blocks. This classic approach, however, has a serious problem. As there may be large numbers of blocks and other elements, the numbers of unknowns and corresponding numbers of available equations become very large and analysis of motion becomes very difficult.

Further, we consider acceleration of each movable part with reference to ground. Most of the time, it is difficult to guess the directional relationship among the accelerations of various entities of the system. In this module, we shall introduce few simplifying methods that allow us to extend the use of the simple framework of static pulley for analyzing motion of movable pulley.

Component pulley system

A pulley system may comprise of smaller component pulley systems. Take the case of a pulley system, which consists of two component pulley systems ("B" and "C") as shown in the figure.

Mixed pulley system



The pulley system comprises of two component pulley systems.

The component pulley system, usually, is a simple arrangement of two blocks connected with a single string passing over the pulley. The arrangement is alike static pulley system except that the pulley (“B” or "C") is also moving along with other constituents of the component systems like string and blocks.

If we could treat the moving pulley system static, then the analysis for the motion of blocks would become very easy. In that case, recall that the accelerations of the blocks connected with single string has equal magnitudes of accelerations, which are oppositely directed. This sense of oppositely directed acceleration, however, is not valid with moving pulley. A block, which appears to have a downward acceleration with respect to static pulley, say 1 m/s^2, will have an upward acceleration of 2 m/s^2 with respect to ground as pulley itself may have an upward acceleration of 3 m/s^2. This situation leads to a simplified framework for determining acceleration.

This simple framework of analysis involves two steps :

- Analyze for accelerations of blocks with respect to moving reference of pulley.
- Use concept of relative acceleration to determine accelerations of blocks with respect to ground.

For understanding the technique, we concentrate on one of the component systems that of pulley "B" as shown in the figure above. Let “ a_1 ” and “ a_2 ” denote accelerations of the two blocks with respect to ground reference. Let “ a_B” denotes the acceleration of pulley “B” with respect to ground. Also, let “ a_{1B} ” and “ a_{2B} ” denote relative accelerations of the two blocks with respect to moving pulley “B”. Analyzing motion of blocks with respect to moving reference of pulley, we have :

Equation:

$$a_{1B} = -a_{2B}$$

Applying concept of relative motion, we can expand relative accelerations as :

Equation:

$$a_{1B} = a_1 - a_B$$

Equation:

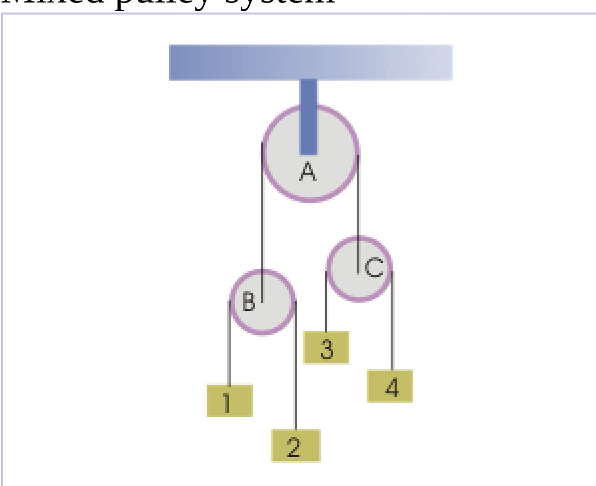
$$a_{2B} = a_2 - a_B$$

where accelerations on the right hand side of the equation are measured with reference to ground. Clearly, using these expansions, we can find accelerations of blocks with respect to ground as required.

Example

Problem : In the arrangement shown the accelerations of blocks "1", "2" and "3" are 2 m/s^2, 8 m/s^2 and 2 m/s^2 respectively in upward direction. The masses of the pulleys and string are negligible. Friction is absent everywhere. Find the acceleration of the block "4".

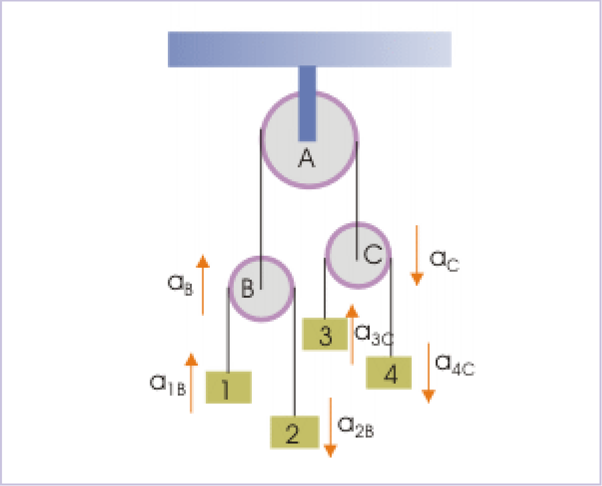
Mixed pulley system



Three blocks have absolute accelerations in upward direction.

Solution : The accelerations of three blocks are given with respect to ground. For analysis in moving frames of pulleys, we assign directions of relative accelerations as shown in the figure below. The pulley “B” and “C” are hanging from fixed pulley, “A”. If pulley “B” moves up, then pulley “C” moves down. The blocks “1” and “2”, in turn, are hanging from moving pulley “B”. If relative acceleration of “1” is up, then relative acceleration of “2” is down as they are connected by one string. Similarly, the blocks “3” and “4” are hanging from moving pulley “C”. If relative acceleration of “3” is up, then relative acceleration of “4” is down as they are connected by one string.

Mixed pulley system



Directions of relative accelerations of the blocks are shown.

Let us now consider upward direction as positive. Since relative acceleration of block "1" is equal and opposite to relative acceleration of block "2", we have :

$$a_{1B} = -a_{2B}$$

Expanding in terms of absolute accelerations, we have :

$$\Rightarrow a_1 - a_B = -(a_2 - a_B) = a_B - a_2$$

Putting given values,

$$\Rightarrow 2 - a_B = a_B - 8$$

$$\Rightarrow 2a_B = 10$$

$$\Rightarrow a_B = 5 \quad m/s^2$$

Now, pulleys “B” and “C” are connected through a string passing over a static pulley “A”. The accelerations of the moving pulleys “B” and “C” are, therefore, equal in magnitude but opposite in direction. From the analysis above, the pulley “B” has upward acceleration of 5 m/s^2. It, then, follows that pulley “C” has acceleration of 5 m/s^2 in downward direction,

$$a_C = -a_B = -5 \quad m/s^2$$

Again following the earlier reasoning that relative accelerations are equal and opposite, we have :

$$a_{3C} = -a_{4C}$$

$$\Rightarrow a_3 - a_C = -(a_4 - a_C) = a_C - a_4$$

Putting values,

$$\Rightarrow 2 - (-5) = -5 - a_4$$

$$\Rightarrow a_4 = -5 - 7 = -12 \quad m/s^2$$

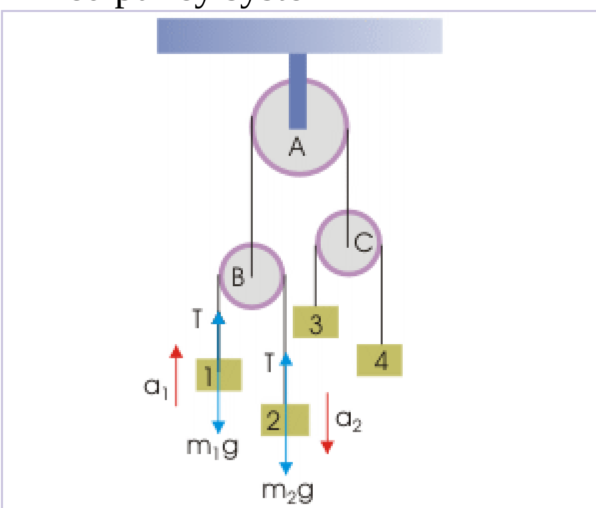
The negative sign shows that the block “4” moves down i.e. opposite to the positive reference direction, which is upward.

Limitation of analysis in moving frame of reference

Analysis of accelerations of blocks attached to moving block is based on the fact that blocks are attached to a taut inflexible string. These accelerations of blocks are accelerations with respect to moving pulley, which is itself accelerating. We can not ,however, use these accelerations directly to analyze forces on the blocks as moving pulley system is an accelerated frame of reference. We need to determine acceleration with respect to ground i.e. inertial frame of reference in order to apply Newton's second law of motion.

We, therefore, need to first determine acceleration with respect to ground using relation for relative motion. Considering the moving pulley "B" and attached blocks "1" and "2", we have :

Mixed pulley system



Blocks "1" and "2" have equal and opposite relative accelerations

$$T - m_1 g = m_1 a_1 = m_1 (a_{1B} + a_B)$$

$$m_2 g - T = m_2 a_2 = -m_2 a_1 = -m_2 (a_{1B} + a_B)$$

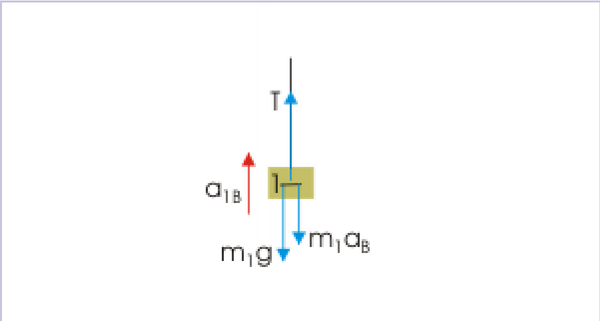
There is yet another approach for analyzing forces in accelerating frame of reference. We apply the concept of pseudo force. We shall study about

pseudo force subsequently in a separate module. Here, we shall limit our discussion to only the working steps as applied in the analysis of motion of moving pulley.

We assume a pseudo force on the body being studied in the accelerated frame. The magnitude of pseudo force is equal to the product of mass and acceleration of the frame of reference. It acts in the direction opposite to the acceleration of the frame of reference. While writing equation of motion, we also incorporate this force. But, we use acceleration of the body with respect to accelerated frame of reference - not with respect to inertial frame. This scheme is easily understood with an example. Considering the case as above, let us analyze the motion of block "1". Let us assume that moving pulley is accelerating upwards.

Here, pseudo force is :

Pseudo force



Force analysis in accelerated frame of reference

$$F_s = m_1 a_B$$

The pseudo force is acting in downward direction as the moving pulley "B" is accelerating upward. Now, applying Newton's second law of motion, we have :

$$T - m_1 g - m_1 a_B = a_{1B}$$

$$T - m_1 g = m_1(a_{1B} + a_B) = m_1 a_1$$

Thus, we see that analyzing motion in accelerated frame with pseudo force is equivalent to analyzing motion in inertial frame.

Advantages of analysis in moving frame of reference

The consideration of relative acceleration as against absolute acceleration for analysis using constraint relation (see Pulleys) has many advantages. We enumerate these advantages here as :

1 : The relative accelerations of blocks with respect to moving pulley are equal in magnitude, but opposite in direction. This is based on the fact that blocks are attached with a single string that passes over moving pulley. This simplifies analysis a great deal.

On the other hand, if we refer accelerations to the ground, then we can not be sure of the directions of accelerations of the blocks as they depend on the acceleration of the pulley itself. It is for this reason that we generally assume same direction of absolute accelerations of the blocks. If the assumption is wrong, then we get negative value of acceleration after analysis, showing that our initial assumption about the direction was wrong and that the acceleration is actually opposite to that assumed. This technique was illustrated in the module on Pulleys .

2 : Sometimes observed values are given in terms of relative reference in the first place. In this situation, we have the easy option to carry out analysis in the accelerated reference itself. Otherwise, we would be required to convert given values to the ground reference, using concept of relative acceleration and carry out the analysis in the ground reference.

Transition between absolute and moving reference

We enumerated advantages of relative acceleration technique. It does not, however, mean that analysis of motion in ground reference has no virtue. As a matter of fact, we are interested in the values, which are referred to ground reference – measurement of accelerations as seen in the ground reference. Even though, relative acceleration technique allows us to simplify solution; the analysis in accelerated frame essentially yields

values, which are referred to the accelerated frame of pulley. Ultimately, we are required to convert the solution or values of acceleration in the ground reference.

On the other hand, we have seen in earlier module (see Pulleys) that the technique of constraint relation is extremely effective. It relates velocities and accelerations of the elements of the system in the ground reference very elegantly – notwithstanding the complexity of the system.

It emerges from above discussion that two frameworks of analysis have their relative strengths and weaknesses. It is, therefore, pragmatic to combine the strengths of two techniques in our analysis. We may transition between two analysis frameworks, using the concept of relative acceleration in one dimension.

Equation:

$$a_{12} = a_1 - a_2$$

The emphasis on the sequence of subscript almost makes the conversion mechanical. It is always helpful to read the relation : “the relative acceleration of “1” with respect to “2” is equal to the absolute acceleration of “1”, subtracted by the absolute acceleration of “2”.

Working with moving pulleys (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

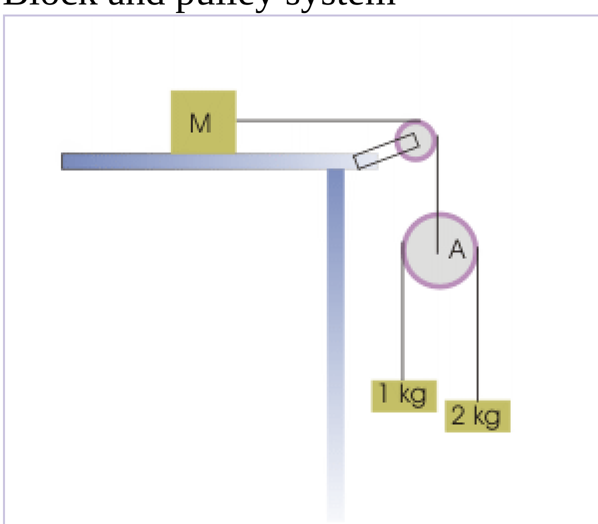
We discuss problems, which highlight certain aspects of the study leading to working with pulleys. The questions are categorized in terms of the characterizing features of the subject matter :

- Block and pulley system
- Incline and pulley system

Block and pulley system

Problem 1 : In the arrangement, find the mass "M" such that block of mass 1 kg moves with uniform velocity. Neglect mass of the pulleys and the string. Also neglect friction between the surfaces.

Block and pulley system

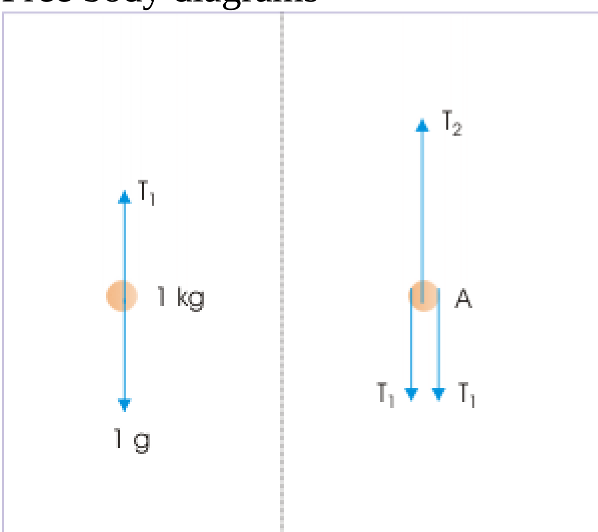


The block on the table is attached to a pulley system.

Solution : The block of mass 1 kg moves with constant velocity with respect to ground. It means that there is no net external force on the block. Let us denote block of mass 1 kg and 2 kg by the subscripts "1" and "2" respectively.

Now, forces on the block of mass 1 kg, are (i) its weight, " $m_1 g$ ", acting downward and (i) tension in the string, say " T_1 ", acting upward. The two forces form a balanced force system as block moves with uniform velocity. Hence,

Free body diagrams



FBD of the block of 1 kg and moving pulley

$$\Rightarrow T_1 = m_1 g = 1X10 = 10 \quad N$$

From the force analysis of the moving mass-less pulley, "A",

$$T_2 = 2T_1$$

Combining two equations,

$$\Rightarrow T_2 = 2X10 = 20 \quad N$$

Tension is the only force on the block, which is placed on the horizontal surface of the table. From the force analysis of the block on the horizontal surface (see FBD of block in the figure below), we have :

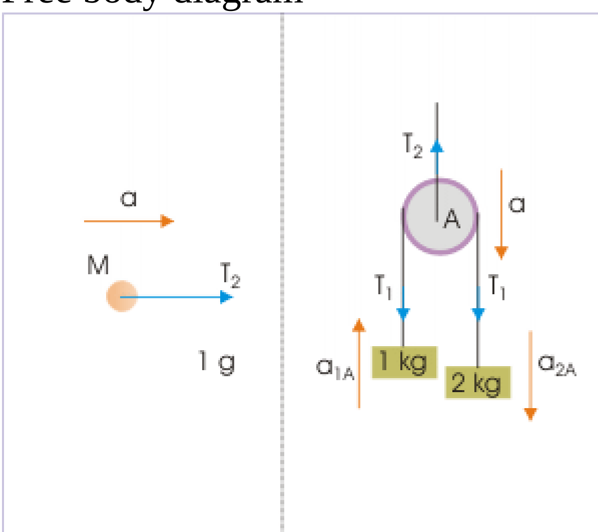
$$T_2 = Ma$$

$$20 = MXa$$

$$a = \frac{20}{M}$$

This is also the acceleration of moving pulley as it is connected to the block on the table with a string. If we consider upward direction as positive, then acceleration of the moving pulley is :

Free body diagram



FBD of the block on the table and forces on the moving pulley system

$$\Rightarrow a_A = -\frac{20}{M} \quad \text{downward}$$

From the expression of acceleration of moving pulley, it is clear that we should get an equation involving acceleration of moving pulley in order to ultimately determine the value of "M" as required. According to question, the block of mass “1 kg” travels with zero acceleration (constant velocity) with reference to ground. The force analysis of this mass, therefore, does not involve acceleration of the moving pulley. Thus, we shall strive to find the acceleration of block of "2 kg" with respect to ground as its expression will involve acceleration of the moving pulley. Since block of mass “1 kg” travels with zero acceleration (constant velocity) with reference to ground, its acceleration with respect to moving pulley is :

We shall, however, take advantage of the fact that acceleration of block of 1 kg has zero acceleration with respect to ground reference. Its acceleration with respect to moving pulley is :

$$a_{1A} = a_1 - a_A$$

$$\Rightarrow a_{1A} = 0 - \left(-\frac{20}{M}\right) = \frac{20}{M} \text{upward}$$

This result is physically verifiable. Since pulley has downward acceleration, the block should have upward acceleration of the same magnitude such that block acceleration with respect to ground is zero. The blocks are connected by the same string that goes over the pulley. It means that relative accelerations of the blocks are equal but opposite in direction (see Working with moving pulleys).

$$\Rightarrow a_{2A} = -a_{1A} = -\frac{20}{M} \text{downward}$$

The acceleration of mass “ m_2 ” with respect to ground can, now, be obtained as :

$$a_{2A} = a_2 - a_A$$

$$a_2 = a_{2A} + a_A = -\frac{20}{M} + \left(-\frac{20}{M}\right) = -\frac{40}{M}$$

From force analysis of block of mass 2 kg, we have :

$$-m_2 g + T_1 = -m_2 a_2$$

$$m_2 g - T_1 = m_2 a_2$$

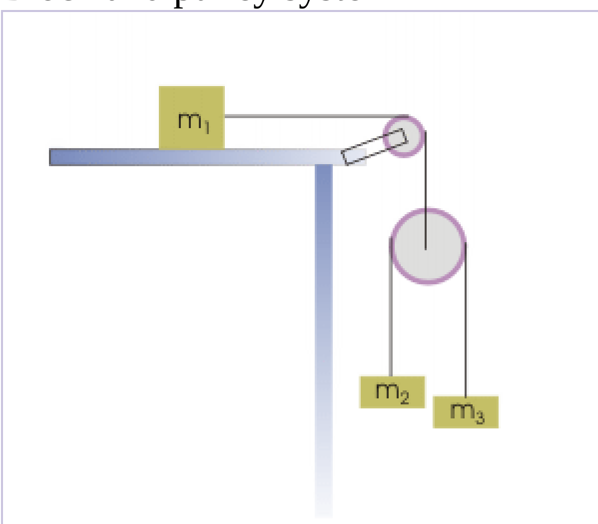
$$\Rightarrow 2X10 - 10 = 2X\frac{40}{M}$$

$$\Rightarrow 10M = 80$$

$$\Rightarrow M = 8kg$$

Problem 2 : For the arrangement as shown in the figure, find the acceleration of block on the horizontal floor. Consider all surfaces “friction-less” and strings and pulleys as “mass-less” for the force analysis.

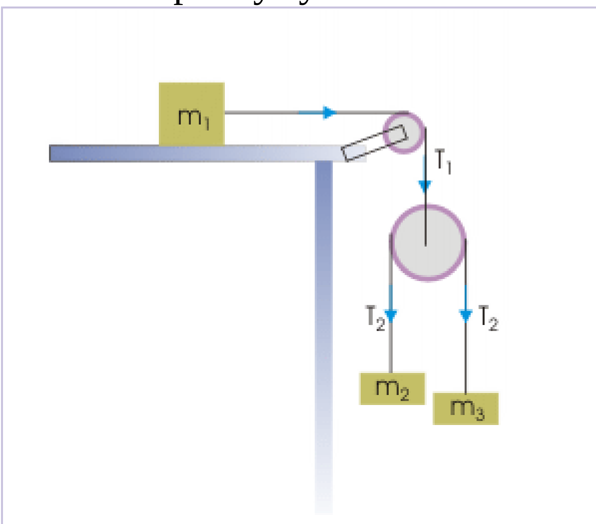
Block and pulley system



The block on the table is attached to a pulley system.

Solution : If we look at the motions of blocks from the reference of moving pulley, then the situation is that of motion of two blocks and a static pulley. Since the blocks are connected by the same string, the accelerations of the blocks with respect to pulley are equal in magnitude but opposite in direction.

Block and pulley system



Tensions in the string are shown.

Let the acceleration of the block of mass, " m_1 ", placed on the horizon, be " a_1 ". Since block on the horizontal surface and the moving pulley are connected by a single piece of string, the acceleration of the moving pulley is same. As such, acceleration of the moving pulley is " a_1 ". Now, let the relative accelerations of the blocks with respect to moving pulley be " a_{21} " and " a_{31}" respectively.

In order to carry out force analysis of the different elements of the system, we are required to express accelerations with respect to ground. This is an important point. Note that Newton's laws of motion is applicable to inertial frame of reference - not to an accelerated frame of reference like that of moving pulley. We can convert "accelerations with respect to accelerating pulley" to "accelerations with respect to ground" by using the concept of relative acceleration. Let block of mass " m_2 " moves down and the block

of mass “ m_3 ” moves up with respect to the moving pulley. Also, let their accelerations with respect to ground be “ a_2 ” and “ a_3 ” respectively. Then,

$$a_{21} = a_2 - a_1$$

$$a_{31} = a_3 - a_1$$

But magnitudes of relative accelerations with respect to moving pulley are equal.

$$|a_{21}| = |a_{31}| = a_r$$

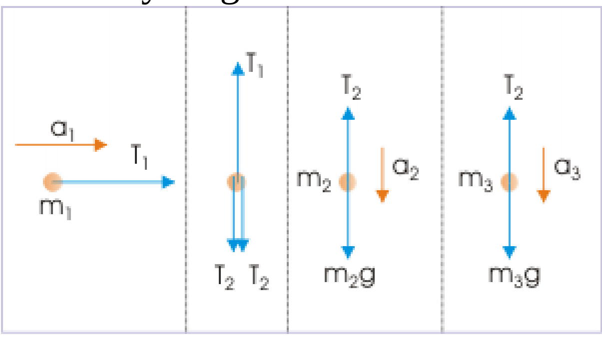
Then, considering downward direction as positive,

$$\Rightarrow a_2 = a_1 + a_{21} = a_1 - a_r$$

$$\Rightarrow a_3 = a_1 + a_{31} = a_1 + a_r$$

Thus, we see that two accelerations are now expressed in terms of one relative acceleration, “ a_r ”, only. The analysis of forces in the ground reference for different elements of the system are :

Free body diagrams



Free body diagrams of block on the table, moving pulley and two blocks are shown.

(i) The block of mass, “ m_1 ” :

$$T_1 = m_1 a_1$$

(ii) For pulley :

$$T_2 = \frac{T_1}{2}$$

Combining two equations,

$$T_2 = \frac{m_1 a_1}{2}$$

(iii) For block of mass, " m_2 " :

$$m_2 g - T_2 = m_2 a_2$$

$$\Rightarrow m_2 g - T_2 = m_2(-a_r + a_1)$$

(iv) For block of mass, " m_3 " :

$$\Rightarrow m_3 g - T_2 = m_3(a_r + a_1)$$

We need to eliminate " a_r " and solve two equations for " T_2 ". For this we multiply first by " m_3 " and second by " m_2 ". The new pair of equations are,

$$\Rightarrow m_2 m_3 g - m_3 T_2 = m_2 m_2 a_1 - m_2 m_3 a_r$$

$$\Rightarrow m_2 m_3 g - m_2 T_2 = m_2 m_3 a_1 + m_2 m_3 a_r$$

Adding, we have :

$$\Rightarrow 2m_2 m_3 g - (m_2 + m_3) T_2 = 2m_2 m_3 a_1$$

Substituting for " T_2 ", we have :

$$\Rightarrow 2m_2 m_3 g - (m_2 + m_3)\frac{m_1 a_1}{2} = 2m_2 m_3 a_1$$

$$a_1\left[2m_2 m_3 + (m_2 + m_3)\frac{m_1}{2}\right] = 2m_2 m_3 g$$

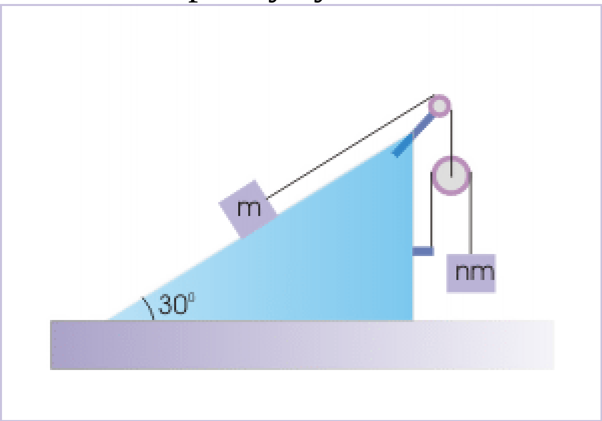
$$\Rightarrow a_1\{4m_2 m_3 + (m_2 + m_3) m_1\} = 4m_2 m_3 g$$

$$\Rightarrow a_1 = \frac{4m_2m_3g}{\{4m_2m_3 + (m_2 + m_3)m_1\}}$$

Incline and pulley system

Problem 3 : Two blocks of mass “m” and “nm” are part of the arrangement involving pulley and incline as shown in the figure. If friction is negligible everywhere and mass of the pulleys and the strings are also negligible, then find the acceleration of the mass “nm”.

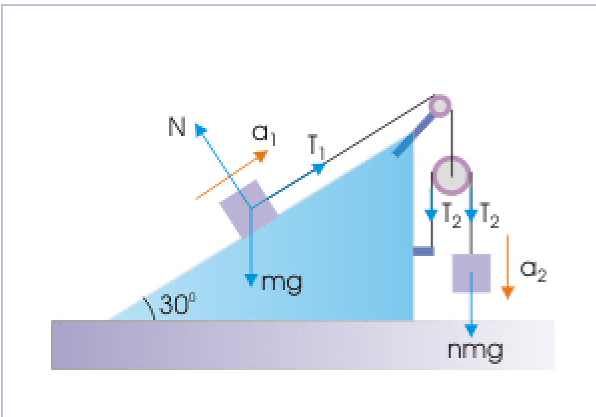
Incline and pulley system



The block on incline has tendency to move down.

Solution : Here, we see that block of mass “m” is connected to the moving pulley by a single string. Hence, their accelerations are equal in magnitude. Let the acceleration of the block of mass “m” be “ a_1 ” and the acceleration of block of mass “nm” be “ a_2 ”. Similarly, let the tensions in the two strings be “ T_1 ” and “ T_2 ” respectively. The force diagrams of the elements of the system are shown here.

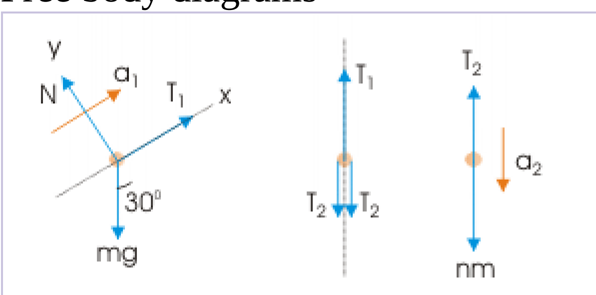
Incline and pulley system



Forces on the elements of the system are shown.

The free body diagrams of block on the incline, moving pulley and hanging block are shown here :

Free body diagrams



Free body diagrams of block on incline, pulley and block hanging from pulley are shown.

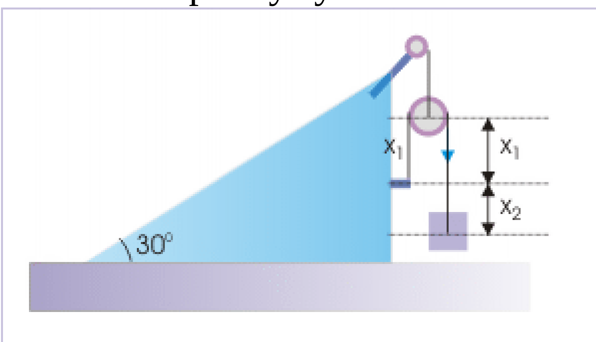
$$T_1 - mg\sin 30^0 = ma_1$$

$$T_1 = 2T_2$$

$$nmg - T_2 = nma_2$$

Now, we need to establish the relation between accelerations of pulley and block of mass “nm” by applying constraint analysis. For this we consider a horizontal reference passing through the fixed end of the string. Using length of string as constraint, we have :

Incline and pulley system



The constrain relation for positions are obtained by measuring positions with respect to a fixed point.

$$x_1 + x_1 + x_2 = L$$

Differentiating w.r.t time twice, we have :

$$\Rightarrow 2a_1 + a_2 = 0$$

Comparing magnitude only,

$$\Rightarrow a_2 = 2a_1$$

We see that if one end of the string is fixed, then the acceleration of the other end of the string passing over a movable “mass-less” pulley is two times the acceleration of the movable pulley. Let,

$$a_1 = a$$

$$a_2 = 2a$$

Substituting for tension and acceleration in the equations of the blocks as derived earlier, we have

$$2T_2 - mg\sin 30^0 = ma$$

$$nmg - T_2 = 2nma$$

Multiplying second equation with 2 and adding to eliminate " T_2 ",

$$2nmg - \frac{mg}{2} = a(m + 4nm)$$

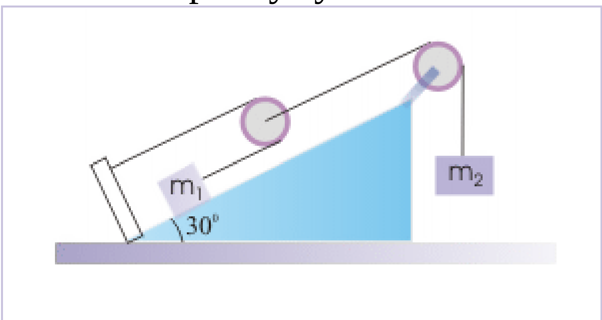
$$\Rightarrow a = \frac{(4n-1)g}{2(1+4n)}$$

The acceleration of the block of mass “nm” is :

$$\Rightarrow a_2 = 2a = \frac{(4n-1)g}{(1+4n)}$$

Problem 4 : In the arrangement shown, the masses are m_1 = 2 kg and m_2 = 4 kg. The mass of the strings and pulleys are negligible. The friction is negligible between block and fixed incline and also between pulley and string. Find the accelerations of two blocks.

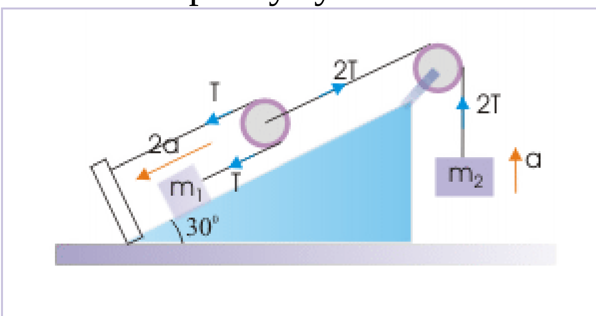
Incline and pulley system



Two Blocks are attached to two moving pulley.

Solution : Considering free body diagram of the moving pulley, we see that tension in the string attached to mass " m_1" is half of the tension attached to the moving pulley. Hence, if "T" be the tension in the string attached to block of mass " m_1 ", then the tension in the string attached to the pulley is "2T".

Incline and pulley system

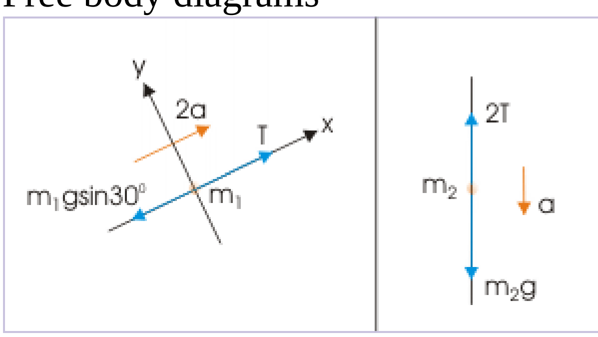


Tensions and accelerations of the elements of the system are shown.

From, the constrain relation as derived in prevous example, we know that the acceleration of the block of mass " m_1 " is twice that of the pulley. Hence, if "a" be the acceleration of the moving pulley, then acceleration of the block of mass " m_1 " is "2a".

The free body diagram of two blocks are shown here. The force analysis of the block of mass " m_1 " is :

Free body diagrams



Free body diagrams of two blocks are shown.

$$T - m_1 g \sin 30^0 = m_1 a_1$$

$$\Rightarrow T - 2X10X\frac{1}{2} = 2X2a$$

$$T - 10 = 4a$$

Similarly, force analysis of the block of mass " m_2 " is :

$$m_2 g - 2T = m_2 a_2$$

$$\Rightarrow 40 - 2T = 4a$$

$$20 - T = 2a$$

Adding two equations, we have :

$$\Rightarrow 20 - 10 = 6a$$

$$\Rightarrow a = \frac{10}{6} = \frac{5}{3}$$

Thus, accelerations of block are :

$$a_2 = a = \frac{5}{3} \quad m/s^2$$

$$a_1 = 2a = \frac{10}{3} \quad m/s^2$$

Acknowledgment

Author wishes to thank John Trautwein for making suggestions to improve this module.

Friction

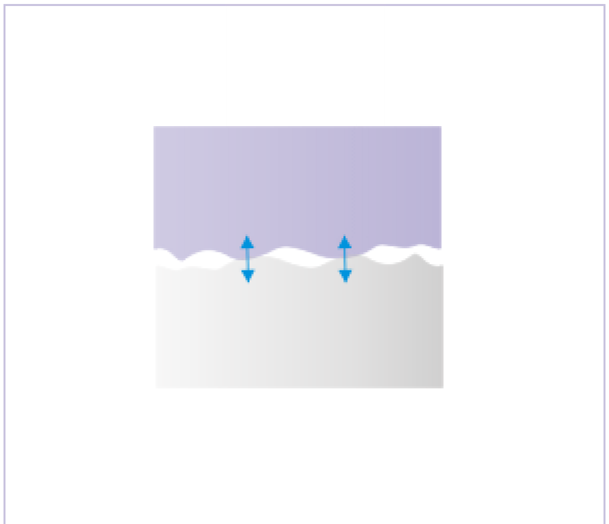
Friction force (also referred simply as friction) plays the role of a dampener to the motion of a body that takes place in contact with another body. All motions involved in our daily life take place in contact with another body or medium. The force of opposition to the motion of a body in relation to another rigid body is called “friction” and in relation to fluid (liquid or gas) medium is called “drag”.

Friction is a huge disadvantage to us. A good part of the energy used in this world goes waste to counteract this force. On the other hand, this force is also responsible for our existence and the very existence of life on the planet. Imagine its absence. How would have we stopped once in motion? Everything would have been perpetually in motion with no control. Could this life have evolved without friction in the first place? Friction is simply indispensable to us.

Genesis of friction

When bodies come in contact with each other, large numbers of atoms of the two surfaces come close to each other, affecting a temporary joint (referred as cold welds in technical parlance) i.e. atoms from the two surfaces become part of one body mass. These joints result from electromagnetic force operating between atoms, when brought sufficiently close. The temporary joints between two surfaces inhibit relative movement between two surfaces.

In general two ordinary surfaces are uneven at microscopic level. The surface is actually formed of small hills and valleys. All points across the surfaces do not come in contact. Still, there are large numbers of such contact points, forming temporary joints.
Surfaces in contact



The surface is actually formed of small hills and valleys.

The weight of the body plays important role in determining friction. When the overlying body has greater mass, it applies greater force at the interface. More points come in contact or become sufficiently close to form joints. Further, all contact points are not cold welded or joints. Due to the weight of the overlying body, more of the contact points become temporary joints or become stronger joints, requiring larger external force to initiate motion. Thus, friction depends on (i) numbers of points in contact (nature of the surfaces in contact) and (ii) normal force at the contact surface which presses them to come closure.

Generally, a smooth polished surface is known to offer smaller friction with respect to a rough surface. When the surface is smooth, then there are more contact points, but corresponding force per point is smaller. In this case, the weight of the block, pressing against the surface beneath, is distributed across larger numbers of contact points. The net result is that there are greater numbers of contact points, but fewer welded joints opposing motion. Thus, a smooth surface offers smaller friction in comparison to rough surface.

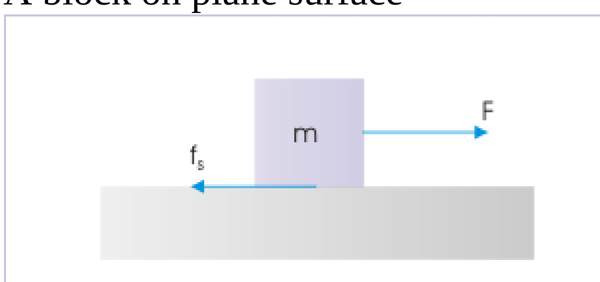
However, if the surfaces are genuinely smooth to perfection and brought together, then there are much greater numbers of contact points, which are already sufficiently close and produce still larger numbers of weld sites. In such case, two bodies become almost inseparable and require a much greater external force to separate two bodies. If the joint is done between very smooth surfaces in the absence of air i.e. vacuum, then the cold welding, at contact sites covering larger contact area, makes the two pieces as one and the bodies are mechanically inseparable.

Static friction

Static friction comes into play when a body tends to move over another body (surface), but the body has yet not been initiated into motion. When we apply a small force to initiate motion of a block, lying on a horizontal surface as shown in the figure; the temporary joints may not yield to the external force. This means that the force is not sufficient enough to break off the welds, which are formed at contact points. In this case, the external force ($F_{||}$) parallel to the contact surface is equal to the static friction, f_s , i.e.

$$f_s = F_{||} = F$$

A block on plane surface

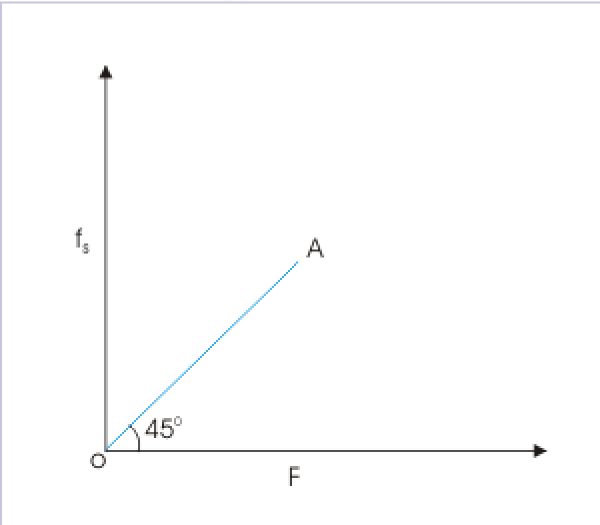


A block is pulled by a horizontal force.

Note that friction force comes into being even before motion is initiated. If we increase the external force in the direction parallel to the contact surface,

then friction force increases so that two forces are equal in magnitude and opposite to each other. It is important to understand that friction responds to the net external force parallel to the contact surface to match the magnitude. In this sense, static friction force is a "self adjusting" force against net external force parallel to the contact force (F||) or the component of external forces along contact surface.

Friction and applied force



Friction is equal to external force parallel to the contact force.

Self adjusting force means that if a 2 N force parallel to contact surface fails to initiate motion, then static friction is 2 N; if a 4 N force parallel to contact surface fails to initiate motion, then static friction is 4 N and so on.

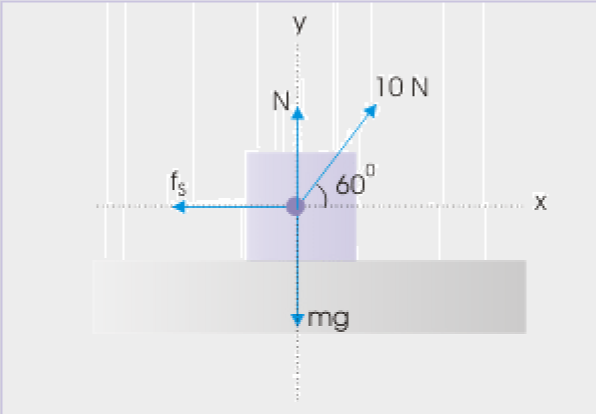
Example:

Problem : A force of 10 N is applied on a block of 10 kg. The force makes an angle 60° with the horizontal. If the block fails to move, then find static friction between two surfaces.

Solution : Here, the block is not initiated into motion by the external force. Now, we know that static friction is equal to the component of net external

force parallel to contact surface :

A block on a plane surface



The block is pulled by an external force of 10 N.

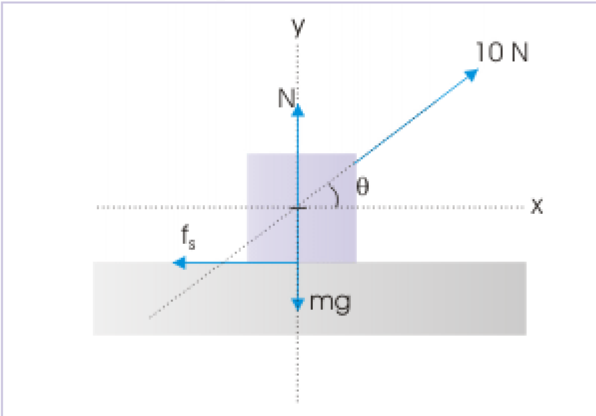
$$f_s = F_{||} = F\cos 60^0 = 10 \text{ x } \frac{1}{2} = 5\ N$$

We must realize following characteristics of the static friction from the discussion so far :

- So long the body does not start moving, the static friction is equal to the component of net external force in the direction parallel to contact surface.
- So long the body does not start moving, the static friction is a self adjusting force i.e. it varies with net external force in the direction parallel to contact surface.

Also, we must realize that friction force (f_s) and component of external force parallel to contact surface ($f_{||}$) are parallel to each other and may not be concurrent in reality. However, we treat them concurrent, while analyzing - for the simple reason that we are considering translational motion. In this situation, the forces can be considered concurrent.

A block on a plane surface



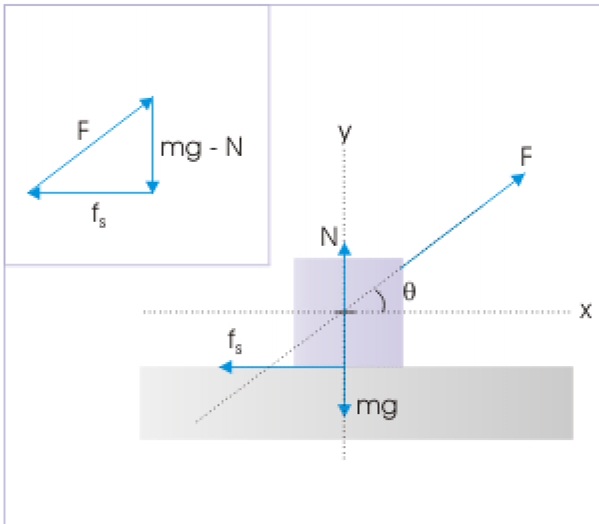
The forces are not concurrent, but are considered concurrent for translational motion.

Note: In this course we shall follow a convention to denote various friction forces. According to this convention, " F_F " denotes friction in general, whose nature is not known; " f_s " denotes self adjusting static friction ; " F_S " denotes limiting or maximum static friction and " F_K " denotes kinetic friction.

Nature of force system

If the body remains at rest under the action of an external force, then the forces on the body are balanced. This means that the forces on the body, including friction, should complete a closed force polygon so that the net force is zero. Let us consider the case of a block of mass “m”, lying on a horizontal surface. There are four forces (i) weight of the block,mg, (ii) normal force, N, (iii) external force (F) and (iv) friction force (f_s).

A block resting on a plane surface



The forces form a closed vector polygon.

Till the body starts moving, the external force and static friction adjust themselves to complete the closed polygon.

Limiting (maximum) static friction

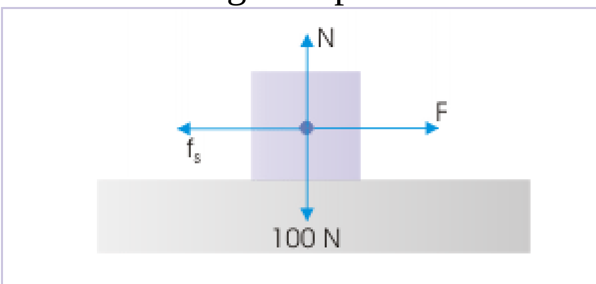
If we continue to increase the component of external force parallel to the contact surface, then for a particular magnitude, the weld joints at the contact points are broken off and the block starts moving. For any particular pair of surfaces, there is a limiting or maximum static friction (F_s).

This limiting friction force between two surfaces can not be specified on the basis of the constitution of the material of bodies alone (such as the iron, lead etc), but its value depends on two additional factors (i) the relative smoothness or roughness of the two surfaces in contact and (ii) normal force between the two surfaces. Normal force is responsible to increase numbers of joints between the surfaces and as such affects friction between surfaces. In order to understand this aspect of friction, let us consider that the mass of block is 10 kg and acceleration due to gravity is 10 m/s^2 .

From force analysis, we observe that normal force on the body is equal to the weight of the block :

$$N = \text{mg} = 10 \text{ X } 10 = 100 \text{ Newton}$$

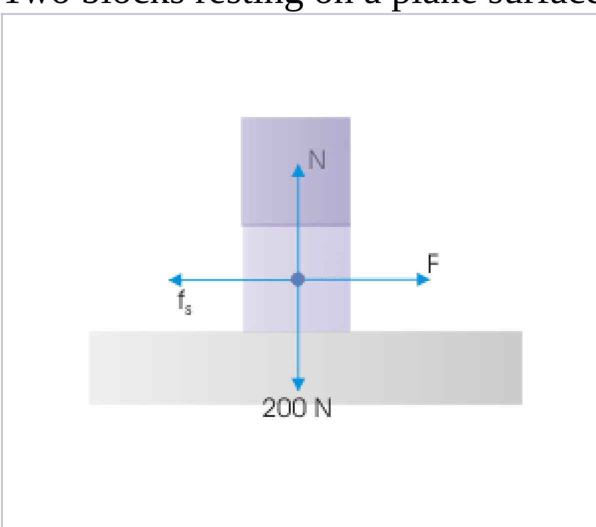
A block resting on a plane surface



The free body diagram superimposed on body systems.

Now, let us put another block of mass of 10 kg over the block on the surface. As a result, the normal force increases to 200 N as shown in the figure below.

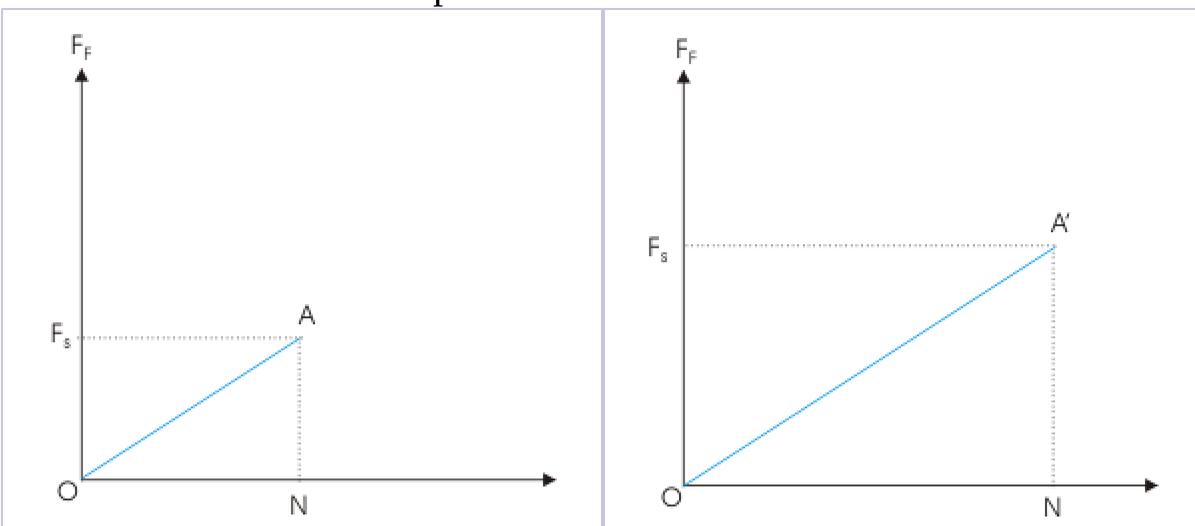
Two blocks resting on a plane surface



Greater normal force means more joints between surfaces.

As the block is now pressing the underlying surface with greater force, more of the contact points are converted into weld sites. Plainly speaking, the atoms at contact points have moved closure, increasing the inter-atomic forces. For the same pair of surfaces, we would, then, need a bigger maximum force (F_s) to initiate motion as shown in the figure below.

Friction .vs. external force plot



Greater normal force extends maximum static friction.

Experimentally, it is found that maximum friction force is proportional to normal force on the block.

$$F_s \propto N$$
$$\Rightarrow F_s = \mu_s N$$

where μ_s is called coefficient of static friction, specific to a given pair of the surfaces in contact. It depends on the nature of the surfaces in contact. Importantly, it neither depends on the normal force unlike friction (force) nor it depends on the area in contact. We should also note that coefficient of friction, μ_s, being equal to $\frac{F_s}{N}$ is a ratio of two forces and is, therefore, dimensionless. Further, coefficient of static friction is defined only for the

maximum static friction, when body is about to move and not for friction before this stage. What it means that we can not estimate self adjusting static friction, using above relation. The self adjusting friction, as we can remember, is equal to the component of external force parallel to the contact surface.

Here, we distinguish maximum static friction force by denoting it with the symbol " F_s ", whereas static friction before motion is denoted by " f_s ".

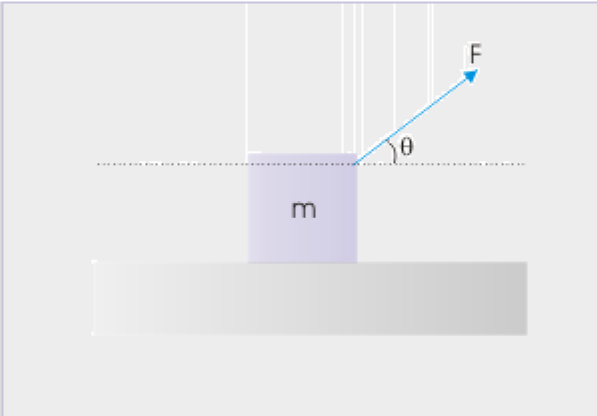
We summarize following characterizing aspects of static (f_s) and maximum static friction (F_s) :

1. Static friction applies in the direction opposite to the component of net external force parallel to contact surface. This, as a matter of fact, is the criteria of deciding direction of friction.
2. Self adjusting static friction (f_s) is equal to the component of net external force parallel to contact surface. It ranges from 0 to a maximum value F_s . Static friction is a self adjusting friction force. It is important to emphasize that static friction, unlike maximum (limiting) static friction, is not given by the expression of coefficient of friction. Note the relation for static friction : $f_s = F_{||}$ (and $f_s \ne \mu_s N$)
3. Maximum static friction is proportional to normal force applied on the body and is a single value (unlike static friction) quantity.
4. The relation between maximum (limiting) static friction and normal force is a scalar relation. This relation connects forces, which are mutually perpendicular to each other.

Example:

Problem : A force "F" is applied on a block of mass "m", making an angle "θ" with the horizontal as shown in the figure. If the block just starts to move with the applied force, find (i) maximum static friction and (ii) static friction coefficient for the two surfaces in contact.

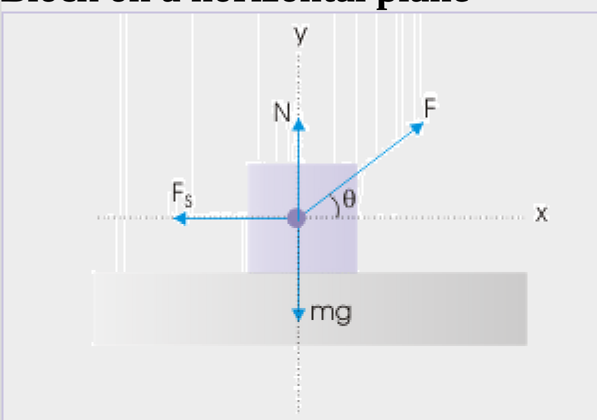
Block on a horizontal plane



The block is pulled by an
external force.

Solution : (i) The maximum static friction is equal to the force parallel to contact surface to initiate the motion. Thus,

Block on a horizontal plane



The block is pulled by an
external force.

$$F_s = F\cos\theta$$

(ii) Coefficient of static friction is ratio of normal force and friction. We, therefore, need to know the normal force on the block. Now, force analysis in y-direction results in following relation for normal force,

$$\sum F_y \Rightarrow N + F\sin\theta - mg = 0$$
$$\Rightarrow N = mg - F\sin\theta$$

Hence,

$$\Rightarrow \mu_s = \frac{F_s}{N} = \frac{F\cos\theta}{mg - F\sin\theta}$$

Kinetic friction

If the external force parallel to contact surface exceeds maximum static friction, then the body starts moving over underlying surface. It does not mean, however, that friction between surfaces disappears. New contact points between surfaces come in contact, some of which are momentarily joined and then broken on continuous basis. The friction force is dropped slightly (almost instantly); but remains constant - independent of the velocity of the body.

We denote kinetic friction force as F_k and corresponding friction coefficient as μ_k . The kinetic friction is related to normal force as :

$$F_k \propto N$$
$$\Rightarrow F_k = \mu_k N$$

The coefficient of kinetic friction is generally independent of the velocity of the body and is practically considered constant.

Motion over a rough surface

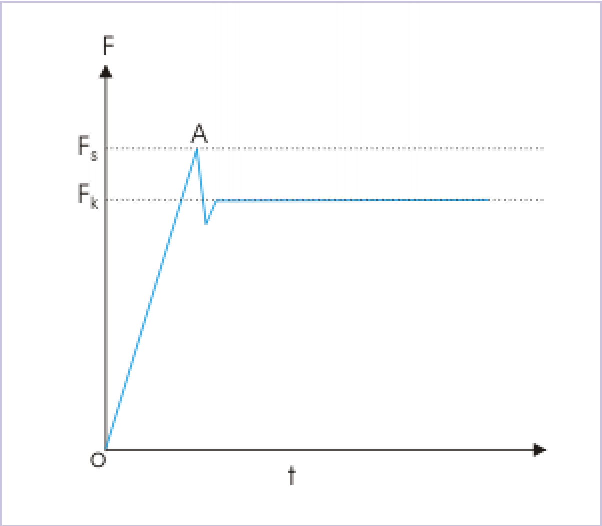
Once the external force along the contact surface exceeds limiting or maximum static friction, the body starts moving on the surface. The nature of motion, subsequent to initiation, depends on the external force. There are two possibilities :

1:Uniform motion :

The external force along the contact surface exactly equals to kinetic friction during the motion. In this case, the body is subjected to a balanced external force system (including friction) having zero net force. As such, the body will continue moving with whatever initial velocity is given to the body. What it means that we apply external force, $F > F_s$, momentarily to impart initial velocity to the body and then subsequently maintain uniform velocity by external force, F, such that $F = F_k$.

A typical friction force - time plot for the situation is shown in the figure. Here, we have considered that external force along the contact surface is increased slowly till it equals maximum static friction, then external force is adjusted simply to maintain velocity of the body.

Freiction .vs. time plot

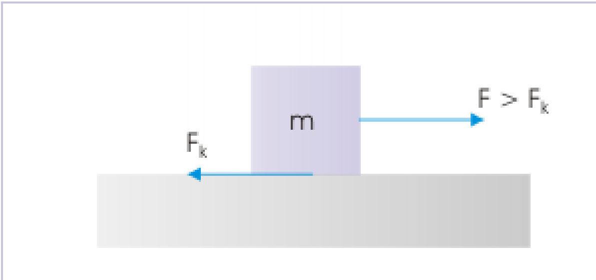


The maximum static friction coefficient, μ_s , and kinetic friction coefficient, μ_k , are usually close values for a pair of given surfaces and may be taken to be equal as an approximation.

2: Accelerated motion :

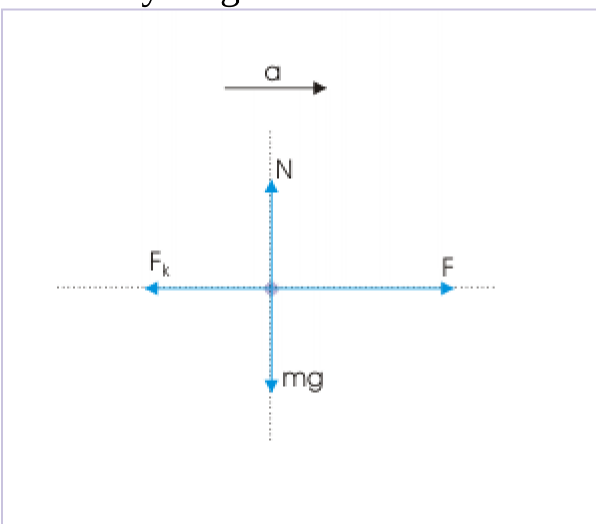
If we maintain external force along the contact surface in excess of kinetic friction, then body undergoes accelerated motion.

Accelerated motion



The free body diagram is shown here in the figure. Let "a" be the acceleration towards right.

Free body diagram



$$\sum F_x = F - F_k = ma$$
$$\Rightarrow a = \frac{F - F_k}{m}$$

Example:

Problem : A block of 10 kg is moving with a speed 10 m/s along a straight line on a horizontal surface at a particular instant. If the coefficient of kinetic friction between block and the surface is 0.5, then find (i) how long does it travel before coming to a stop and (ii) the time taken to come to a stop. Assume g = 10 m/s^2 .

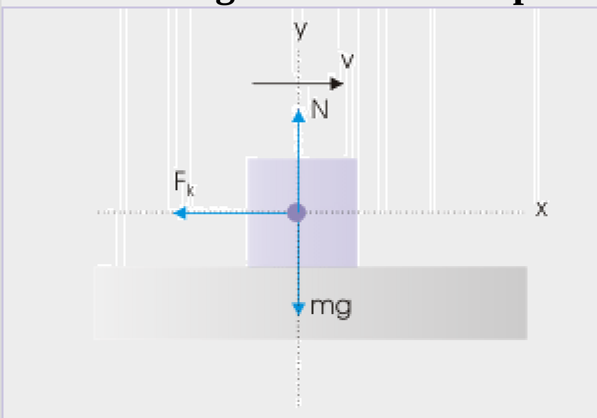
Solution : Kinetic friction is a constant force that acts opposite to the relative velocity of the block with respect to the surface on which it moves.

This constant force results in constant deceleration, whose magnitude is given by :

$$\Rightarrow a = \frac{F_k}{m} = \frac{\mu_k N}{m}$$

Now the free body diagram of the block as superimposed on the body diagram is shown in the figure.

Block moving on a horizontal plane



Friction decelerates the block to a stop.

As there is no motion in y-direction,

$$N = mg$$

In x-direction, the magnitude of acceleration is :

$$\Rightarrow a = \frac{\mu_k N}{m} = \frac{\mu_k mg}{m} = \mu_k g = 0.5 \text{ X } 10{=}5 \;\; m/s^2$$

As the deceleration is constant, we can employ equation of motion to determine the displacement in x-direction.

$$v^2 = u^2 + 2\, ax$$

Here, v = 0, u = 10 m/s and a = -5 m/s^2 .

$$\Rightarrow 0 = 10^2 - 2 \text{ x } 5x$$

$$\Rightarrow x = \frac{100}{10} = 10\,m$$

If “t” is time taken, then using the relation v = u + at, we have :
⇒ 0 = 10 – 5t
⇒ t = 2s

Laws of friction

The properties of friction as explained above are summarized in three laws of friction as :

1: If a force fails to initiate the motion of a body in contact with another surface, then static friction is equal to the component of net external force parallel to the surface of the contact surface.

$$f_s = F_{||}$$

2: The static friction has a maximum value (Fs), which is given by :

$$F_s = \mu_s N$$

where μ_s is the coefficient of static friction and "N" is the magnitude of normal force. The maximum static friction (F_s) and normal force (N) are mutually perpendicular to each other . The coefficient of static friction depends on the nature of the surfaces in contact.

3: If the body begins to move along the surface, then friction force, called kinetic friction, is reduced slightly, which is given by :

$$F_k = \mu_k N$$

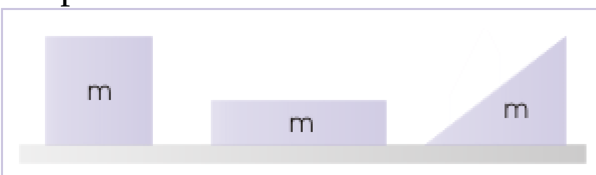
where " μ_k " is coefficient of kinetic friction and depends on the nature of surfaces in contact. Experimentally, it is found that $\mu_k < \mu_s$.

Area of contact and friction

We observe that area of contact does not appear anywhere in our consideration of friction. Though, we might generally believe that a greater contact area should offer greater friction and would be difficult to move.

In order to understand the absence of area in our consideration, let us consider the different shapes of bodies of equal mass in contact with a given surface.

Shapes of bodies and friction



Irrespective of the difference in contact areas, the friction in all three cases is same.

Irrespective of the difference in contact areas, the friction in all three cases is same.

If we recall, friction results from the temporary joints formed between the contact surfaces. Thus, friction depends on (i) the numbers of contact points and (ii) the force inducing joints at these contact points.

The actual contact area may be much less - only up to 40 % of the total surface area in the case of ordinary plane surface. This means that friction does not depend on the total area, but only the part of the area which is actually in contact with the other surface. This fact is incorporated in our consideration through "coefficient of friction" – which represents the mutual characteristic of two surfaces – not of one surface. It largely accounts for the numbers of contact points between two surfaces and hence is characteristic of a pair of surfaces in contact.

On the other hand, formation of joints at the contact points depend on the distribution of normal force at contact points. Normal force, in turn, is distributed across the surface. If the density of contact points throughout is uniform, we can say that formation of joints depends on the normal force per unit area. Normal force is equal to the product of normal force per unit area and area as given here :

$$N = \left(\frac{N}{A}\right)A$$

It is clear that by considering normal force we have implicitly accounted for the area of contact. In the nutshell, we can say in a very general way that (i) coefficient of friction largely accounts for the contact points between a given pair of surfaces and (ii) normal force accounts for the distribution of force at contact points across the whole area.

Friction (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

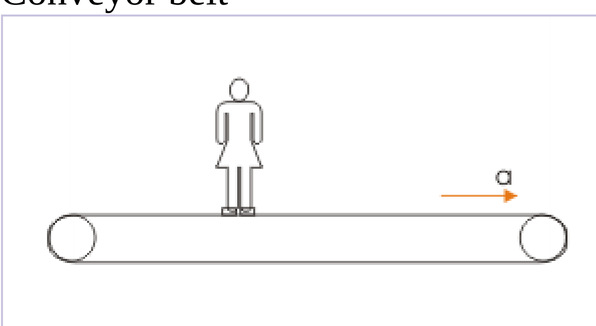
We discuss problems, which highlight certain aspects of the study leading to friction. The questions are categorized in terms of the characterizing features of the subject matter :

- Conveyor belt
- Blocks
- Stack of blocks

Conveyor belt

Problem 1 : At an airport, a girl of 50 kg is standing on a conveyor strip, which transports her horizontally from one point to another. If the conveyor strip has the acceleration of 0.2 m/s^2 and the girl is standing still on with respect to the strip, then what is friction between her shoe and the strip?

Conveyor belt



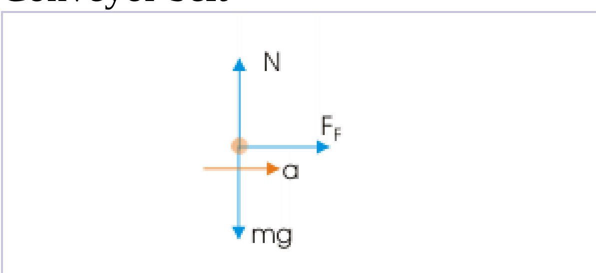
A girl is standing on a moving conveyor strip.

Solution : The girl is standing on an accelerating strip. It is given that she is stand still in the reference of conveyor belt. In the ground reference, however, she is accelerating with the same acceleration as that of conveyor belt.

As seen from the ground reference, the forces in horizontal directions should form an unbalanced force system for her to accelerate with the conveyor strip. In the horizontal direction,

Friction is the only force in the horizontal direction. Hence, it should be in the direction of acceleration. Note that friction on the girl may be less than or equal to the limiting friction, depending on the coefficient of friction between shoe and the strip of the conveyor belt.

Conveyor belt

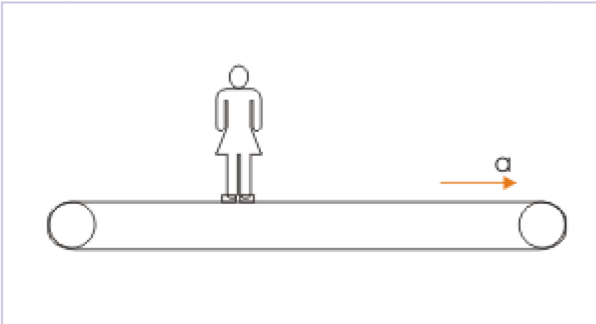


Forces on the girl.

$$F_x = F_F = ma = 50X0.2 = 10N$$

Problem 2 : At an airport, a girl of 50 kg is standing on a conveyor strip, which transports her horizontally from one point to another. If the coefficient of static friction between her shoe and the strip is 0.2, then find the maximum acceleration of the belt for which the girl can remain stand still with respect to the conveyor strip.

Conveyor belt

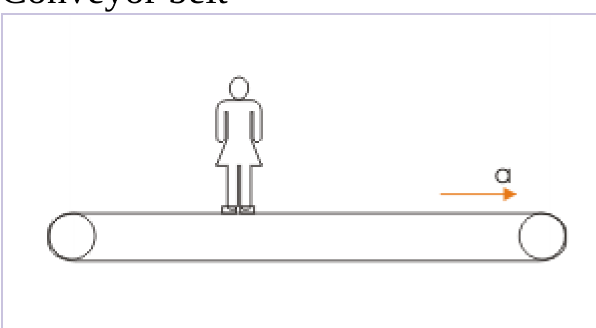


A girl is standing on a moving conveyor strip.

Solution : The girl is standing on an accelerating strip. It is given that she is stand still in the reference of conveyor. In the ground reference, she is accelerating with the same acceleration as that of conveyor strip.

The girl can hold on her position on the conveyor till the friction becomes limiting friction. We, therefore, analyze the forces in horizontal directions with limiting friction,

Conveyor belt



Forces on the girl.

$$F_x = F_s = ma$$

$$\mu N = ma$$

$$a = \frac{\mu N}{m}$$

The forces on the girl in the vertical direction form a balanced force system as there is no motion in that direction,

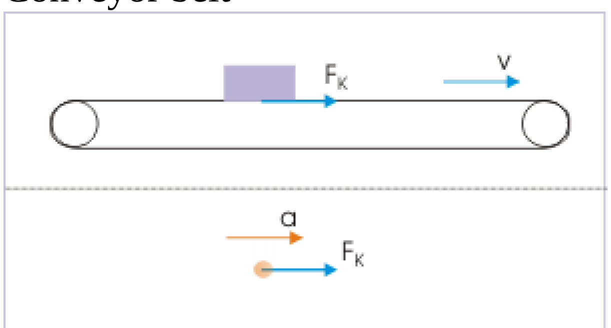
$$F_y = N = mg = 50X10 = 500 \quad \text{Newton}$$

Putting this value in the equation of acceleration, we have :

$$a = 0.2X\frac{500}{50} = 0.2X10 = 2 \quad m/s^2$$

Problem 3 : A box is dropped gently on a conveyor belt moving at a speed “v”. The kinetic coefficient of friction between box and conveyor belt is “μ”. Find the time after which, the box becomes stationary on the belt.

Conveyor belt

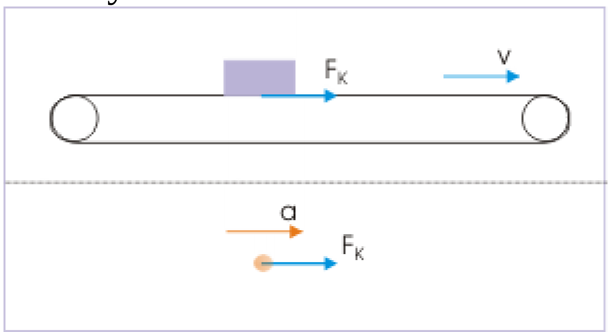


A girl is standing on a moving conveyor strip.

Solution : The initial velocity of box is zero. When we put the box on the belt, the friction on the belt, acts opposite to the direction of motion of the conveyor belt. The friction on the box, opposite to the friction on the belt. acts in the direction of motion of the belt.

The friction force accelerates the box till there is no relative motion between box and belt. In other words, the box accelerates till the box achieves the velocity of the belt i.e. “v”. Since there is relative motion between two bodies, the friction is equal to kinetic friction till their velocities are equal. Thus, acceleration of the box with respect to ground is :

Conveyor belt



A girl is standing on a moving conveyor strip.

$$a = \frac{Fk}{m} = \frac{\mu mg}{m} = \mu g$$

Here, initial velocity is zero, final velocity is “v” and acceleration is “μg”. Thus, applying equation of motion for constant acceleration, we have :

$$v = u + at$$

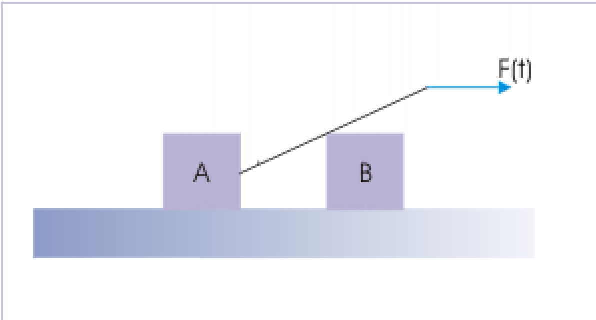
$$v = 0 + \mu gt$$

$$\Rightarrow t = \frac{v}{\mu g}$$

Blocks

Problem 4 : A variable increasing force is applied on a rod in the arrangement shown in the figure. The blocks are identical and are placed over a rough horizontal surface. Determine which of the block will move first?

Two blocks and a rod

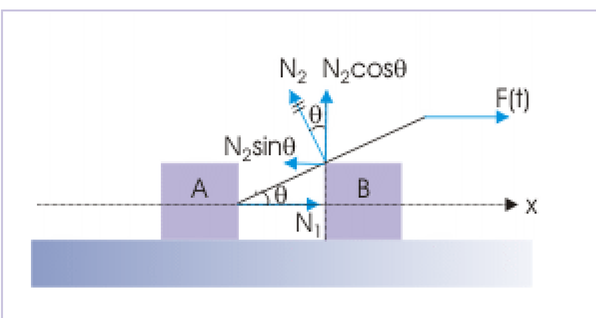


A variable increasing force is applied on a rod.

Solution : The blocks undergo translation in opposite directions. Since blocks are identical, the friction between block and horizontal surface is same for both blocks. It is evident that the block, having greater external force parallel to surface, will overcome friction first.

It is, therefore, evident that we need to analyze forces in the horizontal direction. The forces on the rod are shown in the figure below. The components of forces in horizontal direction form balanced force system before the motion is initiated. From the force analysis in horizontal direction, we have :

Two blocks and a rod



Forces on a rod.

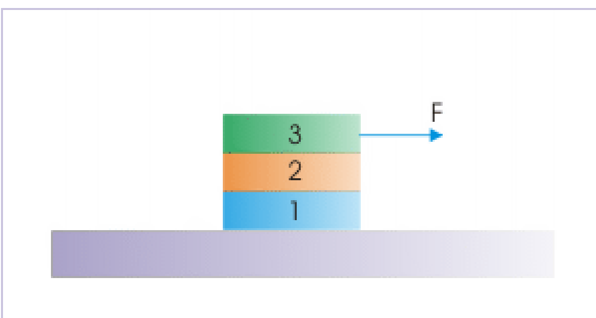
$$\Rightarrow N_2 \sin\theta = N_1 + F$$

The contact normal force are equal and opposite between rod and blocks. The horizontal component of normal force ($N_2 \sin\theta$) as applied by the rod on the block “B” is greater than the horizontal normal force, (N_1) as applied by the rod on the block “A”. It means the external horizontal force on “B” will exceed the limiting friction first. As such, block “B” will move first.

Stack of blocks

Problem 5 : The masses of three blocks stacked over one another are m_1 = 30 kg, m_2 = 10 kg and m_3 = 40 kg. The coefficients of friction between block “1” and ground is μ_0 = 0.1, between “1” and “2” is μ_1 = 0.2 and between “2” and “3” is μ_2 = 0.3. An external force is applied on the block “3” as shown in the figure. What least force is required to initiate motion of any part of the stack?

Three stacked blocks



Blocks of different masses are placed one over other.

Solution : In order to answer this question, we are required to determine limiting static friction at each of the interface. The limiting friction between first block and ground is :

$$F_0 = \mu_0 N = \mu_0(m_1 + m_2 + m_3)g = 0.1(30 + 10 + 40)X10 = 80N$$

The limiting friction between first and second block is :

$$F_1 = \mu_1 N = \mu_1(m_2 + m_3)g = 0.2(10 + 40)X10 = 100N$$

The limiting friction between second and third is :

$$F_2 = \mu_2 N = \mu_2(m_3)g = 0.3X40X10 = 120N$$

If the external force is less than 80 N, then none of the block will move. If external force is equal or greater than 80 N, but less than 100 N, then only the first block (the one in contact with ground) will move as external force on it is equal to the limiting friction between block “1” and the underlying horizontal surface.

Features of friction force

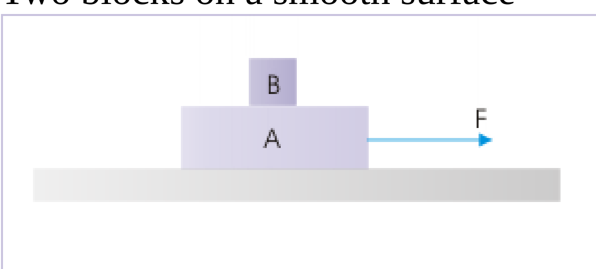
Friction force differs to other forces in many respects. Importantly, friction adjusts to the way external force is applied on the body. This characteristic provides opportunity to manage this force that minimizes our effort to produce acceleration in bodies. At the same time, friction prohibits some geometric restriction on the way we manage external force to produce acceleration. For example, we can not push an object unless external force is applied outside a region called "cone of friction".

Besides, there are certain misplaced notion about friction that it is a negative force, which can not produce acceleration or which can not do positive work. It is generally believed that friction only moderates the acceleration and work of other external forces negatively. As a matter of fact, we shall find that it is actually the friction, which is ultimately driving our car on the road. We shall, however, save for the details on this aspect till we take up the study of rolling motion. We shall, however, make the beginning in this module to answer this question at a basic level.

Can friction cause motion?

To investigate this question, let us consider the combined motion of two blocks system, which is being pulled with an external force such that two blocks move together as shown in the figure. Let us also consider that the block system lies on a smooth horizontal surface, enabling us to neglect friction between block “A” and the underlying horizontal surface.

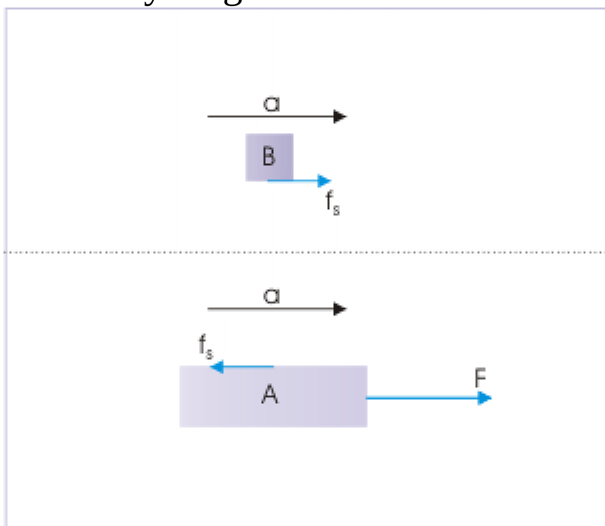
Two blocks on a smooth surface



The blocks slide together on a smooth horizontal plane.

The external forces on the two blocks are shown in the force diagrams in the figure below as seen from the ground reference. The upper block “B” is moving along with lower block “A” due to the friction applied by it.

Free body diagram



Upper block is accelerated by friction.

The friction forces at the interface, which act on two bodies, are action and reaction pair and are equal in magnitude and opposite in direction, in accordance with Newton’s third law of motion. We note that friction on block “B” due to “A” is in the same direction as that of external force “F”. The only external force on block “B”, as seen from the ground, is the friction force, f_s .

The accelerations of the composite block system and individual blocks are same as considered in the beginning. From three corresponding force diagrams, including the one corresponding to composite block system, we have three expressions for the common acceleration :

$$a = \frac{F}{m_A + m_B}$$
$$a = \frac{F - f_s}{m_A}$$
$$a = \frac{f_s}{m_B}$$

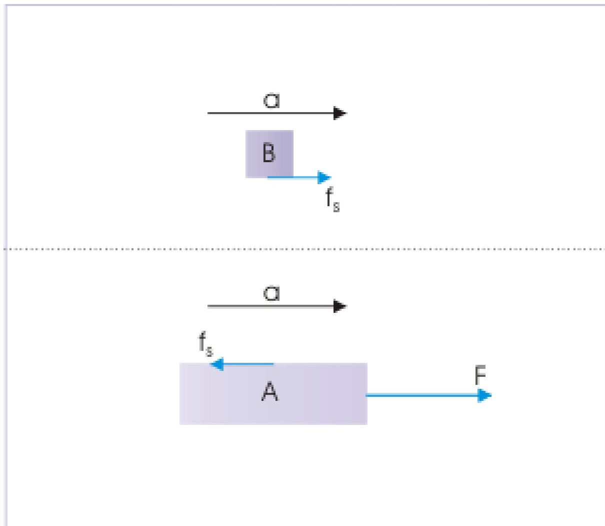
From the last equation, it is clear that static friction, as a matter of fact, caused the motion of block “B” with respect to ground by producing acceleration in the body. Here, measurement of motion (acceleration) is in a frame different to one in which the surface in contact lies. The block is accelerated with respect to ground – not with respect to the reference of block “A”, which applies friction force.

Thus, we see that friction is capable to produce acceleration in a body. It is expected also. This is the very nature of a force. Since friction is a force, it is capable to produce acceleration and do positive work. There is, however, a catch. Friction exists only due to other external forces. It does not exist, if there is no external force on the body. But as far as the question "can friction cause motion?. Answer is yes.

Direction of friction

Direction of friction is opposite to the component of net external force parallel to the contact surface. It is the criteria to decide the direction of friction. There are, however, situations in which external force may not be obvious. In the illustration of the above section, there is no external force on the upper block "B". What would be the direction of friction on block "B"? As a matter of fact, it is the only force (we do not consider vertical normal force as it perpendicular to motion) on block acting parallel to contact surface. Clearly, we can not apply the criteria for determining direction in this case.

We actually analyze the forces on the underneath block "A". The net external force parallel to interface is acting towards right. It, then, follows that the friction on "A" is towards left. Taking cue from this revelation and using Newton’s third law of motion, we decide that the direction of friction on block "B" should be towards right as shown in the figure below.
Free body diagram



Upper block is accelerated by friction.

Alternatively, we can consider relative motion between bodies in contact. In the illustration, we see that block “A” has relative velocity (or tendency to have one) towards right with respect to block “B”. It means that the block “B” has relative velocity (or tendency to have one) towards left with respect to block “A”. Hence, friction force on block “B” is towards right i.e. opposite to relative velocity of “B” with respect to block “A”.

We can formulate a directional rule about the direction of friction as :

“The direction of friction on a body is opposite to the relative velocity of the body (or the tendency to have one) with respect to the body, which is applying friction on it.”

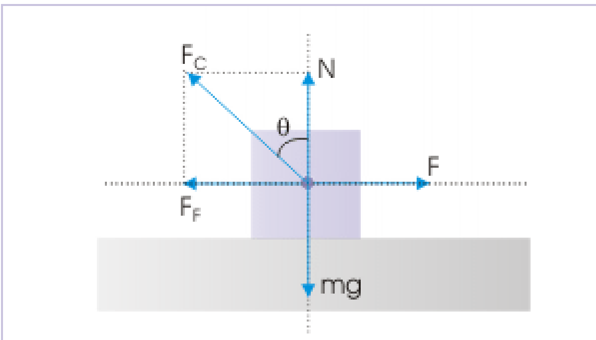
Contact forces revisited

There are two types of contact forces that we encounter when two bodies interact. The contact forces are normal and friction forces. The normal force is a reaction of a body against any attempt (force) to deform it. The ability

of a body to resist deformation also has electromagnetic origin operating at the surface as in the case of friction.

The two contact forces, therefore, can be considered to be manifestation of same inter – atomic forces that apply at the contact interface. The resultant electromagnetic force acts in a direction inclined to the surface. Its component perpendicular to surface is the normal force and component parallel to the surface is friction.

Contact force



Forces on the block

The resultant or net electromagnetic contact force is the vector sum of the two components and is given by :

$$F_C = \sqrt{\left({F_N}^2 + {F_F}^2\right)}$$

Where :

F_C : Resultant contact force

F_N : Normal force, also represented simply as "N"

F_F : Friction force, which can be f_s (static friction) , F_s (maximum static friction) or F_k (kinetic friction)

For the maximum static friction (F_s) and a given normal force (N), the magnitude of resultant contact force is :

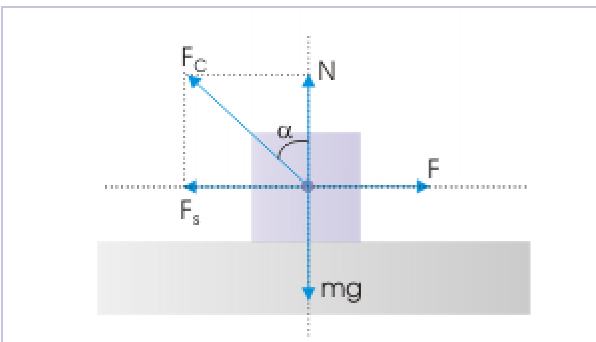
$$F_C = \sqrt{\left({F_N}^2 + {F_F}^2\right)} = \sqrt{\left(N^2 + {F_s}^2\right)}$$
$$\Rightarrow F_C = \sqrt{\left(N^2 + {\mu_s}^2 N^2\right)}$$
$$\Rightarrow F_C = N\sqrt{\left(1 + {\mu_s}^2\right)}$$

This expression represents the maximum contact force between a pair of surfaces.

Angle of friction

The angle of friction for two surfaces in contact is defined as the angle that the maximum contact force makes with the direction of normal force as shown in the figure. From the figure,

Contact force



Forces on the block

$$\tan\alpha = \frac{F_s}{N} = \mu_s$$

The angle of friction, therefore, is :

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{F_s}{N}\right) = \tan^{-1}\left(\mu_s\right)$$

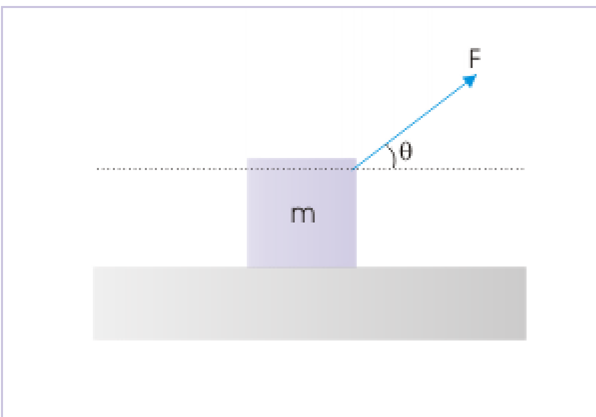
We must understand here that angle of friction is a function of the coefficient of friction between two surfaces. This means that it depends on

the nature of surfaces and is unique for the two surfaces in contact. Specially, we should note that it does not depend on maximum static force or normal force. If we increase “N” (say by putting more weight on the block), then maximum static friction also increases as $F_s = \mu_s N$ and the ratio “ $\frac{F_s}{N}$ ” remains constant.

Minimum external force to initiate motion

External force is required to initiate motion against friction force. For every situation, there is a particular direction in which the requirement of external force to initiate motion is a minimum. Here we consider the case of a block placed on a horizontal surface, which is pulled up by an external force, F, as shown in the figure.

Force on a block



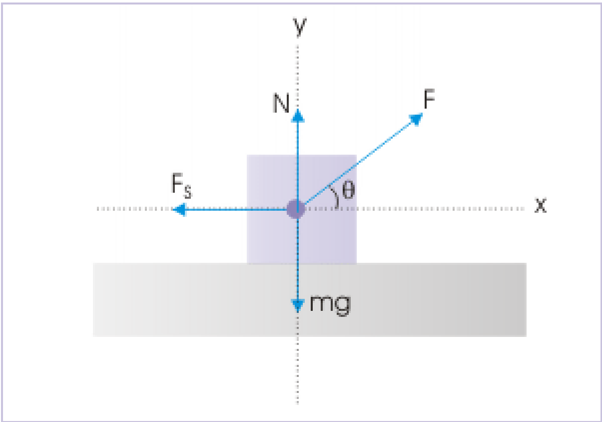
Forces on the block tries to move the block.

At first consideration, we may reason that force would be always least when it is applied exactly opposite to the friction. The force required in such case is given by :

$$F_{||} = \mu_s N = \mu_s mg$$

This consideration, however, is not correct. The external force, besides overcoming maximum static friction, also affects normal force. The component of force in the vertical direction decreases normal force. Since friction is proportional to normal force, an unique orientation will require minimum magnitude of external force.

Force on a block



Forces diagram

When angle "θ" increases (cosθ decreases), the force overcoming maximum static friction decreases :

$$F_{||} = F_x = F\cos\theta$$

On the other hand, normal force, N, decreases with increasing angle (increasing sinθ),

$$N = mg - F\sin\theta$$

Consequently, the maximum static friction decreases as :

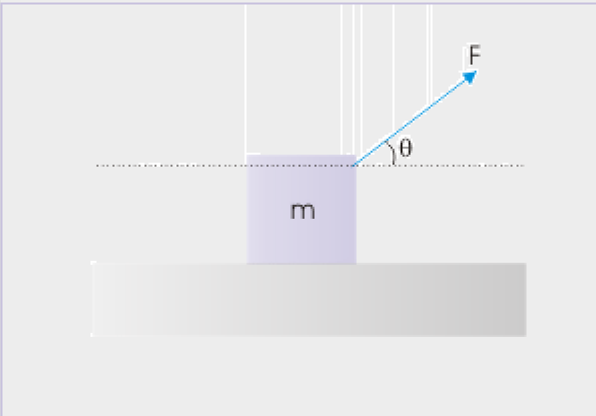
$$F_s = \mu_s N = \mu_s(mg - F\sin\theta)$$

Thus, there is a particular value of the angle "θ" for which magnitude of external force is minimum. We shall work out the particular orientation for this arrangement in the example given here.

Example:

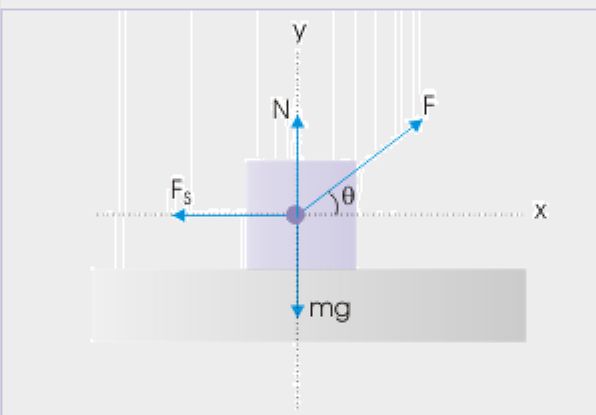
Problem : An external force, "F", is applied on a block at an angle "θ" as shown in the figure. In order to initiate motion, what should be the angle at which magnitude of external force, F, is minimum? Also find the magnitude of minimum force.

Force on a block



Solution : We should obtain an expression of external force as a function of "θ" and then evaluate the function for obtaining the condition for the minimum value. We draw the free body diagram superimposed on the body diagram for the condition of maximum static friction as shown in the figure.

Force on a block



Forces diagram

$$\sum F_x = F\cos\theta - \mu_s N = 0$$
$$\Rightarrow F\cos\theta = \mu_s N$$

and

$$\sum F_y = N + F\sin\theta - mg = 0$$
$$\Rightarrow N = mg - F\sin\theta$$

Combining two equations, we have :

$$\Rightarrow F = \frac{\mu_s mg}{\cos\theta + \mu_s \sin\theta}$$

But, we know that μ_s = tan α, where "α" is the angle of friction.

$$\Rightarrow F = \frac{mg\tan\alpha}{\cos\theta + \tan\alpha\sin\theta}$$
$$\Rightarrow F = \frac{mg\tan\alpha\cos\alpha}{\cos\theta\cos\alpha + \sin\alpha\sin\theta}$$
$$\Rightarrow F = \frac{mg\sin\alpha}{\cos(\theta - \alpha)}$$

The denominator is maximum when cos (θ – α) = 1.

$$\Rightarrow \theta - \alpha = 0$$
$$\Rightarrow \theta = \alpha$$

Corresponding minimum force is :

$$\Rightarrow F_{\text{min}} = mg\sin\alpha$$

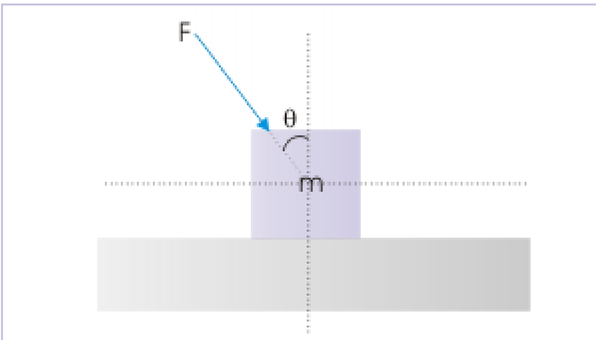
Since
$\tan\alpha = \mu_s; \quad \cos\alpha = 1/\sec\alpha = 1/\sqrt{(1 + \tan^2\alpha)} = 1/\sqrt{(1 + \mu_s^2)}$;
Hence, $\sin\alpha = \sqrt{(1 - \cos^2\alpha)} = \mu_s/\sqrt{(1 + \mu_s^2)}$. Putting this value in the equation,

$$\Rightarrow F_{\text{min}} = \frac{\mu_s mg}{\sqrt{(1 + \mu_s{}^2)}}$$

Cone of friction

Generally a body is moved either by “pulling” or “pushing” corresponding to the very nature of force, which is defined as a “pull” or “push”. We find that existence of friction against relative motion between two bodies puts certain restriction to the ability of external force to move the body.

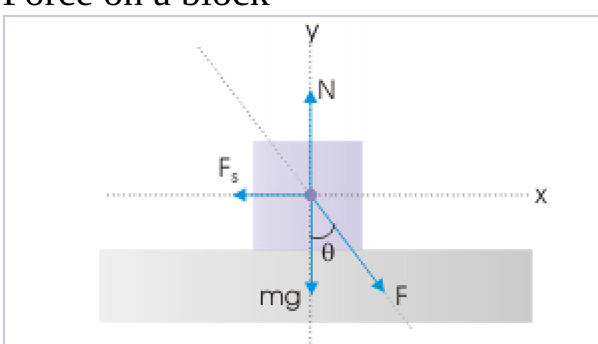
Force on a block



Applied force tries to push the body.

This restriction is placed with respect to application of force to move a body over another surface. To illustrate the restriction, we consider a block lying on a rough horizontal surface. An external force, F, is applied on the body to initiate the motion. The figure here shows the free body diagram superimposed on the body diagram.

Force on a block



Force diagram

The x-direction component of the external force parallel to the contact surface tends to move the body. On the other hand, the y-direction component of the external force in the downward direction increases the normal force, which increases the maximum static friction force. Thus, we see that one of the components of applied force tries to move the body, whereas the other component tries to restrict body to initiate motion.

Friction is equal to maximum static friction to initiate motion,

$$F_s = \mu_s N$$

We analyze for zero net force in both component directions to investigate the nature of external force, "F".

$$\begin{aligned} &\sum F_x = F \sin\theta - \mu_s N = 0 \\ &\Rightarrow F \sin\theta = \mu_s N \end{aligned}$$

and

$$\begin{aligned} &\sum F_y = F \cos\theta + mg - N = 0 \\ &\Rightarrow N = F \cos\theta + mg \end{aligned}$$

Combining two equations, we have :

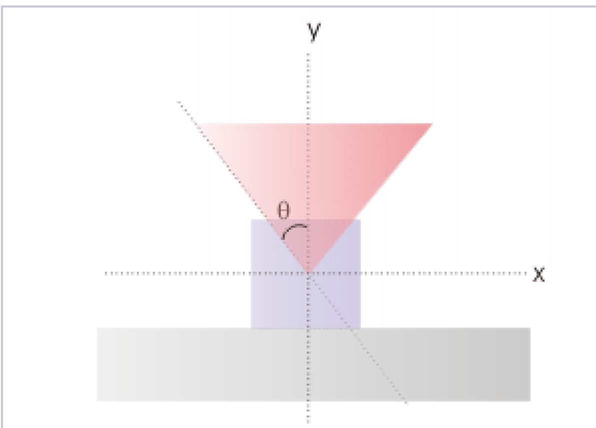
$$\begin{aligned} &\Rightarrow F \sin\theta = \mu_s (F \cos\theta + mg) \\ &\Rightarrow F (\sin\theta - \mu_s \cos\theta) = \mu_s mg \\ &\Rightarrow F = \frac{\mu_s mg}{\sin\theta - \mu_s \cos\theta} \end{aligned}$$

The magnitude of external force is a positive quantity. It is, therefore, required that denominator of the ratio is a positive quantity for the body to move.

$$\Rightarrow \sin\theta - \mu_s \cos\theta \geq 0$$
$$\Rightarrow \tan\theta \geq \mu_s$$
$$\Rightarrow \theta \geq \tan^{-1}(\mu_s)$$
$$\Rightarrow \theta \geq \alpha \text{ (angle of friction)}$$

The body can be pushed for this condition (θ ≥ α). On the other hand, a force applied within the angle of friction, however great it is, will not be able to push the body ahead. A cone drawn with half vertex angle equal to the angle of friction constitutes a region in which an applied external force is unable to move the body.

Cone of friction



Force applied within the angle of friction, however great it is, will not be able to push the body ahead.

Working with friction (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to working with friction. The questions are categorized in terms of the characterizing features of the subject matter/ question :

- Equilibrium
- Acceleration along incline
- Motion with friction
- Contact force on an incline

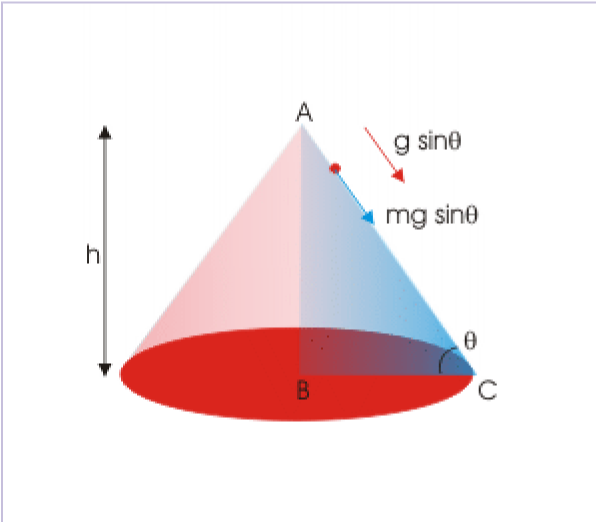
Equilibrium

Stack of blocks

Problem 1 : The coefficient of friction between sand particles of uniform size is "μ". Find the maximum height of the sand pile that can be made in a circular area of radius "r".

Solution : We consider a sand particle on the slant side of the sand pile. We observe here that the particle on the sand pile resembles the situation that of the motion of a particle on an incline of certain angle. The pile will have the maximum height for which the friction between the particles has maximum friction i.e. limiting friction. Beyond this angle, the sand particle will slide down and spill outside the circular base boundary of the sand pile.

Sand pile



The slant side of the sand pile can be treated as an inline.

We know that the angle of incline corresponding to the limiting friction is the angle of repose,

$$\tan\theta = \mu$$

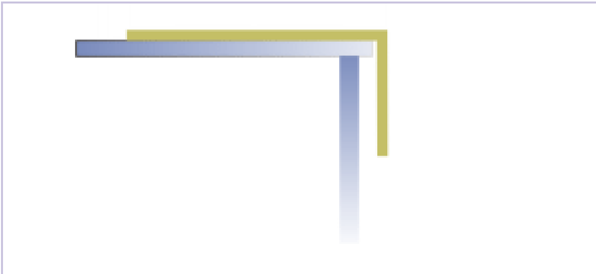
From the geometry of right angle ABC, we have :

$$\tan\theta = \mu = \frac{AB}{BC} = \frac{h_{\max}}{r}$$

$$\Rightarrow h_{\max} = \mu r$$

Problem 2 : A homogeneous chain of length “L” lies on a horizontal surface of a table. The coefficient of friction between table and chain is “μ”. Find the maximum length of the chain, which can hang over the table in equilibrium.

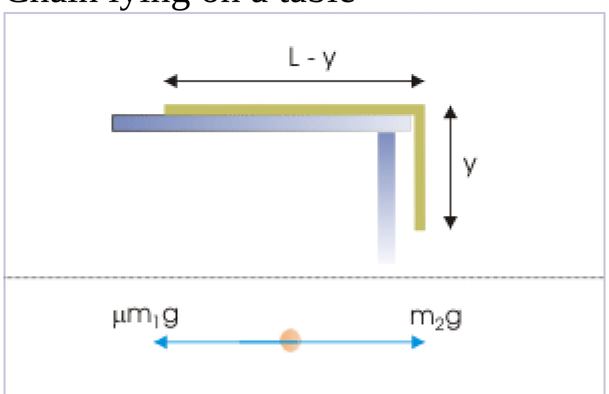
Chain lying on a table



The chain is in equilibrium.

Solution : Here, we have to consider the translational equilibrium of the chain for which net force on the chain should be zero. Now, the chain is homogeneous. It means that the chain has uniform linear mass density. Let it be “λ”. Further, let a length “y” hangs down the table in equilibrium. The length of chain residing on the table, then, is "L – y".

Chain lying on a table



A length "y" of the chain hangs from the table.

The mass of the chain,” m_1 ”, on the table is

$$\Rightarrow m_1 = (L - y)\lambda$$

The mass of the chain,” m_2 ”, hanging is

$$m_2 = y\lambda$$

Free body diagram of the mass of chain, " m_1 ", on the table is shown in the figure above. This mass is pulled down by the weight of the hanging chain. For balanced force condition, the friction force on the chain on the table should be equal to the weight of the hanging chain,

$$\mu m_1 g = m_2 g$$

$$\Rightarrow \mu(L - y)\lambda g = y\lambda g$$

$$\Rightarrow \mu(L - y) = y$$

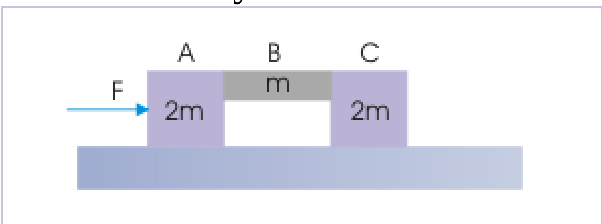
$$\mu L - \mu y = y$$

$$y(1 + \mu) = \mu L$$

$$\Rightarrow y = \frac{\mu L}{(1 + \mu)}$$

Problem 3 : In the arrangement of three blocks on a smooth horizontal surface, the friction between blocks "A" and "B" is negligible, whereas coefficient of friction between blocks "B" and "C" is "μ". Find the minimum force "F" to prevent middle block from sliding down.

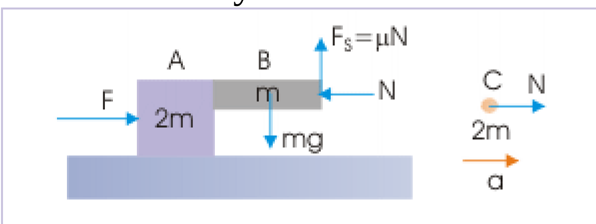
Three blocks system



The blocks are pushed together towards right.

Solution : The block “B” can be held in position by the friction force between “B” and “C”, which operates in vertically upward direction. The friction acts in the upward direction on block “B” as it has the tendency to fall down vertically. We need to evaluate the minimum external force, “F”, which corresponds to a situation when friction in upward direction at the interface is equal to the weight of the block “B”.

Three blocks system



The blocks are in equilibrium.

$$F_S = \mu N$$

The friction force in vertically upward direction at the interface, however, depends on the normal force applied by the block “C” on the block “B”. It is important to understand that “N” is the normal force at the interface between blocks “B” and “C”.

We can find horizontal normal force on block “C”, using second law of motion. Referring to the free body diagram shown in the right side of the figure above.

$$N = m_C a_C$$

Here, $m_C = 2m$. The acceleration “ m_C” is equal to the acceleration of the combined body as three blocks move together. Let “a” be the acceleration of the combined body. Then,

$$N = m_C a_C = 2ma$$

We can find the common acceleration of the blocks, by considering three blocks together. The common acceleration for the combined body is given

by :

$$a = \frac{F}{(3m + m + 2m)} = \frac{F}{6m}$$

Hence,

$$\Rightarrow N = 2mX\frac{F}{6m}$$

$$\Rightarrow N = \frac{F}{3}$$

For preventing the middle block “B” to fall down, the forces in vertical direction on block “B” should form a balanced force system.

$$F_S \mu N = m_B g$$

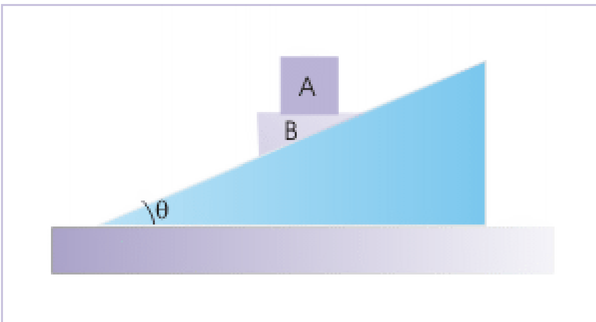
$$\Rightarrow \frac{\mu F}{3} = mg$$

$$\Rightarrow F = \frac{3mg}{\mu}$$

Acceleration along incline

Problem 4 : A block “A” is placed over block “B”, whose base matches with the incline as shown in the figure. The incline is fixed on the horizontal plane. The friction between blocks is “μ”, whereas friction between block “B” and incline is negligible. Find the angle of incline for which block “A” is about to move over block “B”.

Blocks on an incline



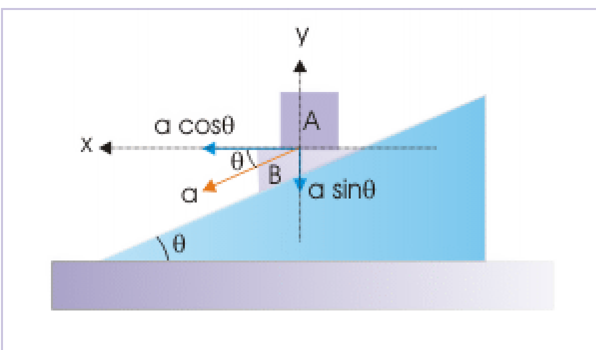
Block "A" is placed on a horizontal surface.

Solution : The block “A” is about to move on the block “B”. It means that the friction between the blocks is equal to limiting friction,

$$F_S = \mu N$$

Now, we need to know the value of “N” in order to determine limiting friction. The block “A” is about to move relative to block “B”. As such, block “A” is not having relative motion with respect “B”. The two blocks move together with common acceleration, “a”, given by :

Blocks on an incline



Acceleration of block "A".

$$a = g \sin \theta$$

Since we require to find normal force "N", we shall consider forces in vertical direction. For this, we need to find acceleration in vertical direction. Hence, resolving acceleration in “x” and “y’ direction, we have :

$$a_x = a\cos\theta = g\sin\theta\cos\theta$$

$$a_y = a\sin\theta = g\sin^2\theta$$

Now, the forces on block “A” in vertical (“y”) direction are (i) its weight and (ii) normal force. Thus,

$$mg - N = ma_y = mg\sin^2\theta$$

$$\Rightarrow N = mg\left(1 - \sin^2\theta\right) = mg\cos^2\theta$$

Putting expression of “N” in the equation of limiting friction, we have :

$$\Rightarrow F_S = \mu mg\cos^2\theta$$

The limiting friction provides the acceleration in horizontal direction (note that blocks have no relative motion between them, but have accelerations with respect to ground),

$$F_S = \mu mg\cos^2\theta = ma_x = mg\sin\theta\cos\theta$$

$$\Rightarrow \tan\theta = \mu$$

$$\theta = \tan^{-1}\mu$$

Motion with friction

Problem 5 : A car, starting from rest, is brought to rest during which it covers a linear distance of 125 m. If the friction between tyre and road is 0.5, then what is the minimum time taken by the car to cover the distance?

Solution : The acceleration or deceleration of the car during travel is determined by the friction. The car will cover the distance in minimum time, if it moves at the maximum acceleration and breaks motion with the

maximum deceleration. The journey takes equal time (say " t_0 ") in two phases of motion for minimum time of journey.

The limiting friction on the car is given by,

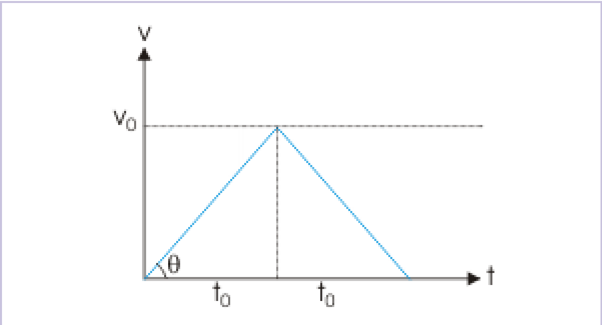
$$F_S = \mu N = \mu mg$$

Acceleration/ deceleration of car corresponding to limiting friction is given by :

$$a = \frac{\mu mg}{m} = \mu g$$

The velocity – time plot corresponding to the situation is given here. From the plot, the maximum speed, " v_0 ", is given by the vertical height of the triangle. The magnitude of acceleration/ deceleration is equal to the slope of velocity – time plot.

Velocity time plot



The car accelerates and decelerates for equal time.

$$\Rightarrow a = \mu g = \tan\theta = \frac{v_0}{t_0}$$

$$\Rightarrow v_0 = \mu g t_0$$

The total displacement during journey is equal to the area of the triangle on velocity – time plot,

$$\Rightarrow 125 = \frac{1}{2} X 2 t_0 X \mu g t_0 = t_0 X 0.5 X 10 t_0 = 5 t_0^2$$

$$\Rightarrow t_0^2 = 25$$

$$\Rightarrow t_0 = 5 \quad s$$

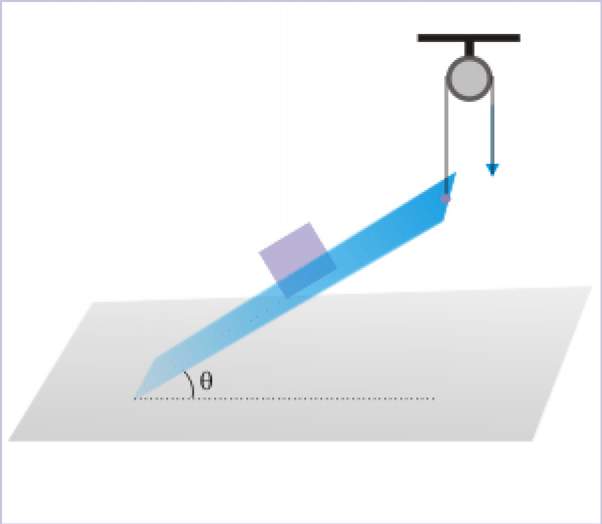
Hence, time taken to cover the distance is :

$$\Rightarrow 2t_0 = 2X5 = 10 \quad s$$

Contact force on an incline

Problem 6 : A block is placed on a rough incline whose angle can be varied as shown in the figure. Draw a plot between contact force and angle of incline (θ) for 0°≤θ≤90° (only show the nature of the plot).

A block on an incline



Angle of incline varies.

Solution : The contact force comprises of normal force and friction. When θ = 90°, there is no normal force. Hence, there is no friction either. The block is under free fall. And,

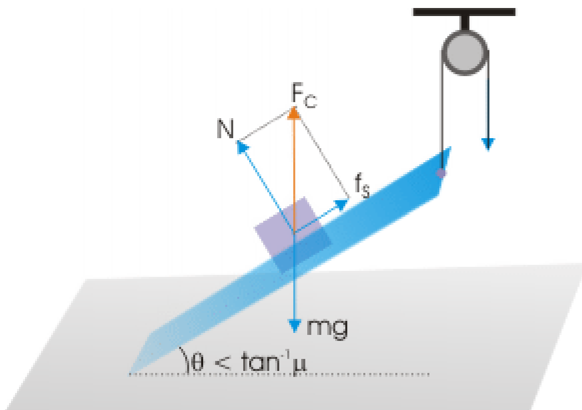
$$F_C = 0$$

We have studied that the block does not slide till the angle of incline is equal to angle of repose. The angle of repose is given by :

$$\Rightarrow \theta = \tan^{-1} \mu$$

Till this angle, the block is stationary and the contact forces (normal and friction forces) and gravity form the balanced force system. It means that contact force is equal to the weight of the block in magnitude. Hence,

A block on an incline



Before motion, the contact force is equal to the weight of the block in magnitude.

$$F_C = mg = \sqrt{(N^2 + f_s^2)}$$

When the angle of incline exceeds the value of angle of repose, then the component of weight parallel to the incline is greater than the friction force. As such, block begins to slide down with kinetic friction. The contact force beyond angle of repose, therefore, is given by :

$$\Rightarrow F_C = \sqrt{\left\{(mg\cos\theta)^2 + (\mu mg\cos\theta)^2\right\}}$$

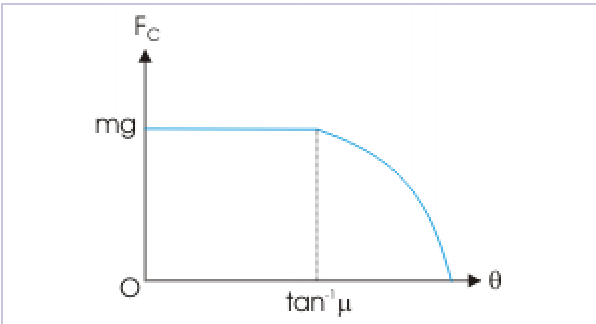
$$\Rightarrow F_C = mg\cos\theta\sqrt{\{1+\mu^2\}}$$

The plot in this region is a portion of cosine curve for

$$\Rightarrow \theta > \tan^{-1}\mu$$

The figure below shows the plot for the complete range.

Contact force



Plot of contact force and angle of incline.

Motion on rough incline plane

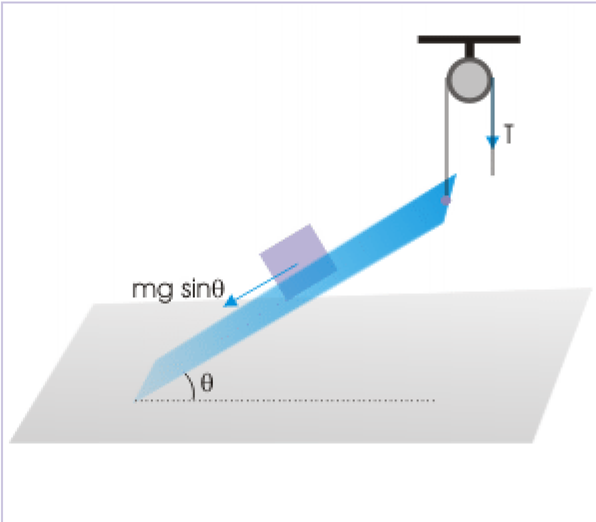
Motion of a block on a rough incline plane is the interplay of different force types and the characterizing features of the incline surface. An incline is a plane surface, whose one end is raised. The raised surface forms an angle “θ” with the horizontal. The block or a body placed on the surface is acted upon by gravity and contact forces (normal and friction forces), besides other forces. In this module, we shall restrict our consideration to force due to gravity and contact forces and shall consider other external force in subsequent module.

Study of motion of a block on an incline is important from the point of view of the measurement of the coefficients of friction.

Static friction and incline

In this section, we consider the motion of a block placed on a stationary incline i.e. incline itself does not move on the horizontal surface. At present, we do not consider any additional force. The forces on the block are (i) weight of the block, mg, (ii) Normal force, N, and (iii) friction, F_F (f_s , F_s or F_k).

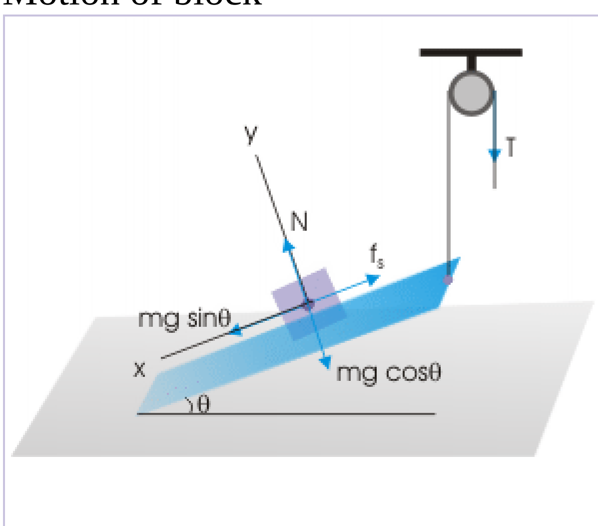
For the sake of illustration, we treat the incline as a plane surface, which can be lifted at one end with certain mechanical arrangement so that the incline angle “θ” with horizontal direction can be varied. This set up enables us to change the gravitational pull on the block in the direction of incline surface and work with incline of different angles for the study of motion on the incline.
Adjustable incline



The incline angle is changed
with string and plley
arrangement.

The forces, in the direction normal to the incline plane, are balanced as there is no motion in that direction. On the other hand, the component of weight parallel to contact surface, mg sinθ, tends to initiate motion of the block along the incline plane. For small value of "θ", the component of force parallel to contact surface is small and is counteracted by friction, acting in the opposite direction such that :

Motion of block



The force parallel to the contact is equal to static friction.

$$f_s = mg \sin\theta$$

Angle of repose

As we increase the angle, θ, the downward force along the incline increases. The friction, in response, also increases till it reaches the maximum value corresponding to limiting (maximum) static friction, F_s . In that situation,

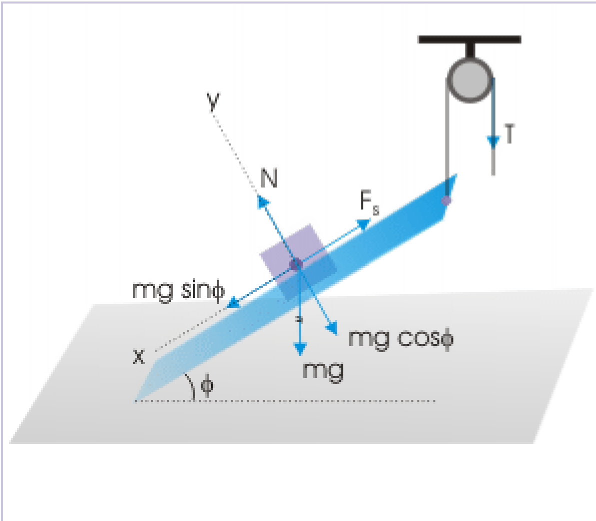
$$F_s = mg \sin\varphi$$

where “φ” is the angle at which the block just starts to slide down. This angle is known as “angle of repose”.

Measurement of coefficient of friction

We can measure coefficient of friction between two surfaces by measuring angle of repose. The free body diagram of the block as superimposed over the block-incline system corresponding to the maximum static friction is shown in the figure.

Measurement of coefficient of static friction



The block is about to slide sown.

$$\sum F_x = mg\sin\varphi - F_s = 0$$
$$\Rightarrow mg\sin\varphi = F_s$$

and

$$\sum F_y = N - mg\cos\varphi = 0$$
$$\Rightarrow mg\cos\varphi = N$$

Combining two equations, we have :

$$\Rightarrow \tan\varphi = \frac{F_s}{N}$$

However, we know that coefficient of static friction is given by :

$$\Rightarrow \mu_s = \frac{F_s}{N}$$

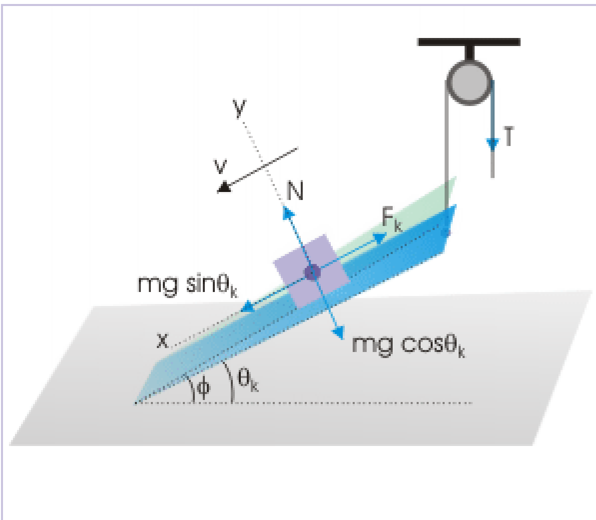
Hence,

$$\Rightarrow \mu_s = \tan\varphi$$

This is an important result as measurement of the angle of repose allows us to determine coefficient of friction between any two given surfaces. It is also evident from the relation above that angle of repose is equal to angle of friction as defined in the earlier module.

This method can also be employed to measure coefficient of kinetic friction as well. For that, we push the block gently over the incline surface and adjust the inclination of the incline or surface such that the block continues to move with uniform velocity. In that situation, the block moves under the balanced force system and kinetic friction is equal to the component of weight along the incline.

Measurement of coefficient of kinetic friction



The block slide sown the plane with uniform velocity.

As such,

$$\Rightarrow \mu_k = \tan\theta_k$$

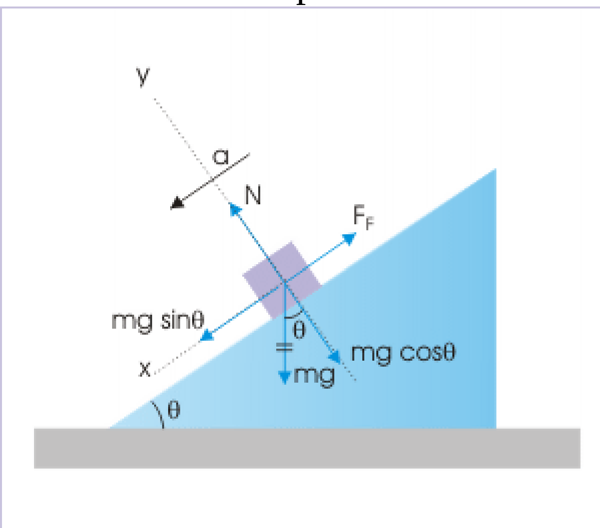
Motion on an incline plane under gravity

A body placed on an incline is subjected to downward pull due to gravity. The downward motion along the incline is, therefore, natural tendency of the body, which is opposed by friction acting in the opposite direction against the component of gravity along the incline. Depending on the relative strengths of two opposite forces, the body remains at rest or starts moving along the incline.

The translational motion of a body on an incline is along its surface. There is no motion in the direction perpendicular to the incline surface. The force components in the direction perpendicular to the incline form a balanced force system.

Let us examine the free body diagram as superimposed on the body systems and shown in the figure. In the direction of incline i.e. x – axis, we have :

Motion on incline plane



Motion of the block is opposed by static friction.

$$\sum F_x = mg\sin\theta - F_F = ma$$

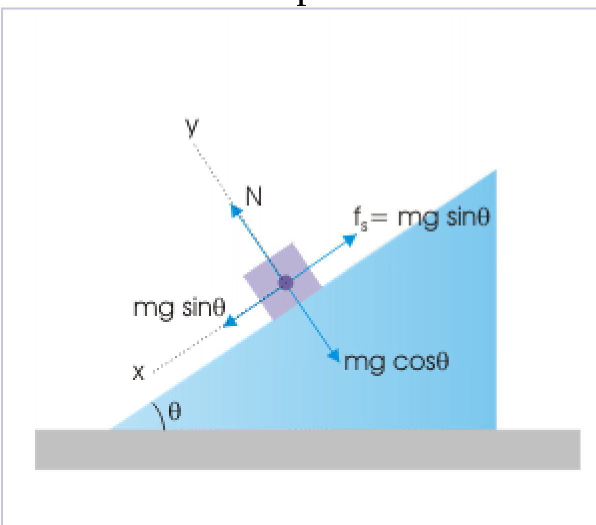
where F_F is the force of friction, which can be static friction, maximum static friction or kinetic friction. In the y-direction,

$$\sum F_y = N - mg\cos\theta = 0$$
$$\Rightarrow N = mg\cos\theta$$

Based on the angle of incline” θ”, there are following possibilities :

Case 1 : The friction is less than the maximum static friction and body is yet to start motion. Here, $\theta < \tan^{-1}(\mu_s)$. In this case, friction is simply equal to component of gravity, parallel to the contact surface. Thus,

Motion on incline plane



Motion of the block is opposed by static friction.

$$F_F = f_s = mg\sin\theta$$

The net component of external forces in x-direction (including friction), therefore, is zero. As such, the block remains stationary.

$$\sum F_x = mg\sin\theta - F_F = mg\sin\theta - mg\sin\theta = 0$$

Case 2 : The friction is equal to the maximum static friction and body is yet to start motion. Here, $\theta = \tan^{-1}(\mu_s)$. In this case also, the component of net external force in x-direction (inclusive of friction) is zero and the block is stationary. Here,

$$\Rightarrow \tan\theta = \tan\varphi = \mu_s$$

Case 3 : The friction is equal to the kinetic friction and body is in motion. Initially, $\theta > \tan^{-1}(\mu_s)$. In this case, friction is given by :

$$F_F = F_k = \mu_k N$$

We must understand here that friction can be equal to kinetic friction only when body has started moving. This is possible when component of gravity, parallel to contact surface is greater than the maximum static friction. However, once this condition is fulfilled, the friction between the surfaces reduces slightly to become equal to kinetic friction. Subsequently, the motion of the body can be either without acceleration (uniform velocity) or with acceleration.

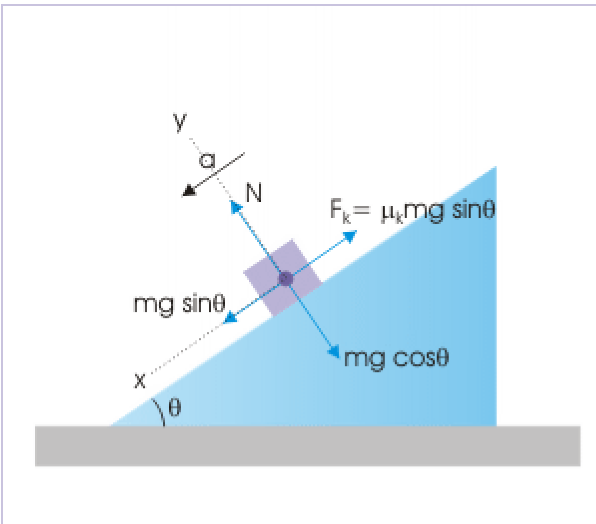
3a: Motion with uniform velocity

This is a case of balanced force system in x-direction. The component of net force in x-direction is zero and the block is moving with constant velocity. Here,

$$\Rightarrow \tan\theta = \mu_k$$

3b: Accelerated motion

This is a case of unbalanced force system in x-direction.
Motion on incline plane



Motion of the block is opposed by kinetic friction.

In x – direction,

$$\sum F_x = mg\sin\theta - F_F = ma_x$$
$$\Rightarrow mg\sin\theta - \mu_k N = ma$$

Putting value of normal force, N, we have :

$$\Rightarrow mg\sin\theta - \mu_k mg\cos\theta = ma$$
$$\Rightarrow a = g\sin\theta - \mu_k g\cos\theta$$

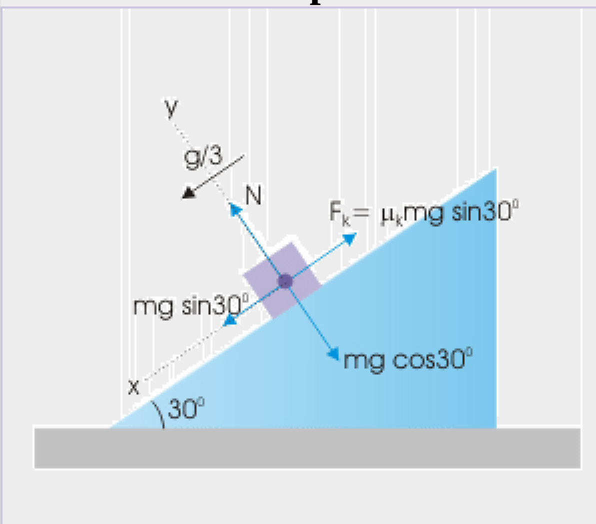
For the given value of "θ" and " μ_k ", acceleration is constant and the block, therefore, moves down with uniform acceleration.

Example:
Problem : A block of mass "m" slides down an incline, which makes an angle of 30° with the horizontal. If the acceleration of the block is g/3, then find the coefficient of kinetic friction. Consider g = 10 m/s^2 .
Solution : The coefficient of kinetic friction is given by :

$$\mu_k = \frac{F_k}{N}$$

In order to evaluate this ratio, we need to find normal and friction forces. The free body diagram of the block as superimposed on the system is shown in the figure.

Motion on incline plane



Motion of the block is opposed by kinetic friction.

In the x-direction,

$$\sum F_x = mg \sin 30^0 - F_k = ma_x$$
$$\Rightarrow mg \sin 30^0 - F_k = m \text{ x } \frac{g}{3}$$
$$\Rightarrow mg \text{ x } \frac{1}{2} - F_k = m \text{ x } \frac{g}{3}$$
$$\Rightarrow F_k = mg \text{ x } \frac{1}{2} - m \text{ x } \frac{g}{3} = \frac{\text{mg}}{6}$$

In the y-direction, there is no acceleration,

$$\sum F_y = mg \cos 30^0 - N = 0$$
$$\Rightarrow N = mg \cos 30^0 = \frac{\sqrt{3}}{2} mg$$

Putting values in the expression of coefficient of friction, we have :

$$\mu_k = \frac{2}{6\sqrt{3}} = \frac{1}{3\sqrt{3}} = 0.192$$

Motion on rough incline plane (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

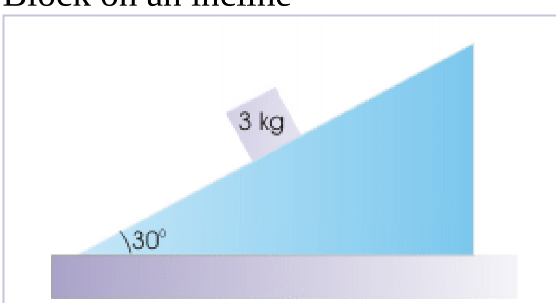
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to motion on rough incline plane. The questions are categorized in terms of the characterizing features of the subject matter :

- Friction
- Incline with different coefficients of friction
- Combined motion of two blocks

Friction

Problem 1 : A block of mass 3 kg is placed on a fixed incline of angle 30°. The coefficients of static and kinetic friction between the block and incline surface are 0.7 and 0.68 respectively. Find the friction acting on the block.
Block on an incline



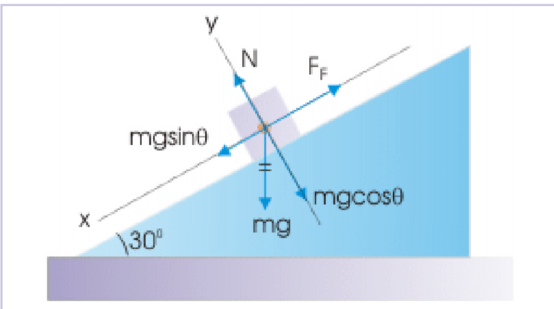
Gravity and friction are the external forces on the block.

Solution : In order to know friction – its magnitude and direction, we need to compare component of external force parallel to incline with the limiting friction. Here, only gravity acts (other than friction) on the block. The component of gravity parallel to incline is :

$$F_{||} = mg \sin 30^0 = 3X10X\frac{1}{2} = 15 \quad N$$

As gravity is the only force pulling the block down, friction acts up as shown in the figure.

Block on an incline



Friction on the block acts up the incline.

The limiting friction is :

$$F_S = \mu_S mg \cos 30^0 = 0.7X3X10X\frac{\sqrt{3}}{2} = 18.19 \quad N$$

It means that pulling force due to gravity is less than limiting friction. Hence, friction is equal to static friction, which is equal to pulling force. The friction, therefore, is :

$$\Rightarrow f_S = F_{||} = 15 \quad N$$

Incline with different coefficients of friction

Problem 2 : An incline has upper first half of its surface perfectly smooth and lower second half of its surface rough. A block beginning from the top of the

incline exactly stops at the bottom of the incline. Find the coefficient of friction between block and lower half of the incline.

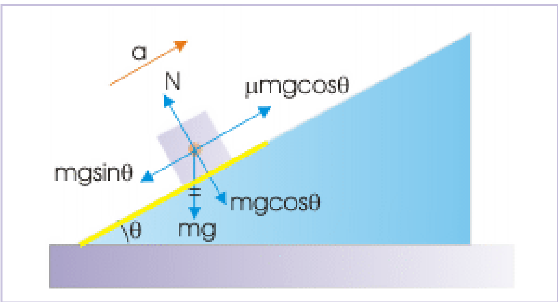
Solution : It is clear from the question that the block accelerates at constant rate in the first half and decelerates at constant rate in the second half of its journey over incline. Further, the journey is divided in two equal halves. The block comes to rest at the bottom of the incline. It means that block decelerates in the second half at the same rate at which it is accelerated in the first half.

Let “a” be acceleration in the first half. As there is no friction in the first half, acceleration is equal to the component of acceleration due to gravity along the incline.

$$a = g \sin \theta$$

In the second half, friction comes into picture. As relative motion between surfaces is involved, we conclude that friction is kinetic friction. The free body diagram of the block for the motion in second half is shown in the figure.

Block on an incline



Forces on the block in the lower half of the incline.

$$\mu mg \cos \theta - mg \sin \theta = ma$$

Note that friction has to be greater than the component of gravity along the incline as net component should be directed up to produce deceleration of the block. It means that had the block started motion on the plane in this section, it would not have moved down without external aid. Substituting for acceleration from consideration in upper half, we have :

$$\Rightarrow \mu mg \cos \theta - mg \sin \theta = mg \sin \theta$$

$$\Rightarrow \mu \cos\theta - \sin\theta = \sin\theta$$

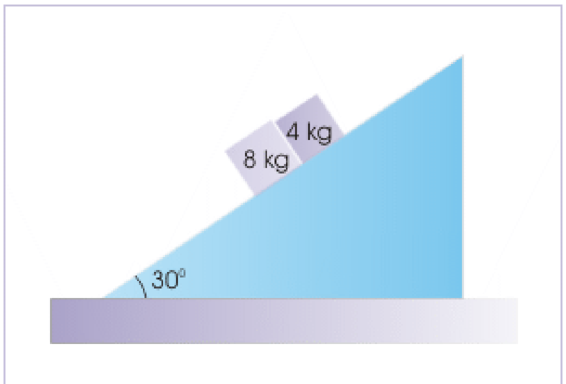
$$\Rightarrow \mu \cos\theta = 2\sin\theta$$

$$\Rightarrow \mu = 2\tan\theta$$

Combined motion of two blocks

Problem 3 : Two blocks “A” and “B” of mass 4 kg and 8 kg respectively are placed side by side on a rough incline of angle 30°. The blocks are released to slide down the incline at the same instant. The coefficients of friction for block “A” and “B” with the incline are 0.2 and 0.3 respectively. Find the accelerations of the blocks.

Two blocks on an incline



The blocks are released simultaneously.

Solution : The incline surface is rough. We need to check relative accelerations of the two blocks to know whether they travel together or separately. The blocks are under the same component of acceleration due to gravity along the incline.

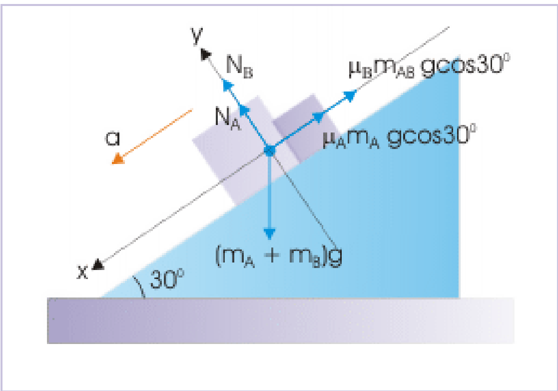
The deceleration due to friction for two blocks are :

$$\frac{\mu_A N}{m_A} = \frac{\mu_A m_A g \cos 30^0}{m_A} = 0.2g\cos 30^0$$

$$\frac{\mu_B N}{m_B} = \frac{\mu_B m_B g \cos 30^0}{m_B} = 0.3g\cos 30^0$$

Clearly, the deceleration due to friction for block "A" is less than that for block "B". This means that block "A" will always press against block "B".

Two blocks on an incline



Forces on the combined body.

Thus, two blocks will move together. Let "a" be the common acceleration. We can analyze the motion treating two blocks as one, having mass of 12 kg. The free body diagram of the combined block is superimposed on the figure. Note that we have combined weight, but not the friction as it is different as coefficients of friction are different for two blocks.

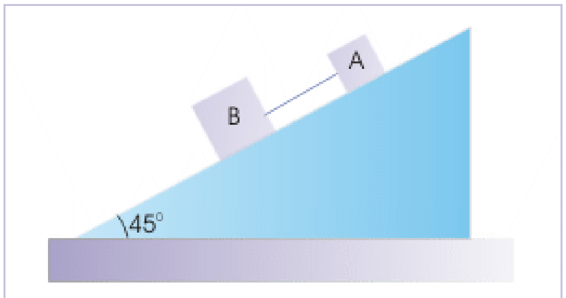
$$(m_A + m_B)g \sin 30^0 - \left(\mu_A m_A g \cos 30^0 + \mu_B m_B g \cos 30^0\right) = (m_A + m_B)a$$

$$\Rightarrow (4+8)10X\frac{1}{2} - \left(0.2X4X10X\frac{\sqrt{3}}{2} + 0.3X8X10X\frac{\sqrt{3}}{2}\right) = (4+8)a$$

$$a = \frac{\left\{60 - \sqrt{3}(4+12)\right\}}{12} = 2.69 \quad m/s^2$$

Problem 4 : Two blocks "A" and "B" have mass 2 kg and 4 kg respectively slide down an incline of angle 45°. They are connected through a "mass-less" rod as shown in the figure. The coefficients of friction between surface and blocks "A" and "B" are 0.75 and 0.25 respectively. Find the tension in the connecting rod.

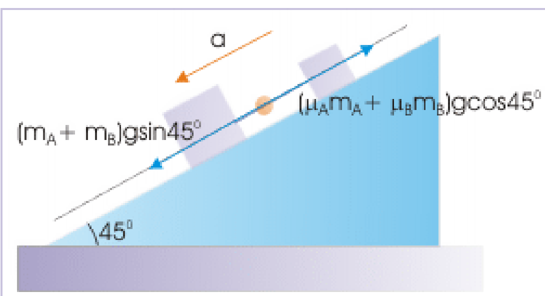
Two blocks on an incline



The blocks are connected by a "mass-less" rod.

Solution : According to question, the blocks along with connecting rod slides down the incline. Since connecting rod is inflexible, the motion of two blocks is constrained. They move together with a common acceleration. Let the common acceleration be "a". In the figure below, we have drawn forces considering combined mass in direction parallel to incline. Note that we have not drawn forces perpendicular to incline to keep the diagram simple. From the analysis of force along the incline, we have :

Two blocks on an incline



Forces on the combined body in direction parallel to incline.

$$(m_A + m_B)g\sin 45^0 - \left(\mu_A m_A g\cos 45^0 + \mu_B m_B g\cos 45^0\right) = (m_A + m_B)a$$

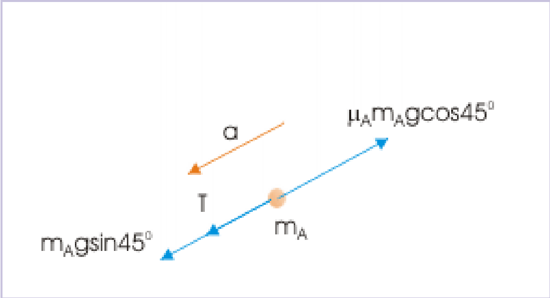
$$\Rightarrow (2+4)10X\frac{1}{\sqrt{2}} - \left(0.75X2X10X\frac{1}{\sqrt{2}} + 0.25X4X10X\frac{1}{\sqrt{2}}\right) = (2+4)a$$

$$\Rightarrow a = \frac{\{42.42 - (10.6 + 7.07)\}}{6}$$

$$\Rightarrow a = \frac{(42.42 - 17.67)}{6} = 4.125 \quad m/s^2$$

In order to find tension in the rod, we consider the free body diagram of block "A".

Free body diagram



Free body diagram of block, "A".

$$T + m_A g\sin 45^0 - \mu_A m_A g\cos 45^0 = m_A a$$

$$\Rightarrow T = m_A a + \mu_A m_A g\cos 45^0 - m_A g\sin 45^0$$

$$\Rightarrow T = 2X4.125 + 0.75X2X10X\frac{1}{\sqrt{2}} - 2X10X\frac{1}{\sqrt{2}}$$

$$\Rightarrow T = 8.25 + 10.6 - 14.14 = 4.71 \quad N$$

Since value of "T" is positive, the assumed direction of tension in the free body diagram is correct.

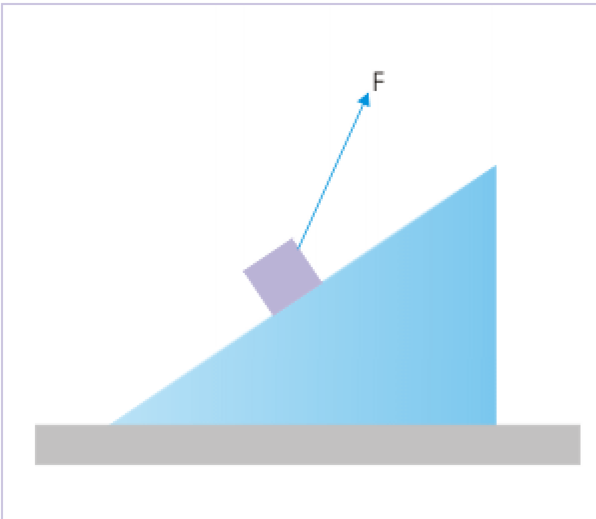
Acknowledgement

Author wishes to acknowledge Mr Lim Swee Sien for pointing out a mistake in the module.

Induced motion on rough incline plane

The natural tendency of a block is to move along the incline in downward direction under the “pull” of gravity. Motion of the block, however, can be induced either up or down along the incline by applying additional force other than three forces (weight, normal force and friction), which always operate on the block.

Induced motion on an incline



Motion of the block can be induced in either up and down direction.

We will study the effect of an additional external force on the motion of a block on the rough incline plane. An additional external force has multiple implications on the state of motion of the block. Some of the important implications are :

- External force can change the normal force (N) and hence maximum static friction for a given pair of surfaces. As a consequence, the force requirement parallel to contact surface to initiate motion will also change.
- External force can change the direction of motion with respect to the incline plane. The block can move either up or down (or the tendency

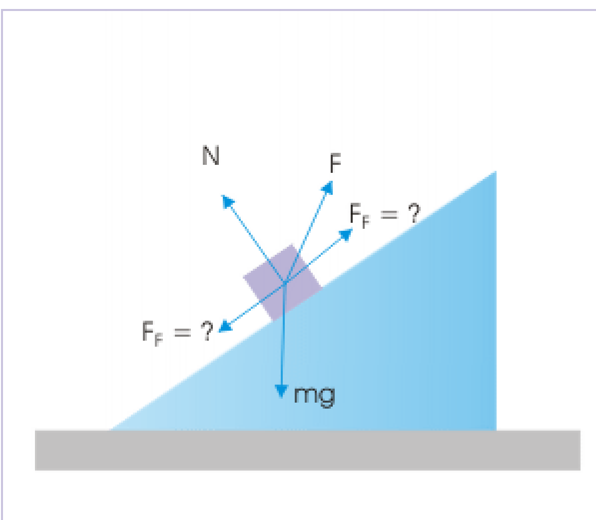
- to move up or down) the incline, depending on the direction of component of net external force at the contact surface.
- Friction is opposite to the direction of the motion (or tendency of motion) of the block with respect to incline.

Most importantly, external force can overcome normal limitations to motion, resulting from requirement to exceed friction. It is so because the limitations are there, because of the fact that force due to gravity (mg), which is responsible for motion, is a constant for a given mass of the block. Such is not the limitation with the applied external force (F). It can have any magnitude and direction.

Direction of friction

The externally unaided force system of three forces (weight, normal force and friction) causes downward motion (or induces tendency for motion in downward direction). On the other hand, the force system that involves additional external force (F) may have component of external forces parallel to contact in either up or down direction. As such, we need to analyze the force system without friction to decide the direction of friction. The figure below highlights this aspect showing question mark against the direction of friction.

Induced motion on an incline



What is the direction of

friction?

In short, there are two possibilities depending on the component of net external force (excluding friction) at the contact. Let ($F_{||}$ denotes the component of net external force (excluding friction) parallel to contact surface. Then :

1. If $F_{||} < 0$, then the block will either have the tendency to move down or there will be downward motion. The friction will then act in the upward direction.
2. If $F_{||} > 0$, then the block will either have the tendency to move up or there will be upward motion. The friction will then act in the downward direction.

Nature of external force and direction of friction

The external force can be distinctly different in one important aspect. The component of external force parallel to the contact surface can either be downward or upward. This distinction of the additional external force provides us with a direct method for determining the direction of friction.

We know that block has a tendency to move down on an incline due to the component of weight parallel to incline in the downward direction. Hence, if the component of additional external force is downward, then the induced motion has downward tendency and friction in upward direction. See the two figures below and orientations of the external force in each case. In either case, component of external force parallel to incline is directed downward. As such, the block is induced to move down.

Nature of applied force

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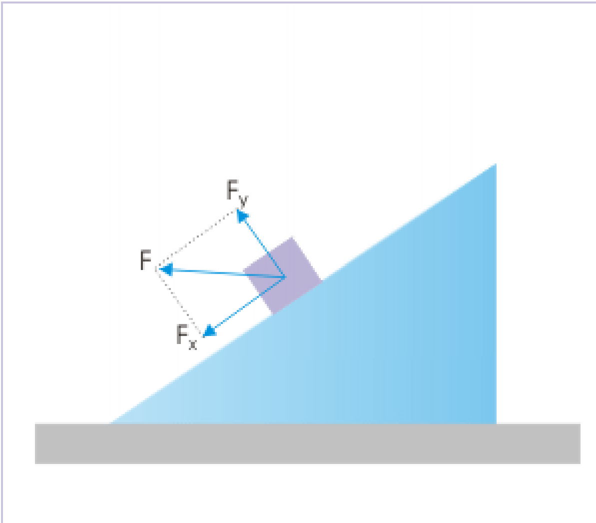
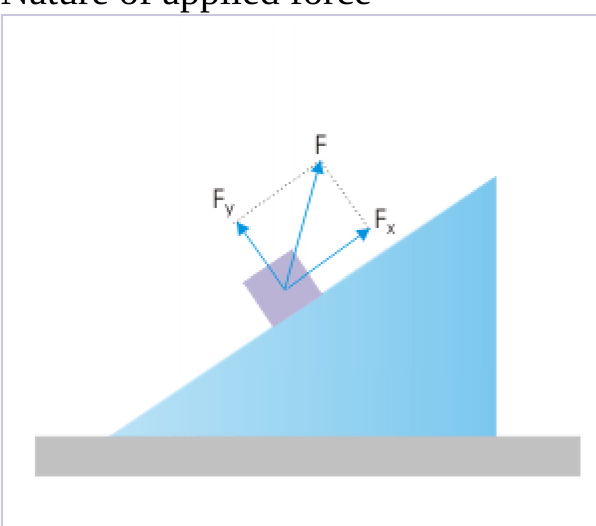


The external force has component along incline in the downward direction.

Such clarity with respect to direction of friction and hence that of the direction of motion is not there, if the component of additional external force parallel to incline is in upward direction. In this case, the sum of the components parallel to incline may be less than or equal to or greater than the maximum static friction. In such situations as shown here in the figure, we need to analyze net force parallel to the contact surface to determine the direction of motion and then that of friction.

Nature of applied force



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The external force has component along incline in the upward direction.

Analysis strategy

Force analysis for induced motion on incline plane will comprise of following steps :

1. Identify the characteristics of applied external force to know the direction of its component along the incline.
2. If the component of external force is down the incline, then motion is induced in downward direction and friction in upward direction.
3. If the component of external force is up the incline, then analyze the external forces (excluding friction) to know the direction of component of net external forces parallel to the incline. Depending on the direction component of net external forces parallel to the incline, determine the direction of friction .
4. After fixing the direction of friction, evaluate the state of friction (static friction, maximum static friction or kinetic friction) from the given set of information.
5. Carry out force analysis along two mutually perpendicular directions, suited to the situation in hand.

Thus, sequential order of the analysis for induced motion on incline plane is as follows :

Additional force characterization --> Direction of motion --> Direction of friction --> Friction state --> Force analysis

Force analysis

In this section, we shall work with different states of friction (static friction, maximum static friction or kinetic friction) for applied additional external force on the block.

Case 1 : Friction equal to kinetic friction

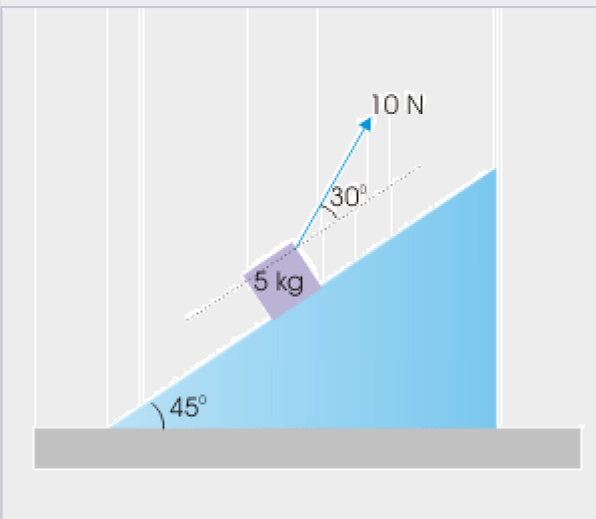
In this case, the block moves on the incline. Net force parallel to contact surface is greater than maximum static friction and friction is equal to kinetic friction.

$$F_F = F_k = \mu_k N$$

Example:

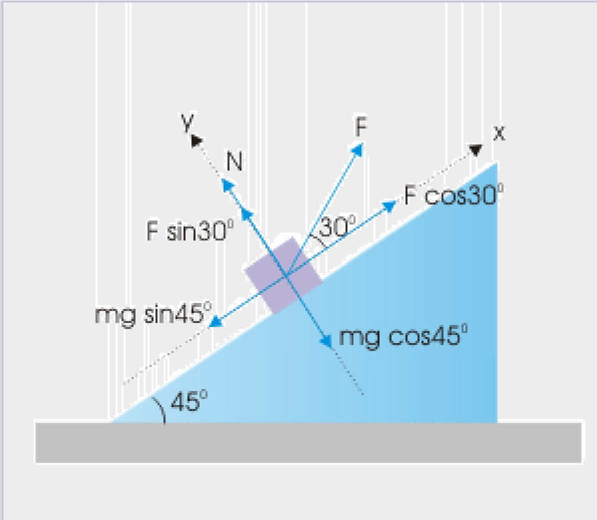
Problem 1 : A block of 5 kg is pulled by a force 10 N making an angle 30° with the incline of angle 45°. If coefficients of static and kinetic frictions are 0.55 and 0.53, then determine friction between block and incline.

Induced motion on an incline



Solution : We observe here that component of additional external force has its component up the incline. We are, therefore, required to carry out force analysis without friction to ascertain the direction of motion. In the figure below, the free body diagram is superimposed on the block and incline system.

Induced motion on an incline



Force analysis in x-direction without friction

The components of forces parallel to contact surface (excluding friction) is :

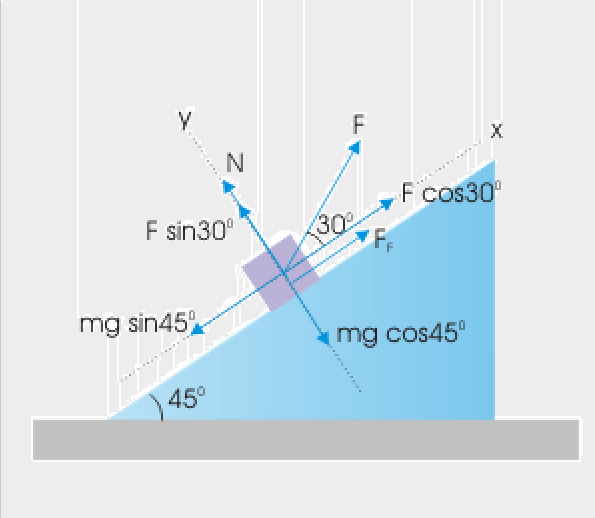
$$\sum F_x = F\cos 30^0 - mg\sin 45^0$$

$$\Rightarrow F_{\text{net}} = 10 \text{ X } \frac{\sqrt{3}}{2} - 5 \text{ X } 10 \text{ X } \frac{1}{\sqrt{2}} = -26.7\,N$$

$$\Rightarrow F_{\text{net}} < 0$$

It means that the block has tendency to move in the downward direction. The friction, therefore, is in up direction. Now, this completes the force system on the block as shown here.

Induced motion on an incline



Force diagram with friction

Note: An additional external force with component parallel to contact up the incline, does not guarantee upward motion. The force of gravity can still be greater than the component of applied additional external force parallel to the contact surface.

Now, our next task is to know the state of friction. To know the state of friction, we need to compare the net force component parallel to contact with maximum static friction. Now, maximum friction force is :

$$F_s = \mu_s N = 0.55N$$

In order to evaluate maximum static friction, we need to know normal force on the block. It is clear from the free body diagram that we can find normal force, "N" by analyzing force in y-direction. As there is no motion in vertical direction, the components in this direction form a balanced force system.

$$\sum F_y = N + F\sin 30^0 - mg\cos 45^0 = 0$$
$$\Rightarrow N = mg\cos 45^0 - F\sin 30^0$$
$$\Rightarrow N = 5 \text{ x } 10 \text{ x } \frac{1}{\sqrt{2}} - 10 \text{ x } \frac{1}{2}$$
$$\Rightarrow N = 30.36 \text{ Newton}$$

The maximum friction force is :

$$\Rightarrow F_s = \mu_s N = 0.55 \text{ x } 30.36 = 16.7\ N$$

Thus magnitude of net component parallel to contact surface (26.7) is greater than maximum static friction (16.7 N). Thus, body will slide down and friction will be equal to kinetic friction. Hence,

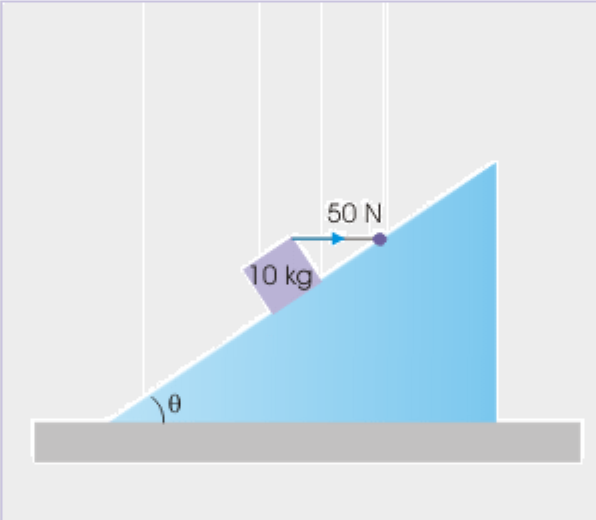
$$\Rightarrow F_F = F_k = \mu_k N = 0.53 \text{ x } 30.36 = 16.1\ N$$

Case 2 : Friction equal to maximum static friction

In this case, net force parallel to contact surface is equal to maximum static friction.

$$F_F = F_s = \mu_s N$$

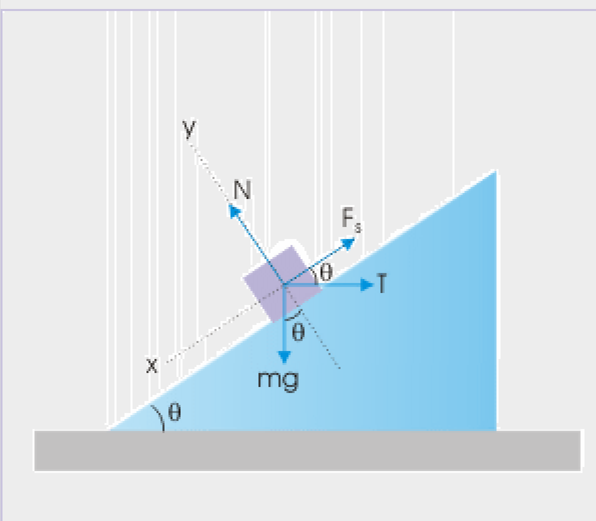
Example:
Problem : A block of 10 kg rests on a rough incline of angle 45°. The block is tied to a horizontal string as shown in the figure. Determine the coefficient of friction between the surfaces, if tension in the string is 50 N. Consider g = 10 m/s^2 .
Induced motion on an incline



Solution : Here, block is stationary. We need to know the motional tendency, had the string were not there.

It is given that the string is taught with a tension of 50 N. Clearly, the block would have moved down had it not been held by the string is taut. The direction of friction, therefore, is upwards. Further, the system is static with taut string. It means that friction has reached maximum static friction. Otherwise, how would have block moved down if not held by the string. Hence, the forces on the block form a balanced force system, including maximum static friction acting upward.

Induced motion on an incline



Forces on the block

$$\sum F_x = mg\sin 45^0 - \mu_s N - T\cos 45^0 = 0$$
$$\Rightarrow \mu_s N = 10 \text{ x } 10 \text{ x } \frac{1}{\sqrt{2}} - 50 \text{ x } \frac{1}{\sqrt{2}} = 50\frac{1}{\sqrt{2}}$$

and

$$\sum F_y = N - mg\cos 45^0 - T\sin 45^0 = 0$$
$$\Rightarrow N = 10 \text{ x } 10 \text{ x } \frac{1}{\sqrt{2}} + 50 \text{ x } \frac{1}{\sqrt{2}} = 150\frac{1}{\sqrt{2}}$$

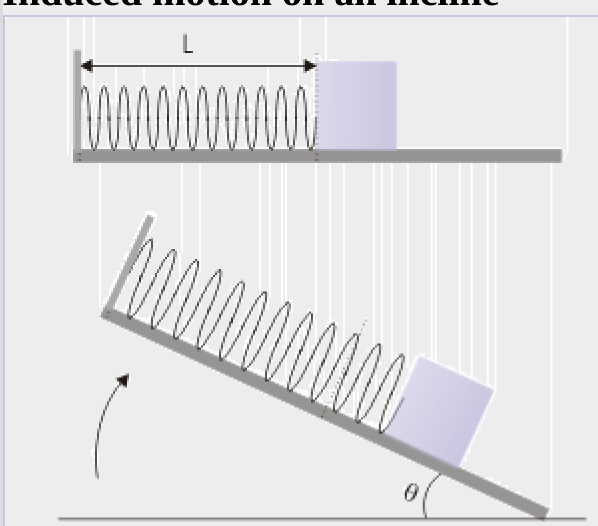
$$\Rightarrow \mu_s = \frac{1}{3}$$

Case 3 : All friction states

Example:

Problem : A block attached with a spring (of spring constant “k”) is placed on a rough horizontal plate. The spring is in un-stretched condition, when plate is horizontal. The plate is, then, raised with one end gradually as shown in the figure. Analyze friction and extension of the spring as angle (θ) increases to 90°.

Induced motion on an incline



Solution : In the initial stages, spring does not apply any force as block remains stationary because of friction. The friction acts in up direction as there is no applied external force due to spring. The magnitude of friction is equal to the component of weight along the incline. The friction and extension in the spring (x) for different ranges of angle “θ” are given here. Note that $\tan^{-1}(\mu_s)$ is equal to angle of repose. This is the angle of inclination of the plate with horizon, when block begins to move down.

(i) For $\theta < \tan^{-1}(\mu_s)$

$$\Rightarrow f_s = mg\sin\theta\,;\ x = 0$$

(ii) For $\theta = \tan^{-1}(\mu_s)$

Till the angle does not reach the value equal to angle of repose, there is no motion of the block.

$$\Rightarrow F_s = \mu_s mg\cos\theta\,;\ x = 0$$

(iii) For $\theta > \tan^{-1}(\mu_s)$

The block starts sliding down. The direction of friction is up along the incline. As the spring stretches, it applies spring force to counteract the net downward force to again bring the block to rest. Before, the block is brought to a stop, the acceleration of the block (“a”) is :

$$\sum F_x = mg\cos\theta - \mu_s mg\cos\theta - kx = ma$$
$$\Rightarrow a = \frac{mg\cos\theta - \mu_s mg\cos\theta - kx}{m}$$

When the block is brought to a stop, the extension in the string x_0 is obtained as under :

$$\sum F_x = mg\cos\theta - \mu_s mg\cos\theta - kx_0 = 0$$
$$\Rightarrow kx_0 = mg\sin\theta - \mu_k mg\cos\theta$$
$$\Rightarrow x_0 = \frac{mg\sin\theta_0 - \mu_s mg\cos\theta_0}{k}$$

Induced motion on rough incline (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

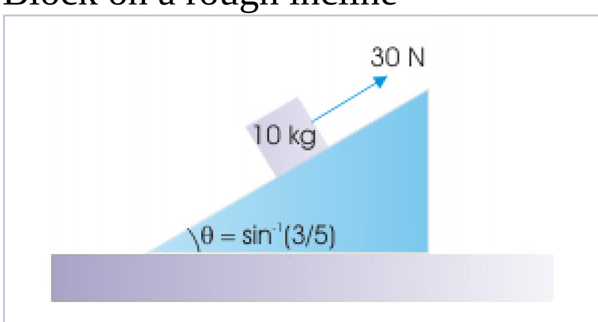
We discuss problems, which highlight certain aspects of the study leading to induced motion on rough incline. The questions are categorized in terms of the characterizing features of the subject matter :

- Nature of friction
- Incline as mechanical device
- Equilibrium

Nature of friction

Problem 1 : A force of 30 N is applied on a block of 10 kg, which is placed on an incline of angle, $\theta = \sin^{-1}(3/5)$. If the coefficient of friction between block and incline is 0.75, then find the friction.

Block on a rough incline



External force of 30 N acts up parallel to incline.

Solution : In order to find friction, we first need to know its direction. Here, the component of gravitational force along the incline is :

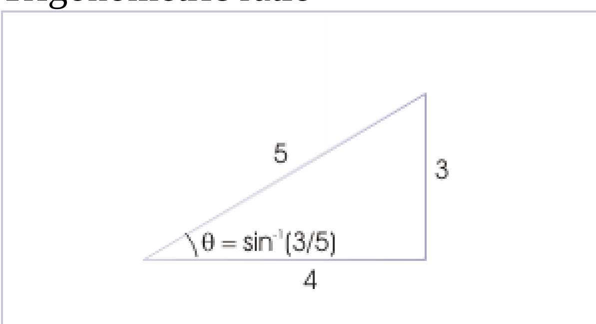
$$mg\sin\theta = 10X10X\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right) = 10X10X\frac{3}{5} = 60 \quad N$$

The limiting friction is :

$$F_S = \mu mg\cos\theta = 0.75X10X10X\cos\theta$$

In order to find the cosine, we use the triangle as shown. Here,

Trigonometric ratio



Cosine is ratio of base and hypotenuse.

$$\Rightarrow \cos\theta = \frac{\sqrt{(5^2 - 3^2)}}{5} = \frac{4}{5}$$

Hence,

$$\Rightarrow F_S = 0.75X10X10X4/5 = 60 \quad N$$

Now, net component of external forces (without friction) along the incline is :

$$\sum F_x = 60 - 30 = 30 \quad N$$

It means that block has the tendency to move down. Therefore, friction acts up. However, friction is not equal to limiting friction. The net component of external forces is 30 N and is less than limiting friction. Therefore, the static friction simply matches the net external force parallel to incline, which is 30 N.

$$f_S = 30 \quad N$$

Incline as mechanical device

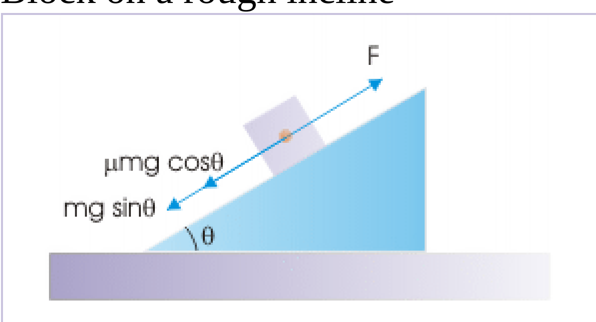
Problem 2 : A block can either be lifted vertically or pulled up along a rough incline of 30°. What should be coefficient of friction between block and incline so that pulling along the incline requires less force parallel to the incline.

Solution : In order to pull the block of mass "m" vertically, the force, " F_1 ", required is equal to the weight of the block,

$$F_1 = mg$$

On the other hand, the net external force parallel to the incline should be greater than the limiting friction to move the block up the incline. The required force is :

Block on a rough incline



Force required to pull up the block along an incline.

$$F_2 = mg\sin\theta + \mu mg\cos\theta$$

For $F_2 < F_1$,

$$\Rightarrow mg\sin\theta + \mu mg\cos\theta < mg$$

$$\Rightarrow \sin\theta + \mu\cos\theta < 1$$

$$\Rightarrow \mu\cos\theta < (1 - \sin\theta)$$

$$\Rightarrow \mu < \frac{(1 - \sin\theta)}{\cos\theta}$$

Putting value of the angle and evaluating, we have :

$$\Rightarrow \mu < \frac{\left(1 - \sin 30^0\right)}{\cos 30^0}$$

$$\Rightarrow \mu < \frac{(1 - 1/2)}{\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \mu < \frac{\left(\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \mu < \frac{1}{\sqrt{3}}$$

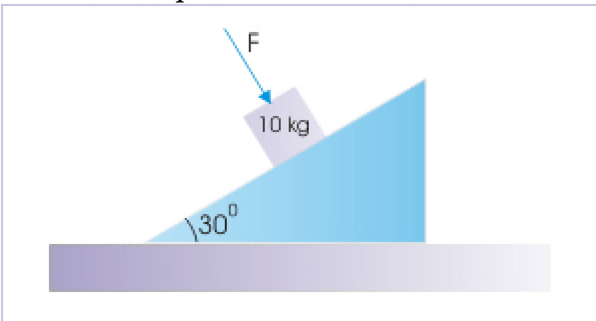
Note: This example clearly demonstrate that (when friction is always present) the pulling along incline may not always involve lesser force as otherwise may be believed.

Equilibrium

Problem 3 : A block of mass 10 kg is placed on a rough incline plane of inclination 30° with an external force “F” applied as shown in the figure.

The coefficient of friction between block and incline is 0.5. Find the minimum force "F" so that block does not slide on the incline.

Block in equilibrium



Force required to pull up the block along an incline.

Solution : We first need to determine the direction of friction so that we could determine the minimum force required for block to remain stationary. Here, the gravitational force down the incline is :

$$mg\sin\theta = 10X10X\sin 30^0 = 10X10X\frac{1}{2} = 50 \quad N$$

The external force "F" has no component along the incline. As such, the component of weight along the incline is the net force parallel to incline. Friction, therefore, acts upward. If external force "F" is not applied, the limiting friction is given by :

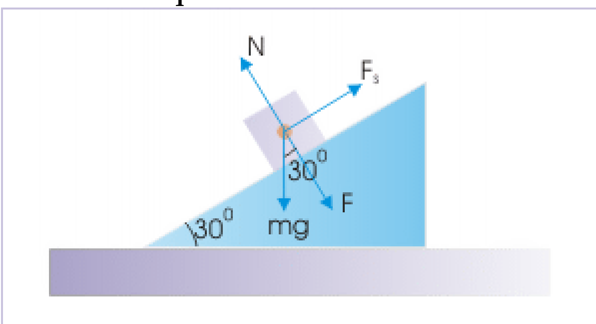
$$F_S = \mu mg\cos\theta = 0.5X10X10X\frac{\sqrt{3}}{2} = 25\sqrt{3} \quad N$$

As the downward gravitational force is greater than limiting friction, the block will slide down.

In order to restrict motion of block, the external force "F" should increase normal force, which in turn increases friction. Let the minimum force be "F" such that block does not side. In this limiting situation, the components

of forces, parallel and perpendicular to incline, form balanced force systems. Free body diagram with external forces is shown in the figure. Considering "x" and "y" axes in parallel and perpendicular directions, we have :

Block in equilibrium



Forces on the block.

$$\sum F_x = mg\sin\theta = \mu N$$

$$\sum F_y = mg\cos\theta + F = N$$

Substituting for “N” in the first equation, we have :

$$\Rightarrow mg\sin\theta = \mu N = \mu mg\cos\theta + \mu F$$

$$\Rightarrow F = \frac{mg\sin\theta - \mu mg\cos\theta}{\mu}$$

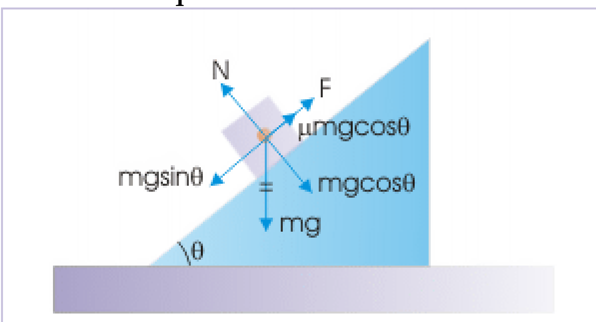
$$\Rightarrow F = \frac{50 - 25\sqrt{3}}{0.5} = 13.4 \quad N$$

Problem 4 : A block of mass “m” is placed on a rough incline of angle “θ”. The downward motion of the block is prevented by applying an upward force, “F”, parallel to incline. If a force of “2F” is required to initiate

motion of the block in upward direction, then determine the coefficient of friction between block and the incline.

Solution : It is given that block has downward motion without any additional force. It means that net component of forces parallel to incline is in downward direction. The downward motion can be prevented if the additional force, parallel to the incline, is equal to the net downward component.

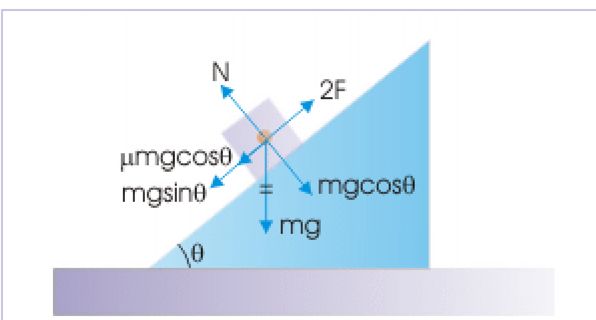
The additional force is equal to net downward component :
Block in equilibrium



Forces on the block.

$$F = mg\sin\theta - \mu mg\cos\theta$$

In order to initiate motion in the upward direction, the additional external force should be equal to the net downward component of forces. As the block tends to move up, the friction, now, acts in the downward direction. Hence,
Block on an incline



Forces on the block.

$$2F = mg\sin\theta + \mu mg\cos\theta$$

Subtracting first equation from the second, we have :

$$\Rightarrow F = 2\mu mg\cos\theta$$

Putting this expression of “F” in the first equation and solving for “μ”,

$$\Rightarrow 2\mu mg\cos\theta = mg\sin\theta - \mu mg\cos\theta$$

$$\Rightarrow 3\mu\cos\theta = \sin\theta$$

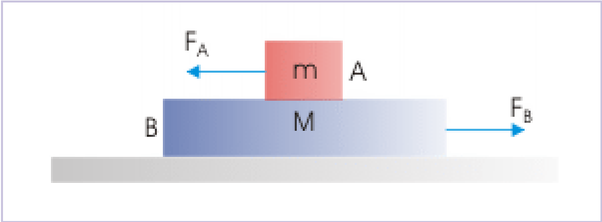
$$\Rightarrow \mu = \frac{\tan\theta}{3}$$

Friction between horizontal surfaces

Friction plays different roles in motion. Consider the case of a block on a plane slab (plank), which, in turn, is placed on a horizontal surface. Typically, friction moderates the motion of a body. In other words, friction retards motion. On the other hand, friction also acts as the “cause” of the motion under certain circumstance. In the role of a “moderator”, friction negates external force. In the role of the “cause” of motion, it is either the sole external force or the greater external force on the body responsible for its motion.

In the "block - plank" set up, we can apply force either on the block or on the plank or on both of them simultaneously. The resulting motion depends on varieties of factors such as friction, external force and location of application of external force etc.

Friction between horizontal surfaces



The block and plank move with respect to each other.

In order to analyze situation as mentioned above, we need to have clear understanding of the way friction works on each of bodies. In this module, we seek to organize ourselves so that we have well thought out plan and method to deal with this kind of motion.

First thing that we need to know is about friction – its magnitude and direction. Secondly, we need to know the nature of friction – whether friction is static, limiting or kinematic. Once, we have complete picture of various friction forces at different interfaces, we are in position to draw the free body diagram and analyze the motion.

Friction between interfaces

We shall consider three cases. In order to keep the matter simple, we make one simplifying assumption that the friction between plank and the horizontal surface is negligible. In other words, the underlying horizontal surface is smooth.

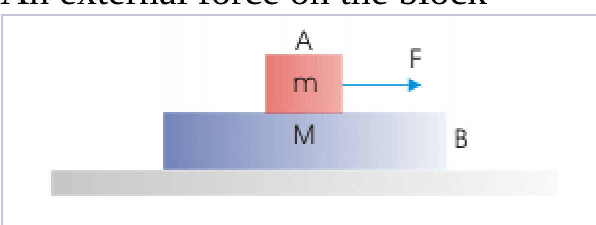
- An external force is applied on the block.
- An external force is applied on the plank.
- One external force each is applied on block and plank.

To keep the description uniform, we refer block as “A”, having mass, “m”, and plank as “B”, having mass, “M”. We shall find that a consistency in denoting block and plank is very helpful in analyzing motion. The analysis of motion for first two cases is similar. It differs only to the extent that point of application of external force changes. Nevertheless, it is interesting to analyze two cases separately to score the differences in two cases.

An external force is applied on the block

The external force is applied on the block as shown in the figure below. There are two possibilities : (i) there is no friction between "A" and "B" or (ii) there is friction between "A" and "B".

An external force on the block

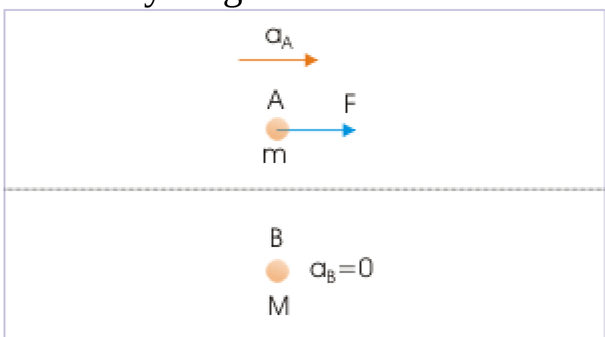


An external force, "F", is applied on the block.

There is no friction between block and plank

We first analyze the motion for a case when all interfaces are “friction – less”. In this situation, the external force on block, “A”, accelerates only the block. As there is no friction between the interface of block and plank, there is no external force on the plank in horizontal direction. As such, plank, “B”, is not accelerated. The free body diagrams of “A” and “B” are as shown in the figure.

Free body diagrams



There is no external force on the plank.

The acceleration of block, “ a_A ”, is :

$$a_A = \frac{F}{m}$$

The acceleration of plank, “ a_B ”, is :

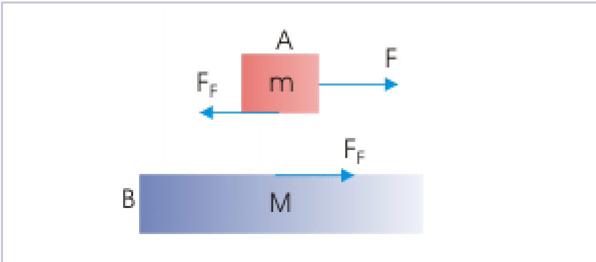
$$a_B = 0$$

There is friction between block and plank

The friction force in this case will influence the motion as it acts as external force on each individual body. The resulting motion of the bodies, however, would depend on the nature of friction (static, limiting or kinetic). In the

figure below, the forces on the block and plank are shown separately for each of them.

Forces on the bodies



Friction is the only external force on the plank.

In order to determine the direction of frictions at the interface, we go by two simple steps. We first consider the body on which external force (F) is applied. The direction of friction on the body (A) is opposite to the external force. We determine friction on the other body (B) by applying Newton's third law of motion. Friction on plank (B) is equal in magnitude, but opposite in direction.

Once direction of friction is known, we need to know the nature of friction. For this, we first calculate limiting friction between the bodies so as to compare it with external applied force "F". The limiting friction is given by :

$$F_S = \mu N = \mu mg$$

Important thing to realize here is that limiting friction depends only on the mass of the block, "m", and is independent of the mass, "M", of the plank. Now, we should compare external force with friction to determine its magnitude. Finally, we analyze motion in following manner :

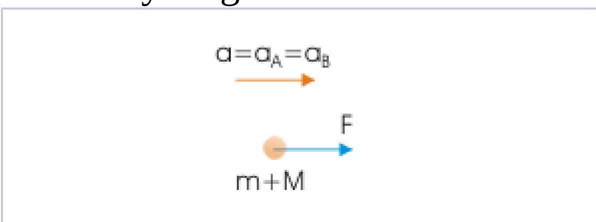
1 : $F \le F_S \quad (F \le \mu_s mg)$

In this case, friction self adjusts to external force, "F". Hence, static friction is given by :

$$\Rightarrow f_s = F$$

As there is no relative motion, the block and plank move together as a single unit. The friction forces at the interface are internal forces for the combined body. The free body diagram of the combined body is shown here.

Free body diagram



Free body diagram of combined body.

The common acceleration of the block and plank, “a”, is :

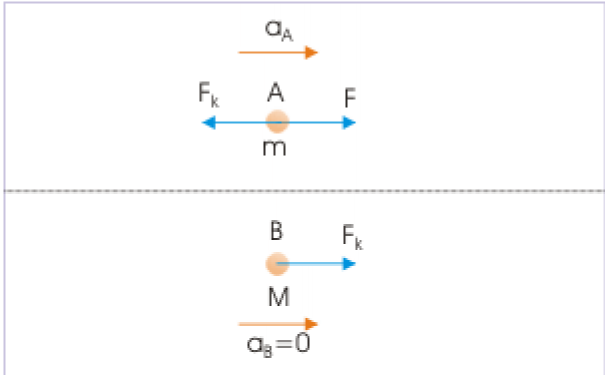
$$a = a_A = a_B = \frac{F}{m+M}$$

2 : $F > F_S \quad (F > \mu_s mg)$

The block and plank move with different accelerations and have relative motion. The friction between the interfaces is, therefore, kinetic friction.

The free body diagrams of block and plank are shown here.

Free body diagrams



Free body diagrams of block and plank.

The acceleration of block, “ a_A ”, is :

$$a_A = \frac{F - \mu_k mg}{m}$$

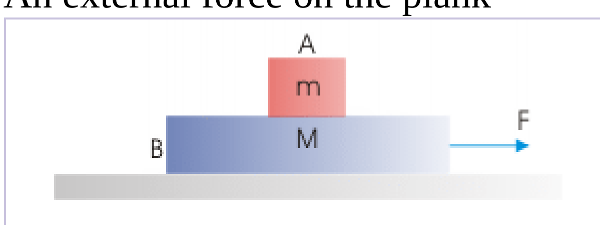
The acceleration of block, “ a_B ”, is :

$$aB = \frac{\mu_k mg}{M}$$

An external force is applied on the plank

The external force is applied on the plank as shown in the figure below. There are two possibilities : (i) there is no friction between "A" and "B" or (ii) there is friction between "A" and "B".

An external force on the plank



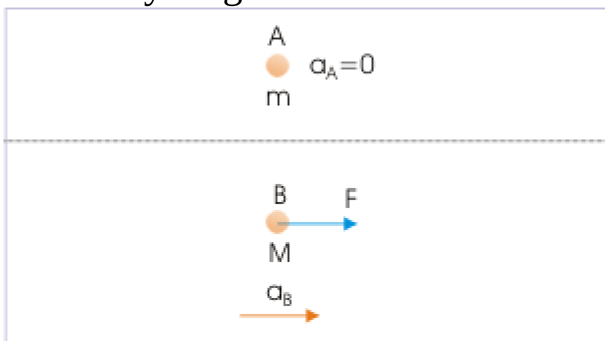
An external force, "F", is

applied on the plank.

There is no friction between block and plank

In this situation, the external force on block, “B”, accelerates only the plank. As there is no friction between the interface of block and plank, there is no external force on the block in horizontal direction. As such, block, “A”, is not accelerated. The free body diagrams of “A” and “B” are as shown in the figure.

Free body diagrams



There is no external force on the block.

The acceleration of block, “ a_A ”, is :

$$a_A = 0$$

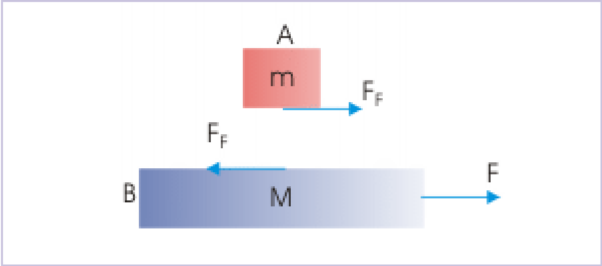
The acceleration of plank, “ a_B ”, is :

$$a_B = \frac{F}{M}$$

There is friction between block and plank

The friction force in this case will influence the motion as it acts as external force on each individual body. The resulting motion of the bodies, however, would depend on the nature of friction (static, limiting or kinetic). In the figure below, the forces on the block and plank are shown separately for each of them.

Forces on the bodies



Friction is the only external force on the block.

In order to determine the direction of frictions at the interface, we go by two simple steps. We first consider the body on which external force is applied. The direction of friction on the body (B) is opposite to the external force. We determine friction on the other body (A) by applying Newton's third law of motion. Friction on block (A) is equal in magnitude, but opposite in direction.

Once direction of friction is known, we need to know the nature of friction. Unlike in the previous case when external force is applied on the block, the situation here is different. The plank carries another mass of block over itself. The external force (F) is not completely used to overcome friction at the interface, but to move the combined mass together. As such, external force can not be directly linked to static friction as in the case when force is applied on the block.

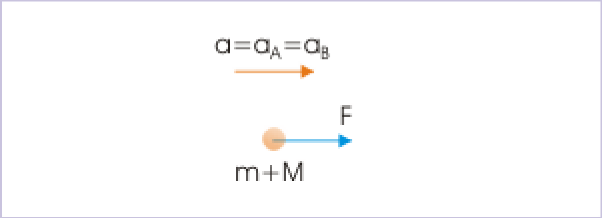
For this reason, we shall adopt a different strategy. We shall assume that friction is static friction. If the analysis of force does not support this

assumption, then we correct the assumption accordingly. Now, the limiting friction is given by :

$$F_S = \mu N = \mu mg$$

Let us assume that " f_S " be the static friction between the surfaces. As there is no relative motion, the block and plank move together as a single unit. The friction forces at the interface are internal forces for the combined body. The free body diagram of the combined body is shown here.

Free body diagram



Free body diagram of combined body.

The common acceleration of the block and plank, “a”, is :

$$a = a_A = a_B = \frac{F}{m+M}$$

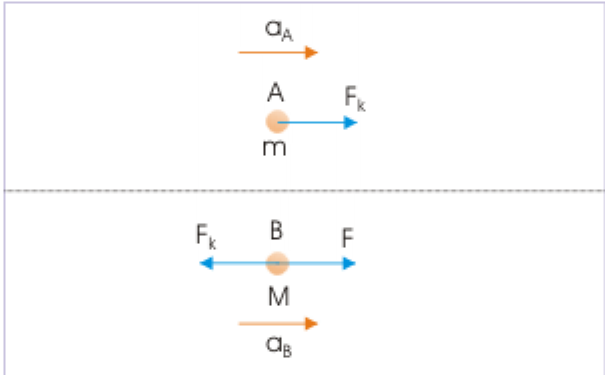
We observe that friction is the only force on the block. Then, applying Newton's second law for the motion of block, we have :

$$f_S = ma$$

Clearly, if " f_S " so calculated is less than or equal to limiting friction, then block and plank indeed move together. However, if " f_S " is greater than limiting friction, then block and plank move with different accelerations. In such case, friction between "A" and "B" is kinetic friction.

The free body diagrams of block and plank are shown here.

Free body diagrams



Free body diagrams of block and plank.

The acceleration of block, “ a_A ”, is :

$$a_A = \frac{\mu_k mg}{m} = \mu_k g$$

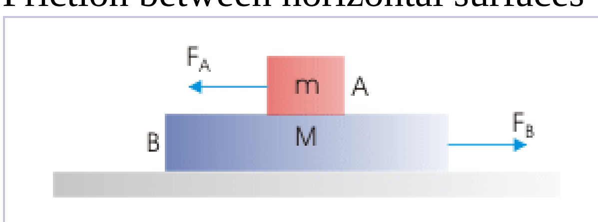
The acceleration of block, “ a_B ”, is :

$$a_B = \frac{F - \mu_k mg}{m}$$

One external force each is applied on block and plank

In this case, external forces act on the block and plank separately. It is, therefore, not possible in this case to compare limiting force with external force as there are two of them, which are acting on two different bodies.

Friction between horizontal surfaces



The block and plank move with

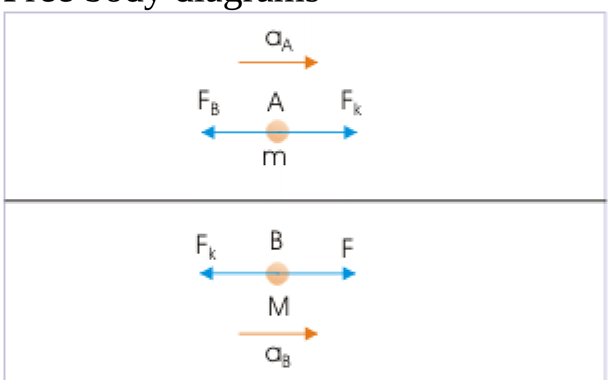
respect to each other.

Here, we adopt the strategy of assuming nature of friction before hand. First, we calculate the limiting friction as before,

$$F_S = \mu N = \mu mg$$

Then, we consider a friction, " f_S ", which is static friction, but whose magnitude is not known. If we are wrong in our assumption as brought out by the analysis, then we will correct our assumption; otherwise not. The free body diagrams of the block and the plank are shown in the figure.

Free body diagrams



Free body diagrams of block and plank.

Since friction is static unknown friction, two bodies move together without any relative motion between them. It means that :

$$a_A = a_B$$

$$\frac{F_B - f_s}{M} = \frac{f_s - F_A}{m}$$

$$\Rightarrow mF_B - mf_s = Mf_s - MF_A$$

$$\Rightarrow f_s(m+M) = mF_B + MF_A$$

$$\Rightarrow f_s = \frac{mF_B + MF_A}{m+M}$$

If the value of " f_s " as calculated, using expression derived above, is less than limiting friction, then we conclude that two bodies are indeed moving together. Otherwise, our assumption were wrong and the bodies are moving with different accelerations. In that case, we calculate accelerations separately as :

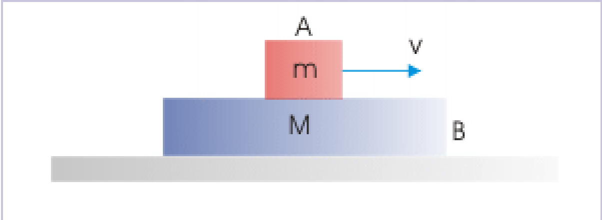
$$\Rightarrow a_A = \frac{f_s - F_A}{m}$$

$$\Rightarrow a_B = \frac{F_B - f_s}{M}$$

Velocity .vs. Force

We have so far analyzed motion, considering external force on the bodies. Under certain condition, if data is provided in the form of velocity of the individual body, then the analysis is simplified significantly. Consider the set up as shown in the figure. At a certain instant, the block is imparted a velocity (alternatively, we have simply allowed a block with certain velocity to slide over the plank underneath).

Friction between horizontal surfaces



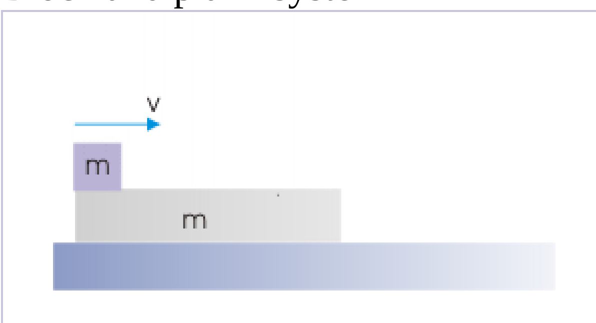
The block and plank move with respect to each other.

An initial velocity to block, here, ensures that there is relative motion at the interface. This, in turn, ensures that the friction between the surfaces is kinetic friction. This means that nature of friction is known and need not be investigated as in the case when external force is applied.

Example

Problem : A block with initial velocity "v" is placed over a rough horizontal plank of the same mass. The plank resides over a smooth horizontal plane. Plot the velocities of the block and the plank with respect to time.

Block and plank system



Initial velocity of block is "v".

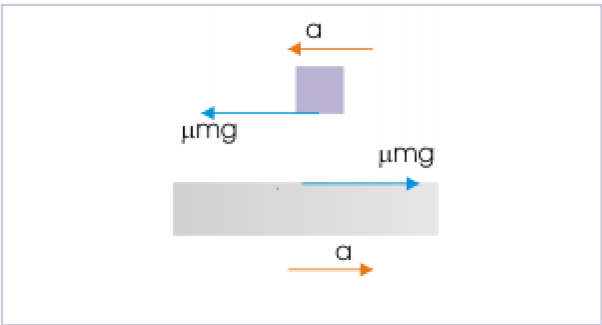
Solution : Let the mass of the block and plank each be “m” and “μ” be the coefficient of kinetic friction between them. Since block is given a velocity with respect to ground, the friction between block and plank is kinetic friction, given by :

$$F_K = \mu N = \mu mg$$

The direction of friction on the block is opposite to the direction of motion. The friction here retards the motion. Let us denote block and plank by subscripts “1” and “2” respectively.The deceleration of the block is given by :

$$a = \frac{\mu mg}{m} = \mu g$$

Forces on the bodies



Friction on two bodies are equal but opposite.

It means that velocity of the block decreases at constant rate with time. The velocity – time plot of block, therefore, is a straight line with negative slope. On the other hand, friction is the only force on the plank in the forward direction. The magnitude of acceleration is same as that of block, because it has the same mass and it is worked by force of same magnitude. Since plank starts from zero velocity, the velocity – time plot of the plank is a straight line of constant slope, starting from the origin of the plot.

The block looses motion due to deceleration, whereas plank acquires motion due to acceleration. A situation comes when speeds of the two entities are equal. In that situation,

$$v_1 = v_2$$

$$v - at = 0 + at$$

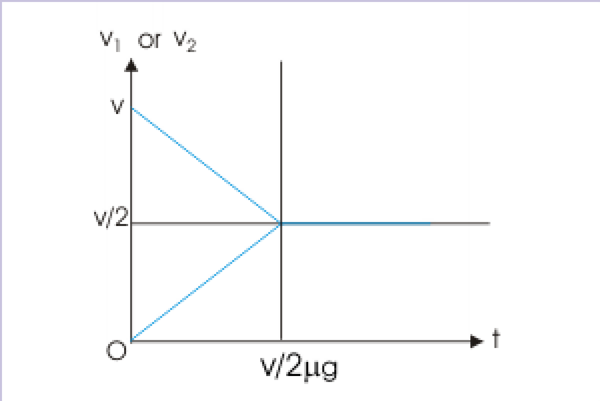
$$t = \frac{v}{2a} = \frac{v}{2\mu g}$$

The common velocity is given by :

$$v_c = v_2 = at = \mu g X \frac{v}{2\mu g} = \frac{v}{2}$$

This means that velocity of the block and plank becomes equal to half of the initial velocity imparted to the block. There is no relative motion between two entities. They simply move together with same velocity as combined mass with the common velocity, “v/2”. The required plot is shown in the figure :

Velocity - time plot



Bodies acquire common velocity after some time.

It must also be understood that the above approximates an ideal condition; the plank and block will eventually stop as there is some friction between plank and the underneath horizontal surface.

Friction between horizontal surfaces (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

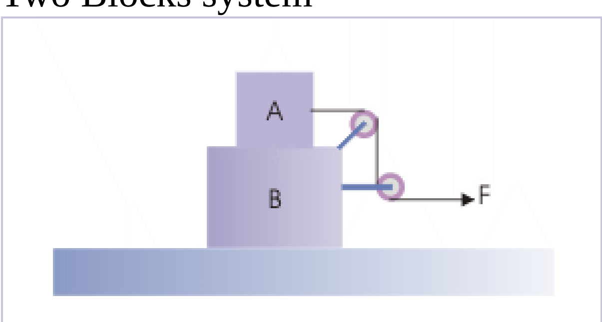
We discuss problems, which highlight certain aspects of the study leading to friction between horizontal surfaces. The questions are categorized in terms of the characterizing features of the subject matter :

- An external force on the block
- An external force on the plank
- One external force each on block and plank
- Block with initial velocity

An external force on the block

Problem 1 : In the arrangement shown in the figure, the mass of the pulleys and strings are negligible. The block “B” rests on the smooth horizontal surface, whereas the coefficient of friction between two blocks is 0.5. If masses of the blocks “A” and “B” are 2 kg and 4 kg respectively, then find the maximum force for which there is no relative motion between two blocks.

Two Blocks system



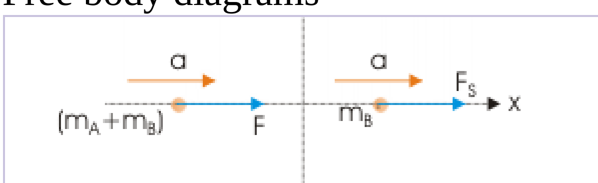
External force is applied on the top block.

Solution : According to the question, two blocks have no relative motion between them. The friction force is less than limiting friction. As the limiting case, the maximum friction between the blocks is equal to limiting friction :

$$F_S = \mu N = \mu m_A g = 0.5X2X10 = 10 \quad N$$

Since blocks have no relative motion between them, they move as a single unit with a common acceleration, say "a". The vertical and horizontal forces by the two brackets holding pulleys balance each other. Hence, these forces are not considered. The free body diagram of the combined body is shown on the left side of the figure. The friction forces between the blocks are internal forces for the combined body and as such not considered. Note that we have not drawn components of forces in vertical direction as the analysis in vertical direction is not relevant here.

Free body diagrams



Blocks move with same accelerations.

$$\Rightarrow a = \frac{F}{(m_A + m_B)} = \frac{F}{6}$$

$$\Rightarrow F = 6a$$

Considering free body diagram of block "B" as shown on the right side of the figure, we have :

$$a = \frac{F_S}{m} = \frac{10}{4} = 2.5 \quad m/s^2$$

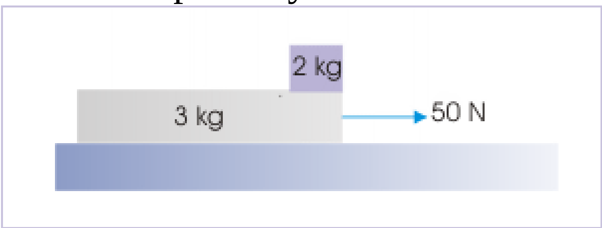
Putting in the expression of force, we have,

$$\Rightarrow F = 6X2.5 = 15 \quad N$$

An external force on the plank

Problem 2 : In the arrangement shown in the figure, mass of block and plank are 2 kg and 3 kg respectively. The coefficient of friction between block and plank is 0.5, whereas there is no friction between plank and underneath horizontal surface. An external force of 50 N is applied on the plank as shown. If the length of plank is 5 m, find the time after which block and plank are separated.

Block and plank system



A force is applied on the plank.

Solution : First, we need to know the nature of the friction between block and plank. If the friction is less than liming friction, then two entities will move together with an acceleration given by :

$$a = \frac{50}{(2+3)} = 10 \quad m/s^2$$

Let us use subscript "1" and "2" to denote block and plank respectively. Since friction is the only force on the block (1), the friction is

$$F_1 = m_1 a = 2X10 = 20 \quad N$$

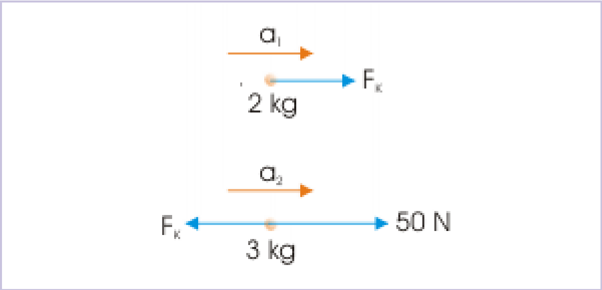
The limiting friction, however, is :

$$F_S = \mu mg = 0.5X2X10 = 10 \quad N$$

It means that our assumption that they move together is wrong. They move with relative velocity between them. The friction, therefore, is equal to kinetic friction and is nearly equal to limiting friction 10 N.

The free body diagram of two entities are shown in the figure.

Free body diagrams



Forces on the block and plank.

$$a_1 = \frac{F_K}{m_1} = \frac{10}{1} = 10 \quad m/s^2$$

$$a_2 = \frac{50 - F_K}{m_2} = \frac{50 - 10}{2} = 20 \quad m/s^2$$

The relative acceleration of 1 (block) with respect 2 (plank) is :

$$a_{12} = a_1 - a_2$$

$$a_{12} = 10 - 20 = -10 \quad m\ /\ s^2$$

Thus, block has relative acceleration in the opposite direction to the direction of individual accelerations i.e. it is directed towards left. Now, initial velocity is zero and total displacement is 5m. Applying equation of motion for constant acceleration, we have :

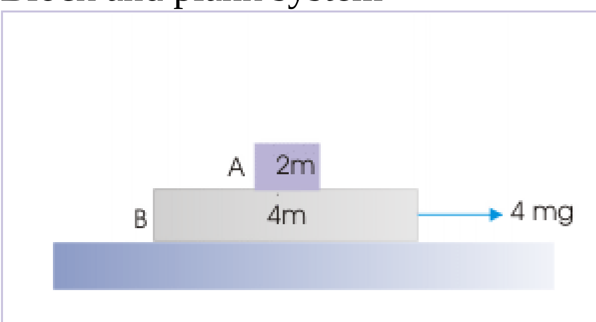
$$x = ut + \frac{1}{2}at^2$$

$$\Rightarrow 5 = 0 + \frac{1}{2}X10t^2$$

$$\Rightarrow t = \sqrt{\frac{10}{10}} = 1 \quad s$$

Problem 3 : In the arrangement shown, a force of 4mg, acts on the plank of mass "4m". The coefficient of friction between block "A" and plank "B" is 0.25, whereas friction between plank and the surface underneath is negligible. Find the acceleration of "A" with respect to ground and its acceleration with respect to block "B".

Block and plank system



An external force is applied on the plank.

Solution : External force acts on the plank "B". As such, it has tendency to move to right with respect to block "A". The friction on the plank is, therefore, directed in the opposite direction i.e. towards left.

According to Newton's third law, the friction acts on block in equal measure but in opposite direction i.e. towards right. We, however, need to know the nature of friction to proceed with the force analysis.

Here, limiting friction is :

$$F_S = \mu mg = 0.25X2mg = \frac{mg}{2}$$

We see here that friction is the only external force on block “A”. We assume here that friction is less than limiting friction (if assumption is wrong then we would get wrong result). In this condition, plank and block will together with a common acceleration given by :

$$\Rightarrow a = \frac{F}{6m} = \frac{4mg}{6m} = \frac{2g}{3}$$

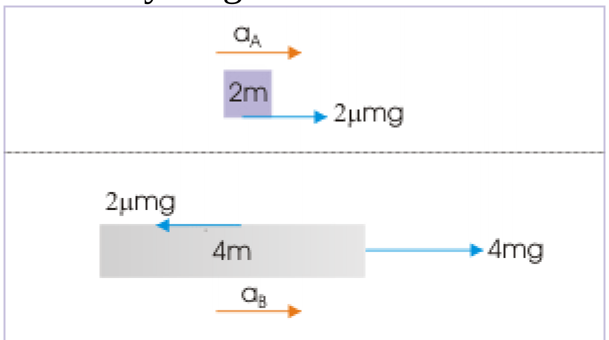
The external force on the block would, then, be :

$$\Rightarrow F_A = 2ma = \frac{2mX2g}{3} = \frac{4mg}{3}$$

Clearly, external force on block “A” is greater than limiting friction, which is not possible as friction is the only external force on it. Friction, we know, can not exceed limiting friction. It means our assumption that block and plank move together has been wrong. It further means that two entities move with relative motion. As such, friction between them is kinetic friction (about equal to limiting friction).

Now, we know the magnitude and direction of friction force operating between the surface and hence, we can draw the free body diagrams of each of the entities as shown in the figure.

Free body diagrams



Forces on block and plank.

The acceleration of the block with respect to ground, " a_A ", is given by :

$$\Rightarrow a_A = \frac{F_S}{2m} = \frac{\frac{mg}{2}}{2\text{m}} = \frac{g}{4}$$

The acceleration of plank with respect to ground, " a_B ", is given by :

$$a_B = \frac{4mg - F_S}{4m} = \frac{4mg - \frac{mg}{2}}{4m} = \frac{7g}{8}$$

The relative acceleration of block "A" with respect to plank, " a_{AB} ", is :

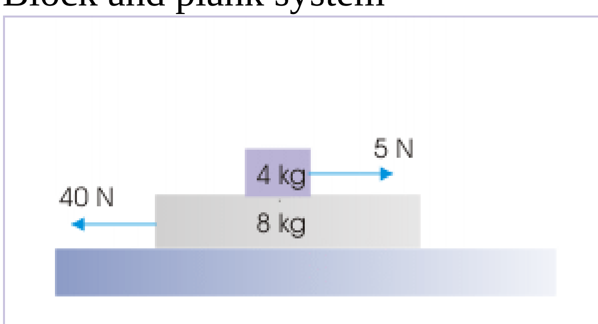
$$\Rightarrow a_{AB} = a_A - a_B = \frac{g}{4} - \frac{7g}{8} = -\frac{5g}{8}$$

The relative acceleration of "A" with respect to be is, thus, directed in the opposite direction of the motion.

One external force each on block and plank

Problem 4 : A block of mass 4 kg is placed on a plank of mass 8 kg. At an instant, two external forces are applied on them as shown in the figure. The coefficient of friction for surfaces between block and plank is 0.5, whereas friction between plank and underneath surface is negligible. Find the friction between block and plank.

Block and plank system



One external force each is

applied on block and plank.

Solution : Considering block, the maximum static friction between block and plank is :

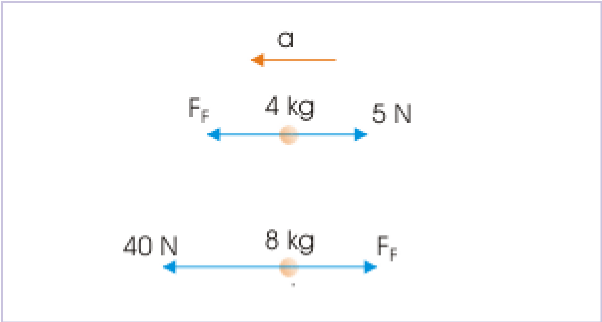
$$F_S = \mu N = \mu mg = 0.5X4X10 = 20N$$

The block and the plank will have relative motion with respect to each other when friction between the surfaces is kinetic friction (about equal to limiting friction). If friction between them is less than limiting static friction, then two entities will move together.

Here, we do not know the magnitude of friction. So we do not know whether two entities move together or not. Let us assume that they move together. If our assumption is wrong, then friction as determined from the force analysis under the assumed condition will equal to limiting friction, otherwise less than it.

Now, net external force on the combined body is towards left. Let the common acceleration be, “a”, and let the friction between block and plank be “ F_F ”. The free body diagrams of block and plank are shown in the figure.

Free body diagrams



Forces on block and plank.

$$a = \frac{F_F - 5}{4} = \frac{40 - F_F}{8}$$

Solving for “ F_F ”, we have :

$$\Rightarrow 8F_F - 40 = 160 - 4F_F$$

$$\Rightarrow 12F_F = 200$$

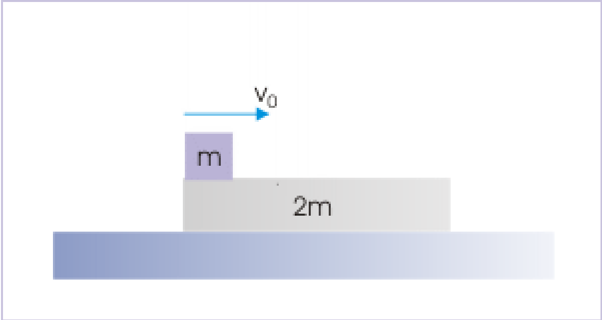
$$\Rightarrow F_F = 16.66N$$

Thus, we see that friction at the interface is actually less than the limiting friction. Our assumption, therefore, was correct and hence, friction between the surface is 16.66 N.

Block with initial velocity

Problem 5 : A plank “B” of mass “2m” is placed over smooth horizontal surface. A block “A” of mass “m” and having initial velocity “ v_0 ” is gently placed over the plank at one of its end as shown in the figure. If the coefficient of friction between “A” and “B” is 0.5, find the acceleration of “B” with respect to “A”.

Block and plank system



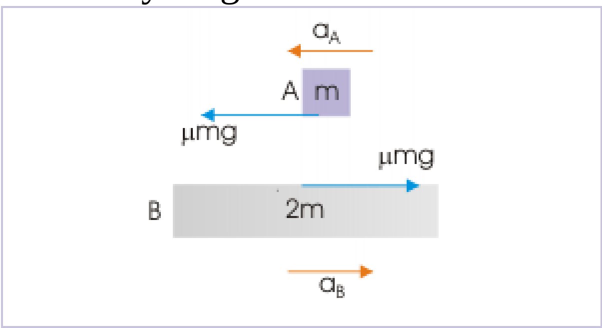
Block is given an initial velocity.

Solution : We see here that block and plank has certain initial relative velocity. It means that friction between “A” and “B” is kinetic friction. The

friction acts opposite to the velocity of the block “A”. On the other hand, friction is the only external force on the plank and acts opposite to the friction on “A”. Clearly, it is the friction that accelerates plank "B".

Let the magnitudes of accelerations of the block and the plank be “ a_A ” and “ a_B ” respectively with respect to ground. The free body diagrams, showing the forces on the block and the plank, are shown here.

Free body diagrams



Forces on block and plank.

The magnitudes of accelerations are :

$$a_A = \frac{\mu mg}{m} = \mu g$$

$$a_B = \frac{\mu mg}{2m} = \frac{\mu g}{2}$$

Two accelerations are in opposite direction. The relative acceleration of “B” with respect to “A” is :

$$a_{BA} = a_B - a_A = \frac{\mu g}{2} - (-\mu g) = \frac{3\mu g}{2}$$

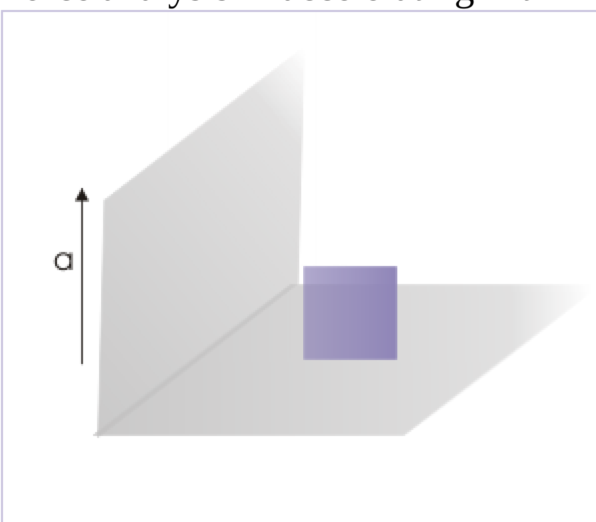
The relative acceleration is in the same direction as that of the acceleration of block “B” i.e. towards right.

Force analysis in accelerated frame

The description, measurement and analysis of motion or force system in inertial and accelerated (non-inertial) frames are different. As the motion of an object in two systems should represent same physical phenomenon, it is desirable that the differences are resolved and made consistent to each other.

Our objective here is to reconcile the differences. For this, we consider a block of mass 10 kg lying on the floor of a lift, which itself is moving up with an acceleration of 2 m/s^2. For the sake of convenience, we consider acceleration due to gravity is equal to 10 m/s^2. Here, we have taken numerical values so that we have the feel of the difference of measurements in inertial and accelerated reference systems.

Force analysis in accelerating lift



A block is at rest with respect to lift.

Inertial frame of reference (ground)

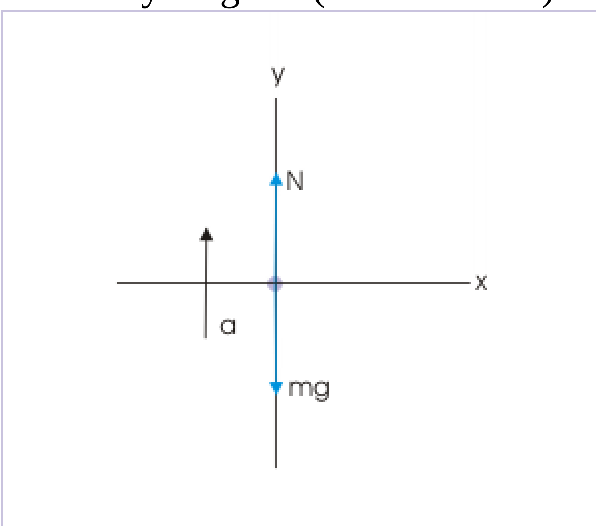
An observer on the ground, who can see the block (let us think that the lift is covered with transparent glass), makes following measurements/

assumptions based on his/her understanding of classical mechanics :

- The block is moving up with an acceleration, a = 2 m/s^2 .
- The acceleration due to gravity is 10 m/s^2 . The acceleration due to gravity in the vicinity of earth surface is a constant and should not change because of the acceleration of the lift.
- The weight of the block is mg = 10 x 10 = 100 N.
- The block is acted upon by a normal force, N, as applied by the floor of the lift in upwards direction.

He/ She analyzes forces, applying Newton's second law of motion. The free body diagram of the block is :

Free body diagram (inertial frame)



$$\sum F_y = N - mg = ma_y = ma$$
$$\Rightarrow N = m(g+a) = 10(10+2) = 120 \text{ Newton}$$

The observer, therefore, concludes (applying Newton's third law) that the force, which the block exerts on the floor is equal in magnitude, N = 120 N as against 100 N, when the lift would have been stationary.

He/ She anticipates that a spring balance will register a value of 120 N as it measures normal force applied on it. We must know that spring balance measures normal force - not the weight. Clearly, the measurement is not same as the weight of the block, mg (10 x10 = 100 N), when lift in

stationary. The normal force is equal to weight only when spring balance is stationary in inertial frame.

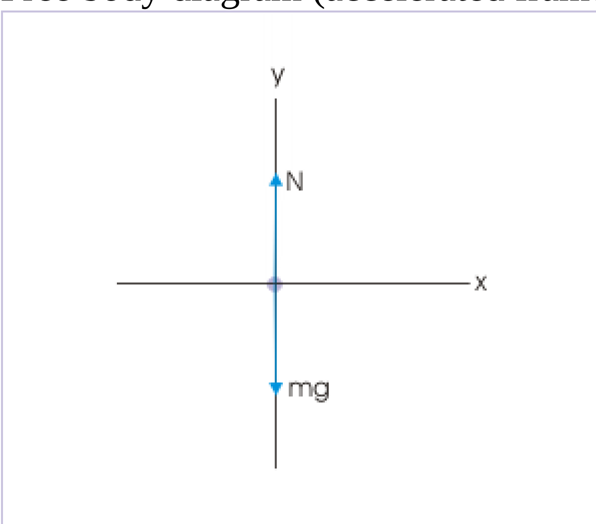
Accelerated frame of reference (lift)

As against above, an observer in the lift makes following measurements/ assumptions based on his/her understanding of classical mechanics :

- The block is at rest.
- The acceleration due to gravity is 10 m/s^2 (This does not change in the vicinity of Earth).
- The weight of the block is mg = 10 x 10 = 100 N. This is based on the fact that mass of the block is 10 kg. Hence, weight, mg = 10 X 10 = 100 N.

He analyzes the force applying Newton's second law of motion. The free body diagram of the block is :

Free body diagram (accelerated frame)



$$\sum F_y = N - mg = 0$$
$$\Rightarrow N = mg = 10 \text{ x } 10 = 100 \text{ Newton}$$

This observer is not capable to perceive and hence measure the acceleration of the lift. He, however, can measure the normal force on the block, say with a spring balance. He/ She find that spring balance actually measures it

to be 120 N! Clearly, there is something with his force analysis. The basic difference in the analysis of two observers arises due to the fact that observer on the ground recognizes block as an accelerated body, whereas observer on the lift recognizes block as stationary.

Resolution of differences

The two observes meet and exchange their observations and discuss the differences. It is obvious that the observer on the ground (inertial frame of reference) prevails as his analysis of force yields the correct value of the normal force in the accelerated lift. Clearly, observer in the accelerated frame of reference (no-inertial) needs to do some adjustment, while applying Newton's second law of motion. The observer of inertial frame of reference points out to the observer in the accelerated frame of reference that the lift was actually being pulled up by an external force and that this external force needs to be accounted in the analysis by the observer in the lift.

They conclude as :

1: Newton’s second law is valid in inertial frame of reference and needs to be modified, when used in the accelerated frame of reference.

2: The body need to be subjected to an additional force, usually referred as pseudo force, to account for the acceleration of the frame of reference. Once this additional force is applied on the body, Newton's second law can be applied as usual even in the non-inertial frame of reference.

3: The application of pseudo force should be carried out in the following manner :

- Determine magnitude of pseudo force," F_S ", by multiplying the mass of the body "m" with the acceleration “a” of the accelerated frame of reference,
- Apply pseudo force on the body in the direction opposite to that of the acceleration “**a**”.

- Analyze the situation, using Newton’s second law as if the frame of reference was an inertial frame of reference.

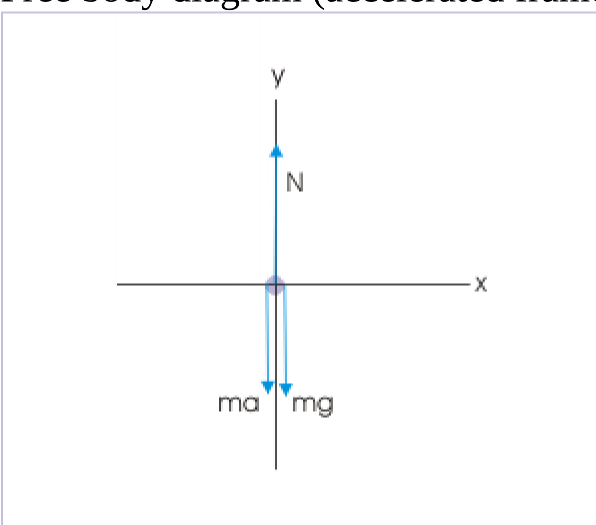
In the nutshell, the pseudo force is given by following vector relation,
Equation:

$$\mathbf{F}_S = -m\mathbf{a}$$

where "**a**" is the acceleration of non-inertial frame of reference.

The two observers also conclude that the spring balance does not measure weight of the block (mg), but the normal force applied on it. Besides they realize that normal contact force depends on the acceleration of the moving reference (container).

Let us check the analysis of the observer in the lift, using the concept of pseudo force. Now, the free body diagram of the block with pseudo force is as given here (the downward forces are shown as two parallel vectors only to show them individually).
Free body diagram (accelerated frame)



$$\sum F_y = N - mg - ma = 0$$
$$\Rightarrow N = m(g + a) = 10(10 + 2) = 120 \text{ Newton}$$

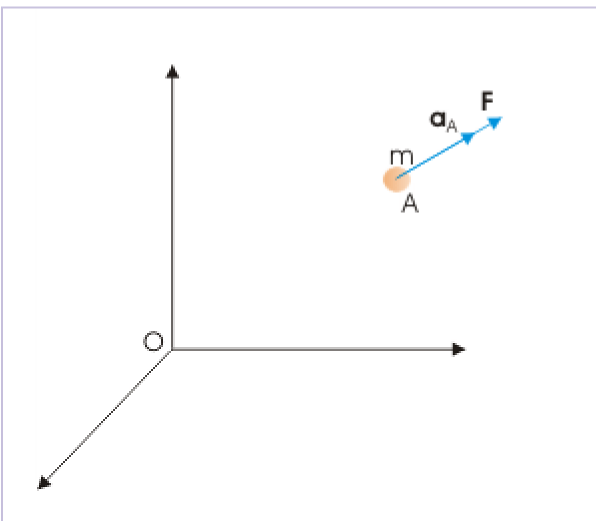
Note that while applying law of motion in the accelerated frame, the observer has not modified his observation that the block is at rest. Adoption of pseudo force technique simply allows reconciling force analysis with the observed value measured in the accelerated system, using Newton’s second law.

Theoretical basis of analysis in accelerated frame

The conception of pseudo force in an accelerated frame of reference is not without basis. We can develop the theoretical framework that justifies using additional force in accelerated frame for applying Newton’s second law.

Here, we consider a body “A” of mass “m”. An external force “**F**” acts on the body as shown. The acceleration of the body with respect to ground, “ $\mathbf{a}_A$ ”, is related to external force as :

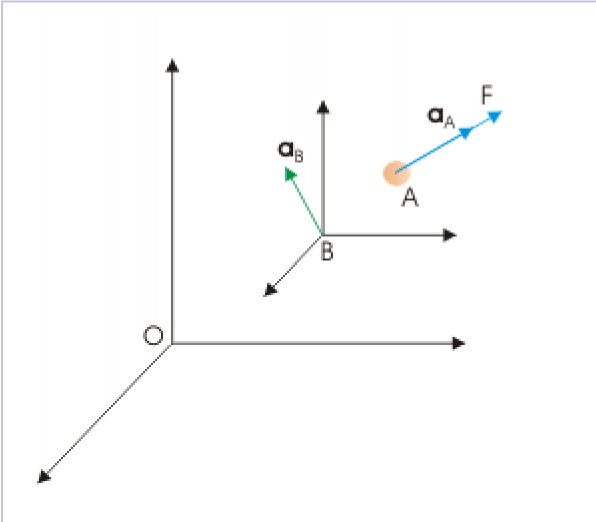
Motion in accelerated frame



$$\mathbf{F} = m\mathbf{a}_A$$

We, now, seek to analyze the motion with respect to a frame of reference, which is moving with acceleration, ” $\mathbf{a}_B$” with respect to ground reference. Let the acceleration of the body with respect to this accelerated frame be “ $\mathbf{a}_{\mathrm{AB}}$” .

Motion in accelerated frame

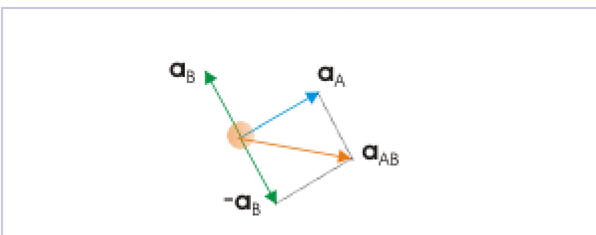


The product of the mass and the acceleration of the body with respect to accelerated reference is given by :

$$\mathbf{F}\prime = m\mathbf{a}_{AB}$$

This product is not equal to the product as obtained in inertial ground frame. This is because, acceleration of the body with respect to accelerated reference " $\mathbf{a}_{\mathrm{AB}}$ " is given as :

Relative acceleration



$$\mathbf{a}_{AB} = \mathbf{a}_A - \mathbf{a}_B$$

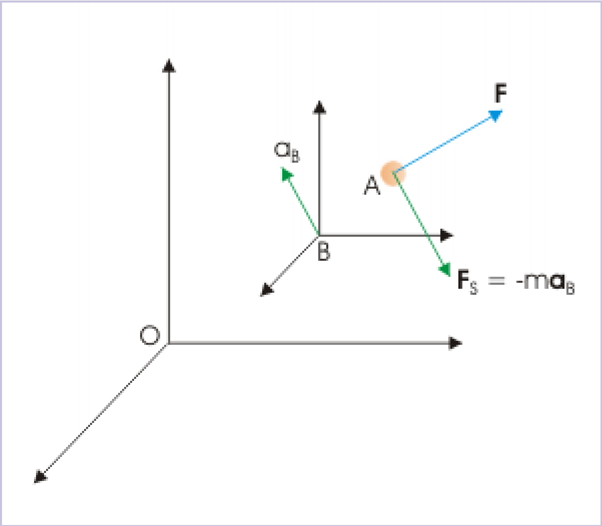
As such, the product in the accelerated frame evaluates to :

$$\Rightarrow \mathbf{F}\prime = m\mathbf{a}_{AB} = m\mathbf{a}_A - m\mathbf{a}_B = \mathbf{F} - m\mathbf{a}_B$$

If Newton's second law of motion is valid in the accelerated frame, then it should connect external force on the body with the acceleration, " $\mathbf{a}_{\mathrm{AB}}$ ", in the accelerated frame of reference. Clearly, this is not the case. However, if we replace external force "**F**" by " $\mathbf{F} - m\mathbf{a}_B$ ", then the modified external

force is equal to the product of mass and acceleration of the body in the accelerated frame.

Motion in accelerated frame



Equation:

$$\mathbf{F} - m\mathbf{a}_B = m\mathbf{a}_{AB}$$

Clearly, we need to apply a force " $a\mathbf{a}_B$ "maB in the direction opposite to the acceleration of the frame of reference “B”. This force is known or termed as “pseudo” force.

Problem 1 : A pendulum is suspended from the roof of a train compartment, which is moving with a constant acceleration “a”. Find the deflection of the pendulum bob from the vertical as observed from the ground and the compartment.

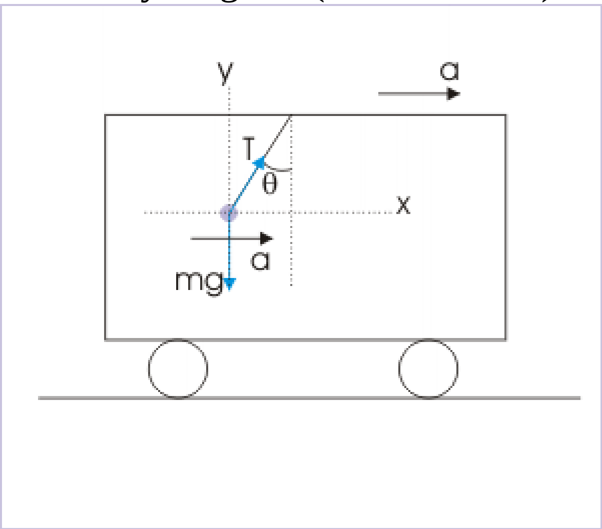
Solution : We analyze the problem from the perspectives of both inertial and accelerated frames.

(i) In the inertial frame of reference,

Free body diagram of pendulum bob

The bob has acceleration “a” towards right. The forces on the bob are (i) weight of the bob, mg, and (ii) tension in the string.

Free body diagram (inertial frame)



$$\sum F_x = T\sin\theta = ma_x$$
$$\Rightarrow T\sin\theta = ma$$

and

$$\sum F_y = T\cos\theta - mg = 0$$
$$\Rightarrow T\cos\theta = mg$$

Combining two equations, we have :

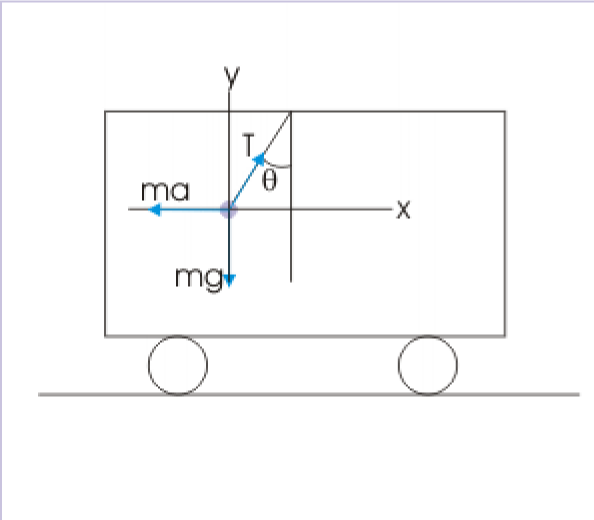
$$\tan\theta = \frac{a}{g}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{a}{g}\right)$$

(ii) In the accelerated frame of reference,

Free body diagram of pendulum bob

The bob is at rest. The forces on the bob are (i) weight of the bob, mg, (ii) tension in the string and (iii) pseudo force ma.

Free body diagram (accelerated frame)



$$\sum F_x = T\sin\theta - ma = 0$$
$$\Rightarrow T\sin\theta = ma$$

and

$$\sum F_y = T\cos\theta - mg = 0$$
$$\Rightarrow T\cos\theta = mg$$

Combining two equations, we have :

$$\tan\theta = \frac{a}{g}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{a}{g}\right)$$

Alternatives

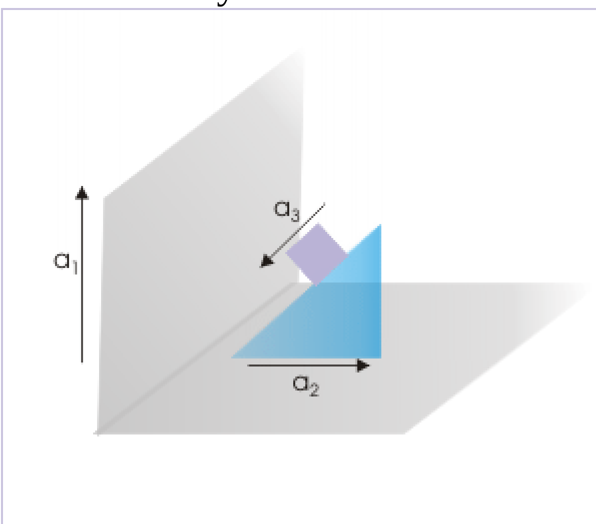
Application of force analysis in accelerated frame of reference may have two approaches. We can analyze using Newton’s law from an inertial frame of reference. Alternatively, we can use the technique of pseudo force and apply Newton’s law right in the accelerated frame of reference as described above.

There is a school of thought that simply denies merit in pseudo force technique. The argument is that pseudo force technique is arbitrary without

any fundamental basis. Further, this is like a short cut that conceals the true interaction of forces with body under examination.

On the other hand, there are complicated situation where inertial frame approach may turn out to be difficult to work with. Consider the illustration depicted in the figure. Here, a wedge is placed on the smooth surface of an accelerated lift. We have to study the motion of the block on the smooth incline surface of the wedge.

Accelerated systems



Multiple accelerations here complicates the situation. The lift is accelerated with respect to ground; wedge is accelerated with respect to lift (as the surface of the lift is smooth) ; and finally block is accelerated with respect to wedge (as wedge surface is also smooth). In this case, it would be difficult to assess or determine attributes of motion by analyzing force in the inertial ground reference. In situation like this, analysis of forces in the non-inertial frame of reference of the lift eliminates one of the accelerations involved.

It must again be emphasized that when we analyze motion with respect inertial frame of reference, then all measurements are done with respect to inertial frame. On the other hand, if we analyze with respect to accelerated frame of reference using concept of pseudo force, then all measurements are done with respect to accelerated frame of reference. For example, if the analysis of force in non-inertial frame yields an acceleration of the block as 5 m/s^2 , then we must know that this is the acceleration of the block with

respect to in incline i.e. accelerated frame of reference - not with respect to ground.

It is generally recommended that we should stick to force analysis in the inertial frame of reference, unless situation warrants otherwise.

Force analysis in accelerated frame (check your understanding)

The questions have been selected to enhance understanding of the topics covered in the module titled " Force analysis in accelerated frame ". All questions are multiple choice questions with one or more correct answers.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

Force analysis in accelerated frame

Exercise:

Problem:

What is the weight of a person of mass “m”, standing in a lift, which is accelerating upward with acceleration “a”.

$$\text{(a) } mg \text{ (b) } m(a+g) \text{ (c) } m(a-g) \text{ (d) } m(g-a)$$

Solution:

The weight of a body is defined as the product of mass and acceleration due to gravitational force. It means that weight of a body is simply equal to the gravitational pull (force) on the body. Since acceleration due to gravity is “g” in the lift (whether accelerating or not), the weight of the person is “mg”. The weight of the person does not change - though the net force on the person has changed.

What is meant here that terms such as “lesser weight”, “greater weight” or “weightlessness” in accelerated motion, are actually loose expressions. These are states, corresponding to change in the “net force” and not the change in the “weight”. The weight remains “mg”.

Strictly speaking, we must use relative terms of weight, only if gravitational force on the body has changed. In accelerated motion, we

can, however, be not incorrect to say that we experience the “feeling” of weightlessness or lesser or greater weight.

Hence, option (a) is correct.

Exercise:

Problem:

A spring balance in an accelerating lift records 30 N for a block of mass 5 kg. It means that :

(a) lift is moving up with increasing speed

(b) lift is moving up with decreasing speed

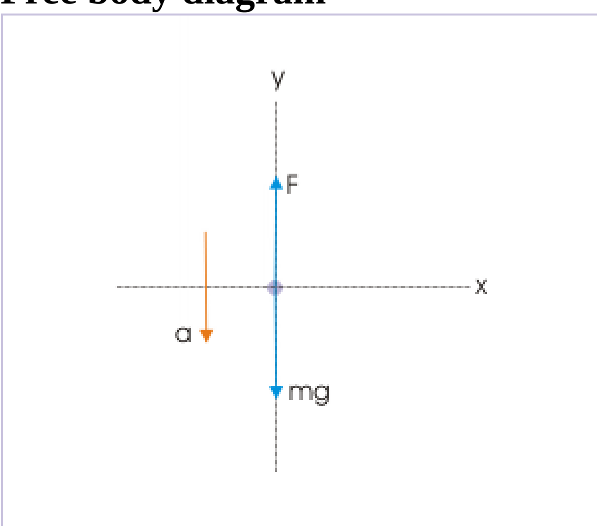
(c) lift is moving down with increasing speed

(d) lift is moving down with decreasing speed

Solution:

The weight of the block is mg = 5 x 10 = 50 N. Since spring balance records lesser weight, it means that spring force (F) is less than the weight of the block. It means that lift is accelerating in downward direction such that :

Free body diagram



$$\sum F_y = mg - F = ma$$
$$\Rightarrow F = m(g - a)$$

An acceleration in downward direction means :

(i) lift is moving down with increasing speed

(ii) lift is moving up with decreasing speed

Hence, options (b) and (c) are correct.

Exercise:

Problem:

A person jumps off from a wall of height “h” with a small box of mass “m” on his head. What is the normal force applied by the box on his head during the fall?

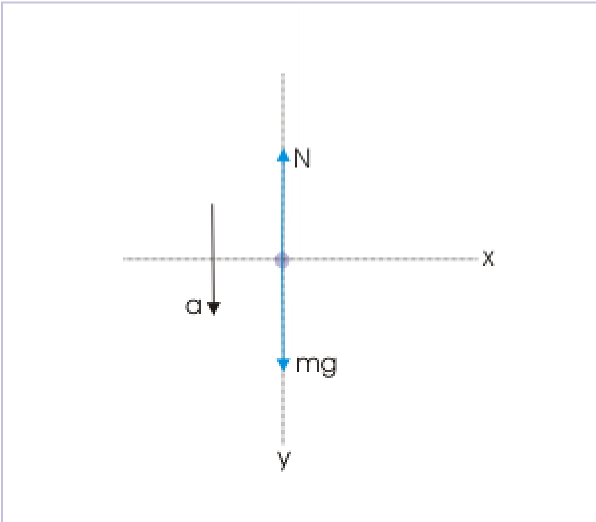
$$\text{(a) } mg \;\; \text{(b) } m(a+g) \;\; \text{(c) } m(a-g) \;\; \text{(d) } \text{zero}$$

Solution:

The motion of the box takes place in relation to the person, who is accelerating down with acceleration “g”. Here, we analyze the problem in the inertial frame of the ground. The forces on the box are (i) its weight, "mg", acting downward and (ii) normal force, "N", acting downward. As the bodies are falling freely under the gravity, the acceleration of each of them is equal to "g" i.e. acceleration due to gravity.

Now, the free body diagram of the box is shown in the figure.

Free body diagram



$$\sum F_y = mg - N = ma$$
$$\Rightarrow N = m(g - a)$$

As a = g,

$$\Rightarrow N = m(g - g) = 0$$

Alternatively, we can analyze the situation in the non-inertial frame of person, who is falling freely under gravity. In this non-inertial reference, box is stationary. The forces on the box in this reference are (i) weight of box, "mg", acting downward (ii) normal force, "N", acting upward and (iii) pseudo force, "mg", applied in the upward direction opposite to the direction of acceleration of the frame of reference.

$$\sum F_y = N + mg - mg = 0$$
$$\Rightarrow N = 0$$

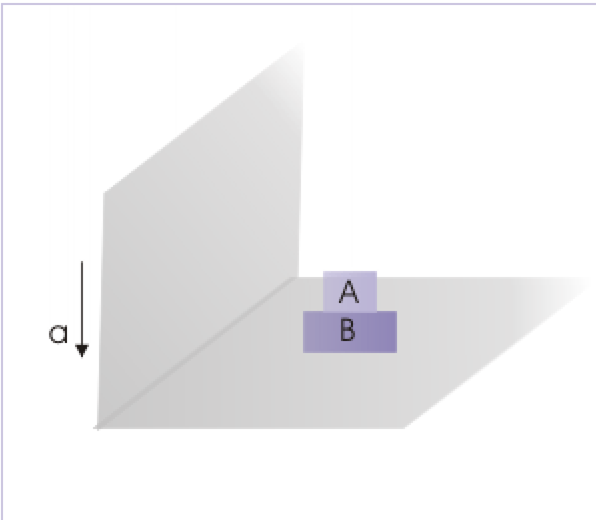
Hence, option (d) is correct.

Exercise:

Problem:

A lift is descending with acceleration 3 m/s^2 as shown in the figure. If the mass of the block "A" is 1 kg, then the normal force (in Newton) applied by block "A" on block "B" is :

Motion in accelerated lift



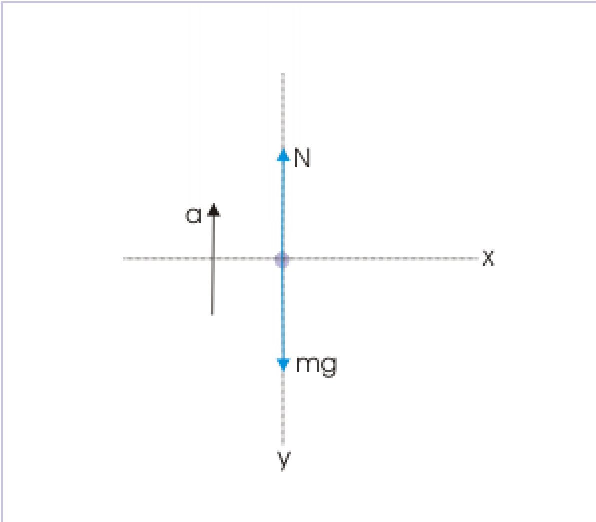
(a) 3 (b) 5 (c) 7 (d) 9

Solution:

It is convenient to carry out force analysis of block "A". For, it is acted by only two forces as compared to three forces for block "B". Since normal forces are equal and opposite in accordance with Newton's third law, it is same whether we determine normal force with respect to "A" or "B".

Free body diagram of "A"

Free body diagram



$$\sum F_y = mg - N = ma$$
$$\Rightarrow N = m(g - a)$$
$$\Rightarrow N = 1 \text{ x } (10 - 3) = 7 \text{ Newton}$$

Alternatively, using the reference of lift, the forces on the block "A" are (i) its weight, "mg", acting downward (ii) Normal force, "N", acting upward and (iii) pseudo force "ma" acting upwards in the direction opposite to that of the acceleration of the lift. In this non-inertial frame, however, the block is stationary. Hence,

$$\sum F_y = N + ma - mg = 0$$
$$\Rightarrow N = m(g - a)$$
$$\Rightarrow N = 1 \text{ x } (10 - 3) = 7 \text{ Newton}$$

Hence, option (c) is correct.

Exercise:

Problem:

A lift of height 2.4 m, starting with zero velocity, is accelerating at 2 m/s^2 in upward direction. A ball of 0.1 kg is dropped from the ceiling of the lift, the moment lift starts its motion. The time taken (in second) by the ball to hit the floor is (consider g = 10 m/s^2) :

$$\text{(a) } 2 \text{ (b) } 0.27 \text{ (c) } 0.63 \text{ (d) } 0.72$$

Solution:

The motion of the ball (say A) is not taking place in contact with lift. The moment it is let go, the only force on the ball is due to gravity. The question, therefore, reduces to motions of two objects, which are moving towards each other to cover a linear distance of 2.4 m, starting with zero speeds. Considering downward direction as positive, the relative acceleration of the ball (A) with respect to lift floor (B) is :

$$a_{\text{AB}} = a_A - a_B$$
$$\Rightarrow a_{\text{AB}} = 10 - \left(-2\right) = 12\, m/s^2$$

Now applying equation for constant acceleration with initial velocity zero,

$$y = \frac{1}{2}at^2$$
$$\Rightarrow 2.4 = \frac{1}{2} \text{ x } 12 \text{ x } t^2$$
$$\Rightarrow t^2 = \frac{4.8}{12} = 0.4$$
$$\Rightarrow t = 0.63\ s$$

Hence, option (c) is correct.

Exercise:

Problem:

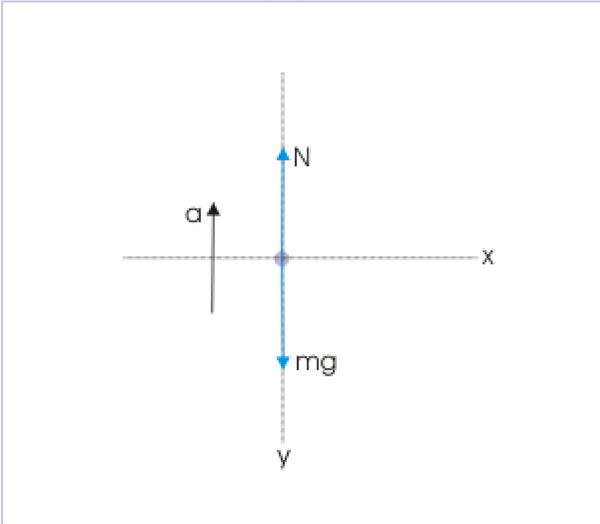
A block hangs from a spring balance in a lift, which is accelerating up at 2 m/s^2 . The balance reads 48 N. Then, the mass of the block (in kg) is (consider g = 10 m/s^2) :

$$\text{(a) } 1 \text{ (b) } 2 \text{ (c) } 3 \text{ (d) } 4$$

Solution:

The balance reads the spring force. The free body diagram is :

Free body diagram



$$\sum F_y = N - mg = ma$$
$$\Rightarrow N = m(a+g)$$
$$\Rightarrow m = \frac{48}{(2+10)} = 4 \text{ kg}$$

Hence, option (d) is correct.

Answers

`1. (a) 2. (b) and (c) 3. (d) 4. (c) 5. (c) 6. (d)`

Motion in accelerated frame (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

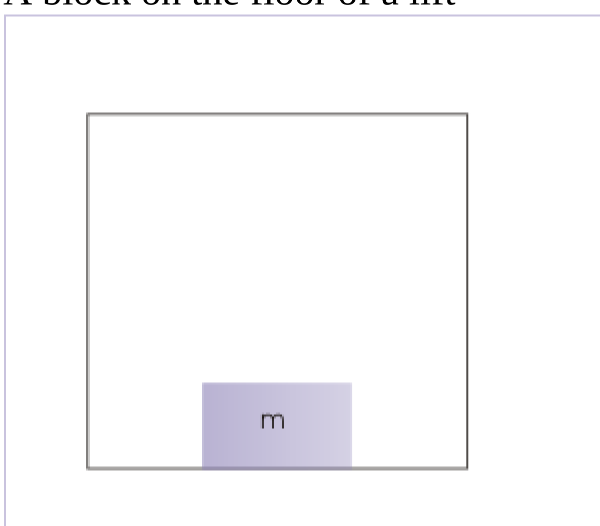
We discuss problems, which highlight certain aspects of the study leading to motion in accelerated frame. The questions are categorized in terms of the characterizing features of the subject matter :

- Motion in a lift
- Motion of an incline on smooth surface
- Motion on an incline

Motion in a lift

Problem 1 : What should be the acceleration "a" of the lift so that a block of mass "m" placed on the floor exerts a normal force of "mg/3" on it.

A block on the floor of a lift



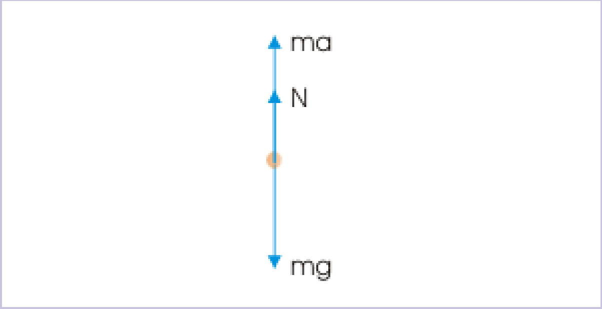
The lift is moving with acceleration.

Solution : The normal force on the floor is less than that exerted by the block when the lift is stationary. It means that lift is accelerating in downward direction.

In order to find acceleration of the lift, we analyze force on the block in the non-inertial frame of lift. The forces of the block are (i) its weight, “mg”, acting downward (ii) Normal force, “N”, acting upward and (iii) pseudo force, “ma” acting upward in the direction opposite to the direction of lift's acceleration.

The free body diagram of the block in the frame of lift is shown in the figure. Analyzing force in the vertical direction, we have :

Free body diagram



The forces on the block.

$$N + ma = mg$$

Substituting for normal force as given in the question,

$$\Rightarrow \frac{Mg}{3} + ma = mg$$

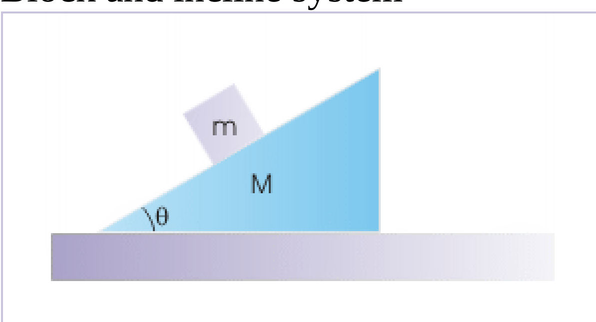
$$\Rightarrow a = g - \frac{g}{3} = \frac{2g}{3}$$

Motion of an incline on smooth surface

The problems here are related to the motion of incline, which is accelerated. The incline serves as the accelerated frame for the motion of block. In this sense, these problems do not exactly pertain to the motion in accelerated frame. We shall, therefore, analyze the motion of incline in inertial ground reference. Problems, involving motion of block in the accelerated frame of incline, are included in the next module.

Problem 2 : In the arrangement shown in the figure, "N" is the normal force applied by the block on the incline of angle "θ". If all surfaces are smooth, then find the acceleration of the incline.

Block and incline system



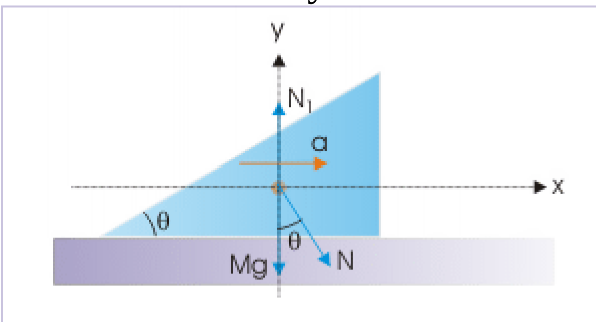
All the surfaces are smooth.

Solution : Since all surfaces are smooth, the incline will accelerate due to the normal force applied by the block.

The forces on the incline are (i) its weight, "mg", (ii) Normal force, "N", applied by the overlying block and (iii) normal force, " N_1 " due to the surface underneath. There is no motion of incline in the vertical direction.

Thus, we need to analyze components of forces in horizontal direction only to find the acceleration in that direction.

Block and incline system



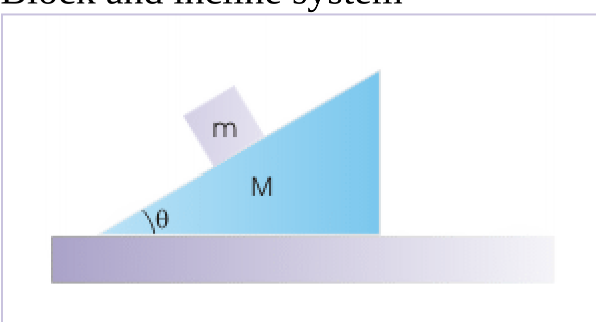
Forces on the incline.

$$\sum F_x \Rightarrow N \sin\theta = Ma$$

$$\Rightarrow a = \frac{N \sin\theta}{M} \quad \text{towards right}$$

Problem 3 : In the arrangement shown in the figure, “N” is the normal force applied by the block on the incline of angle “θ”. If all surfaces are smooth, then find normal force on the incline, as applied by the horizontal surface underneath.

Block and incline system

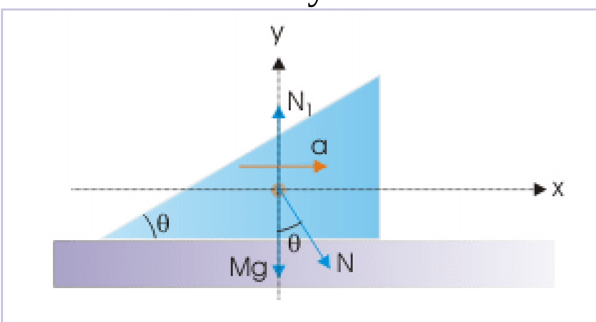


All the surfaces are smooth.

Solution : Since all surfaces are smooth, the incline will accelerate due to the normal force applied by the block.

The forces on the incline are (i) its weight, "mg", (ii) Normal force, "N", applied by the overlying block and (iii) normal force, " N_1 " due to the surface underneath. There is no motion of incline in the vertical direction. Hence, components of forces in vertical direction constitute a balanced force system.

Block and incline system



Forces on the incline.

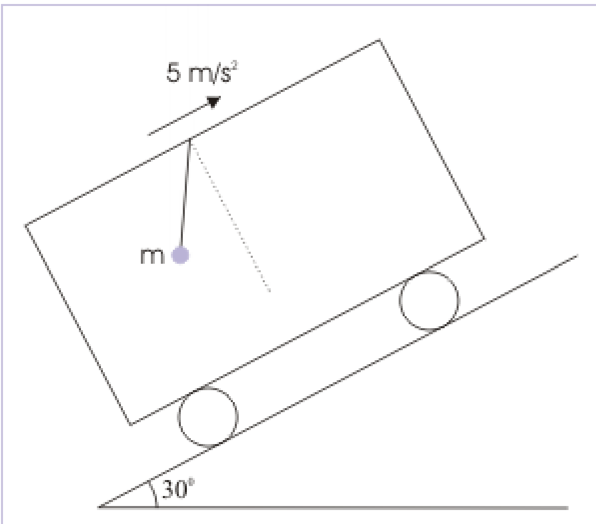
$$\sum F_y = N_1 - Mg + N\cos\theta = 0$$

$$\Rightarrow N_1 = Mg + N\cos\theta$$

Motion on an incline

Problem 4 : A pendulum bob of mass 2 kg hangs from the ceiling of a train compartment. The train is accelerating up an incline terrain at 5 m/s^2 , making an angle 30° with horizontal. Find the angle that the pendulum bob makes with the normal to the ceiling.

Pendulum

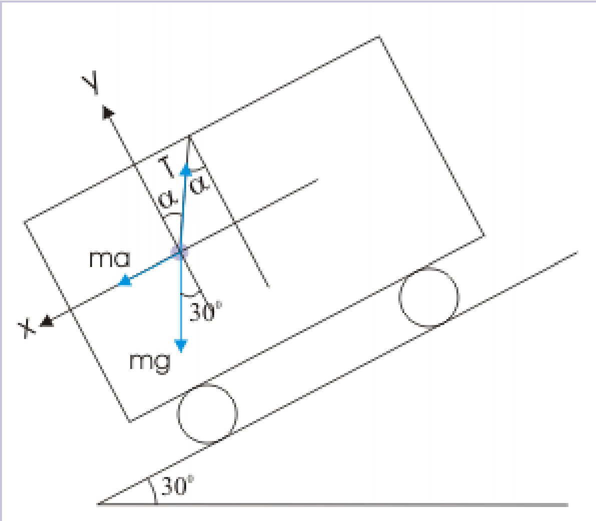


A pendulum bob hangs from the ceiling of a train compartment.

Solution : Let the pendulum bob be at an angle "α" with the normal to the ceiling of the compartment as shown in the figure below. The bob is in a stationary position in the non-inertial frame of the accelerating compartment. We can use this fact in the non-inertial frame. As such, we shall carry out force analysis in the accelerated frame of the compartment.

The forces on the bob are (i) its weight "mg", acting in vertically downward direction (ii) Tension in the string, "T", making an angle "α" with the normal to the ceiling and (iii) pseudo force, "ma", acting down the incline in the direction opposite to the direction of acceleration of frame of reference.

Pendulum



Forces on pendulum bob.

As pseudo force is along the incline, we select coordinate axes parallel and perpendicular to the incline terrain as shown in the figure. FBD as superimposed on the diagram is shown in the figure.

$$\sum F_x \Rightarrow T\sin\alpha = ma + mg\sin 30^0$$

$$\Rightarrow T\sin\alpha = 2X5 + 2X\frac{10}{2} = 20$$

$$\sum Fy \Rightarrow T\cos\alpha = mg\cos 30^0$$

$$\Rightarrow T\cos\alpha = 2X10X\frac{\sqrt{3}}{2} = 10\sqrt{3}$$

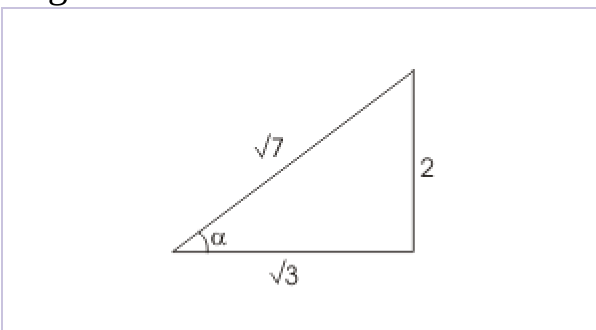
Taking ratio of two equations, we have :

$$\tan\alpha = \frac{2}{\sqrt{3}}$$

From the analysis "x" direction, the tension in the string is :

$$\Rightarrow T = \frac{20}{\sin\alpha}$$

We need to know "sinα" in order to evaluate the expression of tension.
Using the ratio of tangent, we can determine the sine of the angle :
Trigonometric ratio



Sine of the angle.

$$\sin\alpha = \frac{2}{\sqrt{\left\{(2)^2 + \left(\sqrt{3}\right)^2\right\}}} = \frac{2}{\sqrt{7}}$$

Putting in the expression of tension, ”T”,

$$T = \frac{20\sqrt{7}}{2} = 10\sqrt{7} \quad N$$

Motion on accelerated incline plane(application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

The incline plane has two contact or interface surfaces. One is the incline surface, where the block is placed and the other is the base of the incline, which is in contact with the surface underneath. The motion of the block, therefore, may depend on the motion of the incline itself.

We must understand here that there are two ways to look at such situation, which involves motion of the incline itself. We can analyze the motion of the elements of "block and incline" system in either the inertial frame (ground reference) or in the accelerated frame. The two techniques have relative merits in particular situations. As pointed out earlier in the course, we should stick to the analysis from inertial frame to the extent possible unless analysis in accelerated frame gives distinct advantage.

In this module, however, our objective is to learn analysis of motion in accelerated frame. As such, we have selected questions which are better suited to analysis in non-inertial frame of reference.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the motion on accelerated incline plane. The questions are categorized in terms of the characterizing features of the subject matter :

- Coefficient of friction in accelerated frame
- Incline on a smooth horizontal surface
- Incline in an accelerated lift

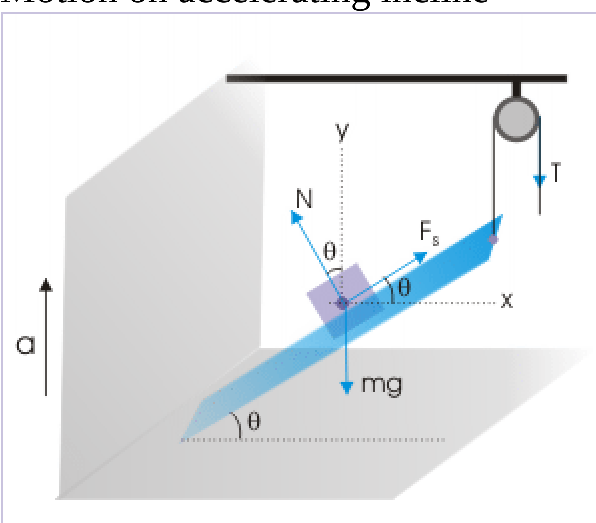
Coefficient of friction in accelerated frame

Problem 1 : A block is placed on a rough incline plane placed on the floor of a lift, which is moving up with acceleration "a". The incline plane is raised at one end to increase the angle of incline slowly till the block is about to move down. Find angle of repose (this is the angle of incline for which block begins to slide).

Solution : Let the angle of incline, when block is about to move down, is "θ". To analyze the forces, we select a coordinate system with axes in horizontal and vertical directions. This orientation of axes takes advantage of the fact that acceleration of the lift is in vertical direction.

The context of motion here is peculiar, in which inertial or non-inertial considerations for force analysis do not make any difference. The reason is that angle of repose is obtained by considering motion in horizontal direction only (as we shall see soon). It does not matter whether we analyze forces without pseudo force (inertial ground reference) or with pseudo force (non-inertial lift reference). Note that if we consider non-inertial analysis, then the pseudo force would be in vertically downward direction and as such would not come in consideration for force analysis in horizontal direction.

Motion on accelerating incline



The incline has acceleration in upward direction.

The component of acceleration of the lift in horizontal direction is zero. The motion of the block in horizontal direction, therefore, is not in accelerated frame of reference. For the condition of balanced net component of forces in horizontal direction :

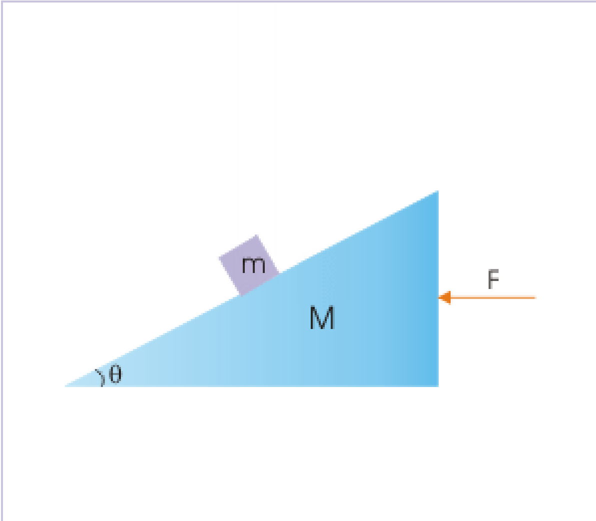
$$\sum F_x = F_s \cos\theta - N \sin\theta = 0$$
$$\Rightarrow \mu_s N \cos\theta - N \sin\theta = 0$$

$$\Rightarrow \tan\theta = \mu_s$$

Thus, we see that angle of repose in the accelerated lift is same as that when the incline is placed on the ground. This result is expected as there is no component of acceleration of the lift in horizontal direction and application of Newton's law remains same as far as force analysis in horizontal direction is concerned.

Incline on a smooth horizontal surface

Problem 2 : A block of mass "m" is kept on a smooth incline of angle "θ" and mass "M". The incline is placed on a smooth horizontal surface. Find the horizontal force "F" required such that block is stationary with respect to incline.
Motion on accelerating incline



The block is stationary on an accelerated incline.

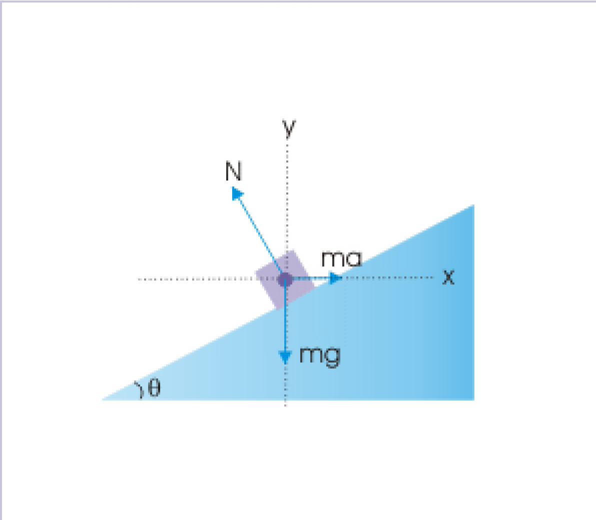
Solution : Since block is stationary with respect to incline, both elements of the system (incline and block) have same acceleration. As such, we can consider "block - incline" to constitute a combined system and treat them as one. As all surfaces are smooth, friction force is zero. Hence, acceleration of the system is given by :

$$a = \frac{F}{(m+M)}$$

We can now analyze forces on the block to find force "F" for the given condition of block being stationary with respect to incline. The block is stationary with reference to incline, but it is accelerated with respect to ground. The condition of "being stationary" is, therefore, given in the context of accelerated frame of incline. Hence, it is more intuitive to carry out analysis in the non-inertial frame of reference.

Three forces on the block, including pseudo force, are shown here in the figure. Further, since two of the three forces i.e weight and pseudo forces are in horizontal and vertical directions, we choose our coordinates accordingly along these directions.

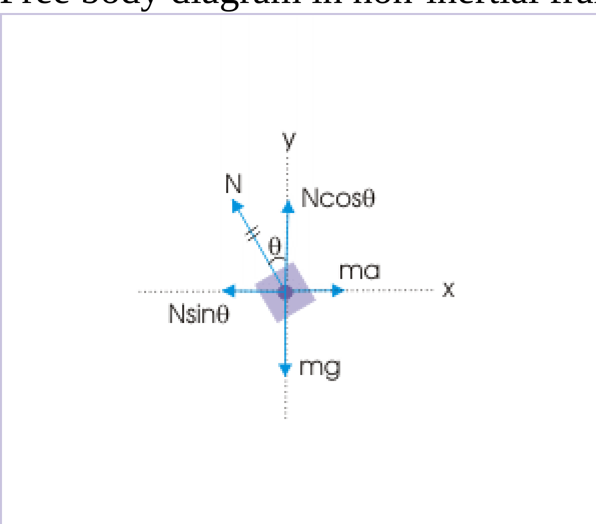
Forces on the block with pseudo force



The block is stationary on an accelerated incline.

The free body diagram of the block with components of forces along the axes is shown here.

Free body diagram in non-inertial frame



The block is stationary on an accelerated incline.

$$\sum F_x = ma - N\sin\theta = 0$$

$$\Rightarrow N\sin\theta = ma$$

$$\sum F_y = N\cos\theta - mg = 0$$

$$\Rightarrow N\cos\theta = mg$$

Taking ratio, we have :

$$\Rightarrow \frac{a}{g} = \tan\theta$$

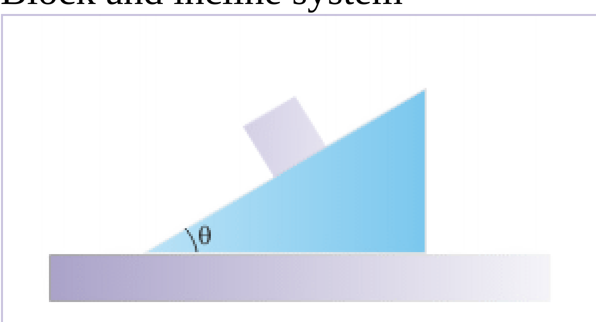
Substituting for "a", we have :

$$\frac{F}{(m+M)g} = \tan\theta$$

$$\Rightarrow F = (m+M)g\tan\theta$$

Problem 3 : A block of mass is placed on a rough incline of angle "θ". The angle of incline is greater than the angle of repose. If coefficient of friction between block and incline is "μ", then what should be acceleration of the incline in horizontal direction such that no friction operates between block and incline.

Block and incline system



The angle of incline is greater

than the angle of repose.

Solution : In order to find the acceleration of the incline for “no friction”, we need to know the net component of forces parallel to the incline. The pulling force here is the component of gravitational force parallel to incline in the downward direction. It is given that the angle of incline is greater than angle of repose. It means that block has started moving downward and force of friction is equal to kinetic friction (about equal to limiting friction).

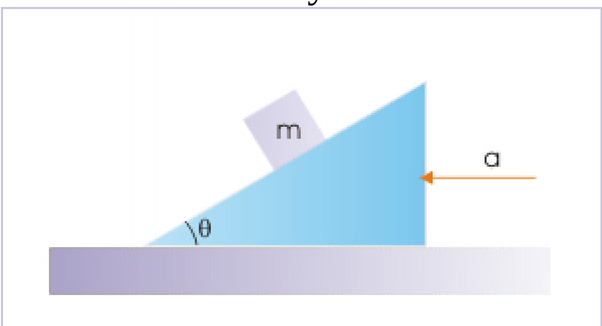
Let the mass of the block be “m”. The net component of force parallel to incline is acting downward and is given by :

$$F = mg \sin\theta - \mu mg \cos\theta$$

The incline moves with a certain acceleration due to forces acting on it. We need to apply a pseudo force to apply Newton's second law of motion. It is clear that pseudo force should have a component along the incline in upward direction so that net component of force (without friction) is zero. This will ensure that there is no friction between interface of block and the incline.

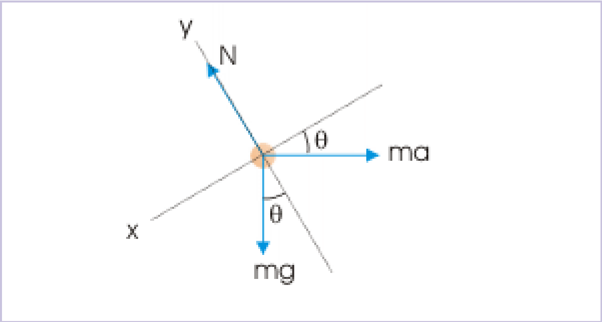
Now, for the pseudo force to have a component along the incline upward direction, the acceleration of the incline should be towards left as indicated in the figure below.

Block and incline system



The angle of incline is greater than the angle of repose.

The free body diagram of the block with pseudo force is shown here.
Free body diagram



The forces on the blocck.

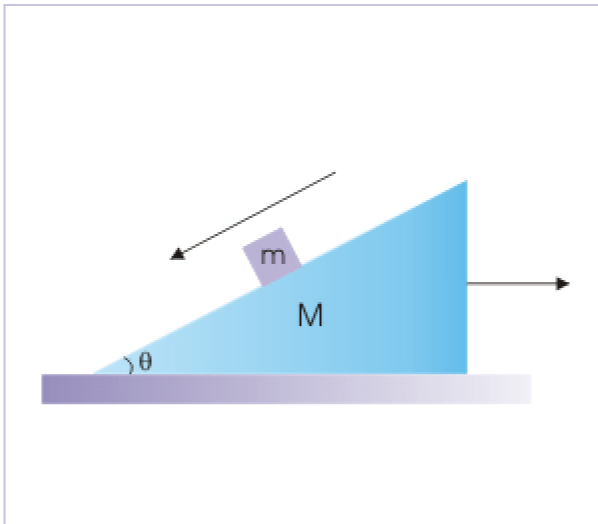
The components of gravitational force and pseudo force form balanced force system so that there is no net component of forces along the incline and hence there is no friction between the interface.

$$mg\sin\theta = ma\cos\theta$$

$$\Rightarrow \tan\theta = \frac{a}{g}$$

$a = g\tan\theta$ towards left

Problem 4 : A block of mass "m" is placed on an incline of angle "θ" and mass "M", which is placed on a horizontal surface. All surfaces are friction - less. Find the acceleration of the block with respect to incline.
Block and incline system

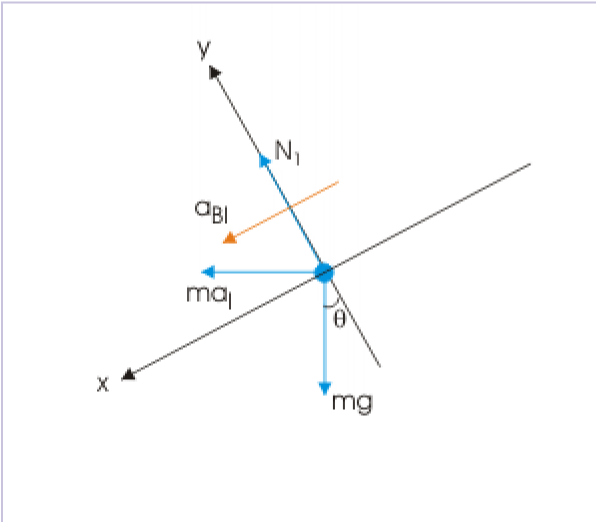


All surfaces are friction-less.

Solution : Here, block and incline both have different accelerations. The block moves down along the incline. The incline, on the other hand, moves right as shown in the figure above. As such, we can not treat two elements as one body and find the their common accelerations with respect to ground.

Let " a_B" , " a_I " and " a_{BI} " respectively denote acceleration of block w.r.t ground, acceleration of the incline w.r.t ground and acceleration of the block w.r.t incline.

We first carry out force analysis on the "block" in the non-inertial frame of incline. Here, the forces on the block are (i) weight of the block, "mg", (ii) normal force due to underlying incline, " N_1 ", and (iii) Pseudo force, " ma_I " applied in the direction opposite to the acceleration of the incline. The free body diagram of the incline in non-inertial reference of incline is shown here.
Free body diagram of block in non-inertial frame



Three forces, including pseudo force, act on the block.

Force analysis on the block yields two relation,

$$\sum F_x = ma_I \cos\theta + mg\sin\theta = ma_{BI}$$

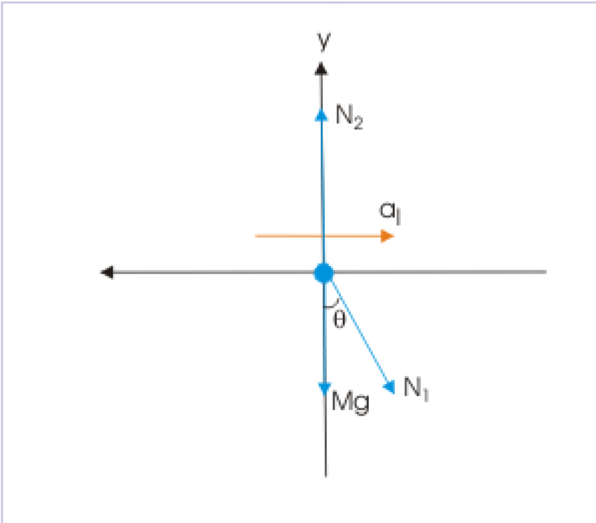
$$\Rightarrow a_{BI} = a_I \cos\theta + g\sin\theta$$

$$\sum F_y = N_1 + ma_I \sin\theta = mg\cos\theta$$

It is clear from the first equation that we need to know " a_I " i.e. the acceleration of incline with respect to ground to determine the value of " a_{BI} ". It is imperative that we carry out force analysis on the "incline" in the inertial frame of reference of ground.

The forces on the incline are (i) weight of the incline, "Mg", (ii) normal force due to overlying block, " N_1 ", and (iii) Normal force due to underneath horizontal surface, " N_2 ". The free body diagram of the incline in inertial ground reference is shown here.

Free body diagram of inline inertial frame



Three forces act on the block.

Force analysis of the incline yields two relations,

$$\sum F_x = N_1 \sin\theta = m a_I$$

$$\sum F_y = N_2 = N_1 \cos\theta + Mg$$

First of the equations is useful that can be combined with the analysis in non-inertial frame of reference. Substituting for " N_1 " in the earlier equation, we have :

$$\Rightarrow \frac{m a_I}{\sin\theta} + m a_I \sin\theta = mg\cos\theta$$

$$\Rightarrow a_I = \frac{(mg\sin\theta\cos\theta)}{\left(M + m\sin^2\theta\right)}$$

We can now determine the required acceleration of the block with respect to incline by substituting the value of " a_I " in the expression of " a_{BI}" :

$$\Rightarrow a_{BI} = \frac{\left(mg\sin\theta\cos{}^2\theta\right)}{\left(M + m\sin{}^2\theta\right)} + g\sin\theta$$

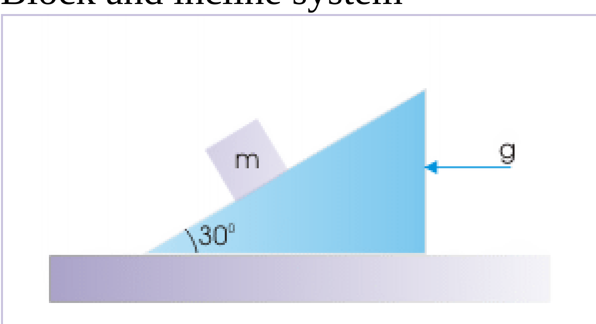
$$\Rightarrow aBI = \frac{\left(mg\sin\theta\cos^2\theta + Mg\sin\theta + m\sin^3\theta\right)}{\left(M + m\sin^2\theta\right)}$$

$$\Rightarrow a_{BI} = \frac{g\sin\theta\left\{M + m\left(\cos^2\theta + \sin^2\theta\right)\right\}}{\left(M + m\sin^2\theta\right)}$$

$$\Rightarrow a_{BI} = \frac{(M + m)g\sin\theta}{\left(M + m\sin{}^2\theta\right)}$$

Problem 5 : A block is placed on an incline, which is accelerated towards left at an acceleration equal to the acceleration due to gravity, as shown in the figure. The angle of incline is 30°. If friction is absent at all contact surfaces, then find the time taken by the block to cover a length of 4 m on the incline.

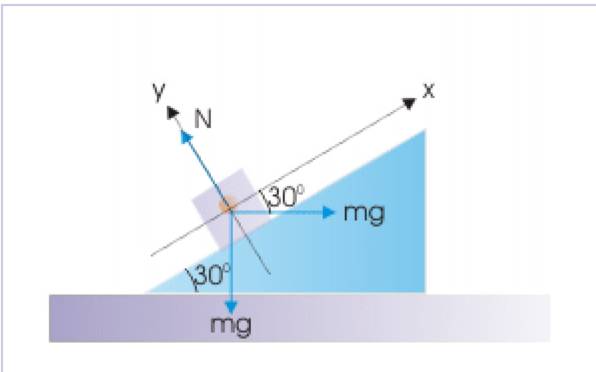
Block and incline system



All surfaces are friction-less.

Solution : Here, initial velocity of block is zero. We need to know the acceleration of the block with respect to incline in order to find the required time to cover a displacement of 4 m along the incline.

Block and incline system



Forces on the block.

Let “a” be the acceleration of the incline with respect to ground and “ a_r ” be the acceleration of the block relative to the incline. We however do not know whether net component of forces parallel to incline is in up or down direction. The free body diagram of the block is superimposed on the block in the figure above. The forces on the block includes pseudo force (mg), acting towards right.

$$F_x = mg\cos\theta - mg\sin\theta$$

$$\Rightarrow F_x = mg\cos 30^0 - mg\sin 30^0$$

$$\Rightarrow F_x = mgX\frac{\sqrt{3}}{2} - mgX\frac{1}{2}$$

$$\Rightarrow F_x = \frac{\left(\sqrt{3}-1\right)mg}{2}$$

Clearly, the net component of forces along the incline is positive. It means that block is moving up. The magnitude of acceleration with respect to incline in upward direction, is :

$$\Rightarrow a_r = \frac{\left(\sqrt{3}-1\right)mg}{2m}$$

$$\Rightarrow a_r = \frac{\left(\sqrt{3}-1\right)g}{2} = 3.66 \quad m/s^2$$

Now, using equation of motion for constant acceleration, we have (initial velocity is zero) :

$$x = \frac{1}{2}at^2$$

$$\Rightarrow 4 = \frac{1}{2}X3.66Xt^2$$

$$\Rightarrow t = \frac{\sqrt{8}}{3.66} = 1.48 \quad s$$

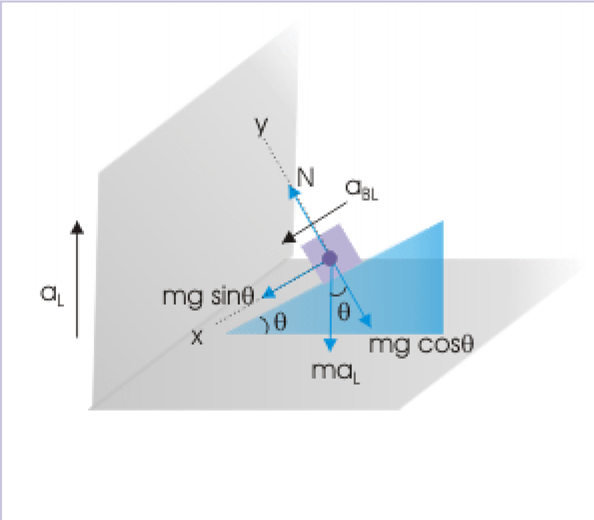
Incline in an accelerated lift

Problem 6 : A block of 2 kg slides down a smooth inclines plane of angle 30° and length 12 m. The incline is fixed at the floor of a lift, which is acceleration up at 2 m/s^2 . Find the time taken by the block to travel from the top to bottom of the incline. Also find the magnitude of acceleration of the block with respect to ground. Consider g = 10 m/s^2 .

Solution : In order to know the time, we need to know the acceleration of the block with respect to incline (not with respect to ground). The block has to travel exactly 12 m in the reference of accelerated lift. The technique of pseudo force, therefore, has distinct advantage over analysis in ground reference as the analysis in accelerated frame will directly give the acceleration of the block with respect to incline. This will allow us to apply equation of motion to find time to cover 12 m in the lift.

Let a_B , a_L and a_{BL} be the acceleration of block w.r.t ground, acceleration of lift (incline) w.r.t ground and acceleration of block w.r.t lift (incline) respectively.

Motion of block on an incline in an accelerated lift



Note that block is acted upon by three forces (i) weight of the block, "mg" (ii) Normal force on the block, "N" and pseudo force, "maL". The axes are drawn parallel and perpendicular to the incline. This is advantageous as acceleration of the block with respect to incline is along the incline. As we have gained experience, free body diagram is not separately drawn. It is shown superimposed on the body system. The figure above also shows the weight of the block resolved along the axes.

$$\sum F_x = mg\sin\theta + ma_L \sin\theta = ma_{\text{BL}}$$
$$\Rightarrow 2 \text{ x } a_{\text{BL}} = 2 \text{ x } 10 \text{ x } \sin 30^0 + 2 \text{ x } 2 \text{ x } \sin 30^0$$
$$\Rightarrow 2a_{\text{BL}} = 2 \text{ x } 10 \text{ x } \frac{1}{2} + 4 \text{ x } \frac{1}{2} = 12$$
$$\Rightarrow a_{\text{BL}} = 6\, m/s^2$$

Now, applying equation of motion,

$$x = ut + \frac{1}{2} a_{\text{BL}}{}^2 = \frac{1}{2} \text{ x } 6 \text{ x } t^2$$
$$\Rightarrow t^2 = \frac{12\text{x}2}{6} = 4$$
$$t = 2\, s$$

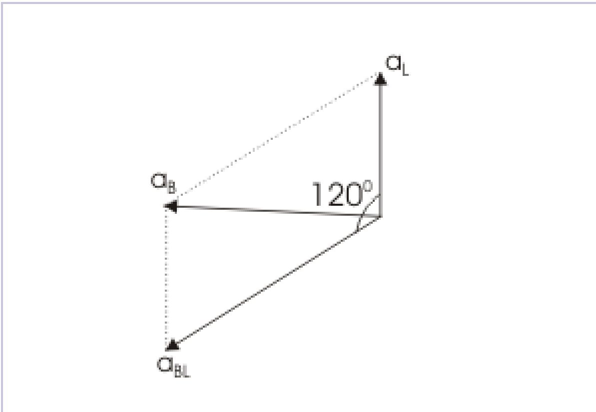
The relative acceleration of the block with respect to incline a_{BL} is related to accelerations of the block and lift w.r.t ground as :

$$\mathbf{a}_{\text{BL}} = \mathbf{a}_B - \mathbf{a}_L$$

The acceleration of block with respect to ground is given as :

$$\Rightarrow \mathbf{a}_B = \mathbf{a}_{\mathrm{BL}} + \mathbf{a}_L$$

Vector addition



Evaluating the right side of the equation,

$$\Rightarrow a_B = \sqrt{(}a_{\mathrm{BL}}{}^2 + a_L{}^2 + 2a_{\mathrm{BL}}a_L \cos 120^0$$
$$\Rightarrow a_B = \sqrt{(}6^2 + 2^2 + 2 \text{ x } 2 \text{ x } 6 \text{ x } -\tfrac{1}{2})$$
$$\Rightarrow a_B = \sqrt{(}36 + 4 - 12) = \sqrt{(}28) = 5.29\, m/s^2$$

Dynamics of circular motion
Uniform circular motion requires a radial force that continuously change its direction, while keeping magnitude constant.

We have already studied the kinematics of circular motion . Specifically, we observed that a force, known as centripetal force, is required for a particle to execute circular motion.

Force is required because the particle executing circular motion needs to change direction continuously. In the case of uniform circular motion (UCM), speed of the particle is constant. The change in velocity is only in terms of change in direction. This needs a force in the radial direction to meet the requirement of the change in direction continuously.

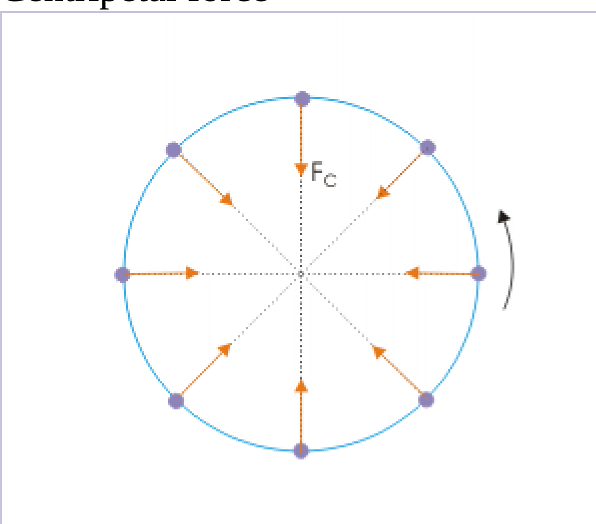
The acceleration is given by :

$$a_r = \frac{v^2}{r} = \omega^2 r$$

The centripetal force is given by :

$$F_C = ma_r = \frac{mv^2}{r} = m\omega^2 r$$

Centripetal force



Uniform circular motion requires a radial force that

continuously change its direction, while keeping magnitude constant.

Centripetal force

The specific requirement of a continuously changing radial force is not easy to meet by mechanical arrangement. The requirement means that force should continuously change its direction along with particle. It is a tall order. Particularly, if we think of managing force by physically changing the mechanism that applies force. Fortunately, natural and many craftily thought out arrangements create situations, in which the force on the body changes direction with the change in the position of the particle - by the very act of motion. One such arrangement is solar system in which gravitational force on the planet is always radial.

Centripetal force is a name given to the force required for circular motion. The net component of external forces which meet this requirement is called centripetal force. In this sense, centripetal force is not a separately existing force. Rather, we should look at this force as component of the external forces on the body in radial direction.

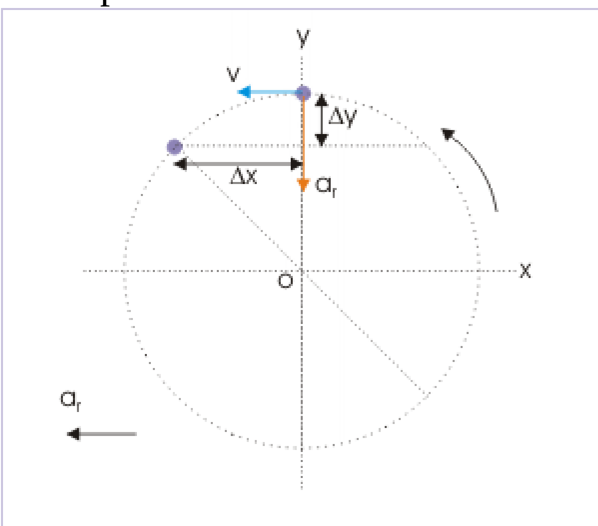
Direction of centripetal force and circular trajectory

There is a subtle point about circular motion with regard to the direction of force as applied on the particle in circular motion. If we apply force on a particle at rest, then it moves in the direction of applied force and not perpendicular to it. In circular motion, the situation is different. We apply force (centripetal) to a particle, which is already moving in a direction perpendicular to the force. As such, the resulting motion from the interaction of motion with external force is not in radial direction, but in tangential direction.

In accordance with Newton's second law of motion, the particle accelerates along the direction of centripetal force i.e. towards center. As such, the particle actually transverses a downward displacement (Δy) with centripetal acceleration; but in the same time, the particle moves sideways (Δx) with constant speed, as the component of centripetal force in the perpendicular direction is zero.

It may sound bizzare, but the fact is that the particle is continuously falling towards the center in the direction of centripetal force and at the same is able to maintain its linear distance from the center, owing to constant side way motion.

Centripetal force



Direction of centripetal force
and circular trajectory

In the figure shown, the particle moves towards center by Δy, but in the same period the particle moves left by Δx. In the given period, the vertical and horizontal displacements are such that resultant displacement finds the particle always on the circle.

$$\Delta x = v \Delta t$$

$$\Delta y = \frac{1}{2} a_r \Delta t^2$$

Force analysis of uniform circular motion

As pointed our earlier, we come across large numbers of motion, where natural setting enables continuous change of force direction with the moving particle. We find that a force meeting the requirement of centripetal force can be any force type like friction force, gravitational force, tension in the string or electromagnetic force. Here, we consider some of the important examples of uniform circular motion drawn from our life experience.

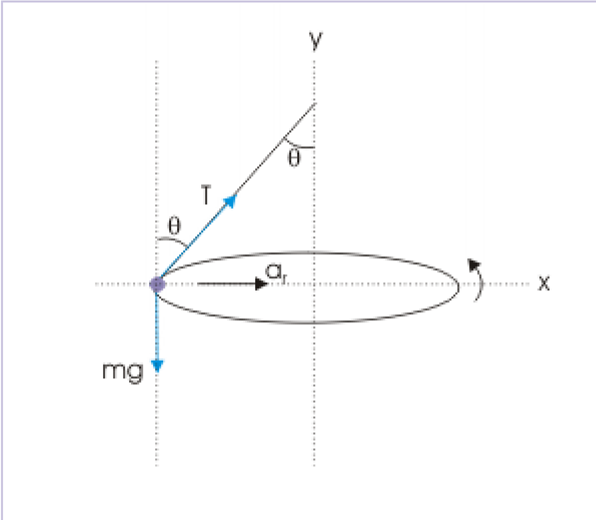
Uniform circular motion in horizontal plane

A particle tied to a string is rotated in horizontal plane by virtue of the tension in the string. The tension in the string provides the centripetal force for uniform circular motion.

We should, however, understand that this force description is actually an approximation, because it does not take into account the downward force due to gravity. As a matter of fact, it is not possible to have a horizontal uniform circular motion (except in the region of zero gravity) by keeping the string in horizontal plane. It is so because, gravitational pull will change the plane of string and the tension in it.

In order that there is horizontal uniform circular motion, the string should be slanted such that the tension as applied to the particle forms an angle with the horizontal plane. Horizontal component of the tension provides the needed centripetal force, whereas vertical component balances the weight of the particle.

Uniform circular motion in horizontal plane



String is not in the plane of circular motion.

$$\sum F_x \Rightarrow T\sin\theta = ma_r = \frac{mv^2}{r}$$

and

$$\sum F_y \Rightarrow T\cos\theta = mg$$

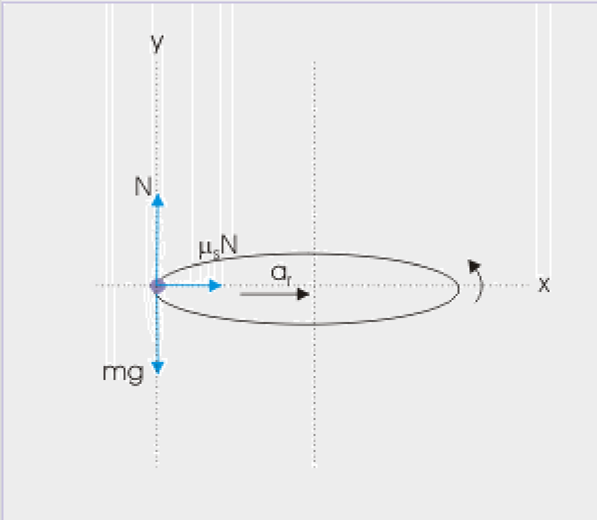
Taking ratio,

$$\Rightarrow \tan\theta = \frac{mv^2}{rg}$$

Example:
Problem : A small boy sits on a horizontal platform of a joy wheel at a linear distance of 10 m from the center. When the wheel exceeds 1 rad/s, the boy starts slipping. Find the coefficient of friction between boy and the platform.
Solution : For boy to be stationary with respect to platform, forces in both vertical and horizontal directions are equal. However, requirement of centripetal force increases with increasing rotational speed. If centripetal

force exceeds the maximum static friction, then boy begins to slip towards the center of the rotating platform.

Horizontal circular motion



In vertical direction,

$$N = mg$$

In horizontal direction,

$$\Rightarrow m\omega^2 r = \mu_s N = \mu_s mg$$
$$\Rightarrow \mu_s = \frac{r\omega^2}{g} = \frac{5x1^2}{10} = 0.5$$

Motion of a space shuttle

A space shuttle moves in a circular path around Earth. The gravitational force between earth and shuttle provides for the centripetal force.

$$mg^{‘} = \frac{mv^2}{r}$$

where g' is the acceleration due to gravity (acceleration arising from the gravitational pull of Earth) on the satellite.

Here, we need to point out an interesting aspect of centripetal force. A person is subjected to centripetal force, while moving in a car and as well when moving in a space shuttle. But the experience of the person in two cases are different. In car, the person experiences (feels) a normal force in the radial direction as applied to a part of the body. On the other hand, a person in the shuttle experiences the "feeling" of weightlessness. Why this difference when body experiences centripetal force in either case?

In the space shuttle, gravity acts on each of the atoms constituting our body and this gravity itself is the provider of centripetal force. There is no push on the body as in the case of car. The body experiences the "feeling" of weightlessness as both space shuttle and the person are continuously falling towards center of Earth. The person is not able to push other bodies. Importantly, gravitational pull or weight of the person is equal to mg' and not equal to zero.

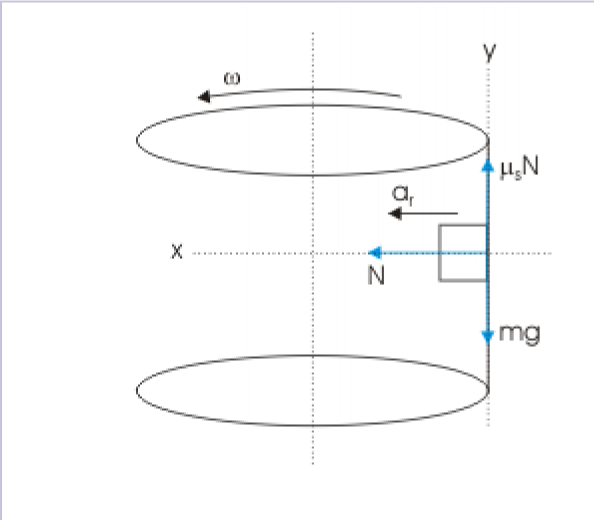
Horizontal circular motion in a rotor

Horizontal rotor holds an object against the wall of a rotating cylinder at a certain angular speed. The object (which could be a person in a fun game arrangement) is held by friction between the surfaces of the object and the cylinder's inside wall. For a given weight of the object, there is a threshold minimum velocity of the rotor (cylinder); otherwise the object will fall down.

The object has a tendency to move straight. As the object is forced to move in a circle, it tends to move away from the center. This means that the object presses the wall of the rotor. The rotor, in turn, applies normal force on the object towards the center of circular path.

$$\sum F_x = N = ma_r = \frac{mv^2}{r}$$

Since friction is linearly related to normal force for a given pair of surfaces (μ_s), it is possible to adjust speed of the rotor such that maximum friction is equal to the weight of the object. In the vertical direction, we have :
Horizontal circular motion in a rotor



As a limiting case, the maximum friction is equal to the weight of the object.

$$\sum F_y = \mu_s N - mg = 0$$
$$\Rightarrow N = \frac{mg}{\mu_s}$$

Combining two equations, we have :

$$\Rightarrow \frac{mv^2}{r} = \frac{mg}{\mu_s}$$
$$\Rightarrow v = \sqrt{\left(\frac{rg}{\mu_s}\right)}$$

This is threshold value of speed for the person to remain stuck with the rotor.

We note following points about the horizontal rotor :

1. The object tends to move away from the center owing to its tendency to move straight.
2. A normal force acts towards center, providing centripetal force
3. Normal force contributes to maximum friction as $F_s = \mu_s N$.

4. Velocity of the rotor is independent of the mass of the object.

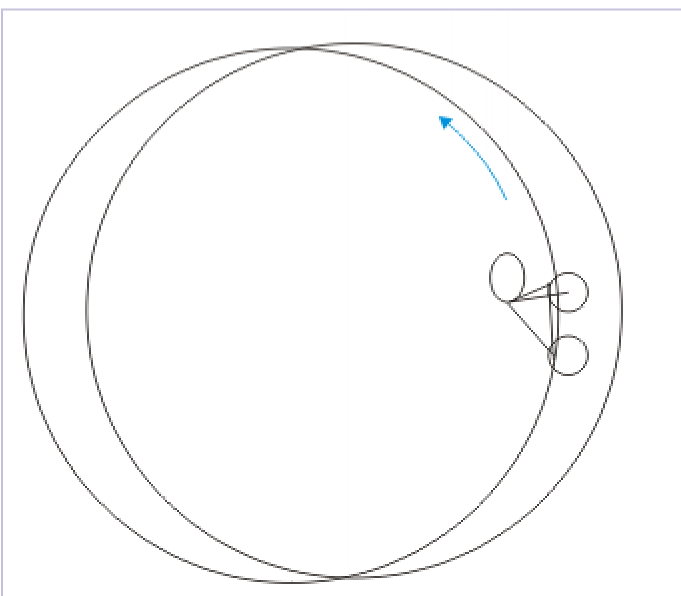
Force analysis of non-uniform circular motion

Motion in vertical loop involves non-uniform circular motion. To illustrate the force analysis, we consider the motion of a cyclist, who makes circular rounds in vertical plane within a cylindrical surface by maintaining a certain speed.

Vertical circular motion

In the vertical loop within a hallow cylindrical surface, the cyclist tends to move straight in accordance with its natural tendency. The curvature of cylinder, however, forces the cyclist to move along circular path (by changing direction). As such, the body has the tendency to press the surface of the cylindrical surface. In turn, cylindrical surface presses the body towards the center of the circular path.

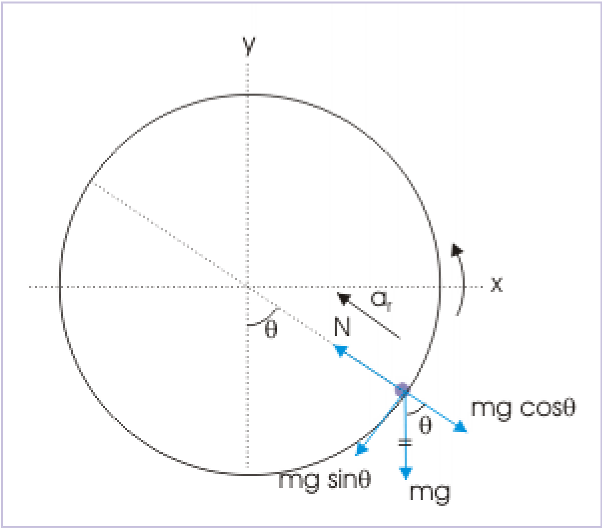
Vertical circular motion



The cyclist executes vertical circular motion along the cylindrical surface.

The free body diagram of the cyclist at an angle “θ” is shown in the figure. We see that the resultant of normal force and component of weight in the radial direction meets the requirement of centripetal force in radial direction,

Vertical circular motion



Force diagram

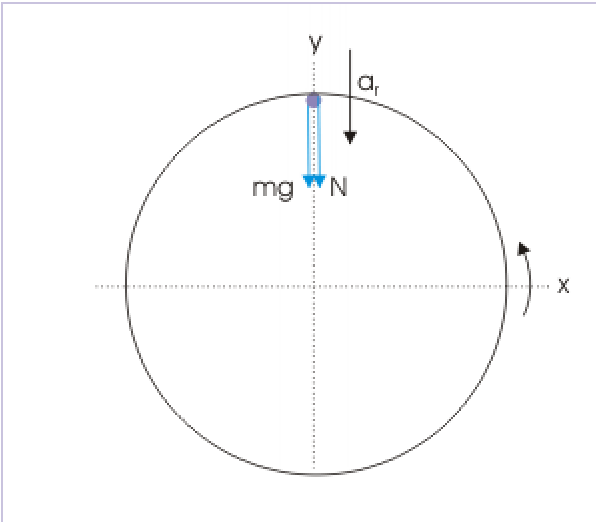
$$N - mg\cos\theta = \frac{mv^2}{r}$$

The distinguishing aspects of circular motion in vertical plane are listed here :

1. Motion in a vertical loop is a circular motion – not uniform circular motion. It is so because there are both radial force (N – mg cosθ) and tangential force (mg sin θ). Radial force meets the requirement of centripetal force, whereas tangential force accelerates the particle in the tangential direction. As a result, the speed of the cyclist decreases while traveling up and increases while traveling down.
2. Centripetal force is not constant, but changing in magnitude as the speed of the cyclist is changing and is dependent on the angle “θ”.

The cyclist is required to maintain a minimum speed to avoid free fall. The possibility of free fall is most stringent at the highest point of the loop. We, therefore, analyze the motion at the highest point with the help of the free body diagram as shown in the figure.

Vertical circular motion



Force diagram at the top

$$N + mg = \frac{mv^2}{r}$$

> **Note:** We can also achieve the result as above by putting the value θ=180° in the equation obtained earlier.

Th minimum speed of the cyclist corresponds to the situation when normal force is zero. For this condition,

$$mg = \frac{mv^2}{r}$$
$$v = \sqrt{(rg)}$$

Vertical motion of a particle attached to a string

This motion is same as discussed above. Only difference is that tension of the string replaces normal force in this case. The force at the highest point is given as :

$$T + mg = \frac{mv^2}{r}$$

Also, the minimum speed for the string not to slack at the highest point (T = 0),

$$mg = \frac{mv^2}{r}$$
$$v = \sqrt{(rg)}$$

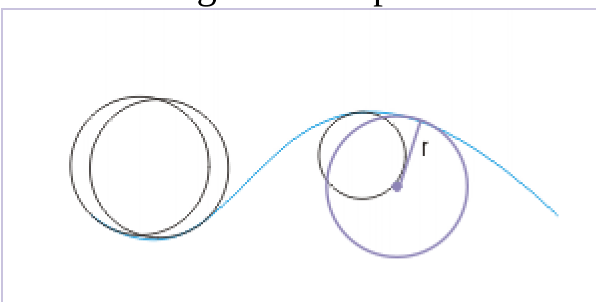
The complete analysis of circular motion in vertical plane involves considering forces on the body at different positions. However, external forces depend on the position of the body in the circular trajectory. The forces are not constant forces as in the case of circular motion in horizontal plane.

We shall learn subsequently that situation involving variable force is best analyzed in terms of energy concept. As such, we will revisit vertical circular motion again after studying different forms of mechanical energy.

Motion along curved path

We all experience motion along a curved path in our daily life. The motion is not exactly a circular motion. However, we can think of curved path as a sequence of circular motions of different radii. As such, motion of vehicles like that of car, truck etc, on a curved road can be analyzed in terms of the dynamics of circular motion. Clearly, analysis is done for the circular segment with the smallest radius as it represents the maximum curvature. It must be noted that "curvature" and "radius of curvature" are inverse to each other.

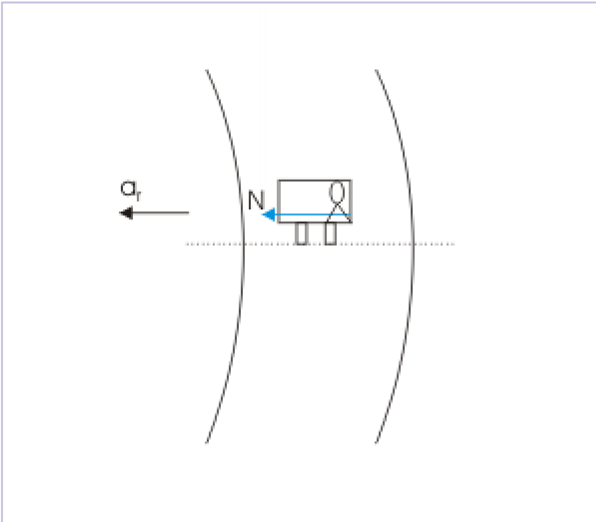
Motion along a curved path



One common experience, in this respect, is the experience of a car drive, which is negotiating a sharp turn. If a person is sitting in the middle of the back seat, she/he holds on the fixed prop to keep the posture steady and move along with the motion of the car. If the person is close to the farther side (from the center of motion) of the car, then she/he leans to the side of the car to become part of the motion of the car.

In either of the two situations, the requirement of centripetal force for circular motion is fulfilled. The bottom of the body is in contact with the car and moves with it, whereas the upper part of the body is not. When she/he leans on the side of the car away from the center of motion, the side of the car applies normal force, which meets the requirement of centripetal force. Finally, once the requirement of centripetal force is met, the complete body is in motion with the car.

Motion along a curved path



The person presses by side of the car.

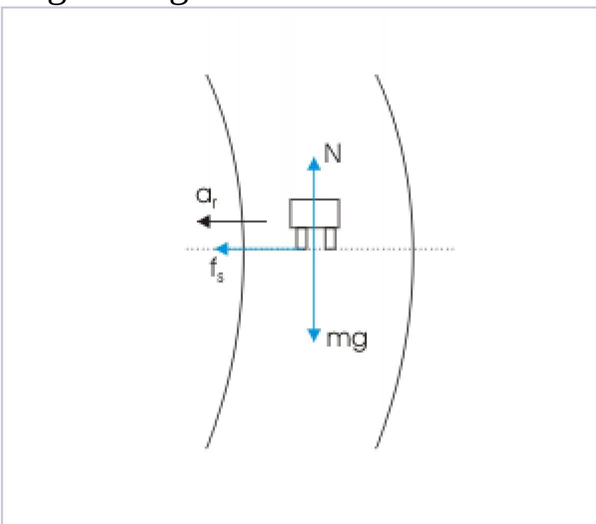
The direction in which the body responds to curved motion is easy to find by just thinking what would be the natural direction of motion of the free part of the body. In the example discussed above, the body seeks to move straight, but the lower part in contact with car moves along curved path having side way component of motion (towards center). The result is that the upper part is away from the center of curvature of the curved path. In order to keep the body upright an external force in the radial direction is required to be applied on the body.

Negotiating a level curve

Let us concentrate on what is done to turn a car to negotiate a sharp turn. We realize that the driver of a car simply guides the steering of the car to move it along the turn. Intuitively, we think that the car engine must be responsible for meeting the requirement of centripetal force. This is right. However, engine does not directly apply force to meet the requirement of centripetal force. It is actually the friction, resulting from the motion caused by engine, which acts towards the center of circular path and meets the requirement of centripetal force.

The body of the car tends to move straight in accordance with its natural tendency? Now, as wheel is made to move side way (by the change in direction), the wheel has the tendency to have relative motion with respect to the road in the direction away from the center of path. In turn, road applies friction, which is directed towards the center.

Negotiating a level curve



The friction meets the requirement of centripetal force.

Important to note here is that we are not considering friction in the forward or actual direction of motion, but perpendicular to actual direction (side way). There is no motion in the side way direction, if there is no side way skidding of the car. In that case, the friction is static friction (f_s) and has not exceeded maximum or limiting friction (F_s). Thus,

$$f_s \le F_s$$
$$\Rightarrow f_s \le \mu_s N$$

There is no motion in vertical direction. Hence,

$$N = mg$$

Combining two equations, we have :

$$\Rightarrow f_s \le \mu_s mg$$

where “m” is the combined mass of the car and the passengers.

In the limiting situation, when the car is about to skid away, the friction force is equal to the maximum static friction, meeting the requirement of centripetal force required for circular motion,

$$\Rightarrow \frac{mv^2}{r} = \mu_s mg$$
$$\Rightarrow v = \sqrt{(\mu_s rg)}$$

We note following points about the condition as put on the speed of the car :

- There is a limiting or maximum speed of car to ensure that the car moves along curved path without skidding.
- If the limiting condition with regard to velocity is not met ($v \le \sqrt{(\mu_s rg)}$), then the car will skid away.
- The limiting condition is independent of the mass of the car.
- If friction between tires and the road is more, then we can negotiate a curved path with higher speed and vice-versa. This explains why we drive slow on slippery road.
- Smaller the radius of curvature, smaller is the limiting speed. This explains why sharper turn (smaller radius of curvature) is negotiated with smaller speed.

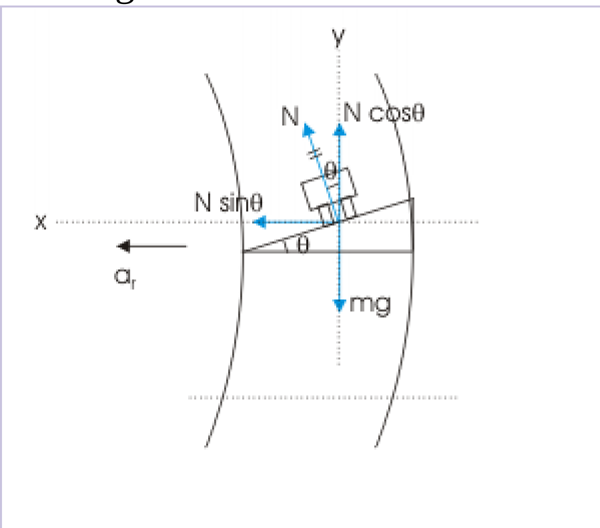
Banking of roads

There are three additional aspects of negotiating a curve. First, what if, we want to negotiate curve at a higher speed. Second, how to make driving safer without attracting limiting conditions as tires may have been flattened (whose grooves have flattened), or friction may decrease due to any other reasons like rain or mud. Third, we want to avoid sideway friction to prolong life of the tires. The answer to these lie in banking of the curved road.

For banking, one side of the road is elevated from horizontal like an incline or wedge. In this case, the component of normal force in horizontal

direction provides the centripetal force as required for the motion along the curved path. On the other hand, component of normal force in vertical direction balances the weight of the vehicle. It is clear that magnitude of normal reaction between road and vehicle is greater than the weight of the vehicle.

Banking of roads



The horizontal component of normal force meets the requirement of centripetal force.

Here, we compute the relation between the angle of banking (which is equal to the angle of incline) for a given speed and radius of curvature as :

$$N\cos\theta = mg$$

and

$$N\sin\theta = \frac{mv^2}{r}$$

Taking ratio,

$$\Rightarrow \tan\theta = \frac{v^2}{rg}$$

$$\Rightarrow v = \sqrt{(rg\tan\theta)}$$

This expression represents the speed at which the vehicle does not skid (up or down) along the banked road for the given angle of inclination (θ). It means that centripetal force is equal to the resultant of the system of forces acting on the vehicle. Importantly, there is no friction involved in this consideration for the circular motion of the vehicle.

Example:

Problem : An aircraft hovers over a city awaiting clearnace to land. The aricraft circles with its wings banked at an angle $\tan^{-1}(0.2)$ at a speed of 200 m/s. Find the radius of the loop.

Solution : The aircraft is banked at an angle with horizontal. Since aircraft is executing uniform circular motion, a net force on the aircraft should act normal to its body. The component of this normal force in the radial direction meets the requirement of centripetal force, whereas vertical component balances the weight of aircraft. Thus, this situation is analogous to the banking of road.

$$\Rightarrow \tan\theta = \frac{v^2}{rg}$$

$$\Rightarrow r = \frac{v^2}{g\tan\theta}$$

$$\Rightarrow r = \frac{200^2}{10\text{x}0.2} = 20000\ m$$

Role of friction in banking

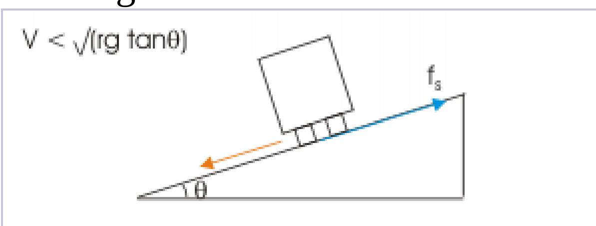
The moot question is whether banking of road achieves the objectives of banking? Can we negotiate the curve with higher speed than when the road is not banked? In fact, the expression of speed as derived in earlier section gives the angle of banking for a particular speed. It is the speed for which

the component of normal towards the center of circle matches the requirement of centripetal force.

If speed is less than that specified by the expression, then vehicle will skid "down" (slip or slide) across the incline as there is net force along the incline of the bank. This reduces the radius of curvature i.e. "r" is reduced - such that the relation of banking is held true :

$$\Rightarrow v = \sqrt{(rg \tan \theta)}$$

Banking of road



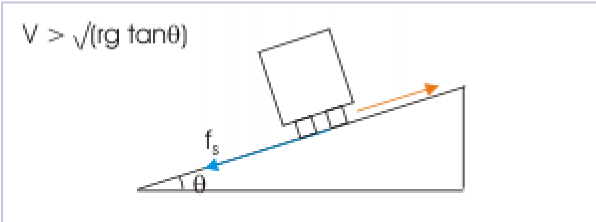
The vehicle has tendency to skid down.

In reality, however, the interacting surfaces are not smooth. We can see that if friction, acting "up" across the bank, is sufficient to hold the vehicle from sliding down, then vehicle will move along the circular path without skidding "down".

What would happen if the vehicle exceeds the specified speed for a given angle of banking? Clearly, the requirement of centripetal force exceeds the component of normal force in the radial direction. As such the vehicle will have tendency to skid "up" across the bank.

Again friction prevents skidding "up" of the vehicle across the bank. This time, however, the friction acts downward across the bank as shown in the figure.

Banking of road



The vehicle has tendency to skid up.

In the nutshell, we see that banking helps to prevent skidding "up" across the bank due to the requirement of centripetal force. The banking enables component of normal force in the horizontal direction to provide for the requirement of centripetal force up to a certain limiting (maximum) speed. Simultaneously, the banking induces a tendency for the vehicle to skid "down" across the bank.

On the other hand, friction prevents skidding "down" as well as skidding "up" across the bank. This is possible as friction changes direction opposite to the tendency of skidding either "up" or "down" across the bank. The state of friction is summarized here :

1: $v = 0; \quad f_S = mg \sin\theta, \quad \text{acting up across the bank}$

2: $v = \sqrt{rg \tan\theta}; \quad f_S = 0$

3: $v < \sqrt{rg \tan\theta}; \quad f_S > 0, \quad \text{acting up across the bank}$

4: $v > \sqrt{rg \tan\theta}; \quad f_S > 0, \quad \text{acting down across the bank}$

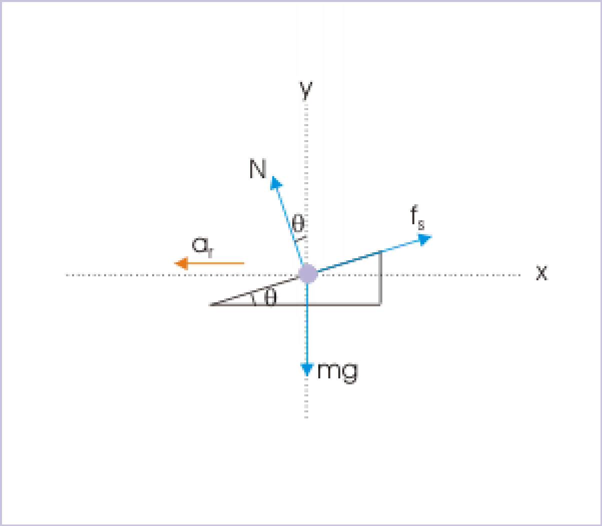
Friction, therefore, changes its direction depending upon whether the vehicle has tendency to skid "down" or "up" across the bank. Starting from zero speed, we can characterize friction in following segments (i) friction is equal to the component of weight along the bank," $\sqrt{mg \sin\theta}$ ", when vehicle is stationary (ii) friction decreases as the speed increases (iii) friction becomes zero as speed equals " $\sqrt{rg \tan\theta}$ " (iv) friction changes

direction as speed becomes greater than " $\sqrt{rg\tan\theta}$ " (v) friction increases till the friction is equal to limiting friction as speed further increases and (vi) friction becomes equal to kinetic friction when skidding takes place.

Skidding down the bank

In this case, the velocity of the vehicle is less than threshold speed " $\sqrt{rg\tan\theta}$ ". Friction acts "up" across the bank. There are three forces acting on the vehicle (i) its weight "mg" (ii) normal force (N) due to the bank surface and (iii) static friction " f_s ", acting up the bank. The free body diagram is as shown here.

Banking of road



The speed is less than the threshold speed.

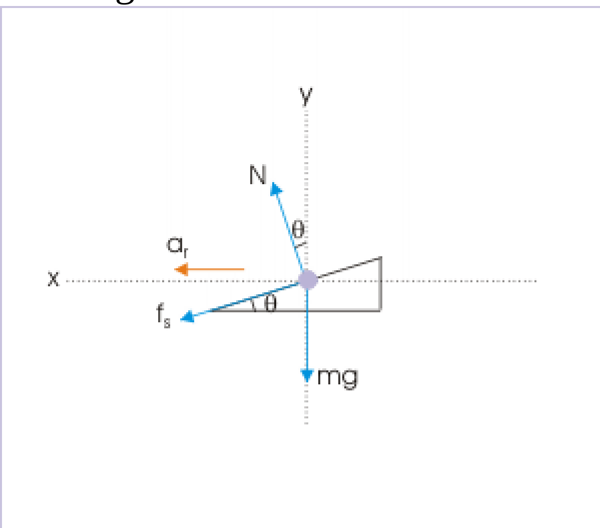
$$\sum F_x \Rightarrow N\sin\theta - f_S\cos\theta = \frac{mv^2}{r}$$

$$\sum F_y \Rightarrow N\cos\theta + f_S\sin\theta = mg$$

Skidding up the bank

In this case, the velocity of the vehicle is greater than threshold speed. Friction acts "down" across the bank. There are three forces acting on the vehicle (i) its weight "mg" (ii) normal force (N) due to the bank surface and (iii) static friction " f_s ", acting down the bank. The free body diagram is as shown here.

Banking of road



The speed is greater than the threshold speed.

$$\sum F_x \Rightarrow N \sin\theta + f_S \cos\theta = \frac{mv^2}{r}$$

$$\sum F_y \Rightarrow N \cos\theta - f_S \sin\theta = mg$$

Maximum speed along the banked road

In previous section, we discussed various aspects of banking. In this section, we seek to find the maximum speed with which a banked curve can be negotiated. We have seen that banking, while preventing upward

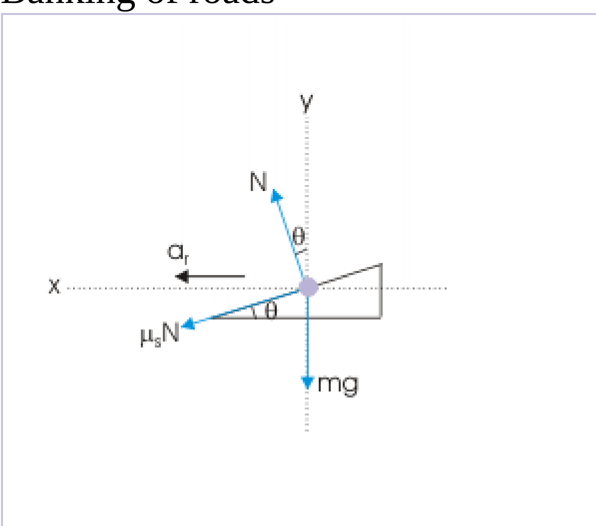
skidding, creates situation in which the vehicle can skid downward at lower speed.

The design of bank, therefore, needs to consider both these aspects. Actually, roads are banked with a small angle of inclination only. It is important as greater angle will induce tendency for the vehicle to overturn. For small inclination of the bank, the tendency of the vehicle to slide down is ruled out as friction between tyres and road is usually much greater to prevent downward skidding across the road.

In practice, it is the skidding "up" across the road that is the prime concern as threshold speed limit can be breached easily. The banking supplements the provision of centripetal force, which is otherwise provided by the friction on a flat road. As such, banking can be seen as a mechanism either (i) to increase the threshold speed limit or (ii) as a safety mechanism to cover the risk involved due to any eventuality like flattening of tyres or wet roads etc. In fact, it is the latter concern that prevails.

In the following paragraph, we set out to determine the maximum speed with which a banked road can be negotiated. It is obvious that maximum speed corresponds to limiting friction that acts in the downward direction as shown in the figure.

Banking of roads



The horizontal component of normal force and friction

together meet the requirement
of centripetal force.

Force analysis in the vertical direction :

$$N\cos\theta - \mu_s N\sin\theta = mg$$
$$\Rightarrow N(\cos\theta - \mu_s\sin\theta) = mg$$

Force analysis in the horizontal direction :

$$N\sin\theta + \mu_s N\cos\theta = \frac{mv^2}{r}$$
$$\Rightarrow N\left(\sin\theta + \mu_s\cos\theta\right) = \frac{mv^2}{r}$$

Taking ratio of two equations, we have :

$$\Rightarrow \frac{g(\sin\theta+\mu_s\cos\theta)}{(\cos\theta-\mu_s\sin\theta)} = \frac{v^2}{r}$$
$$\Rightarrow v^2 = rg\frac{(\sin\theta+\mu_s\cos\theta)}{(\cos\theta-\mu_s\sin\theta)}$$
$$\Rightarrow v = \sqrt{\left\{rg\frac{(\tan\theta+\mu_s)}{(1-\mu_s\tan\theta)}\right\}}$$

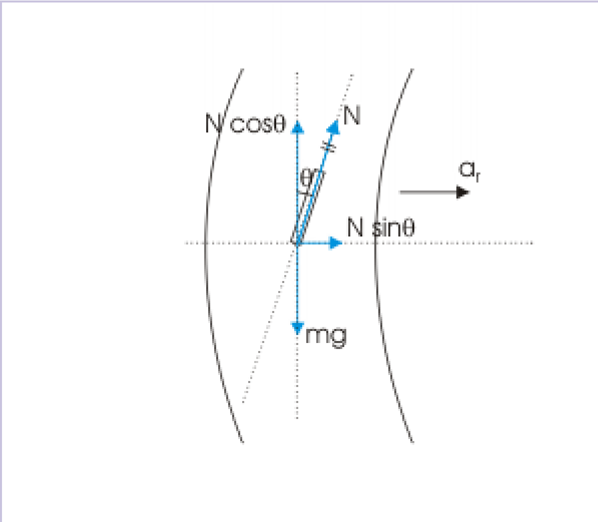
Bending by a cyclist

We have seen that a cyclist bends towards the center in order to move along a circular path. Like in the case of car, he could have depended on the friction between tires and the road. But then he would be limited by the speed. Further, friction may not be sufficient as contact surface is small. We can also see "bending" of cyclist at greater speed as an alternative to banking used for four wheeled vehicles, which can not be bent.

The cyclist increases speed without skidding by leaning towards the center of circular path. The sole objective of bending here is to change the direction and magnitude of normal force such that horizontal component of

the normal force provides for the centripetal force, whereas vertical component balances the "cycle and cyclist" body system.

Banking of roads



The horizontal component of normal force meets the requirement of centripetal force.

$$N \cos \theta = mg$$

$$N \sin \theta = \frac{mv^2}{r}$$

Taking ratio,

$$\Rightarrow \tan \theta = \frac{v^2}{rg}$$
$$\Rightarrow v = \sqrt{(rg \tan \theta)}$$

Work

Work is a general term that we use in our daily life to assess execution or completion of a task. The basic idea is to define a quantity that can be used to determine both "effort" and "result". In physics also, the concept of work follows the same basic idea. But, it is completely "physical" in the sense that it recognizes only force as the "effort" and only displacement as the "result". There is no recognition of mental or any other effort that does not involve physical movement of a body.

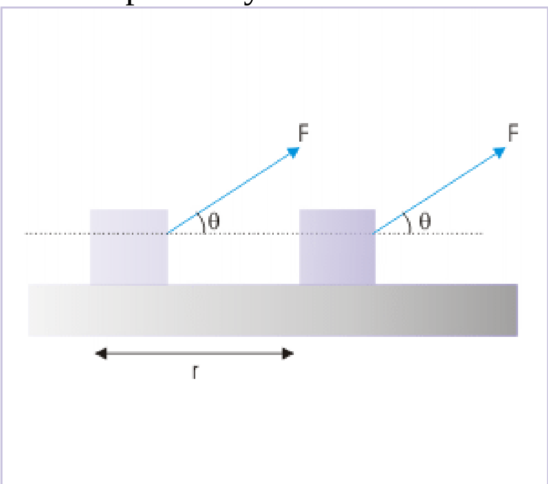
For a constant force (F) applied on a particle, work is defined as the product of "component of force along the direction of displacement" and "the magnitude of displacement of the particle". Mathematically,

$$W = (F\cos\theta)r = Fr\cos\theta$$

where "θ" is angle between force and displacement vectors and "r" is magnitude of displacement. The scalar component of force is also known as the projection of force. SI unit of work is Newton - meter (N-m). This is equivalent to the unit of energy kg $- m^2 s^{-2}$ i.e. Joule (J).

In order to appreciate the working of the formula to compute work, let us consider an example. A block is being pulled by an external force "F" on a smooth horizontal plane as shown in the figure. The work (W) by force (F) is :

A block pulled by external force



External force "F" is a constant force.

$$W = Fr\cos\theta$$

The perpendicular forces i.e. normal force "N" and weight of block "mg" (not shown in the figure) do "no work" on the block as cosθ = cos90° = 0. An external force does the maximum work when it is applied in the direction of displacement. In that case, θ = 0°, cos0° = 1 (maximum) and Work $W_G = Fr$.

Work as vector dot product

Work involves two vector quantities force and displacement, but work itself is a signed scalar quantity. Vector algebra provides framework for such multiplication of vectors, yielding scalar result via multiplication known as dot product. The work done by the constant force as dot product is :

$$W = \mathbf{F}.\mathbf{r} = Fr\cos\theta$$

where "θ" is the angle between force and displacement vectors.

Computation of work

Sign of work

Work is a signed scalar quantity. It means that it can be positive or negative depending on the value of angle between force and displacement. We shall discuss the significance of the sign of work in a separate module. We should, however, be aware that the sign of work has specific meaning for the body on which force works. The sign determines the direction of energy exchange taking place between the body and its surrounding.

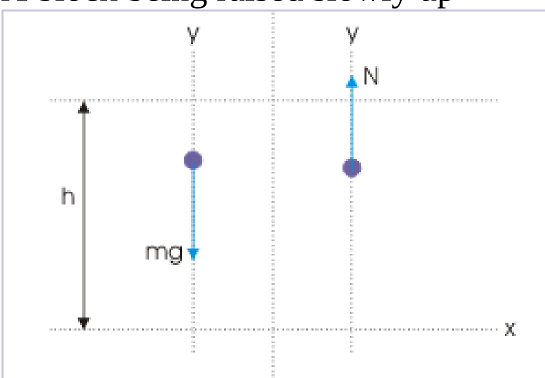
It is clear that the value of "cosθ" decides the sign of work. However, there is an easier method to determine sign of work. We determine the magnitude of work considering projection of force and displacement - without any consideration of the sign. Once magnitude is calculated, we simply check whether the component of force and displacement are in same direction or in opposite direction? If they are in opposite direction, then we put a negative sign before the magnitude of work.

Work by the named force

A body like the block on an incline is subjected to many forces viz weight, friction, normal force and other external forces. Which of the forces do work? Does the work is associated with net force or any of the forces mentioned? In physics, we can relate work with any force or the net force working on the body. The only requirement is that we should mention the force involved. For this reason, we may be required to calculate work by any of the named forces.

It is, therefore, a good idea that we specify the force(s) that we have considered in the calculation of work. To appreciate this point, we consider a block being slowly raised vertically by hand to a height "h" as shown in the figure. As the block is not accelerated, the normal force applied by the hand is equal to the weight of the body :

A block being raised slowly up



Work done by normal force and gravity are different.

$$N = mg$$

For gravity (gravitational force due to Earth), the force and displacement are opposite. Hence, work by gravity is negative.

The work done by gravity is :

$$\Rightarrow W_G = -(mg)h = -mgh$$

For normal force applied by the hand on the block, force and displacement are both in the same direction. Hence, work done is positive :

The work by the hand is :

$$\Rightarrow W_H = N \text{ X } h = mgh$$

Thus, two works are though equal in magnitude, but opposite in sign. It can be easily inferred from the example here that work is positive, if both displacement and component of force along displacement are in the same direction; otherwise negative. It is also pertinent to mention that a subscripted notation for work as above is a good practice to convey the context of work. Finally, we should also note that net force on the body is zero. Hence, work by net force is zero - though works by individual forces are not zero.

Examples

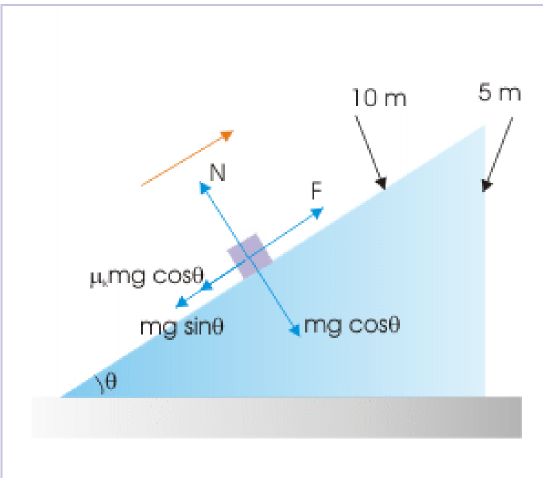
In the discussion above, we have made two points : (i) the sign of work can be evaluated either evaluating "cosθ" or by examining the relative directions of the component of force and displacement and (ii) work is designated to named force. Here, we select two examples to illustrate these points. First example shows computation of work by friction - one of the forces acting on the body. The determination of sign of work is based on evaluation of cosine of angle between force and displacement. Second example shows computation of work by gravity. The determination of sign is based on relative comparison of the directions of the component of force and displacement.

Evaluation of cosine of angle

Problem 1 : A block of 2 kg is brought up from the bottom to the top along a rough incline of length 10 m and height 5 m by applying an external force parallel to the surface. If the coefficient of kinetic friction between surfaces is 0.1, find work done by the friction during the motion. (consider, g = 10 m/s^2).

Solution : We see here that there are four forces on the block : (i) weight (ii) normal force (iii) friction and (iv) force, "F" parallel to incline. The magnitude of external force is not given. We are, however, required to find work by friction. Thus, we need to know the magnitude of friction and its direction. As the block moves up, kinetic friction acts downward. Here, displacement is equal to the length of incline, which is 10 m.

Motion on a rough incline



The forces on the incline

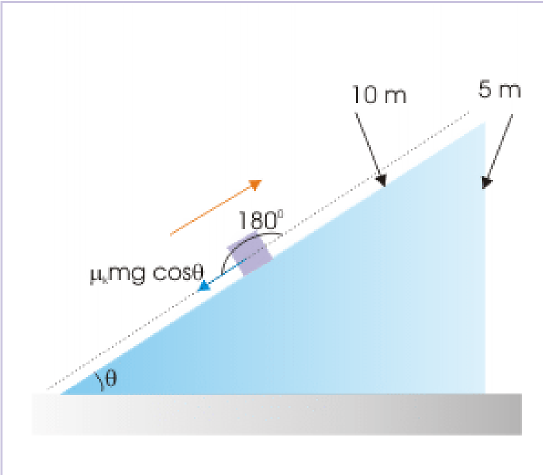
From the figure, it is clear that friction force is given as :

$$F_k = \mu_k N = \mu_k mg\cos\theta$$
$$\text{Here,}\ \ \sin\theta = \frac{5}{10}$$
$$\text{and}\ \ \cos\theta = \sqrt{\left(1 - \sin^2\theta\right)}$$
$$\Rightarrow F = 0.1\ \text{x}\ 2\ \text{x}\ 10\ \text{x}\ \sqrt{\left(1 - \frac{5^2}{10^2}\right)} = 2\ \text{x}\ 0.866 = 1.732\ N$$

To evaluate work in terms of "Frcosφ", we need to know the angle between force and displacement. In this case, this angle is 180° as shown in the figure below.

Note: We denote "φ" instead of "θ" as angle between force and displacement to distinguish this angle from the angle of incline.

Motion on a rough incline



The angle between friction and displacement

$$W = Fr\cos\varphi$$

$$\Rightarrow W_F = Fr\cos\varphi = 1.732 \text{ x } 10 \text{ x } \cos 180^0$$
$$\Rightarrow W_F = 1.732 \text{ x } 10 \text{ x } (-1) = -17.32\ J$$

This example brings out the concept of work by named force (friction). The important point to note here is that we could calculate work by friction even though we were not knowing the magnitude of force of external force, "F". Yet another point to note here is that computation of work by friction is actually independent of - whether block is accelerated or not? In addition, this example illustrates how the evaluation of the cosine of angle between force and displacement determines the sign of work.

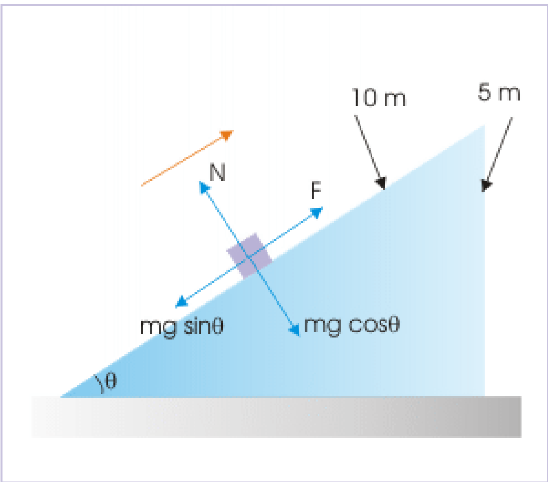
Relative comparison of directions

Problem 2 : A block of 2 kg is brought up from the bottom to the top along a smooth incline of length 10 m and height 5 m by applying an external force parallel to the surface. Find work done by the gravity during the motion. (consider, g = 10 m/s^2).

Solution : In this problem, incline is smooth. Hence, there is no friction at the contact surface. Now, the component of gravity (weight) along the direction of the

displacement is :

Motion on a smooth incline



Forces on the block (except external force)

$$\Rightarrow mg\sin\theta = 2 \text{ x } 10 \text{ x } \frac{5}{10} = 10\ N$$

We, now, determine the magnitude of work without taking into consideration of sign of work. The magnitude of work by gravity is :

$$\Rightarrow W_G = mg\sin\theta \text{ x } r = 10 \text{ x } 10 = 100\ J$$

Once magnitude is calculated, we compare the directions of the component of force and displacement. We note here that the component of weight is in the opposite direction to the displacement. Hence, work by gravity is negative. As such, we put a negative sign before magnitude.

$$\Rightarrow W_g = -100\ J$$

It is clear that this second method of computation is easier of two approaches. One of the simplifying aspect is that we need to calculate cosine of only acute angle to determine the magnitude of work without any concern about directions. We assign sign, subsequent to calculation of the magnitude of work.

Work in three dimensions

We can extend the concept of work to motion in three dimensions. Let us consider three dimensional vector expressions of force and displacement :

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

and

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The work as dot product of two vectors is :

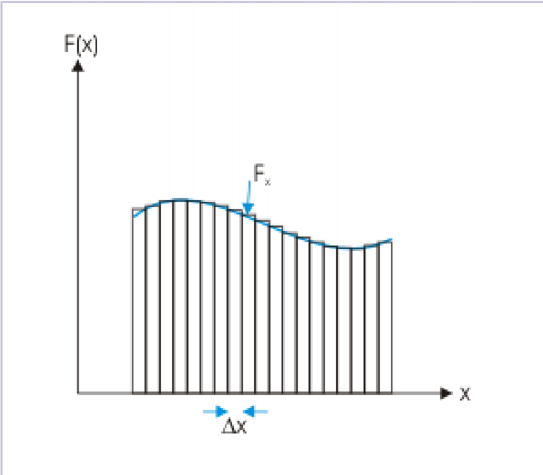
$$W = \mathbf{F}.\mathbf{r} = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}).(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$
$$W = F_x x + F_y y + F_z z$$

Form the point of view of computing work, we can calculate "work" as the sum of the products of scalar components of force and displacement in three mutually perpendicular directions along the axes with appropriate sign. Since respective components of force and displacement are along the same direction, we can determine work in each direction with appropriate sign. Finally, we compute their algebraic sum to determine work by force, "F".

Work by a variable force

We have defined work for constant force. This condition of constant force is, however, not a limitation as we can use calculus to compute work by a variable force. In order to keep the derivation simple, we shall consider force and displacement along same straight line or direction. We have already seen that calculation of work in three dimensional case is equivalent to calculation of work in three mutually perpendicular directions.

A variable force can be approximated to be a series of constant force of different magnitude as applied to the particle. Let us consider that force and displacement are in the same x-direction.
work by variable force



Work is given by the area under the plot.

For a given small displacement (Δx), let

$$F_x$$

be the constant force. Then, the small amount of work for covering a small displacement is :

$$\Delta W = F_x \Delta x$$

We note that this is the area of the small strip as shown in the figure above. The work by the variable force over a given displacement is equal to sum of all such small strips,

$$W = \sum \Delta W = \sum F_x \Delta x$$

For better approximation of the work by the variable force, the strip is made thinner as Δx-->0, whereas the number of stips tends to be infinity. For the limit,

$$W = \lim_{\Delta x \to 0} \sum F_x \Delta x$$

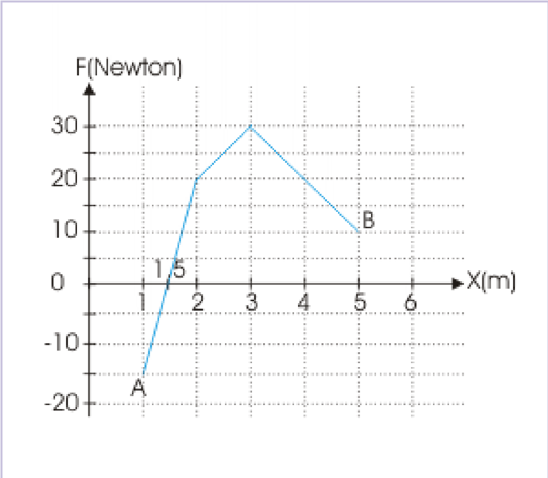
This limit is equal to the area of the plot defined by the integral of force function F(x) between two limits,

$$W = \int_{x_1}^{x_2} F\left(x\right) đx$$

Example

Problem 3 : A particle moves from point A to B along x - axis of a coordinate system. The force on the particle during the motion varies with displacement in x-direction as shown in the figure. Find the work done by the force.

Work by variable force



Force - displacement plot.

Solution : The work done by the force is :

$$W = \int_{x_1}^{x_2} F\left(x\right) đx = \text{Area between plot and x-axis within the limits}$$

Now, the area is :

$$W = -\frac{1}{2} \text{ x } 0.5 \text{ x } 15 + \frac{1}{2} \text{ x } 0.5 \text{ x } 20 + \frac{1}{2} \text{ x } 1 \text{ x } \left(30 + 20\right) + \frac{1}{2} \text{ x } 2 \text{ x } \left(30 + 10\right)$$

$$W = -3.75 + 5 + 25 + 40 = 66.25\ J$$

Acknowledgment

Author wishes to thank Keith for making suggestion to remove calculation error in the module.

Work and energy

Work is an important concept in analyzing physical process. We must know that we have actually made transition to a different analysis framework than that of force, which is based on Newton's laws of motion. What it means that our analysis will ,now, be governed by new set of laws - other than that of laws of motion. Before we discuss details of these laws, we need to first establish new concepts that consitute these laws. Work and energy are two important concepts that will be largely employed for building this new analysis technique.

We have already learned a bit about work. Therefore, we shall introduce the concept of energy first and then discuss relevance of work in the context of energy. Subsequently, we shall accomplish the important step to connect the concept of work with that of energy through a theorem known as "work - kinetic energy theorem" in a separate module.

Energy

Energy is one of two basic quantities (besides mass) that constitute this universe. Because of the basic nature and generality involved with energy, it is difficult to propose an explicit definition of energy, which is meaningful in all situations. We have two options either (i) we define energy a bit vaguely for all situations or (ii) we define energy explicitly in limited context. For all situations, we can say that energy is a quantity that measures the "state of matter".

There are wide verities or forms of energy. The meaning of energy in particular context is definitive. For example, kinetic energy, which is associated with the motion (speed) of a particle, has a mathematical expression to compute it precisely. Similarly, energy has concise and explicit meaning in electrical, thermal, chemical and such specific contexts.

We need to clarify here that our study of energy and related concepts presently deals with particle or particle like objects such that particles composing the object have same motion. We shall extend these concepts

subsequently to group of particles and situation where particles composing object may have different motions (rotational motion).

For easy visualization, we (in mechanical context) relate energy with the capacity of a body to do work ("work" as defined in physics). This definition enables us to have the intuitive appreciation of the concept of energy. The capacity of doing work, here, does not denote that the particle will do the amount of work as indicated by the level of energy. For example, our hand has the capacity to do work. However, we may end up doing "no work" like when pushing a building with our hand.

On the other hand, thermal energy (non-mechanical context) of a body can not be completely realized as work. This is actually one of the laws of thermodynamics. Hence, we may appreciate the connection between energy and work, but should avoid defining energy in terms of the capacity to do work. We shall soon find that work is actually a form of energy, which is in "transit" between different types of energies.

Kinetic energy

One of the most common energy that we come across in our day to day life is the energy of motion. This energy is known as kinetic energy and defined for a particle of mass "m" and speed "v" as :

$$K = \frac{1}{2}mv^2$$

Kinetic energy arises due to "movement" of a particle. The main characteristics of kinetic energy are as follows :

- The expression of Kinetic energy involves scalar quantities mass "m" and speed "v". Importantly, it involves speed i.e. the magnitude of velocity - not vector velocity. Therefore, kinetic energy is a scalar quantity.
- Both mass "m" and speed "v" are positive scalar quantities. Therefore, kinetic energy is a positive scalar quantity. This means that a particle can not have negative kinetic energy.

- Kinetic energy of a particle, at rest, is zero.
- Greater the speed or mass, greater is the kinetic energy and vice-versa.

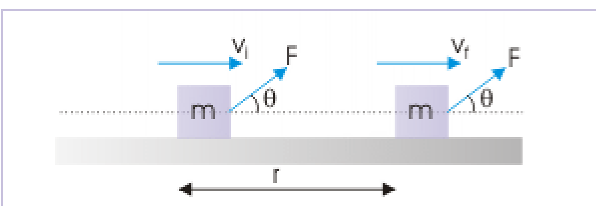
The SI unit of kinetic energy is kg $- m^2 s^{-2}$, which is known as "Joule". Since kinetic energy is a form of energy, Joule (J) is SI unit of all types of energy.

Work and kinetic energy

We have reached the situation when we can attempt to relate work with energy (actually kinetic energy). The relationship is easy to visualize in terms of the motion of a body.

In order to fully appreciate the connection between work and kinetic energy, we consider an example. A force is applied on a block such that component of force is in the direction of the displacement as shown in the figure. Here, work by force on the block is positive. During the time force does positive work, the speed and consequently kinetic energy of the block increases ($K_f > K_i$) as block moves ahead with certain acceleration. We shall know later that the kinetic energy of the block increases by the amount of work done by net external force on it. This is what is known as "work - kinetic energy" theorem and will be the subject matter of a separate module.

Work done on a block



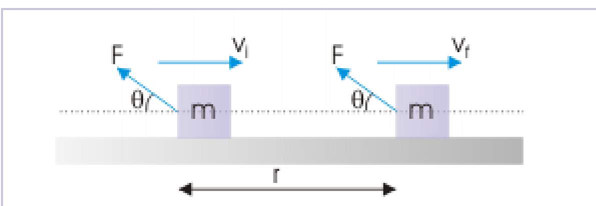
The component of external force and displacement are in same direction.

For the time being, we concentrate on work and determine its relationship with energy in qualitative term. Since kinetic increases by the amount of

work done on the particle, it follows that work, itself, is an energy which can be added as kinetic energy to the block. The other qualification of the work is that it is the "energy" which is transferred by the force from the surrounding to the block. Clearly, force here is an agent, which does the work to increase the speed of the object and hence to increase its kinetic energy. In this sense, "work" by a force is the energy transferred "to" the block, on which force is applied.

We, now, consider the reverse situation as illustrated in the figure below. A force is applied to retard the motion of a block. Here, the component of force is in the opposite direction to the displacement. The work done on the block is negative. The kinetic energy of the block decreases ($K_f < K_i$) by the amount of work done by the force. In this case, kinetic energy is transferred "out" of the block and is equal to the amount of negative work by force. Here, "work" by a force is the energy transferred "from" the block, on which force is applied. Thus, we can define "work" as :

Work done on a block



The component of external force and displacement are in opposite directions.

Work

> Work is the energy transferred by the force "to" or "from" the particle on which force is applied.

It is also clear that a positive work means transfer of energy "to" the particle and negative work means transfer of energy "from" the particle. Further, the term "work done" represents the process of transferring energy to the particle by the force.

Work done by a system of forces

The connection of work with kinetic energy is true for a special condition. The change in kinetic energy resulting from doing work on the body refers to work by net force - not work by any of the forces operating on the body. This fact should always be kept in mind, while attempting to explain motion in terms of work and energy.

Above observation is quite easy to comprehend. Consider motion of a block on a rough incline. As the block comes down the incline, component of gravity does the positive work i.e. transfers energy from gravitational system to the block. As a consequence, speed and therefore kinetic energy of the block increases. On the other hand, friction acts opposite to displacement and hence does negative work. It draws some of the kinetic energy of the block and transfers the same to surrounding as heat. We can see that gravity increases speed, whereas friction decreases speed. Since work by gravity is more than work by fraction in this case, the block comes down with increasing speed. The point is that net change in speed and hence kinetic energy is determined by both the forces, operating on the block. Hence, we should think of relation between work by "net" force and kinetic energy.

As mentioned earlier, a body under consideration may be subjected to more than one force. In that case, if we have to find the work by the net force, then we can adopt either of two approaches:

(i) Determine the net (resultant) force. Then, compute the work by net force.

$$\mathbf{F} = \sum \mathbf{F}_i$$
$$W = \mathbf{F.r} = Fr\cos\theta$$

(ii) Compute work by individual forces. Then, sum the works to compute work by net force.

$$W_i = \mathbf{F}_i.\mathbf{r}_i$$
$$W = \sum W_i$$

Either of two methods yields the same result. But, there is an important aspect about the procedures involved. If we determine the net force first, then we shall require to use free body diagram and a coordinate system to analyze forces to determine net force. Finally, we use the formula to compute work by the net force. On the other hand, if we follow the second approach, then we need to find the product of the components of individual forces along the displacement and displacement. Finally, we carry out the algebraic sum to find the work by the net force.

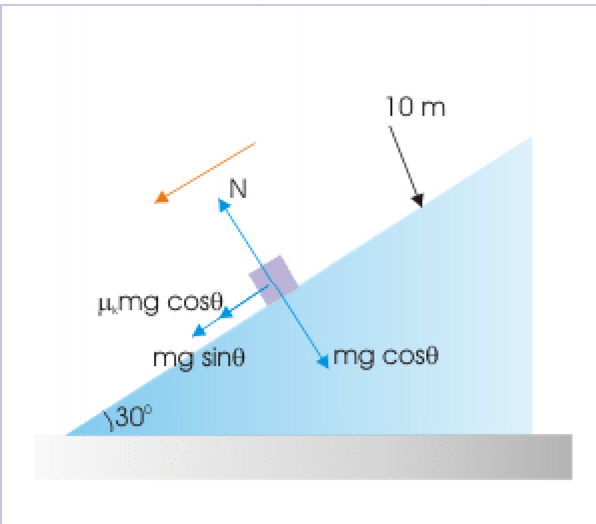
This is a very significant point. We can appreciate this point fully, when the complete framework of the analysis of motion, using concepts of work and energy, is presented. For the time being, we should understand that work together with energy provides an alternative to analyze motion. And, then think if we can completely do away with "free body diagram" and "coordinate system". Indeed, it is a great improvisation and simplification, if we can say so. But then this simplification comes at the cost of more involved understanding of physical process. Meaning of this paragraph will be more clear as we go through the modules on the related topics. For the time being, let us work out a simple example illustrating the point.

Example

Problem 1: A block of mass "10 kg" slides down through a length of 10 m over an incline of 30°. If the coefficient of kinetic friction is 0.5, then find the work done by the net force on the block.

Solution : There are two forces working on the block along the direction of displacement (i) component of gravity and (ii) kinetic friction in the direction opposite displacement.

A block on a rough incline



Work by gravity is positive and work by friction is negative.

The component of gravity along displacement is "mg sinθ" and is in the direction of displacement. Work by gravity is :

$$W_G = mgL\sin 30^\circ = 10 \ \text{X} \ 10 \ \text{X} \ 10 \ \text{X} \ \tfrac{1}{2} = 500 \ J$$

Friction force is " $\mu_K mg\sin\theta$ " amd is in the opposite direction to that of displacement.

$$W_F = -\mu_K mgL\cos 30^\circ = -0.5 \ \text{X} \ 10 \ \text{X} \ 10 \ \text{X} \ 10 \ \text{X} \ \tfrac{\sqrt{3}}{2} = -433 \ J$$

Hence, work by net force is :

$$W_{\text{net}} = W_G + W_F = 1000 - 433 = 567 \ J$$

The important thing to note here is that reference of component of force is with respect to displacement - not with respect to any coordinate direction. This is how we eliminate the requirement of coordinate system to calculate work.

Imgine if we first find the net force. It would be tedius as friction acts along the incline, but gravity acts vertically at angle with the incline. Even if, we find the net force, its angle with displacement would be required to evaluate the expression of work. Clearly, calculation of work for individual force is easier. However, we must keep in mind that we can employ this technique only if we know the forces beforehand.

Anaysis of motion

The concepts of work and energy together are used to analyze motion. The basic idea is to analyze motion such that it does not require intermediate details of the motion like velocity, acceleration and path of motion. The independence from the intermediate details is the central idea that makes work - energy analysis so attractive and elegant. It allows us to analyze motion of circular motion in a vertical plane, motion along paths which are not straight line and host of other motions, which can not be analyzed with laws of motion easily. This is possible because we find that work by certain class of force is independent of path. Further, under certain situations, the analysis is independent of details of attributes like velocity and acceleration.

Path of motion

In this section, let us examine the issue of path independence. Does the work depend on the path of motion? The answer is both yes and no. Even though computation of work involves displacement - a measurement in terms of end points, work is not always free of the path involved. The freedom to path depends on the nature of force. We shall see that work is independent of path for force like gravity. Work only depends on the vertical displacement and is independent of horizontal displacement. It is so because horizontal component of Earth's gravity is zero. However, work by force like friction depends how long (distance) a particle moves on the actual path.

The class of force for which work is independent of path is called "conservative" force; others are called "non-conservative". We can,

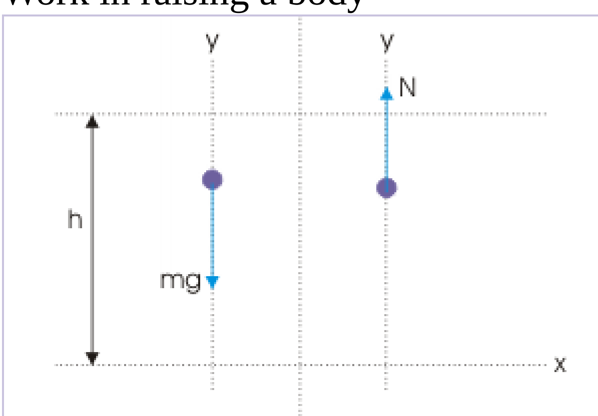
therefore, say that work is independent of path for conservative force and is dependent for non-conservative force. This is the subject matter of a separate module on "conservative force" and as such we would not elaborate the concept any more.

This limited independence may appear to be disadvantageous. A complete independence for all forces would have allowed us to analyze motion without intermediate details in all situations. However, important point here is that major forces in nature are conservative forces - gravitational and electromagnetic forces. This allows us to devise techniques to calculate work by "non-conservative force" indirectly without details, using other concepts (work - kinetic energy theory) that we are going to develop in next module.

Details of motion

Let us check out on the details of the motion. Does work depend on whether a body is moved with acceleration or without acceleration? We can have a look at the illustration, in which we raise a block slowly against gravity through a certain height. Here, net force on the block is zero. Hence, work by net force is zero.

Work in raising a body



Work by net force is zero.

Now, let us modify the illustration a bit. We raise the block with certain constant velocity through the same height. As velocity is constant, it means that there is no acceleration and net force on the block is zero. Hence, work by net force is zero. However, if we raise the block with some acceleration, would it affect the amount of work by net force? The presence of acceleration means net force and as such, work by net force is not zero. Greater net force will mean greater work and acceleration.

In order to get the picture, we now look at the illustration from a different perspective. What about the work by component forces like gravity or normal force as applied by the hand? Work by individual force is multiplication of force and the displacement. Since gravity is constant near the surface, work by gravity is constant for the given displacement. As a matter of fact, work by gravity is only dependent on vertical displacement.

When the block is raised slowly or with constant velocity, the net force is zero. In these circumstances, the work by normal force is equal to work by gravity. This is an important deduction. It allows us to compute work by either force without any reference to velocity and acceleration.

We see that the work by conservative force is independent of the details of motion. Under certain situations, when work by net force is zero, we can determine work by other force(s) in terms of work by conservative force. Thus, the basic idea is to make use of the features of conservative force to simplify analysis such that our consideration is independent of the details of motion.

Role of Newton's laws of motion

The discussion so far leads us to identify distinguishing features of laws of motion at one hand and work-energy concepts on the other - as far as analysis of motion is concerned.

The main distinguishing feature of the application of laws of motion is that we should know the details of motion to analyze the same. This feature is both a strength and a weakness. The strength in the sense that if we have to

know the details like acceleration, then we are required to analyze motion in terms of laws of motion. "Work - energy" does not provide details.

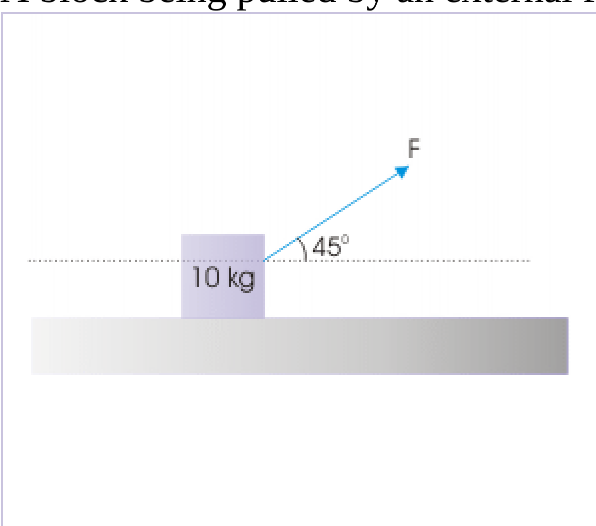
On the other hand, if we have to know the broad parameters like energy or work, then "Work - energy" provides the most elegant solution. As a matter of fact, we would find that analysis by "Work - energy" is often supplemented with analysis by laws of motion to obtain detailed results.

Clearly, when both frameworks are used in tandem, we get the best of both worlds. The example here highlights this aspect. Carefully, note how two concepts are combined to achieve the result.

Example

Problem 2 : A block of 10 kg is being pulled by a force "F" applied at an angle 45° as shown in the figure. The coefficient of kinetic friction between the surfaces is 0.5. If the block moves with constant velocity, calculate the work done by the applied force in moving the block by 1.5 m.

A block being pulled by an external force

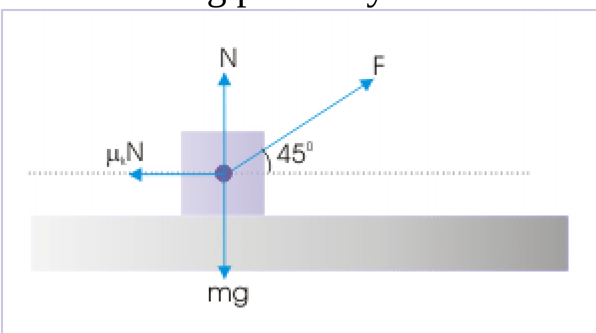


A constant force pulls the block horizontally.

Solution : We had earlier made the point that calculation of work does not require force analysis. This is indeed the case when we know the force. In this case, however, we do not know force and as such we can not do away with free body diagram and coordinate system.

In order to find work by applied force, "F", we are required to know the force. We can know force by analyzing force system on the block. Here, velocity is constant. This means that the block is moving with a constant speed along a straight line. As there is no acceleration involved, the forces on the blocks are balanced. Now the free body diagram for the balanced force system is shown here :

A block being pulled by an external force



Free body diagram drawn on the body system.

$$\sum F_x = F\cos\theta - \mu_k N = 0$$
$$\Rightarrow F\cos\theta = \mu_k N$$

and

$$\sum F_y = N + F\sin\theta - mg = 0$$

Combining two equations, we have :

$$F\cos\theta = \mu_k(mg - F\sin\theta)$$

$$F = \frac{\mu_k mg}{\cos\theta + \mu_k \sin\theta}$$

Work done by external force, F, :

$$\Rightarrow W = \mathbf{F.r} = Fr\cos 45^0 = \frac{\mu_k mgr\cos 45^0}{\cos 45^0 + \mu_k \sin 45^0}$$

$$\Rightarrow W = \frac{0.5\text{x}10\text{x}10\text{x}1.5\text{x}\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} + 0.5\text{x}\frac{1}{\sqrt{2}}}$$

$$\Rightarrow W = 50\ J$$

Work and energy (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

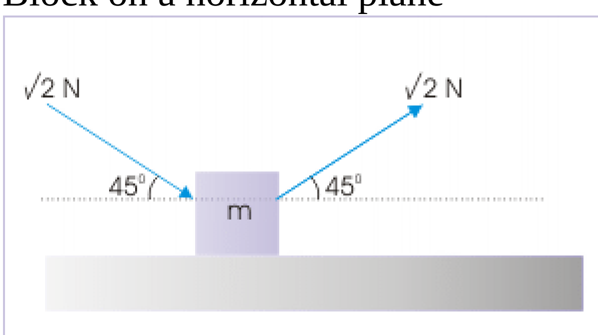
We discuss problems, which highlight certain aspects of the study leading to energy and work. The questions are categorized in terms of the characterizing features of the subject matter :

- Work by a system of forces
- Work by two/three dimensional force
- Work by variable force
- Kinetic energy

Work by a system of forces

Problem 1 : Two forces are applied on a block in the manner as shown in the figure. If the displacement of block on the horizontal surface is 10 m, find the work done by two forces.

Block on a horizontal plane



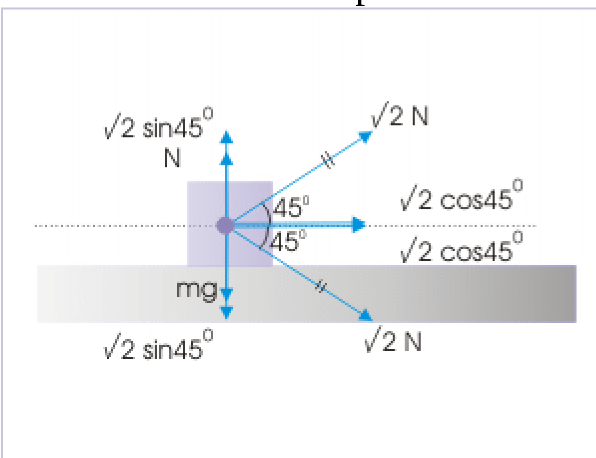
Solution : We only need to consider horizontal components of forces for computing work. Vertical components of forces do not contribute to work,

being perpendicular to the displacement.

We can solve this problem in two ways : either (i) we compute work by the horizontal components of each force separately and then add them to get the total work, or (ii) we first find the net horizontal component of the forces and then compute the work done. Here, we shall use the later technique.

Free body diagram is shown in the figure.

Block on a horizontal plane



The net component of forces in the horizontal direction is :

$$\sum F_x = F\cos 45^0 + F\cos 45^0 = 2F\cos 45^0$$

$$\sum F_x = 2 \text{ x } \sqrt{2} \text{ x } \frac{1}{\sqrt{2}} = 2\,N$$

Work done is :

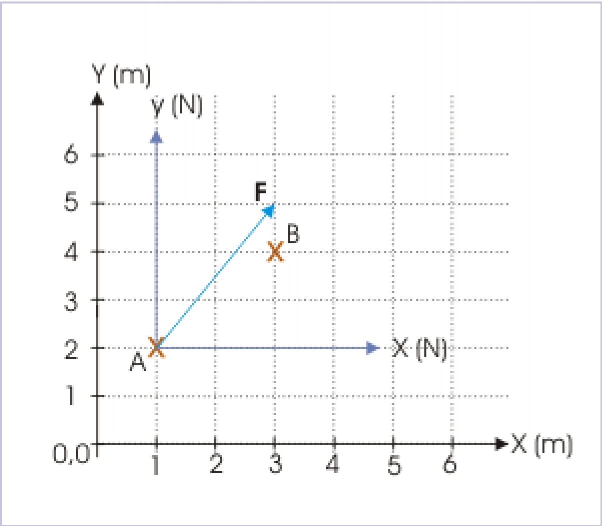
$$W = 2 \text{ x } 10 = 20\,J$$

Work by two/three dimensional force

Problem 2 : A force (2**i** + 3**j**) Newton acts on a particle. If force and displacement both are are in same plane, then find work done by the force in moving particle from (1 m,2 m) to (3 m,4 m) along a straight line.

Solution : We can infer from the expression of force, which involves two unit vectors, that it acts in xy - plane. Since force and displacement both are in same plane, the displacement of the particle is also in xy-plane.

Force and displacement



The figure above shows force vector (2**i** + 3**j**) and displacement end points A (1,2) and B(3,4). Note that we have used two coordinate systems - one for displacement and one for force.

Now, the work in two dimensions is given by :

$$W = F_x x + F_y y$$

Here,

$$F_x = 2N; \quad F_y = 3N; \quad x = 3 - 1 = 2 \quad m; \quad y = 4 - 2 = 2 \quad m$$

$$W = 2x2{+}3x2{=}10\ J$$

Problem 3 : A particle is acted upon by a two dimensional force (2**i**+3**j**). The particle moves along a straight line in xy - plane, defined by the equation, ax + 3y = 5, where "a" is a constant. If work done by the force is zero, then find "a".

Solution : As both force and displacement are non-zero, the work can be zero, only if force and displacement are perpendicular to each other. In order to compare the directions of force and displacement, we write equation of a straight line, which is parallel to line of action of force. From the expression of the force, it is clear that its slope, " m_1 ", is given by :

$$m_1 = \frac{y}{x} = \frac{3}{2}$$

The equation of a straight line parallel to the line of action of force is :

$$y = m_1 x + c = \frac{3}{2} x + c$$

Now, rearranging equation of trajectory as given in the question, we have :

$$y = -\frac{a}{3} + \frac{5}{3}$$

Let " m_2 " be the slope of the line representing trajectory. Then,

$$m_2 = -\frac{a}{3}$$

For two lines to be perpendicular, the product of slopes is $m_1 \text{ X } m_2 = -1$,

$$\left(\frac{3}{2}\right)\left(-\frac{a}{3}\right) = -1$$
$$\Rightarrow a = 2$$

Work by variable force

Problem 4 : A particle is acted by a variable force in xy - plane, which is given by ($2x\mathbf{i} + 3y^2\mathbf{j}$) N. If the particle moves from (1 m,2 m) to (3 m,4 m), then find the work done by the force.

Solution : The force is a two dimensional variable force. We use integration method to find work in two mutually perpendicular directions and then sum them to find the total work.

$$W_x = \int_1^3 2x dx$$
$$\Rightarrow W_x = \left[x^2\right]_1^3 = 9 - 1 = 8\ J$$

and

$$W_y = \int_2^4 3y^2 dy$$
$$\Rightarrow W_y = \left[y^3\right]_2^4 = 64 - 8 = 56\ J$$
$$\Rightarrow W = W_x + W_y = 8 + 56 = 64\ J$$

Kinetic energy

Problem 5 : A body, having a mass of 10 kg, moves in a straight line with initial speed of 2 m/s and an acceleration of 1 m/s^2 . Find the initial time rate of change of kinetic energy.

Solution : Kinetic energy of the particle is given by :

$$K = \frac{1}{2}mv^2$$

We need to obtain a relation for "dK/dt" to obtain the required rate. Now,

$$\frac{dK}{dt} = \frac{1}{2}m \text{ x } 2v\frac{dv}{dt}$$
$$\Rightarrow \frac{dK}{dt} = mav$$

Putting values, we have :

$$\frac{dK}{dt} = 10 \text{ X } 1 \text{ X } 2 = 20\ J$$

Problem 6 : If the ratio of the magnitude of linear momentum and kinetic energy of a particle at any given instant is inversely proportional to time, then find the nature of motion.

Solution : The ratio of linear momentum and kinetic energy of a particle at any given instant is :

$$\frac{p}{K} = \frac{mv}{\frac{1}{2}mv^2} = \frac{2}{v}$$

However, it is given that the ratio of linear momentum and kinetic energy is inversely proportional to time.

$$\frac{p}{K} \propto \frac{1}{t}$$

Comparing two equations,

$$\frac{2}{v} \propto \frac{1}{t}$$

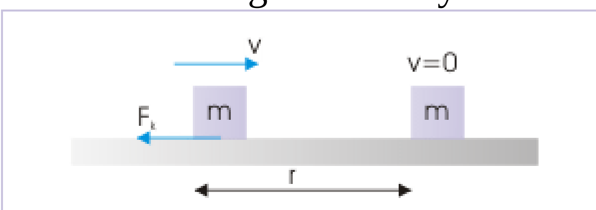
$$\Rightarrow v \propto t$$

We know that "v = u + at" for motion with constant acceleration. In this case, $v \propto t$, which is same as the relation obtained above. Hence, motion considered in the question is characterized by constant acceleration.

Work - kinetic energy theorem
The kinetic energy of a particle changes by the amount of work done on it.

Work is itself energy, but plays a specific role with respect to other forms for energy. Its relationship with different energy forms will automatically come to the fore as we investigate them. In this module, we shall investigate the relationship between work and kinetic energy.

To appreciate the connection between work and kinetic energy, let us consider a block, which is moving with a speed "v" in a straight line on a rough horizontal plane. The kinetic friction opposes the motion and eventually brings the block to rest after a displacement say "r".
A block is brought to rest by friction



Friction applies in opposite direction to displacement .

Here, kinetic friction is equal to the product of coefficient of kinetic friction and normal force applied by the horizontal surface on the block,

$$F_k = \mu_k N = \mu_k mg$$

Kinetic friction opposes the motion of the block with deceleration, a, :

$$a = \frac{F_k}{m} = \frac{\mu_k mg}{m} = \mu_k g$$

Considering motion in x-direction and using equation of motion for deacceleration, ${v_2}^2 = {v_1}^2 - 2ar$, we have :

$$0 = v^2 - 2ar$$
$$v^2 = 2ar = 2\mu_k gr$$

Thus, kinetic energy of the block in the beginning of motion is :

$$K = \frac{1}{2}mv^2 = \frac{1}{2} \text{ x } m2\mu_k gr = \mu_k mgr$$

A close inspection of the expression of initial kinetic energy as calculated above reveals that the expression is equal to the magnitude of work done by the kinetic friction to bring the block to rest from its initial sate of motion. The magnitude of work done by the kinetic friction is :

$$W_F = F_k r = \mu_k mgr = K$$

This brings up to a new definition of kinetic energy :

Kinetic energy

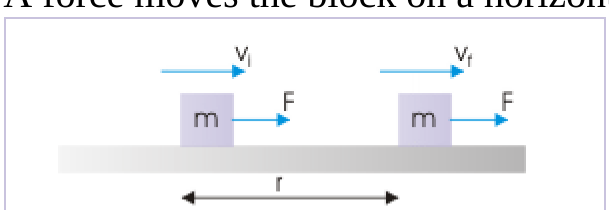
> Kinetic energy of a particle in motion is equal to the amount of work done by an external force to bring the particle to rest.

Work - kinetic energy theorem

Work - kinetic energy theorem is a generalized description of motion - not specific to any force type like gravity or friction. We shall, here, formally write work - kinetic energy theorem considering an external force. The application of a constant external force results in the change in kinetic energy of the particle. For the time being, we consider a "constant" external force. At the end of this module, we shall extend the concept to variable force as well.

Let v_i be the initial speed of the particle, when we start observing motion. Now, the acceleration of the particle is :

A force moves the block on a horizontal surface



Force does work on the block.

$$a = \frac{F}{m}$$

Let the final velocity of the particle be v_f. Then using equation of motion, ${v_f}^2 = {v_i}^2 + 2ar$,

$${v_f}^2 - {v_i}^2 = 2\left(\frac{F}{m}\right)r$$

Multiplying each term by 1/2 m, we have :

$$\frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_i}^2 = \frac{1}{2}m \text{ x } 2\left(\frac{F}{m}\right)r = Fr$$

$$\Rightarrow K_f - K_i = W$$

This is the equation, which is known as work - kinetic energy theorem. In words, change in kinetic energy resulting from application of external force(s) is equal to the work done by the force(s). Equivalently, work done by the force(s) in displacing a particle is equal to change in the kinetic energy of the particle. The above work - kinetic energy equation can be rearranged as :

$$\Rightarrow K_f = K_i + W$$

In this form, work - kinetic energy theorem states that kinetic energy changes by the amount of work done on the particle. We know that work can be either positive or negative. Hence, positive work results in an increase of the kinetic energy and negative work results in a decrease of the kinetic energy by the amount of work done on the particle. It is emphasized here for clarity that "work" in the theorem refers to work by "net" force - not individual force.

Work - kinetic energy theorem with multiple forces

Extension of work - kinetic energy theorem to multiple forces is simple. We can either determine net force of all external forces acting on the particle,

compute work by the net force and then apply work - kinetic energy theorem. This approach requires that we consider free body diagram of the particle in the context of a coordinate system to find the net force on it.

$$\mathbf{F}_N = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + ... + \mathbf{F}_n$$

Work - kinetic energy theorem is written for the net force as :

$$\Rightarrow K_f - K_i = \mathbf{F}_N \mathbf{.} \mathbf{r}$$

where $\mathbf{F}_N$ is the net force on the particle. Alternatively, we can determine work done by individual forces for the displacement involved and then sum them to equate with the change in kinetic energy. Most favour this second approach as it does not involve vector consideration with a coordinate system.

$$\Rightarrow K_f - K_i = \sum W_i$$

Application of Work - kinetic energy theorem

Work - kinetic energy theorem is not an alternative to other techniques available for analyzing motion. What we want to mean here is that it provides a specific technique to analyze motion, including situations where details of motion are not available. The analysis typically does not involve intermediate details in certain circumstances. One such instance is illustrated here.

In order to illustrate the application of "work - kinetic energy" theory, we shall work with an example of a block being raised along an incline. We do not have information about the nature of motion - whether it is raised along the incline slowly or with constant speed or with varying speed. We also do not know - whether the applied external force was constant or varying. But, we know the end conditions that the block was stationary at the beginning of the motion and at the end of motion. So,

$$\Rightarrow K_f - K_i = 0$$

It means that work done by the forces on the block should sum up to zero (according to "work - kinetic energy" theorem). If we know all other forces but one "unknown" force and also work done by the known forces, then we are in position to know the work done by the "unknown" force.

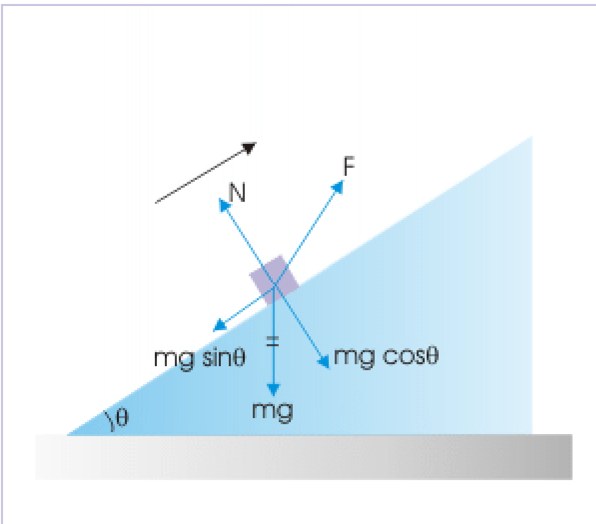
We should know that application of work-kinetic energy theorem is not limited to cases where initial and final velocities are zero or equal, but can be applied also to situations where velocities are not equal. We shall discuss these applications with references to specific forces like gravity and spring force in separate modules.

Example

Problem : A block of 2 kg is pulled up along a smooth incline of length 10 m and height 5 m by applying an external force. At the end of incline, the block is released to slide down to the bottom. Find (i) work done by the external force and (ii) kinetic energy of the particle at the end of round trip. (consider, g = 10 m/s^2).

Solution : This question is structured to bring out finer points about the "work - kinetic energy" theorem. There are three forces on the block while going up : (i) weight of the block, mg, and (ii) normal force, N, applied by the block and (iii) external force, F. On the other hand, there are only two forces while going down. The external force is absent in downward journey. The force diagram of the forces is shown here for upward motion of the block.

A block on an incline



Known forces acting on the block

(i) work done by external force (W_{F}) during round trip

The most striking aspect about the external force, "F", is that we do not know either about its magnitude or about its direction. We have represented the external force on the force diagram with an arbitrary vector. Further, external force, F, acts only during up journey. Note that the block is simply released at the end of upward journey. It means that we need to find work by external force only during upward journey.

$$W_F = W_{\mathrm{F(up)}} + W_{\mathrm{F(down)}} = W_{\mathrm{F(up)}} + 0 = W_{\mathrm{F(up)}}$$

The kinetic energies in the beginning and at the end of "upward" motion are zero. Note the wordings of the problem that emphasizes this. From "work - kinetic energy" theorem, we can coclude that sum of the work done by all three forces is equal to zero during upward motion. It is, therefore, clear that if we know the work done by the other two forces (normal force and gravity), then we shall find out the work done by the external force, "F", as required.

Work done by the forces normal to the incline is zero. It follows then that we do not need to consider normal force. According to "work-kinetic energy" theorem, the sum of work done by gravity and external force for upward motion is zero :

$$W_{\text{G(up)}} + W_{\text{F(up)}} = 0$$

Thus, we need to compute work done by the gravity in order to compute work by the external force "F". Now, the component of weight parallel to incline is directed downward. It means that it (gravity) does negative work on the block while going up. The component of gravity along incline is "mg sinθ", acting downward. Work done by gravity during upward journey is :

$$W_{\text{G(up)}} = Fr = -mg \sin\theta \ \text{ X } \ L = -mgL \sin\theta$$
$$W_{\text{G(up)}} = -2 \text{ x } 10 \text{ x } 10 \text{ x } \left(\frac{5}{10}\right) = -100\ J$$

Hence, work done by the external force, "F", during the round trip is :

$$W_F = W_{\text{F(up)}} = -W_{\text{G(up)}} = -\left(-100\right) = 100\ J$$

(ii) Kinetic energy at the end of round trip :

Initial kinetic energy of the block is zero. The kinetic energy is increased by the work done by the forces acting on the block. According to work - kinetic energy theorem :

$$K_f - K_i = K_f - 0 = W_{\text{(round-trip)}}$$

Total work done during round trip by external force, "F", is 100 J as computed earlier. Total work done during round trip by gravity is :

$$W_{\text{G(roundtrip)}} = -mgL \sin\theta + mgL \sin\theta = 0$$

Hence, total work done during round trip is :

$$W_{\text{roundtrip}} = W_{\text{F(roundtrip)}} + W_{\text{G(roundtrip)}} = 100 + 0 = 100\ J$$

$$\Rightarrow K_f = 100\ J$$

For understanding purpose, we again emphasize that total work done during upward motion is zero as block is stationary in the beginning and at the end during motion up the incline. The positive work done by external force, "F", is canceled by the exactly same amount of negative work by gravity. The net work is done in only downward motion by the gravity, whereupon kinetic energy of the block increases.

Finally, we should note that this example was specially designed to illustrate calculation of work by unknown force indirectly, using "work-kinetic energy" theorem. This application, however, depends on the specific situation. For example, had the incline been rough, we would be required to consider friction force as well. This friction, in turn, would depend on the external force. As such, we would not be in a position to calculate work by "unknown" force indirectly as in this case. Clearly, in that situation, we would need to know the external force as well to apply "work-kinetic energy" theorem.

Context of work - kinetic energy theorem

We summarize important aspects of work - kinetic energy theorem as :

1 : Generally, significance of "work-kinetic energy" theorem is not fully appreciated. Often, it is convenient and more intuitive to use laws such as "conservation of energy" in general or "conservation of mechanical energy" in mechanics. But we should know that mathematical formulations of these generalized conservation laws, as a matter of fact, are extension of this theorem.

2 : We should understand that kinetic energy and work are general concept not limited to any class of forces such as conservative forces. For this particular reason, we selected some examples involving friction (non-conservative force) to emphasize that the "work-kinetic energy" theorem is not limited to conservative force.

3 : The "work-kinetic energy" theorem considers works by all the forces working on the body or equivalently work by the "net" force. Unlike force

analysis, application of "work-kinetic energy" theorem neither requires coordinate system nor free body diagram.

4 : Application of "work-kinetic energy" theorem is particularly suited for situations in which there is no change in the kinetic energy. This, sometimes, allows us to calculate work by even "unknown" force.

Work - kinetic energy theorem and variable force

We derived mathematical expression of work-kinetic energy theorem in relation to constant force. We restricted our consideration to constant force to keep the discussion simple.

Here, we set out to establish work - energy theorem for an external force "F", which varies with displacement (r). We consider a particle of mass "m" moving along a straight line under the application of force "F". The work done in moving the particle by a small linear distance, "dr", is :

$$đW = \mathbf{F}.\,đ\mathbf{r} = F đr \cos\theta$$

Let us also consider that force and displacement are in the same direction. This linear consideration is a convenient set up for even two or three dimensional cases. In general case also, we can compute work and kinetic energy in three mutually perpendicular directions of the coordinate system considering components of force and displacement in respective directions. Each of the computation in mutually perpendicular directions is a linear consideration.

$$\Rightarrow đW = F đr \cos 0^0 = F đr$$

Now,

$$\mathrm{F} = m\ \mathrm{a}$$

Hence,

$$\Rightarrow đW = ma đr$$
$$\Rightarrow đW = m\left(\frac{đv}{đt}\right)đr$$
$$\Rightarrow đW = m\left(\frac{đr}{đt}\right)đv = mvđv$$

Integrating for limits corresponding to initial and final position of the particle,

$$W = \int đW$$
$$W = \int_{v_1}^{v_2} mvđv$$

$$W = \frac{1}{2}m\left({v_f}^2 - {v_i}^2\right)$$

$$\Rightarrow W = K_f - K_i$$

Work - kinetic energy theorem (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to work - kinetic energy theorem. The questions are categorized in terms of the characterizing features of the subject matter :

- Constant force
- Variable force
- Maximum kinetic energy

Constant force

Problem 1 : A bullet traveling at 100 m/s just pierces a wooden plank of 5 m. What should be the speed (in m/s) of the bullet to pierce a wooden plank of same material, but having a thickness of 10m?

Solution : Final speed and hence final kinetic energy are zero in both cases. From "work- kinetic energy" theorem, initial kinetic energy is equal to work done by the force resisting the motion of bullet. As the material is same, the resisting force is same in either case. If subscript "1" and "2" denote the two cases respectively, then :

For 5 m wood plank :

$$0-\frac{1}{2}m{v_1}^2=-Fx_1$$
$$\frac{1}{2}m100^2=F\ \text{X}\ 5$$

For 10 m wood plank :

$$0 - \frac{1}{2} m {v_2}^2 = -F x_2$$
$$\frac{1}{2} m {v_2}^2 = F \text{ X } 10$$

Taking ratio of two equations, we have :

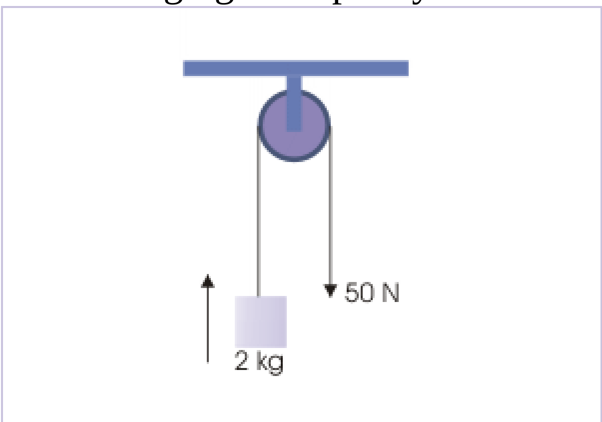
$$\frac{{v_2}^2}{100^2} = \frac{F\text{x}10}{F\text{x}5} = 2$$

$$\Rightarrow {v_2}^2 = 2 \text{ x } 10000 = 20000$$

$$\Rightarrow v_2 = 141.4 \, m/s$$

Problem 2 : A block of 2 kg is attached to one end of a string that passes over a pulley as shown in the figure. The block is pulled by a force of 50 N, applied at the other end of the string. If change in the kinetic energy of the block is 60 J, then find the work done by the tension in the string.

Block hanging from pulley

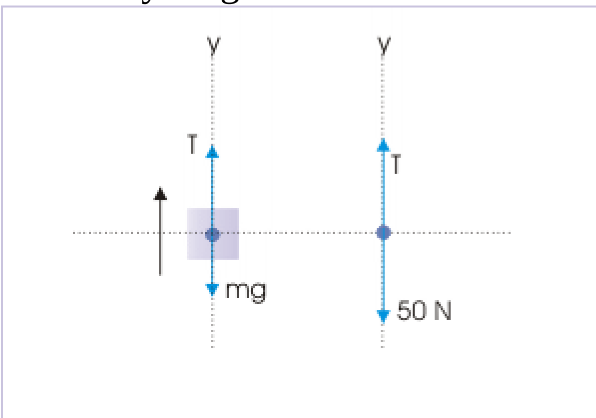


Solution : We need to know tension and the displacement to find the work by tension. We note here that change in kinetic energy is given. Using "work - kinetic energy" theorem, we find work by the net force - not by the tension alone - on the block :

$$W = \Delta K = 60 \, J$$

In order to find tension and net force on the block, we draw free body diagram as shown in the figure.

Free body diagram



$$T = 50\ N$$

and

$$\sum F_y = T - mg$$
$$\sum F_y = 50 - 2 \text{ x } 10 = 30\ N$$

Let the vertical displacement be "y". Then, the work done by the net force is :

$$30y = 60$$
$$y = 2\ m$$

Now, we have values of tension and displacement of the block. Hence, work done by the tension in the string :

$$W_T = T \text{ x } y = 50 \text{ x } 2 = 100\ J$$

Note that this example illustrates two important aspects of analysis, using "work - kinetic energy" theorem : (i) work equated to change in kinetic energy is work by net force and (ii) if details like the value of tension and displacement are not known, then we need to employ force analysis to find quantities used for the calculation of work by individual force.

Problem 3 : A block of 1 kg, initially at 10 m/s, moves along a straight line on a rough horizontal plane. If its kinetic energy reduces by 80 % in 10 meters, then find coefficient of kinetic friction between block and horizontal surface.

Solution : From the given data, we can find the change in kinetic energy and hence work done by friction (which is the only force in this case). We do not consider weight or normal force as they are perpendicular to the direction of displacement. Here, $K_f = 0.2K_i$. Applying "Work - kinetic energy" theorem,

$$W = K_f - K_i = 0.2K_i - K_i = -0.8K_i$$

Now,

$$K_i = \frac{1}{2}m{v_i}^2 = \frac{1}{2}\text{ x 1 x 100} = 50\ J$$

Combining two equations,

$$W = -0.8\text{ x }50 = -40\ J$$

Now, work by the friction is related to friction as :

$$W = -F_K r$$

and friction is related to normal force as :

$$F_K = \mu_K N$$

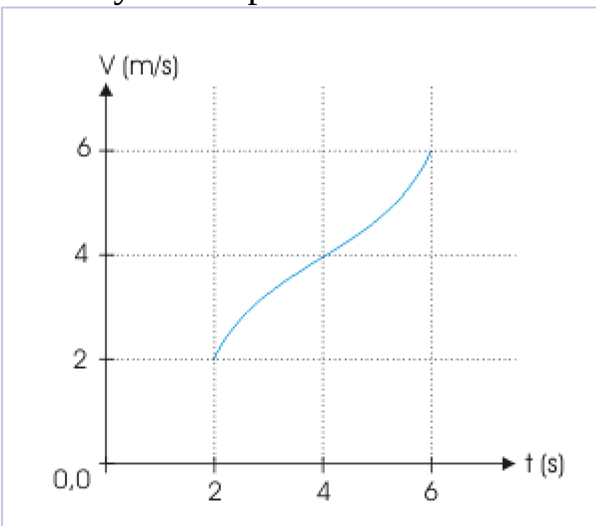
Combining two equations, we have :

$$W = -\mu_K N r$$
$$\mu_K = -\frac{W}{Nr} = -\frac{W}{mgr}$$
$$\mu_K = -\frac{-40}{1\text{x}10\text{x}10} = 0.4$$

Variable force

Problem 4 : Velocity - time plot of the motion of a particle of 1 kg from t = 2s to t = 6s, is as shown in figure. FInd the work done by all the forces (in Joule) on the particle during this time interval.

Velocity - time plot



Solution : We are required to find work by all the forces, operating on the particle. No information about force is given. However, states of motion at end points can be read from the given plot. The "work - kinetic energy" theorem is independent of intermediate detail. From the plot. we have : v_i = 2 m/s and v_f = 6 m/s. Now, applying theorem for the end conditions, we get the work by all forces, acting on the particle :

$$W = K_f - K_i$$

$$W = \frac{1}{2}m\left({v_f}^2 - {v_i}^2\right)$$

$$\Rightarrow W = \frac{1}{2} \text{ x } 1 \text{ x } \left(6^2 - 2^2\right) = 0.5 \text{ x } \left(36 - 4\right)$$

$$\Rightarrow W = 16\ J$$

Problem 5 : The displacement of a particle of 0.1 kg moving along x-axis is a function of time as x (in meters) = 2 t^2 + t. Find the work done (in Joule) by the net force on the particle during time interval t = 0 to t = 5 s.

Solution : Here, displacement is given as a function of time. This enables us to determine speed at given time instants. This, in turn, enables us to determine change in kinetic energy and hence work by the net force on the particle. Now,

$$v = \frac{dx}{dt} = 4t + 1$$

Speed at t = 0,

$$\Rightarrow v_i = 1\, m/s$$

Speed at t = 5 s,

$$\Rightarrow v_f = 21\, m/s$$

Applying work-kinetic energy theorem, we get work done by the net force as :

$$W = K_f - K_i$$

$$W = \frac{1}{2} m\left(v_f{}^2 - v_i{}^2\right)$$
$$\Rightarrow W = \frac{1}{2} \text{ x } 0.1 \text{ x } \left(21^2 - 1^2\right) = 22\, J$$

Maximum kinetic energy

Problem 6 : A particle of 0.1 kg is at rest when x = 0. A force given by the function F(x) = ($9 - x^2$) N, is applied on the particle. If displacement is in meters, then find the maximum kinetic energy (in Joule) of the particle for x > 0.

Solution : The given force is a function of displacement. We can use integral form to determine work. This work can, then, be equated to the change in kinetic energies, using "work - kinetic energy" theorem. Once we have the relation of kinetic energy as a function of "x", we can differentiate and apply the condition for maximum kinetic energy. Now,

$$W = \int F(x) dx = \int (9 - x^2) dx$$
$$\Rightarrow W = 9x - \frac{x^3}{3}$$

The particle is initially at rest. It means that its initial kinetic energy is zero. Now, applying "work - kinetic energy" theorem, we have :

$$K_f - K_i = K_f = K$$
$$\Rightarrow K = 9x - \frac{x^3}{3}$$

$$\Rightarrow \frac{dK}{dt} = 9 - \frac{3x^2}{3} = 9 - x^2$$

For $\frac{dK}{dt} = 0$, we have :

$$\Rightarrow x = -3\,m \text{ or } +3\,m$$

But for x > 0, x = 3 m. Second derivative of kinetic enregy should be negative for this value of "x = 3" so that kinetic energy at this point is maximum.

$$\frac{d^2K}{dt^2} = -2x = -2x3 = -6$$

Thus, kinetic energy at x = 3 m is maximum for x > 0. It is given by :

$$K = 9x - \frac{x^3}{3} = 9 \text{ x } 3 - \frac{3^3}{3} = 18\,J$$

Work by gravity

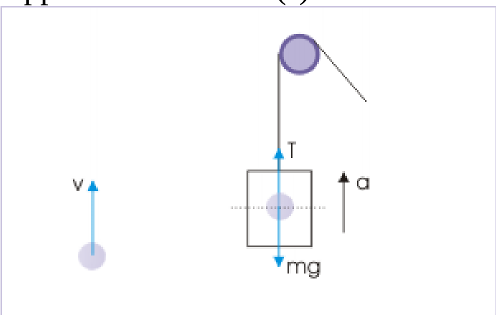
Gravity is common to all motions. The most striking aspect of work by gravity stems from the fact that the force due to gravity is constant. As such, value of work is constant for a given displacement irrespective of the state of motion of the particle (i.e velocity of particle).

We consider two different situations for the analysis of work :

- Particle is subjected to single force due to gravity.
- Particle is subjected to force due to gravity and other forces.

Two situations are different in their manifestations. For example, projecting an object in vertical direction with initial velocity is different to a situation in which the object (say a lift) is pulled up by the rope attached to it. In the first case, only force that influences the motion of the object is the force due to gravity, whereas the motion of the object in the second case is determined by the force of gravity and tension in the string pulling the object. The situation, in the first case, is described by initial velocity. On the other hand, we will need force analysis of the forces with a free body diagram to know the forces and resulting acceleration. These two situations are shown in the figure.

Application of force (s)



First figure shows an object thrown at a speed. Second figure shows lift being pulled up by the tension in the rope.

There are many such situations in which force(s) other than garvity plays a role in the motion of an object. When we lift a book from the table, two forces act on the book (i) weight of the book and (ii) force applied by the hand. Note that gravity is always present and is common to all situations.

Particle is subjected to single force due to gravity.

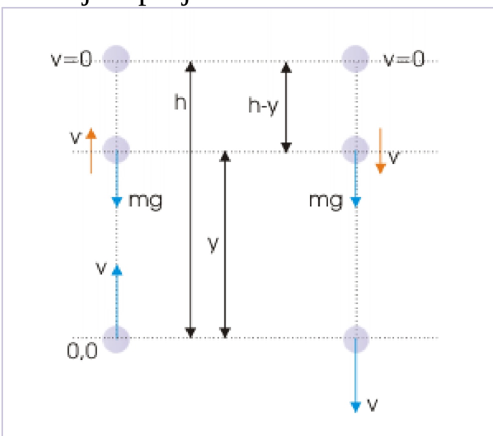
We discuss here two different profile of motions (i) vertical and (ii) inclined motion :

Vertical motion

When we throw a particle like object with an initial velocity vertically upwards, the force due gravity decelerates the motion by doing a negative work on the object. For a vertical displacement, y, the work done is :

$$W_{\mathrm{G(up)}} = (-mg)r = -mgy$$

An object projected in vertical direction



Up and down motions of a vertically projected object

Kinetic energy of the particle is correspondingly reduced by the amount of work :

$$K_{\mathrm{f(up)}} = K_{\mathrm{i(up)}} + W_{\mathrm{G(up)}} = K_{\mathrm{i(up)}} - mgy$$

It is clear from "work-kinetic energy" theorem that initial kinetic energy is equal to "mgh", where "h" is the maximum height.

$$K_{\mathrm{f(up)}} = K_{\mathrm{i(up)}} + W_{\mathrm{G(up)}} = mgh - mgy = mg(h - y)$$

During this upward motion, force due to gravity transfers energy "from" the particle. This process continues till the particle reaches the maximum height, when kinetic energy is reduced to zero (y = h):

$$K_{\mathrm{f(up)}} = 0$$

A reverse of this description of motion takes place in downward motion. The work by gravity is positive in the downward motion. During this part of motion, the kinetic energy of the particle increases by the amount of work :

$$W_{\mathrm{G(down)}} = mgr = mg\left(h - y\right)$$

where “h” is the maximum height attained by the particle. During this downward motion, force due to gravity transfers energy "to" the particle. As a consequence, kinetic energy increases, which is given by :

$$K_{\mathrm{f(down)}} = K_{\mathrm{i(down)}} + W_{\mathrm{G(up)}} = K_{\mathrm{i(down)}} + mg\left(h - y\right) = 0 + mg\left(h - y\right) = mg\left(h - y\right)$$

The process continues till the particle returns to the initial position. Its velocity increases and becomes equal to the velocity of projection on its return to the initial position. For y = 0,

$$K_{\mathrm{f(down)}} == mgh$$

As such, kinetic energy, at the end of round trip, is equal to that in the beginning.

$$\Rightarrow K_{\mathrm{f(down)}} = K_{\mathrm{i(up)}}$$

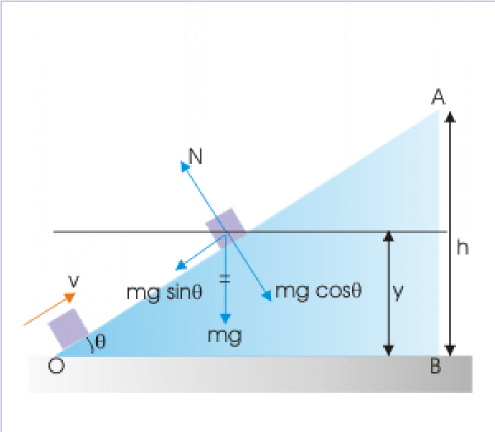
For the round trip, net work by gravity is zero. Net transfer of energy “to” or “from” the particle is zero. Initial kinetic energy of the particle is retained at the end of round trip.

Important aspect of this description of motion under gravity is that the end results are specific to gravity. The same result will not hold for motion, which is intervened by force like friction. We shall know that the difference arises due to the difference in the nature of two forces. We shall discuss these difference in detail in a separate module on conservative force.

Motion along incline

We shall, now, consider projection of a block on a smooth incline with initial velocity "v". Only force acting on the block is force due to gravity. The component of force is mg sinθ, which opposes motion in upward direction and aids motion in downward direction. Work done for going up the incline for a displacement “y” is :

Motion along incline



The component of gravity along incline is acting downward.

$$W_{\mathrm{G(up)}} = \left(-mg\sin\theta\right) \text{ x } r$$

where "r" is any intermediate length along the incline covered by the block. Considering trigonometric ratio of sine of the angle of incline, we have :

$$\Rightarrow y = r\sin\theta$$

Hence,

$$W_{\mathrm{G(up)}} = -mgy$$

This is an important result as it underlines that the work depends only on the vertical component of displacement i.e. the displacement along the direction in which force is applied - not on the component of displacement in the direction perpendicular to that of force. Work by gravity for horizontal displacement is zero as force and displacement are at right angle. This independence of work by gravity from horizontal displacement is further reinforced with the concept of conservative force and corresponding concept of potential energy as discussed in the corresponding modules.

The initial kinetic energy of the block decreases as force due to gravity transfers energy from the block :

$$K_{\mathrm{f(up)}} = K_{\mathrm{i(up)}} - mgy$$

This process continues till the particle reaches the maximum height, when kinetic energy is reduced (speed is zero) to zero :

$$K_{\mathrm{f(up)}} = 0$$

A reverse of this description of motion takes place in downward motion. The work by gravity is positive (but same in magnitude) and kinetic increases by the amount of work :

$$W_{\mathrm{G(down)}} = mgr = mg\left(h - y\right)$$

where “h” is the maximum height attained by the particle.

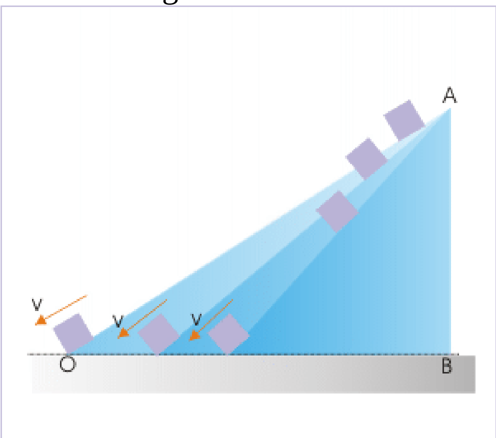
The process continues till the particle returns to the initial position, when its velocity is same as that in the beginning. As such kinetic energy at the end of round trip is equal to that in the beginning.

$$\Rightarrow K_{\mathrm{f(down)}} = K_{\mathrm{i(up)}}$$

For the roundtrip, net work by gravity is zero. Net transfer of energy “to” or “from” the particle is zero. Initial kinetic energy of the particle is retained at the end of round trip.

These results are exactly the same as for the vertical motion discussed in the previous section. Interestingly, work along any incline that reaches same height is same. The objects released from a given height will have same velocity and kinetic energy, irrespective of the length of incline involved.

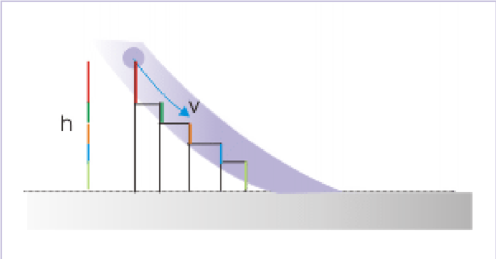
Motion along incline



Blocks have same velocity and kinetic energy on reaching ground.

The work, kinetic energy and velocity are independent of path between end points, provided the vertical height is same in all cases. This is even true for curved path. Any curved path can be considered to be series of linear path having horizontal and vertical components. As horizontal component is not relevant, vertical components sum up to the net height.

Motion along curved path



Vertical components sum up the net height.

We must, however, emphasize that this is not a general result, but specific to gravity and other conservative forces (we shall define this concept in a separate module.). For example, if the surfaces are rough, then block will not be able to retain its kinetic energy on the return to initial position of projection due to friction, which is not a conservative force.

In the nutshell, we have learned following characterizing features of the motion of block acted upon by gravity alone :

- Gravity is a constant force. Hence, work by gravity is constant for a given displacement.
- The particle retains the kinetic energy on return to same vertical position.
- Work by gravity for horizontal displacement is zero as force and displacement are at right angle.
- Work by gravity is independent of path details and depends only on the vertical component of displacement.

Particle is subjected to gravity and other forces

We must understand that work by force due to gravity for a given displacement is independent of the presence of other forces. The work done by gravity remains same. When other forces are also present, "work-kinetic energy" theorem has following form :

$$\Rightarrow K_f - K_i = W_G + W_F$$

where W_G and W_F are the work done by the gravity and other force(s).

Here, work done by other external force(s) may be analyzed with respect to following different conditions :

1. Initial and final speeds are zero.
2. Initial and final speeds are same.
3. Initial and final speeds are different.

In the first two cases, initial and final kinetic energy are same. Hence,

$$\Rightarrow K_f - K_i = W_G + W_F = 0$$
$$\Rightarrow W_F = -W_G$$

This is an important result. The work done by other force(s) is same as that by gravity, but with a negative sign (we have already discussed such case in previous module). This feature of motion, when end conditions are same, facilitates computation of work by other forces a great deal, as we need not be concerned of the external force other than gravity. We simply compute the work by gravitational force. The work by other external force(s) is equal to the work by gravity with a negative sign.

In third case, kinetic energies at end points are not same. The difference in kinetic energy during a motion is equal to the net work done by all forces. In this case, we can not consider work by other forces to be equal in magnitude to that by gravity. Instead, we would require to find individual force by force analysis and then compute work by each force individually.

Example:

Problem : A lift, weighing 1000 kg, is raised vertically by a string - pulley arrangement with an upward acceleration of 1 m/s^2 . If the initial speed of the lift is 2 m/s, find the speed attained by the lift after traveling a vertical distance of 10 m.

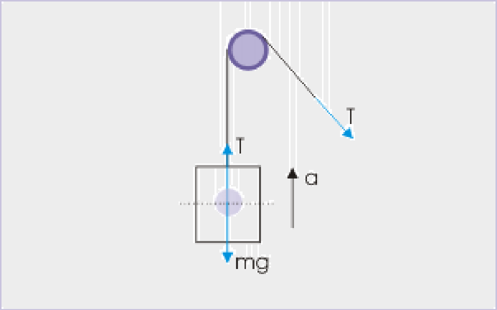
Solution : Here, lift is moving with upward acceleration of 1 m/s^2. It means that velocities in the beginning and end are not same. Therefore, work done by gravity and tension is not zero.

Now, work done by gravity is :

$$\Rightarrow W_G = -mgy = -1000 \text{ x } 10 \text{ x } 10 = -100000 \; J$$

In order to know the work done by the tension, we need to know the tension. Note here that work by tension is not equal to work by gravity as in earlier case. From, free body diagram, we have :

Free body diagram



$$\Rightarrow T - mg = ma$$
$$\Rightarrow T = m(g+a) = 1000 \text{ x } (10+1) = 11000 \; N$$

Work done by tension in the string is :

$$\Rightarrow W_T = T \text{ x } y$$
$$\Rightarrow W_T = 11000 \text{ x } 10 = 110000 \; J$$

Using “work - kinetic energy” theorem, we have :

$$K_f - K_i = W_G + W_F = -100000 + 110000 = 10000 \; J$$
$$\Rightarrow \frac{1}{2} m\left({v_f}^2 - {v_i}^2\right) = 10000$$

$$\Rightarrow {v_f}^2 = \frac{2\text{x}10000}{m} + {v_i}^2$$

$$\Rightarrow {v_f}^2 = 20 + 2^2$$
$$\Rightarrow {v_f}^2 = 24$$
$$\Rightarrow v_f = 4.9\, m/s$$

Work by spring force

Spring is an arrangement, which is capable to apply force on a body. As we will see, spring force is similar to gravity in many important ways. At the same time, it is different to gravity in one important way that it applies a variable force unlike gravity.

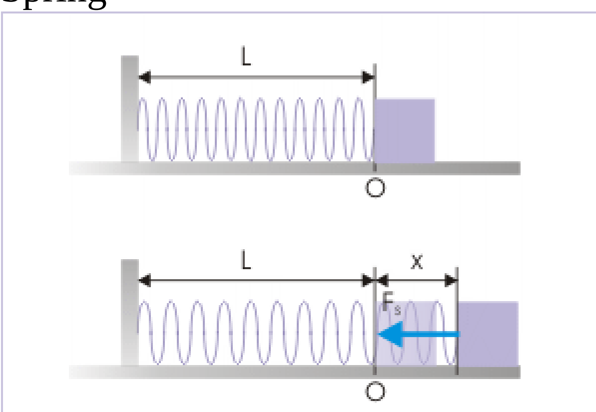
In this module, we shall first outline the characterizing aspects of spring and then develop an expression for work. Subsequently, we shall analyze situations first with spring force alone and then along with other external force(s) on similar line as in the case of work by gravity.

Spring force

Spring force is given by the expression :

$$F = -kx$$

Spring



Spring force

where k is spring constant, measured experimentally for a particular spring. The value of k (=-F/x) measures the stiffness of the spring. A high value indicates that we would require to apply greater force to change length of the spring.

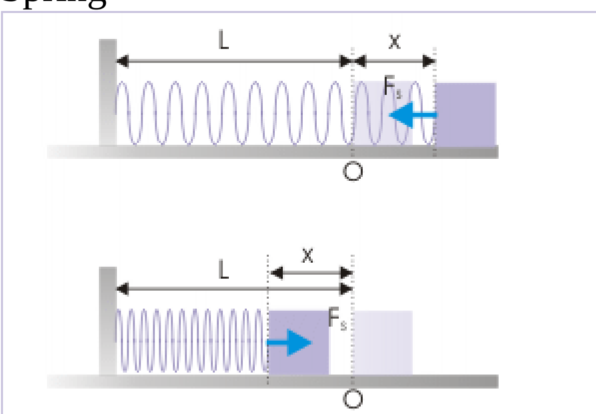
This relationship, defining spring force, is also known as “Hooke’s law”, which is valid under two specific conditions :

1. The mass of spring is negligible (and can be neglected).
2. There is no dissipative force (like friction) involved with spring.

The attribute “x” is measured from the position of “neutral length” or “relaxed length” - the length of spring corresponding to situation when spring is neither stretched nor compressed. We shall identify this position as origin of coordinate reference.

The negative sign in the expression indicates an important aspect of the nature of spring force that it is always opposite to the direction of the displacement. It means that it is always directed towards the position at neutral length i.e. origin. See the figure to visualize how spring force is always directed towards origin.

Spring



Spring force is always directed towards origin

Context of spring force

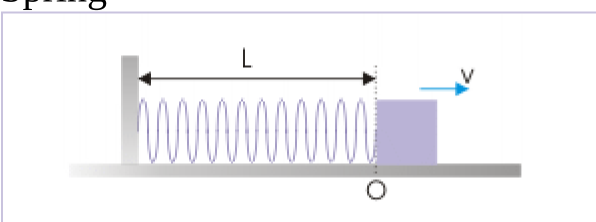
The forces on the body (block) attached to a horizontal spring may include friction, if the surfaces between block and the surface is rough. For the sake

of simplicity, however, we shall consider the surface to be smooth. The force due to gravity and normal force are normal to the motion and do not influence force analysis here. As such, we will not consider either friction or gravity, while calculating work for the motion on smooth horizontal surface.

Just like the case of gravity, there are two situations with respect to application of force on a body by the spring.

1: In one case, the body is given a jerk, say towards right and the body is let go. In this case, the only force on the block is spring force. This situation is similar to vertical projection of an object with an initial speed. This case of spring force, therefore, is specified by an initial speed, preferably at the origin as shown in the figure.

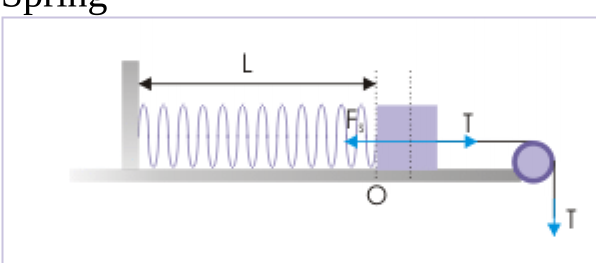
Spring



Only force on the block is spring force.

2: In other case, there is external force(s) in addition to spring force. The horizontal force is continuously applied on the block as it moves along the surface. This situation is similar to the case of pulling a lift vertically by string and pulley arrangement. In this case, we may be required to analyze external force with the help of “free body diagram”.

Spring



There is other force (tension, T)
besides spring force.

Work by spring force

Now, we are set to obtain an expression for the work done by the spring. We give the block a jerk to the right as discussed earlier. Let the extension of the spring be “x”. Then,

$$F = -kx$$

As the force is variable, we can not use the expression “Frcosθ” to determine work. It is valid for constant force only. We need to apply expression, involving integration to determine work :

$$W_S = \int F\left(x\right) dx$$

Let x_i and x_f be the initial and final positions of the block with respect to origin, then

$$W_S = \int_{x_i}^{x_f} -kx dx$$
$$\Rightarrow W_S = -k\left[\frac{x^2}{2}\right]_{x_i}^{x_f}$$
$$\Rightarrow W_S = \frac{1}{2}k\left({x_i}^2 - {x_f}^2\right)$$

If block is at origin, then $x_i = 0$ and $x_f = x$ (say), then

$$\Rightarrow W_S = -\frac{1}{2}kx^2$$

When block is subjected to the single force due to spring, the work is given by above expressions. During motion towards right, the spring force on the block acts opposite to the direction of motion. Its magnitude increases with increasing displacement. Thus, spring force does negative work transferring

energy "from" the block. As a consequence, kinetic energy of the block (speed) decreases.

This process continues till the velocity and kinetic energy of the block are zero. Spring force, then, pulls the block towards the mean position i.e. origin in the negative x - direction. The spring force, now, is in the direction of motion. It does the positive work on the block transferring energy to the block.

The process continues till the particle returns to the initial position, when its velocity is same as that in the beginning. As there is no dissipative force like friction, kinetic energy of the block on return, at mean position i.e. origin, is equal to that in the beginning.

$$\Rightarrow K_f = K_i$$

For this round trip, net work by spring force is zero. Net transfer of energy “to” or “from” the particle is zero. Initial kinetic energy of the particle is retained at the end of round trip. Thus, we can see that the motion under spring is lot similar as that of motion under gravity.

Can we guess here - what will happen to block hereafter? The block has velocity towards left. As such, it will move past the origin. However, spring force will come into picture immediately with compression in the spring and pull the block in the opposite direction i.e towards origin. In doing so, the spring force draws kinetic energy "from" the system. The process continues till the block stops towards an extreme position on the left.

Thereafter, the spring force accelerates the block in the opposite direction i.e. towards origin. If there is no dissipation of energy involved (an ideal condition), this process continues and the block oscillates about the origin.

Example:

Problem : A block of 1 kg is attached to the spring and is placed horizontally with one end fixed. If spring constant is 500 N/m, find the

work done by the horizontal force to pull the spring slowly through an extension of 10 cm.

Solution : Though, it is not explicitly stated, but it can be inferred from the word "slowly" that the block is pulled without any kinetic energy. This is a situation, when initial and final speeds are zero. This means that initial and final kinetic energies are zero (equal). Hence, work done by the two forces (external applied force and spring force) is zero. In this condition, work done by the horizontal force is equal to the work done by the spring force, but opposite in sign. Now work done by the spring force is :

$$W_S = -\frac{1}{2} \text{ x } 500 \text{ x } 0.1^2$$
$$W_S = -2.5\ J$$

Thus, work done by the horizontal force is :

$$W_S = 2.5\ J$$

We must note that this situation is exactly similar to that of gravity. In order to find the work by any other external force, we calculate work by spring force. In the nutshell, we measure work by other external force indirectly using a known work by spring force.

Motion with spring and other force(s)

We must understand that work by spring force, for a given displacement, is independent of the presence of other forces. The work done by spring remains same.

For this situation, “work-kinetic energy” theorem has following form :

$$\Rightarrow K_f - K_i = W_S + W_F$$

where W_S and W_F are the work done by the spring force and other applied force(s) respectively.

Here, work done by other external force(s) may be analyzed with respect to following different conditions :

1. Initial and final speeds are zero.
2. Initial and final speeds are same.
3. Initial and final speeds are different.

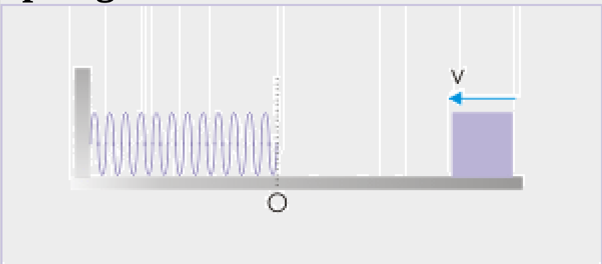
In the first two cases, initial and final kinetic energy are same. Hence,

$$\Rightarrow K_f - K_i = W_S + W_F = 0$$
$$\Rightarrow W_F = -W_S$$

This is an important result. This means that we can simply compute the work done by spring force and assign the same preceded by a negative sign as the work done by other force(s). Such situation can arise when external force displaces the block and extends the spring such that end velocities are either zero or same.

In third case, kinetic energies at end points are not same. However, work done by spring force remains same as before. Thus, the difference in kinetic energy during a motion is attributed to the net work as done by spring and other forces.

Example:
Problem : A block of 1 kg with a speed 1 m/s hits a spring placed horizontally as shown in the figure. If spring constant is 1000 N/m, find the compression in the spring.
Spring and block

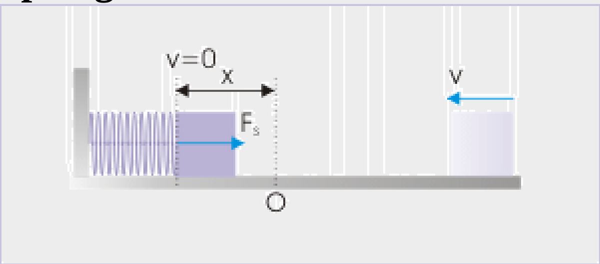


Spring is compressed by the

striking block.

Solution : When block hits the spring, it is compressed till the block stops. Here, we see that kinetic energies at the beginning and at the point when block stops are not same. Note that the only force acting on the block is due to spring. Hence,

Spring and block



At the end block comes to a stop momentarily.

$$K_f - K_i = W_S$$

$$\Rightarrow W_S = K_f - K_i = 0 - \frac{1}{2}mv^2$$
$$\Rightarrow W_S = -0.5 \text{ x } 1 \text{ x } 1^2 = -0.5\, J$$

Now, work by spring is :

$$\Rightarrow W_S = -\frac{1}{2}kx^2$$
$$\Rightarrow W_S = -0.5 \text{ x } 1000 \text{ x } x^2 = -500 \text{ x } x^2$$

Combining two values, we have :

$$\Rightarrow -250 \text{ x } x^2 = -0.5$$

$$\Rightarrow x^2 = \frac{0.5}{500} = 0.001$$
$$\Rightarrow x = 0.032\, m$$

Work by gravity and spring force (check your understanding)

The questions have been selected to enhance understanding of the topics covered in the modules titled " Work by gravity " and " Work by spring force " . All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

Understanding level (Work by gravity and spring force)

Exercise:

Problem:

An object released from the top of a rough incline of height "h" reaches its bottom and stops there. If coefficient of kinetic friction between surfaces is 0.5, then what should be the work done by the external force to return the object along the incline to its initial position ?

$$\text{(a)}\ \ 1.5\text{ mgh}\ \ \text{(b)}\ \ 2\text{mgh}\ \ \text{(c)}\ \ 3.5\text{ mgh}\ \ \text{(d)}\ \ 4\text{ mgh}$$

Solution:

Speeds at initial and final poistions are same. Therefore, total work done by gravity and friction during downward motion is zero according to "work-kinetic energy" theorem. It means that work done during downward motion by gravity (mgh) equals work done by friction (-mgh), which is opposite in sign. During upward motion, both gravity and friction oppose motion of the object. Hence, work by

external force must equal to the magnitude of negative works by gravity and friction.

$$W_{\text{Ext}} = W_{\text{Gravity}} + W_{\text{Friction}}$$
$$\Rightarrow W_{\text{Ext}} = 2W_{\text{Gravity}}$$
$$\Rightarrow W_{\text{Ext}} = 2mgh$$

Note: The work by external force is positive as direction of force and displacement are same.

Hence, option (b) is correct.

Exercise:

Problem:

Two springs, having spring constants k_1 and k_2 ($k_1 > k_2$), are stretched by same external force. If W_1 and W_2 be the work done by the external force on two springs, then :

$$\text{(a) } W_1 = \frac{k_1}{k_2} W_2 \text{ (b) } W_1 < W_2 \text{ (c) } W_1 > W_2 \text{ (d) } W_1 = W_2$$

Solution:

Work done by the external force in stretching spring by "x" is :

$$W = \frac{1}{2} k x^2$$

We need to eliminate extension "x" in terms of "F" and "k" in order to make the comparison for same force. Now, external force on the spring is :

$$F = kx$$

Combining two equations, we have :

$$W = \frac{F^2}{2k}$$

For same external force,

$$\Rightarrow \frac{W_1}{W_2} = \frac{k_2}{k_1}$$

As $k_1 > k_2$,

$$\Rightarrow W_1 < W_2$$

Hence, option (b) is correct.

Exercise:

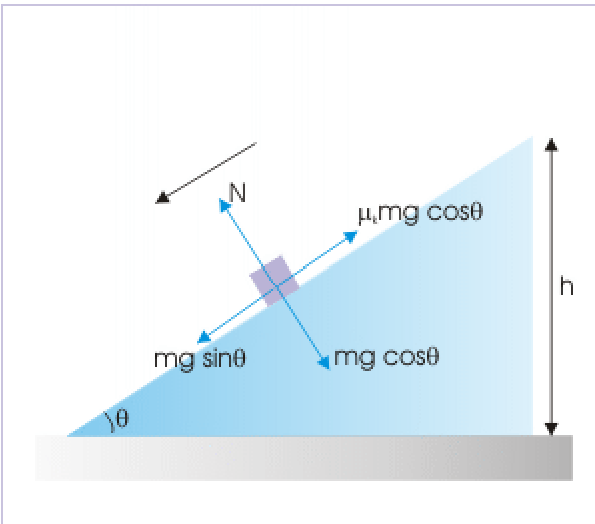
Problem:

A block is released from the height "h" along a rough incline. If "m" is the mass of the block, then speed of the object at the bottom of the incline is proportional to :

$$\text{(a) } m^0 \text{ (b) } m^1 \text{ (c) } m^2 \text{ (d) } m^3$$

Solution:

We need to obtain an expression for the speed "v". This relation can be obtained, using "work - kinetic energy" theorem :
Block on an incline



$$\frac{1}{2}mv^2 = mgl\sin\theta - \mu_k mgl\cos\theta$$
$$\Rightarrow \frac{1}{2}v^2 = gl\sin\theta - \mu_k gl\cos\theta$$

Clearly, the speed at the bottom of the incline is independent of mass "m".

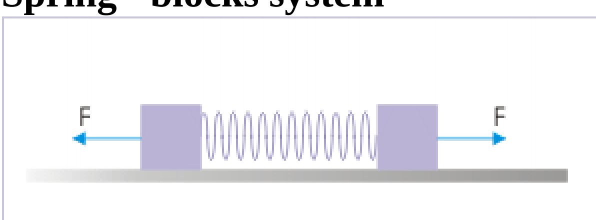
Hence, option (a) is correct.

Exercise:

Problem:

Two blocks of equal masses are attached to ends of a spring of spring constant "k". The whole arrangement is placed on a horizontal surface and the blocks are pulled horizontally by equal force to produce an extension of "x" in the spring. The work done by the spring on each block is :

Spring - blocks system



$$\text{(a)}\ \frac{1}{2}kx^2\ \text{(b)}\ \frac{1}{4}kx^2\ \text{(c)}\ -\frac{1}{2}kx^2\ \text{(d)}\ -\frac{1}{4}kx^2$$

Solution:

The work done by a spring force for an extension "x" is :

$$W_S = -\frac{1}{2}kx^2$$

However, spring does this work on blocks of same mass, which are pulled by equal horizontal force. The work by spring is, thus, equally divided on two blocks. Hence, work done by the sping on each block is :

$$\text{W'}_S = -\frac{1}{4}kx^2$$

Hence, option (d) is correct.

Exercise:

Problem:

An external force does 4 joule of work to extend it by 10 cm. What additional work (in Joule) is required to extend it further by 10 cm ?

$$\text{(a)}\ 4\ \text{(b)}\ 8\ \text{(c)}\ 12\ \text{(d)}\ 16$$

Solution:

The first part of the question can be used to determine the spring constant. The work done by external force is :

$$W = \frac{1}{2}kx^2$$

$$\Rightarrow k = \frac{2W}{x^2}$$
$$\Rightarrow k = \frac{2\text{x}4}{0.1^2} = 800\ N/m$$

Now, work done to extend the spring by a total of 20 cm,

$$\Rightarrow W = \frac{1}{2}\ \text{x}\ 800\ \text{x}\ 0.2^2 = 16\ J$$

Thus, additional work required is 16 - 4 = 12 J

Hence, option (c) is correct.

Application level (Work by gravity and spring force)

Exercise:

Problem:

Rain drop falls from a certain height and attains terminal velocity before hitting the ground. If radius of rain drop is "r", then work done by the forces (gravity and viscous force) on the rain drop is proportional to :

$$\text{(a) } r \;\; \text{(b) } r^3 \;\; \text{(c) } r^5 \;\; \text{(d) } r^7$$

Solution:

Let "v" be the terminal speed. The change in kinetic energy ($\frac{1}{2}mv^2$) is equal to the work done by gravity and viscous force. Gravity does the positive work, whereas viscous force does negative force. When the drop attains terminal velocity, then viscous force equals the weight of the rain drop,

$$F = mg$$

$$\Rightarrow 6\pi\eta rv = mg$$

$$\Rightarrow v = \frac{mg}{6\pi\eta r}$$

$$\Rightarrow v = \frac{\frac{4}{3}\pi r^3\rho g}{6\pi\eta r}$$

$$\Rightarrow v \propto r^2$$

$$\Rightarrow v = Ar^2$$

where "A" is a constant. Now, work done by the forces :

$$W = \frac{1}{2}mv^2$$
$$\Rightarrow W = \frac{1}{2} \text{ x } \frac{4}{3}\pi r^3 \rho A^2 r^4$$
$$\Rightarrow W \propto r^7$$

Hence, option (d) is correct.

Answers

`1. (b) 2. (b) 3. (a) 4. (d) 5. (c) 6. (d)`

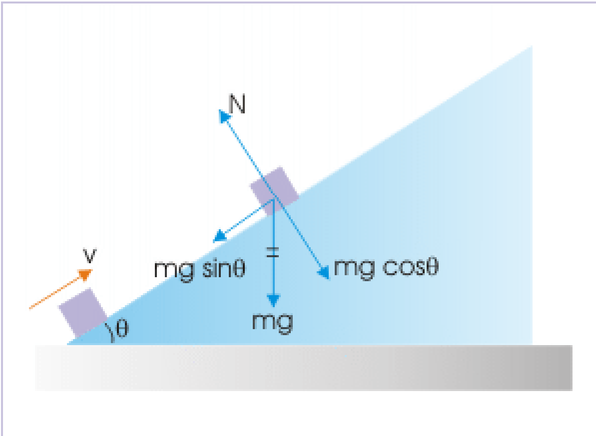
Conservative force
Total work done by conservative force in a closed path motion is zero.

Forces differ in one important aspect. They differ in the way they transfer energy from an object. All forces are consistent and similar in their character in transferring energy "from" an object in motion. They actually differ in their ability to return the energy "to" the object, when motion is reversed. One class of force conserves energy in the form of potential energy and transfers the same when motion is reversed. Additionally, these forces return the energy to object in equal measure with respect to the energy taken from the object. This means that motion of an object interacted by conservative force(s), involving reversal of motion end up regaining its initial motion. The class of forces that conserve energy for reuse is called "conservative" forces. Similarly, the class of forces that do not conserve energy for reuse are called "non-conservative" force.

We have noticed that forces such as gravitation force, elastic force, electromagnetic forces etc conform to the requirement of energy reuse as decribed above. Hence, these forces are conservative forces.

Understanding conservative force

In order to understand the perspective of conservative force, we consider a block of mass "m", which is projected up a smooth incline. Here, interactiion involves gravitational force. The force due to gravity, opposing the motion, is mg sinθ along the incline. Let us consider that the block travels a total length "L" along the incline before returning to the point of projection.
A block projected on a smooth incline plane



The force due to gravity does
the work on the block.

Work done by gravity during motion in upward direction is :

$$W_{\mathrm{G(up)}} = -mgL\sin\theta$$

Work done by gravity during motion in downward direction is :

$$W_{\mathrm{G(down)}} = mgL\sin\theta$$

Thus, we find that :

$$W_{\mathrm{G(up)}} = -W_{\mathrm{G(down)}}$$

This condition, when stated in general terms, specify whether the interacting force is conservative or not. In general, let W_1 and W_2 be the work done by the interacting force during motion and reversal of motion respectively. Then, the inetracting force is conservative force, if :

$$\Rightarrow W_1 = -W_2$$

Work done along closed path is zero :

$$\Rightarrow W = W_1 + W_2 = 0$$

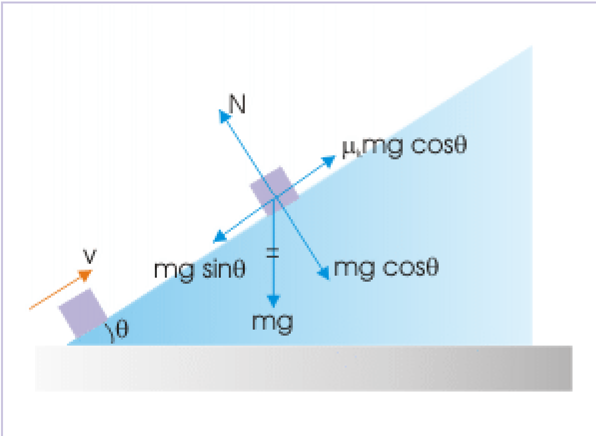
We summarize important points about the motion, which is interacted by conservative force :

- The object regains initial motion (kinetic energy) on return to initial position in a closed path motion.
- Conservative force transfers energy "to" and "from" an object during a closed path motion in equal measure.
- Conservative force tansfers energy between kinetic energy of the object in motion and the potential energy of the system interating with the object.
- Work done by conservative force is equal to work done by it on reversal of motion.
- Total work done by conservative force in a closed path motion is zero.

Understanding non-conservative force

It is also inferred from the discussion above that the other class of forces known as "non-conservative" force do not meet the requirement as needed for being conservative. They transfer energy "from" the object in motion just like conservative force, but they do not transfer this energy "to" the potential energy of the system to regain it during reverse motion. Instead, they transfer the energy to the system in an energy form which can not be used by the force to transfer energy back to the object in motion. Friction is one such non-conservative force.

Let us, now, consider the motion of a block projected on a rough incline instead of a smooth incline. Here, force due to gravity and friction act on the block. During up motion force due to gravity and friction together oppose motion and does negative work on the block. As block is subjected to greater force than earlier, the block travels a lesser distance (L'). Works done by force due to gravity and friction during upward motion are :
A block projected on a rough incline plane



The force due to gravity and friction do work on the block.

$$W_{\mathrm{G(up)}} = -mg\mathrm{L'}\sin\theta$$

$$W_{\mathrm{F(up)}} = -\mu_k mg\mathrm{L'}\sin\theta$$

Total work done during upward motion is :

$$W_{\mathrm{up}} = -mg\mathrm{L'}(\sin\theta + \mu_k\cos\theta)$$

On the other hand, force due to gravity does positive work, whereas friction again does negative work during downward motion. Works done by force due to gravity and friction during downward motion are :

$$W_{\mathrm{G(down)}} = mg\mathrm{L'}\sin\theta$$

$$W_{\mathrm{F(down)}} = -\mu_k mg\mathrm{L'}\sin\theta$$

Total work done during downward motion is :

$$W_{\mathrm{down}} = mg\mathrm{L'}(\sin\theta - \mu_k\cos\theta)$$

Total work during the motion along closed path is,

$$W_{\mathrm{total}} = -2mg\mathrm{L'}\mu_k\cos\theta$$

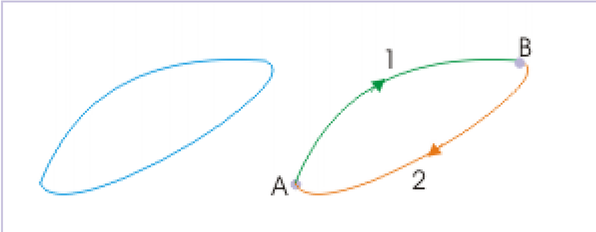
Here, net work in the closed path motion is not zero as in the case of motion with conservative force. As a matter of fact, net work in a closed path motion is equal to work done by the non-conservative force. We observe that friction transfers energy from the object in motion to the "Earth - Incline - block" system in the form of thermal energy - not as potential energy. Thermal energy is associated with the motion of atoms/ molecules composing block and incline. Friction is not able to transfer energy "from" thermal energy of the system "to" the object, when motion is reversed in downward direction. In other words, the energy withdrawn from the motion is not available for reuse by the object during its reverse motion.

We summarize important points about the motion, which is interacted by conservative force :

- The speed and kinetic energy of the object on return to initial position are lesser than initial values in a closed path motion.
- Non - conservative force does not transfers energy "from" the system "to" the object in motion.
- Non - conservative force transfers energy between kinetic energy of the object in motion and the system via energy forms other than potential energy.
- Total work done by non-conservative force in a closed path motion is not zero.

Path independence of conservative force

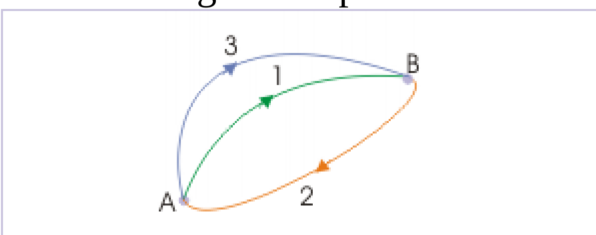
We have already seen that work done by the gravity in a closed path motion is zero . We can extend this observation to other conservative force systems as well. In general, let us conceptualize what we have learnt here about conservative force. We imagine a closed path motion. We imagine this closed path motion be divided in two motions between points "A" and "B". Starting from point "A" to point "B" and then ending at point "A" via two work paths named "1" and "2" as shown in the figure. As observed earlier, the total work by the conservative force for the round trip is zero :
Motion along closed path



$$W = W_{\text{AB1}} + W_{\text{BA2}} = 0$$

Let us now change the path for motion from A to B by another path, shown as path "3". Again, the total work by the conservative force for the round trip via new route is zero :

Motion along closed path



$$W = W_{\text{AB3}} + W_{\text{BA2}} = 0$$

Comparing two equations,

$$W_{\text{AB1}} = W_{\text{AB3}}$$

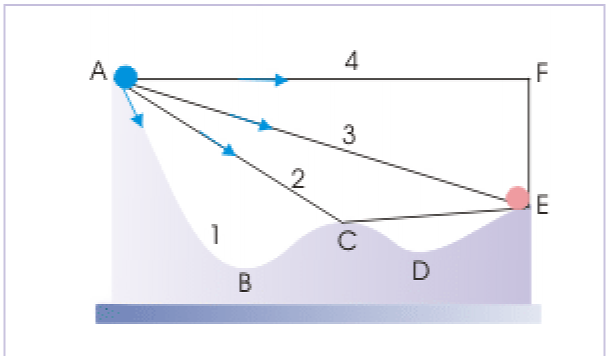
Similarly, we can say that work done for motion from A to B by conservative force along any of the three paths are equal :

$$W_{\text{AB1}} = W_{\text{AB2}} = W_{\text{AB3}}$$

We summarize the discussion as :

1: Work done by conservative force in any closed path motion is zero. The word "any" is important. This means that the configuration of path can be shortest, small, large, straight, two dimensional, three simensional etc. There is no restriction about the path of motion so long only conservative force(s) are the ones interacting with the object in motion.

Motion along closed path



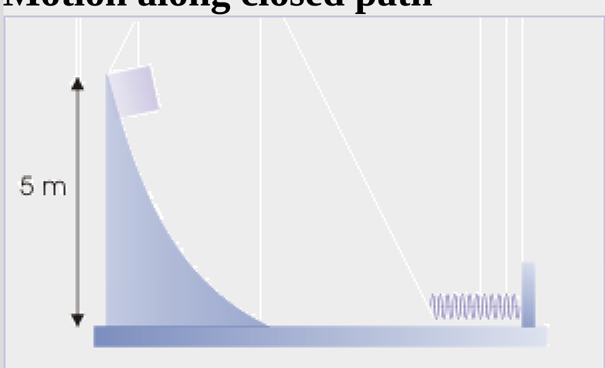
2: Work done by conservative force(s) is independent of the path between any two points. This has a great simplifying implication in anlyzing motions, which otherwise would have been tedious at the least. Four paths between "A" and "E" as shown in the figure are equivalent in the context of work done by conservative force. We can select the easiest path for calculating work done by the conservative force(s).

3: The system with conservative(s) force provides a mechanical system, where energy is made available for reuse and where energy does not become unusable for the motion.

Example:

Problem : A small block of 0.1 kg is released from a height 5 m as shown in the figure. The block following a curved path tansitions to a linear horizontal path and hits the spring fixed to a wedge. If no friction is involved and spring constant is 1000 N/m, find the maximum compression of the spring.

Motion along closed path



Solution : Here we shall make use of the fact that only conservative force is in play. We need to know the speed of the block in the horizontal section of the motion, before it strikes the spring. We can use "work - kinetic energy" theorem. But, the component of gravity along the curved path is a variable force. It would be difficult to evaluate the work done in this section, using integration form of formula for work.
Knowing that work done is independent of the path, we can evaluate the workdone for motion along a vertical straight path.

$$W = mgh$$

According "work - kinetic energy" theorem,

$$K_f - K_i = W$$

Here, speed at the time of release is zero. Thus, K_i is zero. Let the speed of the block on the horizontal section before hitting the spring be "v", Then,

$$\Rightarrow \frac{1}{2} mv^2 = mgh$$

Now, we switch our consideration to the compression of the spring. Let the maximum compression be "x". Then according "work - kinetic energy" theorem,

$$K_f - K_i = W$$

Here, speed at maximum compression is zero. Thus, K_f is zero.

$$\Rightarrow -\frac{1}{2} mv^2 = -\frac{1}{2} kx^2$$

Combining equations :

$$\Rightarrow mgh = \frac{1}{2} kx^2$$

$$\Rightarrow x = \sqrt{\left(\frac{2mgh}{k}\right)}$$

$$\Rightarrow x = \sqrt{\left(\frac{\text{2x0.1x5x5}}{1000}\right)} = 0.1\, m = 10 \text{ cm}$$

Note : This problem can be solved with simpler calculation using the concept of the conservation of mechanical energy. Energy concept will eliminate intermediate steps involving "work-kinetic energy" theorem. This point will be elaborated in the module on conservation of mechanical energy.

Mathematical form

The two famous conservative forces viz gravitational and coulomb forces obey inverse square law. It, sometimes, leads us to think that a force needs to conform to this mathematical form to be conservative. As a matter of fact, this is not the requirement. The only requirement is that evaluation of work for a force is independent of intermediate details. It means that we should be able to determine work by merely providing the initial and final values.

Potential energy

Potential energy refers to the energy arising from the arrangement of a system of objects (particles), which interact with each other. Here, interaction means that objects apply force on each other. A change in the arrangement brings about a change in the forces interacting on the objects and the associated potential energy.

It is not very difficult to understand how energy can be associated with the arrangement of objects within a system. It is an unsaid rule, but it is a fact that objects (matter) always exist under the influence of different forces and as such objects interact with each other. One or combination of forces is actually the reason that we have a system of objects together. When the arrangement of the objects changes in the arrangement, there is change in the potential energy of the system . We facilitate the study of change in arrangement by an unique design. Generally, we isolate the object whose motion is to be studied from the rest of system that applies force on the object.

This definition of potential energy as quoted above is not definitive in the sense that the same can not be used for quantitative measurement. We shall formulate explicit mathematical expression in this module for potential energy for the general case. There are various forms of potential energy : gravitational, elastic, electromagnetic and such others potential energies. The meaning and definition of potential energy in the specific context is built upon this general definition.

Unlike kinetic energy, we do not refer or measure absolute value of potential energy. We generally work with the difference in potential energies between two states. Which of the configuration of particles can be considered to possess zero potential energy? One theoretical zero energy reference is the distribution of given particles at infinity. It is expected that no force operates on them and hence their potential energy is considered zero there. However, we can consider any other configuration as zero reference for our convenience. We consider, for example, ground level as zero potential reference in the case of gravitation. The main point is that measurement of change in potential energy does not change with the choice of reference.

From the discussion so far, we underline two characterizing aspects of potential energy :

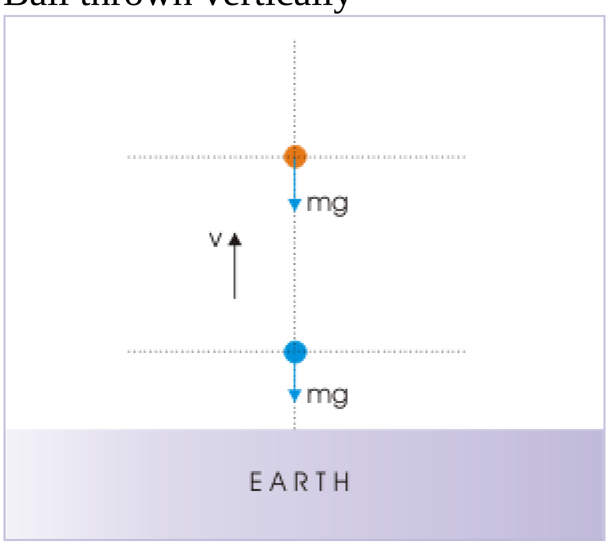
1. Potential energy is associated with a system of objects - not with a single object.
2. Generally, potential energy of a particle at infinity is considered zero. In practice, we can refer measurement of potential energy with respect to certain datum.

Importantly, the above two features of potential energy is different to kinetic energy, which has an absolute value and is referred to a single object.

Change in potential energy

In order to derive a general expression for change in potential energy, we refer to the vertical motion of a ball, projected upward. As the ball moves up, initial kinetic energy of the ball decreases by the amount of negative work done by the gravity. In the meantime, however, separation between Earth and ball also increases. Consequently, the potential energy of "Earth - ball" system increases by the same amount. In effect, gravity transfers energy "from" the kinetic energy of the ball "to" the potential energy of the "Earth - ball" system.

Ball thrown vertically



Gravity transfers energy between object and system.

In the reverse motion i.e. when ball starts moving downward, gravity transfers energy "from" potential energy of the "Earth - ball" system "to" the kinetic energy of the ball. The transfer of energy in "up" and "down" motion is same in quantity. This means that exchange of energy between "ball" and "Earth - ball" system is replicated in reverse motion.

A positive work on the object means that energy is being transferred from the system to the object. Consequently, the potential energy of the system decrease. As such, final potential energy of the system is lesser than initial potential energy. On the other hand, a negative work on the object means that energy is being transferred from the object to the system. Consequently, the potential energy of the system increases. As such, final potential energy of the system is greater than initial potential energy.

Between any two instants, the change in potential energy of the "Earth - ball" system is negative of the work done by the gravity on the "ball".

$$\Delta U = -W$$

The result so obtained for gravitational force can be extended to any conservative force. The change in potential energy of a system, therefore, is defined "as the negative of the work done by conservative force". Note the wordings here. We have defined potential energy of a system - not to an individual entity. In practice, however, we refer potential energy to a particle with an implicit understanding that we mean a system - not an individual entity.

We must note following important aspects of potential energy as defined above :

1: We have derived this expression in the context of gravitational force, but is applicable to a class of forces (called conservative forces), which transfer energy in above manner. It is important to understand that there are non-

conservative forces, which transfer energy to the system in other energy forms like heat etc. - not in the form of potential energy.

2: This is the expression for change in potential energy, which assumes different forms in different contexts (for different conservative forces).

3: We should realize that we deal here with the "change" in potential energy - not with "unique" value of potential energy corresponding to a specific configuration. This can be handled with the help of a known reference potential energy. The measurement of potential energy with respect to a chosen reference value enables us to assign a unique value to potential energy corresponding to a specific system configuration. We shall further elaborate this aspect while discussing specific potential energies.

4: The change in potential energy differs to kinetic energy in one aspect that it can assume negative value as well. We must remember that kinetic energy is always positive. A positive change in potential energy means that final potential energy is greater than initial potential energy. On the other hand, a negative value of change in potential energy means that final potential energy is lesser than initial potential energy.

5: Potential energy is defined only in the context of conservative force. For the sake of clarity, we should also realize that the conservative force doing work is internal to the system, whose potential energy is determined. We can rewrite the defining relation of potential energy that work refers to work by conservative as :

$$\Delta U = -W_C$$

Change in potential energy for constant and variable forces

The relationship as given above forms the basic relation that is used to develop specific relations for different force systems. For constant conservative force, work is defined as :

$$W = F_C r \cos\theta$$

Substituting in the equation, the change in potential energy involving constant force is :

$$\Rightarrow \Delta U = -F_C r \cos\theta$$

For variable force having displacement in the same direction, the work is obtained as :

$$\Rightarrow W = \int F_C \left(r\right) dr$$

The change in potential energy is equal to the integral evaluated over two appropriate limits corresponding to displacement :

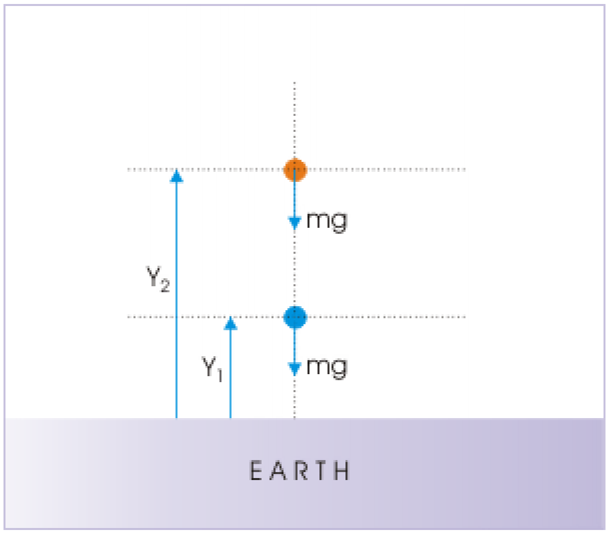
$$\Rightarrow \Delta U = -\int F_C \left(r\right) dr$$

In mechanical context, we deal with gravitational and elastic potential energies. Elastic potential energy involves any configuration, which can be deformed elastically like spring. In this module, we will drive specific expressions of potential energy for these two systems.

Gravitational potential energy

We shall evaluate expression for potential energy for the vertical motion of a ball, which is thrown up with a certain speed. The gravity "-mg" works against the motion. Importantly, gravity is a constant force and as such we can use the corresponding form for the change in potential energy. For a displacement from a height y_1 to y_2 , the change in potential energy for the "Earth - object" system is :

Ball thrown vertically



Gravity transfers energy between object and system.

$$\Rightarrow \Delta U = -\left(-mg\right)\left(y_2 - y_1\right)\cos 0^0$$
$$\Rightarrow \Delta U = mg(y_2 - y_1)$$

The change in potential energy is, thus, directly proportional to the vertical displacement. It is clear that potential energy of "Earth - ball" system increases with the separation between the elements of the system . Evaluation of change in gravitational potential energy requires evaluation of change in displacement.

Motion of an object on Earth involves "Earth - object" as the system. Thus, we may drop reference to the system and may assign potential energy to the object itself as the one element of the system i.e. Earth is common to all systems that we deal with on Earth. This is how we say that an object (not system) placed on a table has potential energy of 10 J. We must, however, be aware of the actual context, when referring potential energy to an object.

We should also understand here that we have obtained this relation of potential energy for the case of vertical motion. What would be the

potential energy if the motion is not vertical - like for motion on a smooth incline? The basic form of expression for the potential energy is :

$$\Rightarrow \Delta U = -Fr\cos\theta$$

Here, we realize that the right hand expression yields to zero for θ = 90°. Now, any linear displacement, which is not vertical, can be treated as having a vertical and horizontal component. The enclosed angle between gravity, which is always directed vertically downward, and the horizontal component of displacement is 90°. As such, the expression of change in potential energy evaluates to zero for horizontal component of displacement. In other words, gravitational potential energy is independent of horizontal component of displacement and only depends on vertical component of displacement.

This is an important result as it renders the calculation of gravitational potential lot easier and independent of actual path of motion. The change in gravitational potential energy is directly proportional to the change in vertical altitude. As a matter of fact, this result follows from the fact that the work done by the gravitational force is directly proportional to the change in vertical displacement.

Reference potential energy

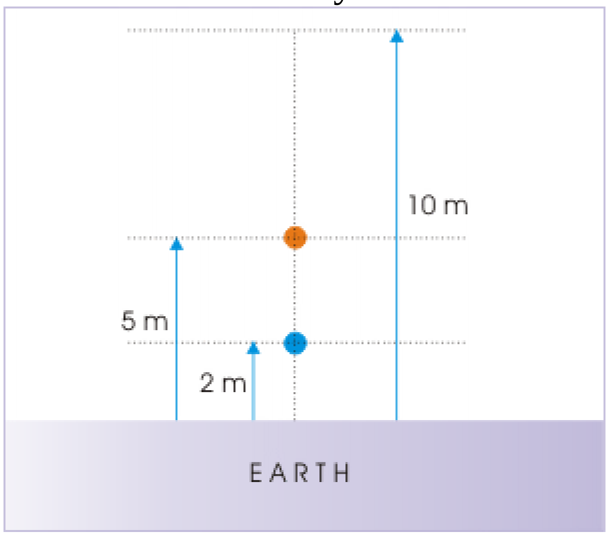
In order to assign unique value of potential energy for a specific configuration, we require to identify reference gravitational potential energy, which has predefined value - preferably zero value. Let us consider the expression of change in potential energy again,

$$\Rightarrow \Delta U = mg(y_2 - y_1)$$

The important aspect of this relation is that change in displacement is not dependent on the coordinate system. Whatever be the reference, the difference in displacement $(y_2 - y_1)$ remains same. Thus, we are at liberty as far as choosing a gravitational potential reference (and preferably assign

zero value to it). Let us consider here the motion of ball which rises from $y_1 = 2\,m$ to $y_2 = 5\,m$ with reference to ground. Here,

Ball thrown vertically



Change in potential energy is independent of choice of reference.

$$\Rightarrow \Delta U = mg(y_2 - y_1)$$
$$\Rightarrow \Delta U = mg(5 - 2) = 3mg$$

Let us, now consider if we measure displacement from a reference, which is 10 m above the ground. Then,

$$y_1 = -(10 - 2) = -8\,m$$
$$y_2 = -(10 - 5) = -5\,m$$

and

$$\Rightarrow \Delta U = mg(y_2 - y_1)$$
$$\Rightarrow \Delta U = mg\{-5 - (-8)\} = 3mg$$

If we choose ground as zero gravitational potential, then

$$y_1 = 0,\ U_1 = 0,\ y_2 = h\ \text{(say)},\ \text{and}\ U_2 = U$$

Gravitational potential energy of an object (i.e. "Earth - ball" system) at a vertical height "h" is :

$$\Rightarrow U_2 - U_1 = mg(y_2 - y_1)$$
$$\Rightarrow U - 0 = mg(h - 0)$$
$$\Rightarrow U = mgh$$

We must be aware that choice of reference does not affect the change in potential energy, but changes the value of potential energy corresponding to a particular configuration. For example, potential energy of the object at ground in ground reference is :

$$U_g = mgh = mg \text{ x } 0 = 0$$

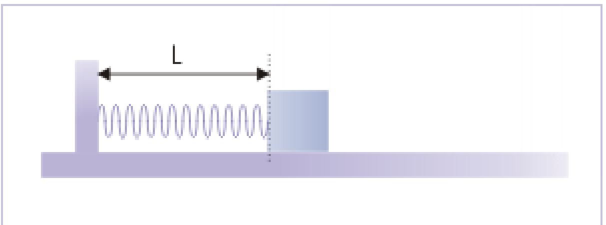
The potential energy of the object at ground in reference 10 m above the ground is :

$$\text{U'}_g = mgh = mg \text{ x } -10 = -10mg$$

Thus, we must be consistent with the choice of reference potential, when analyzing a situation.

Spring potential energy

Spring force transfers energy like gravitational force through the system of potential energy. In this sense, spring force is also a conservative force. When we stretch or compress a spring, the coils of the spring are accordingly extended or compressed, reflecting a change in the arrangement of the elements within the spring and the block attached to it. The spring and the block attached to the spring together form the system here as we consider that there is no friction involved between block and the surface. The relative change in the spatial arrangement of this system indicates a change in the potential energy of the system.
Spring block system



Energy is exchanged between block and "spring - block" system

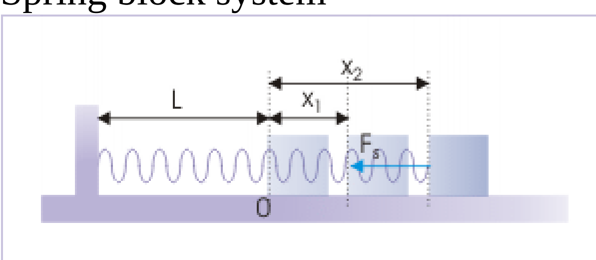
When we give a jerk to the block right ways, the spring is stretched. The coils of the spring tend to restore the un-stretched condition. In the process of this restoration effort on the part of the spring elements, spring force is applied on the block. This force tries to pull back the spring to the origin or the position corresponding to relaxed condition. The spring force is a variable force. Therefore, we use the corresponding expression of potential energy :

$$\Rightarrow \Delta U = -\int F(x)dx$$

$$\Rightarrow \Delta U = -\int(-kx)dx$$

Integrating between two positions from the origin of the spring,

Spring block system



Energy is exchanged between block and "spring - block" system

$$\Rightarrow \Delta U = \int_{x_i}^{x_f} kx dx$$

$$\Rightarrow \Delta U = k\left[\frac{x^2}{2}\right]_{x_i}^{x_f}$$

$$\Rightarrow\Rightarrow \Delta U = \frac{1}{2}k\left({x_f}^2 - {x_i}^2\right)$$

Considering zero potential reference at the origin,
$x_i = 0,\ U_i = 0,\ x_f = x\ (\text{say}),\ U_f = U\ (\text{say})$, we have :

$$\Rightarrow U = \frac{1}{2}kx^2$$

The important aspect of this relation is that potential energy of the block (i.e. "spring - block" system) is positive for both extended and compressed position as it involves squared terms of "x". Also, note that this relation is similar like that of work except that negative sign is dropped.

Absolute potential energy

Absolute potential energy is defined with reference to infinity.

There is a disconcerting aspect of potential energy. In the earlier module titled “Potential energy”, we defined “change in potential energy” – not the potential energy itself!

We assigned zero gravitational potential reference for Earth’s gravitation to the ground level and zero elastic potential energy to the neutral position of the spring. The consideration of zero reference potential energy enabled us to define and assign potential energy for a unique position – not to the difference of positions. This was certainly an improvement towards giving meaning to absolute value of potential energy of a system. In this module, we shall broaden the reference and aim to define absolute potential energy for a particular configuration of a system in general.

Reference at Infinity

The references to ground for gravitation or a neutral position for a spring are essentially local context. For example, gravitation is not confined to Earth system only. What if we want to refer potential energy value to an object on the surface of our moon? Would we refer its potential energy in reference to Earth’s ground?

We may argue that we can have moon’s ground as reference for the object on its surface. But this will also not serve purpose as there might be occasions (as always is in the study of the motions of celestial bodies) where we would need to compare potential energies of systems belonging to Earth and moon simultaneously. The point is that the general concept of potential energy can not be bounded to a local reference. We need to expand the meaning of reference, which is valid everywhere.

Now, we have seen that change in potential energy is equal to negative of work by conservative force. So existence of potential energy is related to existence of conservative force. Can we think a situation in which this conservative force is guaranteed to be zero. There is no such physical reference, but there is a theoretical possibility of such eventuality. Let us

have a look at the Newton's law of gravitation (this law will be discussed subsequently). The force of gravitation between two particles, " m_1 " and " m_1 " is given by :

$$F = \frac{G m_1 m_2}{r^2}$$

As $r \rightarrow \infty, \quad F \rightarrow 0$. As there is no force on the particle, there is no work involved. Hence, we can conclude that a system of two particles at a large (infinite) distance has zero potential. As infinity is undefined, we can think of system of particles at infinity, which are separated by infinite distances and thus have zero potential energy.

Theoretically, it is also considered that kinetic energy of the particle at infinity is zero. Hence, mechanical energy of the system of particles, being equal to the sum of potential and kinetic energy, is also zero at infinity.

Infinity appears to serve as universal zero reference. The measurement of potential energy of any system with respect to this zero reference is a unique value for a specific configuration of the system. Importantly, this is valid for all conservative force system and not confined to a particular force type like gravitation.

Definition of potential energy

Having decided the universal zero reference, we are now in position to define potential energy, using the expression obtained for the change in potential energy :

$$\varDelta U = U_2 - U_1 = -W_C$$

If we set initial position at infinity, then $U_1 = 0$. Let us denote potential energy of a system to be "U" for a given configuration. Then,

$$\Rightarrow U - 0 = -W_C$$

$$\Rightarrow U = -W_C = -\int_{\infty}^{0} F_C dr$$

Hence, we can now define potential energy as given here :

Potential energy

> The potential energy of a system of particles is equal to “negative” of the work by the conservative force as a particle is brought from infinity to its position in the presence of other particles of the system.

We should understand that the work by conservative force is independent of path and hence no reference is made about path in the definition. This work has a unique value. Hence, it gives a unique value of potential to the system of particles.

Distribution of potential energy

There is a peculiar aspect of the definition of potential energy, presented above. It defines potential energy of the system of particles in terms of work on a “single” particle. This peculiarity can be explained as it defines work in the presence of other particles and as such accounts for the forces operating on the particle due to their presence.

This definition, however, is not clear about how potential energy is distributed among the particles in the system. The value of potential energy does not throw any light on this aspect. As a matter of fact, it is not possible to segregate potential energy for the individual constituents of the system. Potential energy, therefore, belongs to all of them – not to any one of them.

Potential energy and external force

The potential energy is defined in terms of work by conservative force and zero reference potential at infinity. It is equal to the “negative” of work by conservative force :

$$\Rightarrow U = -W_C = -\int_{\infty}^{0} F_C dr$$

Can we think to express this definition of potential energy in terms of external force? In earlier module, we have analyzed the motion of a body, which is raised by hand slowly to a certain vertical height. The significant point of this illustration was the manner in which body was raised. It was, if we can recall, raised slowly without imparting kinetic energy to the body being raised. It was described so with a purpose. The idea was to ensure that work by the external force (in this case, external force is equal to the normal force applied by the hand) is equal to the work by gravity.

Since speed of the body is zero at the end points, "work-kinetic energy" theorem reduces to :

$$W = K_f - K_i = 0$$

$$W = 0$$

This means that work by "net" force is zero. It follows, then, that works by gravity (conservative force) and external force are equal in magnitude, but opposite in sign.

$$\Rightarrow W = W_C + W_F = 0$$

$$W_F = -W_C$$

Under this condition, the work by external force is equal to negative of work by conservative force :

$$\Rightarrow W_F = -\int F_C dr$$

where " F_C " is conservative force. It means that if we work on the particle slowly without imparting it kinetic energy, then work by the external force is equal to negative of the work by conservative force. In other words, work by external force without a change in kinetic energy of the particle is equal

to change in potential energy only. Equipped with this knowledge, we can define potential energy in terms of external force as :

Potential energy
: The potential energy of a system of particles is equal to the work by the external force as a particle is brought from infinity slowly to its position in the presence of other particles of the system.

The context of work in defining potential energy is always confusing. There is, however, few distinguishing aspects that we should keep in mind to be correct. If we define potential energy in terms of conservative force, then potential energy is equal to “negative” of work by conservative force. If we define potential energy in terms of external force, then potential energy is simply equal to work by external force, which does not impart kinetic energy to the particle.

Potential energy and conservative force

Potential energy is unique in yet another important respect. Unlike other forms of energy, potential energy is directly related to conservative force. We shall establish this relation here. We know that a change in potential energy is equal to the negative of work by gravity,

$$\Delta U = -F_C \Delta r$$

For infinitesimal change, we can write the equation as,

$$\Rightarrow dU = -Fc\, dr$$

$$\Rightarrow F_C = -\frac{dU}{dr}$$

Thus, if we know potential energy function, we can find corresponding conservative force at a given position. Further, we can see here that force – a vector – is related to potential energy (scalar) and position in scalar form. We need to resolve this so that evaluation of the differentiation on the right yields the desired vector force.

As a matter of fact, we handle this situation in a very unique way. Here, the differentiation in itself yields a vector. In three dimensions, we define an operator called “grad” as :

$$\text{grad} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)$$

where "

$$\frac{\partial}{\partial x}$$

" is partial differentiation operator. This is same like normal differentiation except that it considers other dimensions (y,z) constant. In terms of “grad”,

$$\Rightarrow \mathbf{F} = -\,\text{grad}\ U$$

The example given here illustrates the operation of “grad”.

Example

Problem 1: Gravitational potential energy in a region is given by :

$$U(x, y, z) = -\left(x^2 y + y z^2\right)$$

Find gravitational force function.

Solution : We can obtain gravitational force in each of three mutually perpendicular directions of a rectangular coordinate system by differentiating given potential function with respect to coordinate in that direction. While differentiating with respect to a given coordinate, we consider other coordinates as constant. This type of differentiation is known as partial differentiation.

Thus,

$$F_x = -\frac{\partial}{\partial x} = -\frac{\partial}{\partial x} - \left(x^2 y + yz^2\right) = 2xy$$

$$F_y = -\frac{\partial}{\partial y} = -\frac{\partial}{\partial y} - \left(x^2 y + yz^2\right) = x^2 + y^2$$

$$F_z = -\frac{\partial}{\partial z} = -\frac{\partial}{\partial z} - \left(x^2 y + yz^2\right) = 2yz$$

Hence, required gravitational force is given as :

$$\Rightarrow \mathbf{F} = -\operatorname{grad} U$$

$$\mathbf{F} = -\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)U$$

$$\mathbf{F} == 2xy\mathbf{i} + \left(x^2 + y^2\right)\mathbf{j} + 2yz\mathbf{k}$$

This example illustrates how a scalar quantity (potential energy) is related to a vector quantity (force). In order to implement partial differentiation by a single operator, we define a differential vector operator “grad” a short name for “gradient” as above. For this reason, we say that conservative force is equal to gradient of potential energy.

Potential energy values

Evaluation of the integral of potential energy is positive or negative, depending on the nature of work by conservative force.

$$U = -W_C = -\int_{\infty}^{0} F_C \mathbb{d} r$$

The nature of work by the conservative force, on the other hand, depends on whether force is attractive or repulsive. The work by attractive force like gravitation and electrostatic force between negative and positive charges do “positive” work. In these cases, component of force and displacement are in

the same direction as the particle is brought from infinity. However, as a negative sign precedes the right hand expression, potential energy of the system operated by attractive force is ultimately negative.

It means that potential energy for these conservative forces would be always a negative value. The important thing is to realize that maximum potential energy of such system is “zero” ay infinity.

On the other hand, potential energy of a system interacted by repulsive force is positive. Its minimum value is “zero” at infinity.

Note: We shall not work with numerical examples or illustrate working of different contexts presented in this module. The discussion, here, is limited to general theoretical development of the concept of potential energy for any conservative force. We shall work with appropriate examples in the specific contexts (gravitation, electrostatic force etc.) in separate modules.

Conservation of energy
When only conservative forces interact, the mechanical energy of an isolated system can not change.

Work - kinetic energy theorem is used to analyze motion of a particle. This theorem, as pointed out, is a consideration of energy for describing motion of a particle in general, which may involve both conservative and non-conservative force. However, “work-kinetic energy” theorem is limited in certain important aspects. First, it is difficult to apply this theorem to many particle systems and second it is limited in application to mechanical process – involving motion.

Law of conservation of energy is an extension of this theorem that changes the context of analysis in two important ways. First, it changes the context of energy from a single particle situation to a system of particles. Second, law of energy conservation is extremely general that can be applied to situation or processes other than that of motion. It can be applied to thermal, chemical, electrical and all possible processes that we can think about. Motion is just one of the processes.

Clearly, we are embarking on a new analysis system. The changes in analysis framework require us to understand certain key concepts, which have not been used before. In this module, we shall develop these concepts and subsequently conservation law itself in general and, then, see how this law can be applied in mechanical context to analyze motion and processes, which are otherwise difficult to deal with. Along the way, we shall highlight advantages and disadvantages of the energy analysis with the various other analysis techniques, which have so far been used.

In this module, the “detailed” treatment of energy consideration will be restricted to process related to motion only (mechanical process).

Mechanical energy

Mechanical energy of a system comprises of kinetic and potential energies. Significantly, it excludes thermal energy. Idea of mechanical energy is that it represents a base line (ideal) case, in which a required task is completed with minimum energy. Consider the case of a ball, which is thrown upward with certain initial kinetic energy. We analyze motion assuming that there is no air resistance i.e. drag on the ball. For a given height, this assumption represents the baseline case, where requirement of initial kinetic energy for the given height is least. Mechanical energy is expressed mathematically as :

$$E_M = K + U$$

We can remind ourselves that potential energy arises due to “position” of the particle/ system, whereas kinetic energy arises due to “movement” of the particle/ system.

Other forms of energy

Different forms of energy are subject of individual detailed studies. They are topics of great deliberation in themselves. Here, we shall only briefly describe characteristics of other forms of energy. One important aspect of other forms of energy is that they are simply a macroscopic reflection of the same mechanical energy that we talk about in mechanics.

On a microscopic level or still smaller level, other forms of energy like thermal, chemical, electrical and nuclear energies are actually the same potential and kinetic energy. It is seen that scale of dimension involved with energy changes the ultimate or microscopic nature of mechanical energy as different forms of energy.

Hence, “thermal” energy is actually a macroscopic reflection of kinetic energy of the atoms and molecules. On the other hand, “internal” energy is potential and kinetic energy of the particles constituting the system. Similarly, various forms of electromagnetic energy (electrical and magnetic energy) are actually potential and kinetic energy. Besides, field energy like radiation energy does not involve even matter.

Nevertheless, analysis of energy at macroscopic level requires that we treat these forms of energy as a different energy with respect to mechanical energy, which we associate with the motion of the system

Process

In mechanics, we are concerned with motion – a change in position. On the other hand, energy is a very general concept that extends beyond change in position i.e. motion. It may involve thermal, chemical, electric and such other changes called processes. For example, a body may not involve motion as a whole, but atoms/molecules constituting the body may be undergoing motion all the time. For example, work for gas compression does not involve locomotion of the gas mass. It brings about change in internal and heat energy of the system.

Clearly, we need to change our terminology to suit the context of energy.

From the energy point of view, a motion, besides involving a change in position, also involves heat due to friction. For example, heat is produced, when a block slides down a rough incline. Thus, we see that even a process involving motion (mechanical process) can involve energy other than mechanical energy (potential and kinetic energy). The important point to underline is that though “mechanical energy” excludes thermal energy, but “mechanical process” does not.

System

A system comprises of many particles, which are interacted by different kinds of force. It is characterized by a boundary. We are at liberty to define our system to suit analysis of a motion or process. Everything else other than system is “surrounding”.

The boundary of the system, in turn, is characterized either to be “open”, “closed” or “isolated”. Accordingly, a system is open, closed or isolated. We shall define each of these systems.

In an “open” system, the exchange of both “matter” and “energy” are permitted between the system and its surrounding. In other words, nothing is barred from or to the system.

A “closed” system, however, permits exchange of energy, but no exchange of matter. The exchange of energy can take place in two ways. It depends on the type of process. The energy can be exchanged in the form of “energy” itself. Such may be the case in thermal process in which heat energy may flow “in” or “out” of the system. Alternatively, energy can be transferred by “work” on the system or by the system. In the nutshell, transfer of energy can take place either as “energy” or as “work”.

An “isolated” system neither permits exchange of energy nor that of matter. In other words, everything is barred “to” and “from” the system.

Interestingly, there is no exchange of mass in both “closed” and “isolated” system. Barring system involving nuclear reaction, the conservation of mass in these systems means that total numbers of atoms remain a constant.

Mechanical system

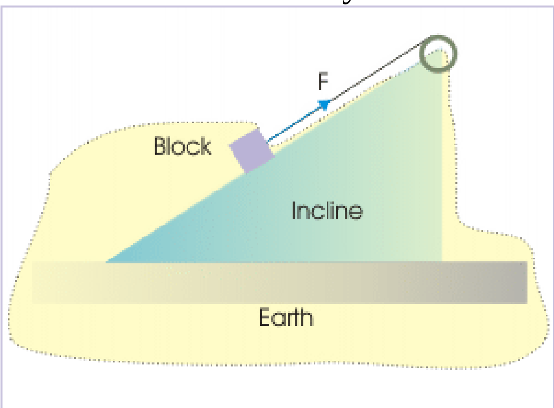
System types are defined with respect to “energy” and “matter”. However, we are required to know about the role of external force on a system in mechanics. Remember internal forces within a system sum up to zero. Does external force anyway change the system type?

In order to answer this question, we seek to know what does an external force do? For one, it tries to change the motion of the system and hence does the work. So if we apply the force on an isolated system, work will change energy of the system – even if system is isolated! Actually, there is nothing wrong in the definition of an isolated system. It says that energy and matter both are not exchanged. We know that work is just a form of “energy” is transit. Hence, work on the system is also barred by the definition.

We have actually brought this context with certain purpose. Our mechanical system is more akin to the description as given above. We need to know the role of force. Now, the question is what would we call such a system – a closed system or an isolated system.

Here, we consider an example of mechanics to bring out this point. Let us consider a block, which slides down a rough incline. Here, “Earth-incline-block” system is an isolated system as shown in the figure with an enclosed area.

“Earth-incline-block” system



“Earth-incline-block” system
plus an external force

Let us, now, consider an external force, which pushes the block up as shown in the figure. The work by force increases potential energy of the system, may increase kinetic energy of the block (if brought up with an acceleration) and produces heat in the isolated system.

In the nutshell, an external force on an isolated system is equivalent to "closed system", which allows exchange of energy with surrounding only via work by an external force. So it is a special case of "closed" system. Remember that energy exchange in "closed" system takes place either as "energy" or "work". In our mechanical context, the "closed" system allows exchange of energy as "work" only.

Interpretation of system type

The definitions of three system types are straight forward, but its physical visualization is not so easy. In the following paragraphs, we bring out important points about a system type.

1: Any system on Earth is linked with it – whether we say so or not. We have seen that "potential energy" of a particle as a matter of fact belongs to the system of "Earth – particle" system – not only to the particle. Since Earth is comparatively very large so that its mechanical energy does not change due to motion of a particle, we drop the Earth reference. It means if we draw a boundary for a system constituting particles only – we implicitly mean the reference to Earth and that Earth is part of the system boundary.

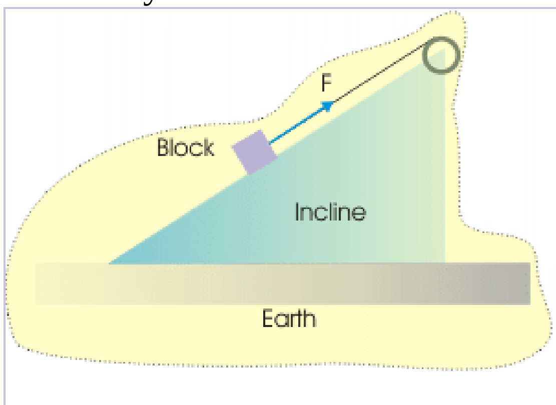
2: We should emphasize that system is not bounded by a physical boundary. We can understand this point by considering a "particle", which is projected vertically up from the surface of Earth. What would be the type of this "Earth-particle" system? It is an "open" system, if we think that particle is dragged by air. Drag is a non-conservative force. It takes out kinetic energy and transfers the same as heat or sound energy. The system, therefore, allows exchange of energy with the air, which physically lies in between "particle" and "Earth" of the system. However, if we assert that there is negligible air resistance, then we are considering the system as an isolated system. Clearly, it all depends what attributes we assign to the boundary in accordance with physical process.

3: We need to quickly visualize system type in accordance with our requirement. To understand this, let us get back to the example of "Earth-incline-block" system.

We can interpret boundary and hence system in accordance with the situation and objective of analysis.

In an all inclusive scenario, we can consider “Earth-incline-block” system plus the “agent applying external force” as part of an “isolated” system.

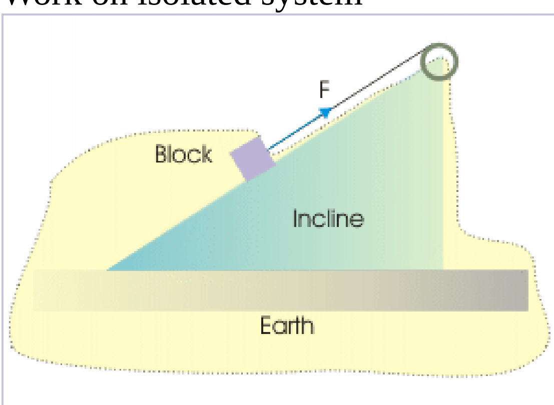
Isolated system



“Earth-incline-block” system
plus an external force

We can relax the boundary condition a bit and define a closed system, which allows transfer of energy via work only. In that case, “Earth-incline-block” system is “closed” system, which allows transfer of energy via “work” – not via any other form of energy. In this case, we exclude the agent applying external force from the system definition.

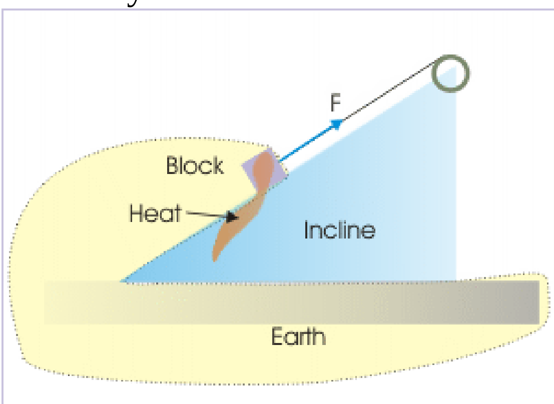
Work on isolated system



“Earth-incline-block” system
plus an external force

In the next step, we can define a proper “closed” system, which allows exchange of energy via both “work” and “energy”. In that case, “Earth – block” system constitutes the closed system. External force transfers energy by doing work, whereas friction, between block and incline, produces heat, which is distributed between the defined system of “Earth-block” and the “incline”, which is ,now, part of the surrounding. This system is shown below :

Closed system



“Earth-block” system

4: We see that energy is transferred between system and surrounding, if the system is either open or closed. Isolated system does not allow energy transfer. Clearly, system definition regulates transfer of energy between system and surrounding – not the transfer that takes place within the system from one form of energy to another. Energy transfers from one form to another can take place within the system irrespective of system types.

Work – kinetic energy theorem for a system

The mathematical statement of work – kinetic energy theorem for a particle is concise and straight forward :

$$W = \Delta K$$

The forces on the particle are external forces as we are dealing with a single particle. There can not be anything internal to a particle. For calculation of work,

we consider all external forces acting on the particle. The forces include both conservative (subscripted with C) and non-conservative (subscripted with NC) forces. The change in the kinetic energy of the particle is written explicitly to be equal to work by external force (subscripted with E) :

$$W_E = \Delta K$$

When the context changes from single particle to many particles system, we need to redefine the context of the theorem. The forces on the particle are both internal (subscripted with I)and external (subscripted with E) forces. In this case, we need to calculate work and kinetic energy for each of the particles. The theorem takes the following form :

$$\Rightarrow \sum W_E + \sum W_I = \sum \Delta K$$

We may be tempted here to say that internal forces sum to zero. Hence, net work by internal forces is zero. But work is not force. Work also involves displacement. The displacement of particles within the system may be different. As such, we need to keep the work by internal forces as well. Now, internal work can be divided into two groups : (i) work by conservative force (example : gravity) and work by non-conservative force (example : friction). We, therefore, expand "work-kinetic energy" theorem as :

$$\Rightarrow \sum W_E + \sum W_C + \sum W_{NC} = \sum \Delta K$$

Generally, we drop the summation sign with the understanding that we mean "summation". We simply say that both work and kinetic energy refers to the system – not to a particle :

$$\Rightarrow W_E + W_C + W_{NC} = \Delta K$$

However, work by conservative force is equal to negative of change in potential energy of the system.

$$W_C = -\Delta U$$

Substituting in the equation above,

$$\Rightarrow W_E - \Delta U + W_{NC} = \Delta K$$

$$\Rightarrow W_E + W_{NC} = \Delta K + \Delta U = \Delta E_{\text{mech}}$$

Work by non-conservative force

One of the familiar non-conservative force is friction. Work by friction is not path independent. One important consequence is that it does not transfer kinetic energy as potential energy as is the case of conservative force. Further, it only transfers kinetic energy of the particle into heat energy, but not in the opposite direction. What it means that friction is incapable to transfer heat energy into kinetic energy.

Nonetheless, friction converts kinetic energy of the particle into heat energy. A very sophisticated and precise set up measures this energy equal to the work done by the friction. We have seen that gravitational potential energy remains in the system. What about heat energy? Where does it go? If we consider a “Earth-incline-block” system, in which the block is released from the top, then we can visualize that heat so produced is distributed between “block” and “incline” (consider that there is no radiation loss). Next thing that we need to answer is “what does this heat energy do to the system?” We can infer that heat so produced raises the thermal energy of the system.

$$W_{NC} = -\mathit{\Delta} E_{\text{thermal}}$$

Note that work by friction is negative. Hence, we should put a negative sign to a positive change in thermal energy in order to equate it to work by friction.

Putting this in the equation of “Work-kinetic energy” expression, we have :

$$\Rightarrow W_E = \mathit{\Delta} K + \mathit{\Delta} U + \mathit{\Delta} E_{\text{thermal}} = \mathit{\Delta} E_{\text{mech}} + \mathit{\Delta} E_{\text{thermal}}$$

Law of conservation of energy

We now turn to something which we have not studied so far, but we shall employ those concepts to complete the picture of conservation of energy in the most general case.

Without going into detail, we shall refer to a consideration of thermodynamics. Work on the system, besides bringing change in the kinetic energy, also brings about change in the “internal” energy of the system. Similarly, combination of internal and external forces can bring about change in other forms of energy as well. Hence, we can rewrite “Work-kinetic energy” expression as :

$$\Rightarrow W_E = \mathit{\Delta} E_{\text{mech}} + \mathit{\Delta} E_{\text{thermal}} + \mathit{\Delta} E_{\text{others}}$$

This equation brings us close to the formulation of "conservation of energy" in general. We need to interpret this equation in the suitable context of system type. We can easily see here that we have developed this equation for a system, which allows energy transfers through work by external force. Hence, context here is that of special "closed" system, which allows transfer of energy only through work by external force. What if we choose a system boundary such that there is no external force. In that case, closed system becomes isolated system and

$$W_E = 0$$

Putting this in "work – kinetic energy" expression, we have :

$$\Rightarrow \Delta E_{\text{mech}} + \Delta E_{\text{thermal}} + \Delta E_{\text{others}} = 0$$

If we denote "E" to represent all types of energy, then :

$$\Rightarrow \Delta E = 0$$

$$\Rightarrow E = \text{constant}$$

Above two equations are the mathematical expressions of conservation of energy in the most general case. We read this law in words in two ways corresponding to above two equations.

Definition 1: The change in the total energy of an isolated system is zero.

Definition 2: The total energy of an isolated system can not change.

From above two interpretations, it emerges that "energy can neither be created nor destroyed".

Relativity and conservation of energy

Einstien's special theory of relativity establishes equivalence of mass and energy. This equivalence is stated in following mathematical form.

$$E = mc^2$$

Where "c" is the speed of light in the vacuum. An amount of mass, "m" can be converted into energy "E" and vice – versa. In a process known as mass annihilation of a positron and electron, an amount of energy is released as,

$$e^{+} + e^{-} = hv$$

The energy released as a result is given by :

$$E = 2 \ \text{X} \ 9.11 \ \text{X} \ 10^{-31} \ \text{X} \ 3 \ \text{X} \ \left(10^{8}\right)^{2} = 163.98 \ \text{X} \ 10^{-15} = 1.64 \ \text{X} \ 10^{-13} J$$

Similarly, in a process known as “pair production”, energy is converted into a pair of positron and electron as :

$$hv = e^{+} + e^{-}$$

This revelation of equivalence of “mass” and “energy” needs to be appropriately interpreted in the context of conservation of energy. The mass – energy equivalence appears to contradict the statement that “energy can neither be created nor destroyed”.

There is, however, another perspective. We may consider “mass” and “energy” completely exchangeable with each other in accordance with the relation given by Einstein. When we say that “energy can neither be created nor destroyed”, we mean to include “mass” also as “energy”. However, if we want to be completely explicit, then we can avoid the statement. Additionally, we need to modify the statements of conservation law as :

Definition 1: The change in the total equivalent mass - energy of an isolated system is zero.

Definition 2: The total equivalent mass - energy of an isolated system can not change.

Alternatively, we have yet another option to exclude nuclear reaction or any other process involving mass-energy conversion within the system. In that case, we can retain the earlier statements of conservation law with the qualification that process excludes “mass-energy” conversion.

In order to maintain semblance to pre-relativistic revelation, we generally retain the form of conservation law, which is stated in terms of “energy” with an implicit understanding that we mean to include “mass” as just another form of “energy”.

Nature of motion and conservation law

So far we have studied motion in translation. We need to clarify the context of conservation of energy with respect to other motion type i.e. rotational motion. In subsequent modules, we shall learn that there is a corresponding “work-energy theorem”, “potential energy concept” and actually a parallel system for analysis for rotational motion. It is, therefore, logical to think that constituents of the isolated system that we have considered for development of conservation of energy can possess rotational kinetic and potential energy as well. Thus, conservation law is all inclusive of motion types and associated energy. When we consider potential energy of the system, we mean to incorporate both translational and rotational potential energy. Similar is the case with other forms of energy – wherever applicable.

However, if we a have a system involving translational energy only, then it allows us to consider rigid body as point mass equivalent to particle of the system. This is a significant simplification as we are not required to consider angular aspect of motion and hence energy associated with angular motion.

System types and conservation law

It may appear that conservation law is subject to system definition. Certainly it is not. We state conservation law in the context of an isolated system for our convenience. We can as well state the law for “open” and “closed” system. Not only that we can have a statement of conservation law considering “universe” as the only system.

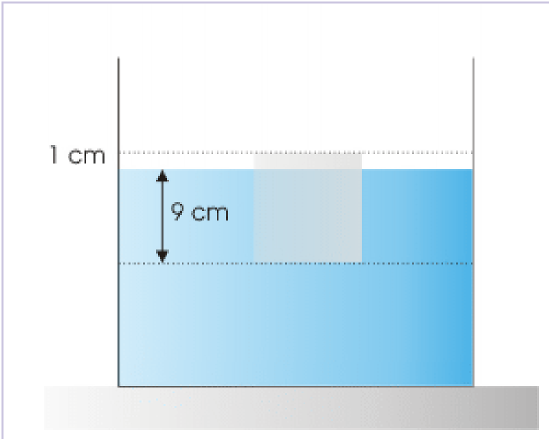
Actually, the statement of conservation law as “energy can neither be created nor destroyed”, applies to all systems including universe. For system like “open” or “closed” systems, which allow exchange of energy, we can think in terms of “transfer” of energy. A statement may be phrased like “change in the energy of the system is equal to the energy transferred “to” or “from” the system”.

We can be quite flexible in the application of conservation law with the help of “accounting” concept. We can consider “energy” as “money” in our account. Our account is credited or debited by the amount we deposit or withdraw money. Similarly, the energy of the system increases by the amount of energy supplied to the system and decreases by the amount of energy withdrawn form the system.

Example

Problem 1: An ice cube of 10 cm floats in a partially filled water tank. What is the change in gravitational potential energy (in Joule) when ice completely melts (sp density of ice is 0.9) ?

Ice cube in a tank



The ice cube is 90% submerged in the tank.

Solution :

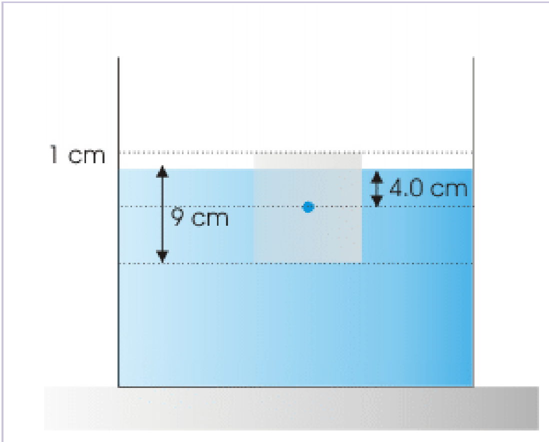
This question has been included with certain purpose. Though we have not studied "phase change" in the course up to this point, but we can apply our understanding broadly to understand this question. Along the way, we shall point out relevance of this question for the conservation of energy.

Now, gravitational potential energy will change if there is change in the water level or the level of center of mass of ice mass.

The ice cube is 90 % submerged in the water body as its specific density is 0.9. When it melts, the volume of water is 90 % of the volume of ice. Clearly, the melted ice occupies volume equal to the volume of submerged ice. It means that level of water in the tank does not change. Hence, there is no change in potential energy, as far as the water body is concerned.

However, the level of ice body changes after being converted into water. Its center of mass was 4.0 cm below the water level in the beginning, as shown in the figure.

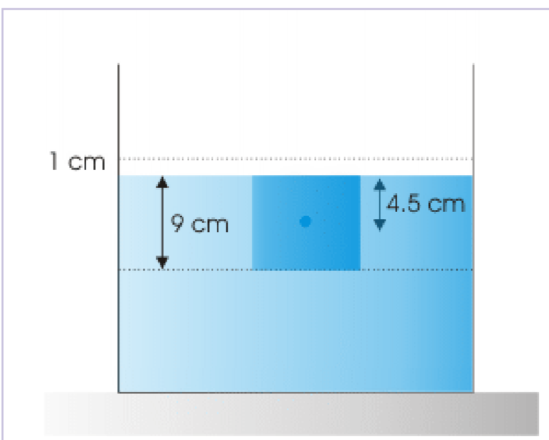
Ice cube in a tank



The center of mass of the ice cube is 4 cm below water level.

When ice converts in to water, the center of the converted water body is 4.5 cm below the same water level. Thus, there is a change of level by 0.5 cm. The potential energy of the ice, therefore, decreases :

Ice cube in a tank



The center of mass of the ice cube is 4.5 cm below water level.

$$\Delta U = -mg\Delta h = -V\rho g\Delta h$$

$$\Rightarrow \mathit{\Delta} U = -0.1^3 x 0.9 \text{ X } 10 \text{ X } 0.5 = -0.045 \quad J$$

We need to account for this energy change. The gravitational energy of the system of “water-ice” can not decrease on its own. We shall come to know that phase change is accompanied by exchange of heat energy. The ice cube absorbs this heat mostly from the water body and a little from surrounding atmosphere. If we neglect energy withdrawn from the atmosphere (only 10 % is exposed), then we can say that energy is transferred from potential energy of “ice-water” system to the internal energy of the system. As such, there is a corresponding increase in the internal energy of the system. This transfer of energy forms take place as “heat” to the ice body. Hence, this is merely a transfer of energy of the system from one form to another.

Here, we do not intend to prove the exactness of change in potential energy with the change in the internal energy in the system. But, the point about accounting of energy, in general, is illustrated by this example.

Note: We shall not work additional problems involving other forms of energy at this juncture. We shall, however, work with the application of conservation of energy in the mechanical context in a separate module. Also, we should know that first law of thermodynamics is a statement of law of conservation energy that includes heat as well. Therefore, study of first law of thermodynamics provides adequate opportunity to work with situations in non-mechanical context.

Conservation of energy in mechanical process
Conservation of energy in mechanical process is a subset of the law of conservation of energy.

We have discussed formulation of conservation of energy in general. In this module, we shall discuss conservation law as applied to mechanical process. Typically, a mechanical process involves three different kinds of energy - kinetic, potential and heat. We shall, however, restrict treatment of heat as an equivalent to work by friction.

On the other hand, the process generally involves gravitational, elastic and friction forces. These forces transfer energy involving different energy forms by doing work on the objects of the system.

Depending on the context of situation, we use either closed or isolated system. Further, mechanical process is generally limited to exchange of energy within or outside the system via “work”. In short, we can describe a mechanical process in terms of following elements :

- System : closed or isolated
- Energy : kinetic, potential and heat (work by friction)
- Forces : gravitational, elastic and friction
- Transfer of energy : work by forces across and within the system

In order to facilitate systematic study of mechanical process, we shall study three different situations in progressive simplicity of the process :

1. A mechanical process with external force on the system.
2. A mechanical process without external force on the system.
3. A mechanical process without external force on the system and without non-conservative internal force (absence of friction).

In this module, we shall discuss the first two processes. Third process represents the ideal mechanical process. The corresponding conservation law is known as “conservation of mechanical energy”. This law is being dealt in a separate module.

Mechanical process with external force on the system

One implication of external force on the system is that we are dealing with special “closed” system, which permits exchange of energy with surrounding via “work by external force” only. The form of conservation law for the mechanical process is :

$$W_E = \mathit{\Delta} E_{\text{mech}} + \mathit{\Delta} E_{\text{thermal}}$$

However, we pointed out that we shall limit change in thermal energy to “heat energy”, which is equal to the negative of work by friction only. Hence, we can rewrite the equation as :

$$\Rightarrow W_E = \mathit{\Delta} E_{\text{mech}} - W_F$$

$$\Rightarrow W_E + W_F = \mathit{\Delta} E_{\text{mech}}$$

$$\Rightarrow W_E + W_F = \mathit{\Delta} K + \mathit{\Delta} U$$

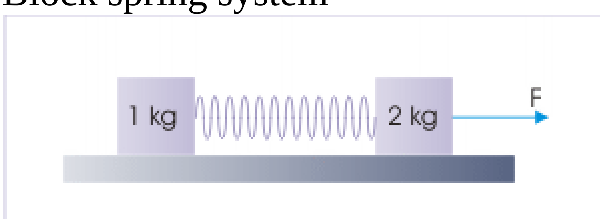
In words, we can put the conservation of energy for mechanical process as :

“Work by external force on the system and work by friction within the system is equal to the change in potential and kinetic energy of the system.”

Example

Problem 1: Two blocks of 1 kg and 2 kg are connected by a spring as shown in the figure. If spring constant is 500 N/m and coefficient of static and kinetic friction between surfaces are each 0.5, then what minimum constant horizontal force, F, (in Newton) is required to just initiate block on the left ? (Consider g = 10 m/s^2).

Block spring system



An external force is applied to

initiate block on left.

Solution : The system consists of two blocks and one spring. The system is subjected to an external force. Thus, system is "closed" system, which allows energy exchange with system via work by external force. As friction is also involved, the corresponding energy statement is :

$$\Rightarrow W_E + W_F = \Delta K + \Delta U$$

We are required to find minimum force. We need to understand the implication of this phrase. It is clear that we can apply external force in such a manner that block of "2 kg" acquires kinetic energy by the time block of "1 kg" is initiated in motion. Alternatively as a base case, we can apply external force gradually and slowly in increasing magnitude till the block of "1 kg" is initiated in motion. In this case, the block of "2 kg" does not acquire kinetic energy. This mode of application of external force represents the situation when minimum external force will be required to initiate block of "1 kg". Hence,

$$\Delta K = 0$$

Motion of block is constrained in horizontal direction. There is no change of vertical elevation. Hence, there is no change in gravitational potential energy. However, spring is stretched from its neutral state. As a consequence, there is change in elastic potential energy of the spring. Let "x" be the extension in the spring by the time block of "1 kg" is initiated in motion. Then, the total change in potential energy is :

$$\Delta U = \frac{1}{2}kx^2$$

The work by friction is done only on the right block of "2 kg" :

$$W_F = -\mu m_2 g x$$

On the other hand, the work by external force is :

$$W_E = Fx$$

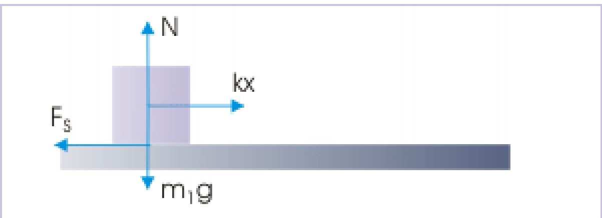
Putting all these values in the equation of conservation of energy :

$$\Rightarrow Fx - \mu m_2 gx = \frac{1}{2}kx^2$$

$$\Rightarrow F - \mu m_2 g = \frac{1}{2}kx$$

Clearly, we can not solve this equation as there are two unknowns, “F” and “x”. We, therefore, make use of the fact that spring force on the block of mass “1 kg” is equal to maximum static friction,

Forces on the block



The spring force is equal to maximum static friction.

$$kx = \mu m_1 g$$

Combining two equations, we have :

$$\Rightarrow F - \mu m_2 g = \frac{1}{2}\mu m_1 g$$

$$\Rightarrow F = \mu g\left(m_1 + \frac{m_2}{2}\right) = 0.5X10\left(1 + \frac{2}{2}\right)X10 = 10 \quad N$$

Note: It is interesting to note that force required to initiate left block is independent of spring constant.

Mechanical process without external force on the system

Since no external force operates on the system, there is no work by external force. The system under consideration is, therefore, an isolated system. The form of conservation law for general mechanical process is further reduced as :

$$W_F = \Delta K + \Delta U$$

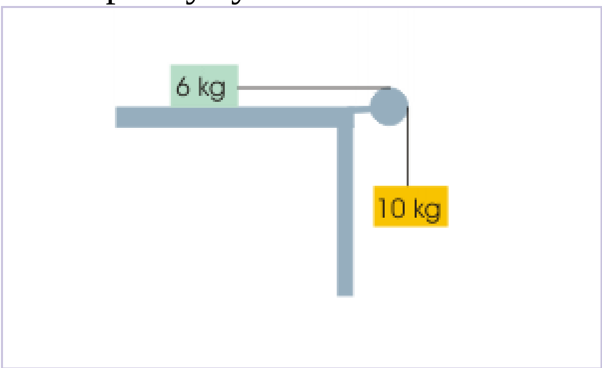
In words, we can put the conservation of energy for mechanical process under given condition as :

"Work by friction within an isolated system is equal to the change in potential and kinetic energy of the system."

Example

Problem 2: In the arrangement shown, the block of mass 10 kg descends through a height of 1 m after being released. The coefficient of friction between block and the horizontal table is 0.3, whereas pulley is friction-less. Considering string and pulley to be "mass-less", find the speed of the blocks.

Block pulley system



Blocks move by 1 m.

Solution : Here, friction is present as internal force to the system. Hence, we use the form of conservation law as :

$$W_F = \Delta K + \Delta U$$

Let us denote 6 kg and 10 kg blocks with subscript “1” and “2”. There is no change of height for the block on the table. Only change in gravitational potential energy is due to change in the height of the block hanging on the other end of the string. Thus, change in potential energy is :

$$\Delta U = m_1 gh + m_2 gh = 0 - 10x10x1 = -100 \quad J$$

Two blocks are constrained by a taut string. It means that both blocks move with same speed. Let the speed of the blocks after traveling “1 m” be "v". Now, initial kinetic energy of the system is zero. Therefore, the change in kinetic energy is given by :

$$\Delta K = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2$$

$$\Rightarrow \Delta K = \frac{1}{2} X6Xv^2 + \frac{1}{2} X10Xv^2$$

$$\Rightarrow \Delta K = 8v^2$$

Friction works only on the block lying on the table. Here, work by friction is given as :

$$W_F = -\mu Nx = -\mu m_1 gx = -0.3X6X10X1 = -18 \quad J$$

Putting in the equation of conservation of energy,

$$\Rightarrow -18 = 8v^2 - 100$$

$$\Rightarrow v^2 = \frac{82}{8}$$

$$\Rightarrow v = 3.2 \quad m/s$$

Conservation of mechanical energy
When only conservative force interacts within the system, the mechanical energy of an isolated system can not change.

Conservation of mechanical energy is description of an ideal mechanical process. It is characterized by absence of non-conservative force like friction. There is no external force either on the system. Therefore, the elements of this ideal process are :

- System : isolated
- Energy : kinetic and potential
- Forces : gravitational and elastic
- Transfer of energy : No transfer of energy across the system.

Characteristics of an ideal mechanical process

Conservation of mechanical energy applies to a mechanical process in which external force and non-conservative internal forces are absent.

There is no external force on the system. Hence, work by external force is zero. There is no exchange or transfer of energy across the system. Therefore, we use an isolated system to apply conservation of energy. We should, however, note that transfer of energy from one form to another takes place within the system, resulting from work done by internal force.

There is no “non-conservative” force like friction in the system. It means that there is no change in thermal energy of the system. The internal forces are only conservative force. This ensures that transfer of energy takes place only between kinetic and potential energy of the isolated system. Since potential energy is regained during the process, there is no dissipation of energy.

As there is no dissipation of energy involved, the system represents the most energy efficient reference for the particular process. One of the most striking feature of this system is that only force working in the system is conservative force. This has great simplifying effect on the analysis. The work by conservative force is independent of path and hence calculation of

potential energy of the system is path independent as well. The independence of path, in turn, allows analysis of motion along paths, which are not straight.

Conservation of mechanical energy

The statement of conservation of energy for the ideal mechanical process is known as “conservation of mechanical energy". The equation for the mechanical process is :

$$W_E + W_F = \Delta K + \Delta U$$

Here,

$$W_E = 0$$

$$W_F = 0$$

Hence, for the isolated system,

$$\Rightarrow \Delta K + \Delta U = 0$$

$$\Rightarrow \Delta E_{\text{mech}} = 0$$

$$\Rightarrow E_{\text{mech}} = 0$$

This is what is known as conservation of mechanical energy. We can interpret this equation in many ways and in different words :

1: When only conservative forces interact within an isolated system, sum of the change in kinetic and potential energy between two states is equal to zero.

$$\Delta K + \Delta U = 0$$

2: When only conservative forces interact within an isolated system, sum of the kinetic and potential energy of an isolated system can not change.

$$\Delta K = -\Delta U$$

$$\Rightarrow K_f - K_i = -(U_f - U_i)$$

$$\Rightarrow K_i + U_i = K_f + U_f$$

3: When only conservative forces interact within an isolated system, the change in mechanical energy of an isolated system is zero.

$$\Delta E_{\mathrm{mech}} = 0$$

4: When only conservative forces interact within an isolated system, the mechanical energy of an isolated system can not change.

$$E_{\mathrm{mech}} = 0$$

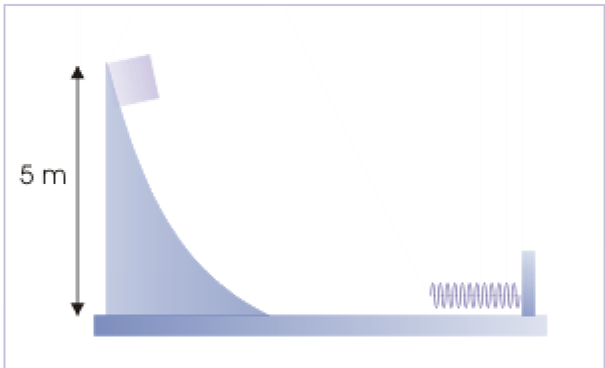
We should be aware that there are two ways to apply conservation law. We can apply it in terms of energy for initial (subscripted with “i”) and final (subscripted with “f”) states or in terms of “change” in energy. This point will be clear as we work with examples.

Here, we shall work an example that we had previously analyzed with the help of "work-kinetic energy" theorem. This will help us to understand (i) what are the elements of ideal mechanical process (ii) how conservation principle works and (iii) how does it help simplify analysis.

Example

Problem : A small block of 0.1 kg is released from a height 5 m as shown in the figure. The block following a curved path transitions to a linear horizontal path and hits the spring fixed to a wedge. If no friction is involved and spring constant is 1000 N/m, find the maximum compression of the spring.

Motion of a block



Solution : Here block, curved incline and spring form isolated system. Earth is implicit element of the system. We may argue why incline and why not horizontal surface? Curved incline (its shape) determines the component of gravity driving block downward. Hence, it is included in the system. Horizontal surface also applies force on block, but in normal direction to the motion. Now, as friction is not involved, the horizontal surface is not involved in the exchange of energy.

Two of the elements i.e. block and spring are subjected to motion. Application of conservation principle, thus, involves more than one object unlike analysis based on laws of motion. As friction is not involved, we can apply the conservation of mechanical energy. Now, initial energy of the system is :

$$\Rightarrow E_i = K + U = 0 + mgh = mgh = 0.1 \text{ x } 10 \text{ x } 5 = 5\ J$$

When the compression in the spring is maximum, kinetic energies of both block and spring are zero. Let "x" be the maximum compression,

$$\Rightarrow E_f = K + U = 0 + \frac{1}{2}kx^2 = \frac{1}{2}kx^2$$
$$\Rightarrow E_f == 0.5 \text{ x } 1000x^2 = 500x^2$$

According to conservation of mechanical energy :

$$\Rightarrow 500x^2 = 5$$
$$\Rightarrow x = 0.1\,m = 10\ \text{cm}$$

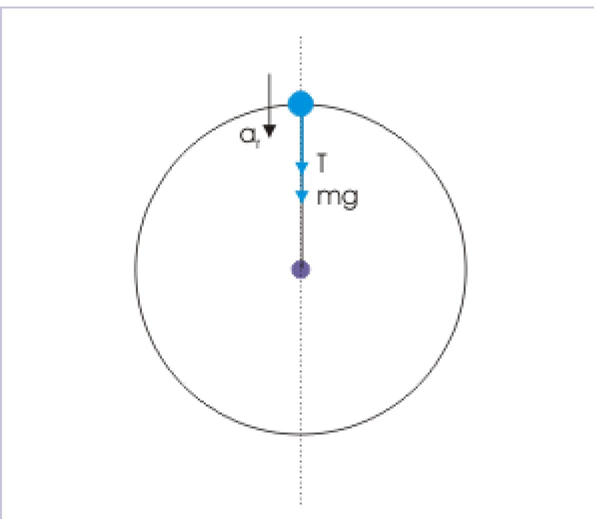
Note: We can compare this solution with the one given in the module titled " Conservative forces ".

Vertical circular motion

Study of this vertical motion was incomplete earlier as we were limited by constant force system and linear motion in most of the cases. On the other hand, vertical circular motion involved variable force and non-linear path. The analysis of the situation is suited to the energy analysis. Let us consider that a small object of mass "m" is attached to a string and is whirled in a vertical plane.

Force analysis at the highest point gives the minimum speed required so that string does not slack. It is important to understand that this is the most critical point, where string might slack. The forces in y-direction :

Vertical circular motion



The object maintains a minimum speed.

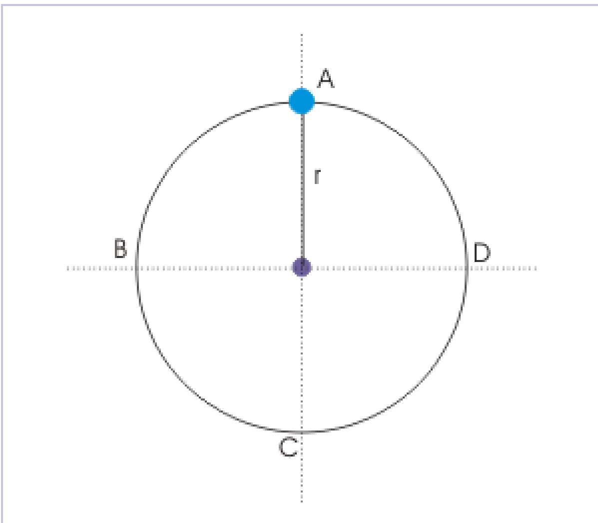
$$\sum F_y = mg + T = \frac{mv^2}{r}$$

For the limiting case, when T = 0,

$$\Rightarrow mg = \frac{mv^2}{r}$$
$$\Rightarrow v = \sqrt{(gr)}$$

This expression gives the limiting value of the speed of particle in vertical circular motion. If the speed of the particle is less than this limiting value, then the particle will fall. Once this value is known, we can find speed of the particle at any other point along the circular path by applying conservation of mechanical energy. Here, we shall find speed at two other positions - one at the bottom (C) and other at the middle point (B or D).

Vertical circular motion



The object maintains a minimum speed.

In order to analyze mechanical energy, we need to have a reference zero gravitational potential energy. For this instant case, we consider the horizontal level containing point "C" as the zero reference for gravitational potential energy. Now, mechacnial energy at points "A", "C" and "D" are :

$$\Rightarrow E_A = K + U = \frac{1}{2} m {v_A}^2 + mg \text{ x } 2r = \frac{1}{2} mgr + 2mgr = \frac{5}{2} mgr$$

$$\Rightarrow E_C = K + U = \frac{1}{2} m {v_C}^2 + 0 = \frac{1}{2} m {v_C}^2$$

$$\Rightarrow E_D = K + U = \frac{1}{2} m {v_D}^2 + mgr$$

From first two equations, we have :

$$\Rightarrow v_C = \sqrt{(5gr)}$$

From last two equations, we have :

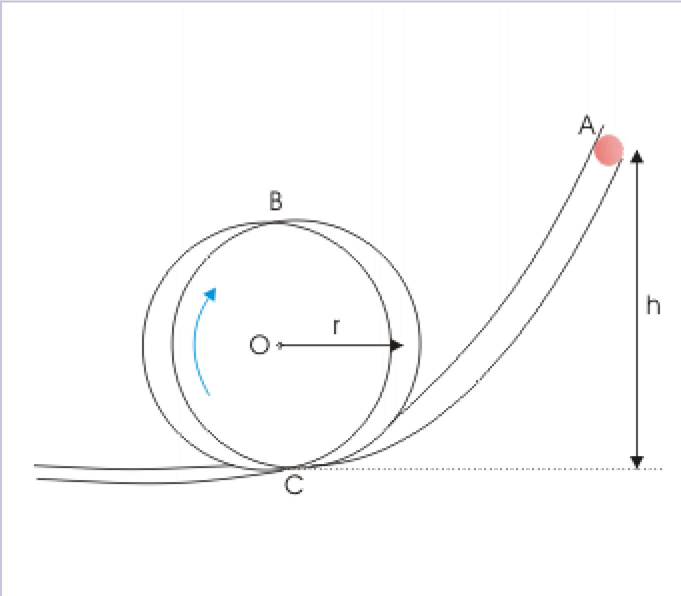
$$\Rightarrow v_D = \sqrt{(3gr)}$$

This shows how energy analysis can be helpful in situations where force on the object is not constant and path of motion is not straight. However, this example also points out one weak point about energy analysis. For example, the initial condition for minimum speed at the highest can not be obtained using energy analysis and we have to resort to force analysis. What it means that if details of the motion are to be ascertained, then we have to take recourse to force analysis.

Example

Problem 2: The ball is released from a height “h” along a smooth path shown in the figure. What should be the height “h” so that ball goes around the loop without falling of the track ?

Vertical circular motion



The ball does not fall off the track.

Solution : The path here is smooth. Hence, friction is absent. No external force is working on the "Earth – ball" system. The only internal force is gravity, which is a conservative force. Hence, we can apply conservation of mechanical energy.

The most critical point where ball can fall off the track is "B". We have seen that the ball need to have a minimum speed at "B" as given by :

$$v_B = \sqrt{gr}$$

Corresponding to this requirement at "B", the speed of the ball can be obtained at bottom point "C" by applying conservation of mechanical energy. The speed of the ball as worked out earlier is given as :

$$v_C = \sqrt{5gr}$$

In order to find the height required to impart this speed to the ball at the bottom, we apply law of conservation of energy between "A" and "C".

$$K_i + U_i = K_f + U_f$$

Considering ground as zero reference gravitational potential, we have :

$$\Rightarrow 0 + mgh = \frac{1}{2}mv_C^2 + 0$$

$$\Rightarrow v_C^2 = \frac{2mgh}{m} = 2gh$$

$$\Rightarrow v_C = \sqrt{2gh}$$

Equating two expressions of the speed at point "C", we have :

$$\Rightarrow 5gr = 2gh$$

$$\Rightarrow h = \frac{5r}{2}$$

Conservation of energy (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the conservation of energy. The questions are categorized in terms of the characterizing features of the subject matter :

- Particle
- Projectile
- Pulley - block system
- Incline – block - spring system
- Vertical circular motion

Particle

Problem 1: The potential energy of a particle of mass "m" is given by " $\frac{1}{2}Ax^2$ ". If non-conservative force is not involved and total mechanical energy is "E", then find its speed at "x".

Solution : External force and non-conservative force are absent here. Hence, we can apply conservation of mechanical energy. Total mechanical energy of the particle is constant,

$$E = U + K$$

$$\Rightarrow K = E - U$$

$$\Rightarrow K = E - \frac{1}{2}Ax^2$$

$$\Rightarrow \frac{1}{2} m v^2 = E - \frac{1}{2} A x^2$$

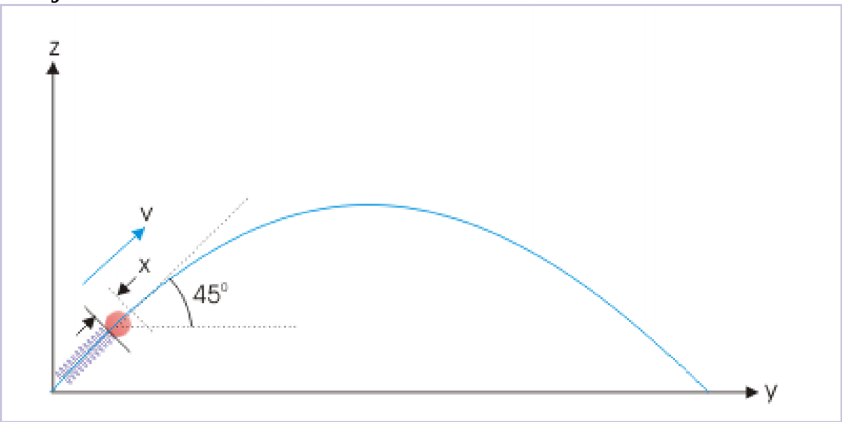
$$\Rightarrow v = \sqrt{\left\{ \frac{(2E - Ax^2)}{m} \right\}}$$

Projectile

Problem 2: A ball of 10 gm is fired from a spring gun of spring constant 500 N/m. The spring is initially compressed by 5 cm and is, then, released. Find the maximum horizontal distance (in meter) traveled by the bullet.

Solution : We are required to find maximum horizontal distance. The horizontal range of a projectile from the ground at an angle "θ" is given by :

Projectile



A ball is fired from spring gun.

$$R = \frac{v^2 \sin 2\theta}{g}$$

For maximum horizontal range, the angle of projection should be 45°. The maximum range is :

$$\Rightarrow R = \frac{v^2}{g}$$

In order to find speed of projection, we shall employ conservation of mechanical energy. The gun and ball forms an isolated system. Initially, kinetic energy of the bullet is zero, whereas initial elastic potential energy of the spring is " $\frac{1}{2}kx^2$ ". Let "v" be the speed when ball breaks the contact with spring. Now, applying conservation of mechanical energy,

$$K_i + U_i = K_f + U_f$$

Here,

$$K_i = 0$$

$$U_i = \frac{1}{2}kx^2$$

$$K_f = \frac{1}{2}mv^2$$

$$U_f = 0$$

Thus,

$$\Rightarrow \frac{1}{2}kx^2 = \frac{1}{2}mv^2$$

$$\Rightarrow v^2 = \frac{kx^2}{m}$$

Putting this expression in the expression of horizontal range, the maximum horizontal range is :

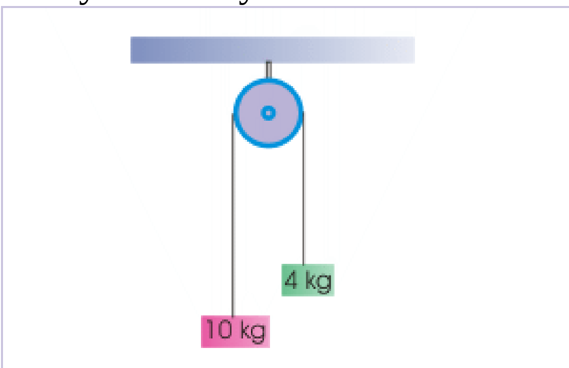
$$\Rightarrow R = \frac{kx^2}{mg}$$

$$\Rightarrow R = \frac{500X(0.05)^2}{0.01X10} = \frac{500x0.0025}{0.1} = \frac{1.25}{0.1} = 12.5 \quad m$$

Pulley - block system

Problem 3: In the arrangement shown, the lighter block ascends a height of 2 m after being released. Find the speed of the blocks. Consider string to be mass-less and no friction at the contact.

Pulley - block system



The blocks move vertically by 2m.

Solution : Since friction is absent, only gravity interacts with the elements of the system. We can consider arrangement as isolated system. No external force acts on the system. Remember that Earth is part of the system and gravitational force is internal to the system. Hence, the situation fulfills all conditions for applying conservation of mechanical energy.

Now, two blocks are constrained by a taut string. It means that both blocks move with same speed. Let us denote 4 kg and 10 kg blocks with subscript “1” and “2”. Note that data is given for the change in height. Hence, it is easier to determine change in the potential energy.

$$\Delta U = m_1 gh + m_2 gh = 4X10X2 - 10X10X2 = -120 \quad J$$

Now, the change in kinetic energy is given by :

$$\Delta K = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2$$

$$\Rightarrow \Delta K = \frac{1}{2} X4Xv^2 + \frac{1}{2} X10Xv^2$$

$$\Rightarrow \Delta K = 7v^2$$

Now applying conservation of mechanical energy, we have :

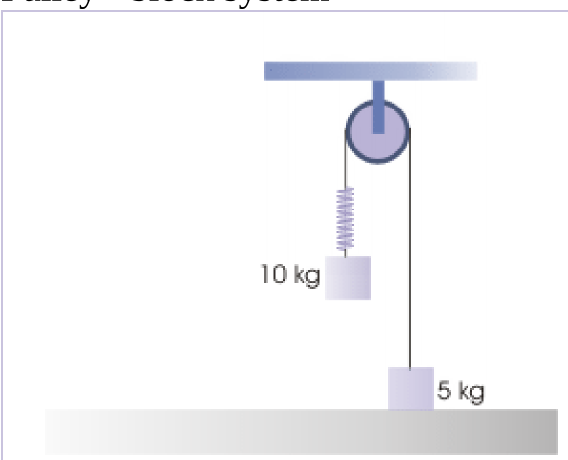
$$\Delta K + \Delta U = 0$$

$$\Rightarrow 7v^2 - 120 = 0$$

$$\Rightarrow v = 4.14 \quad m/s$$

Problem 4: The block of mass 10 kg is released in the arrangement consisting of various elements as shown in the figure. If spring constant is 50 N/m, then find the speed of 10 kg block, when block of 5 kg is about to leave the ground. (Consider g = 10 m/s^2)

Pulley - block system



The block resting on the floor is about to be lifted.

Solution : Before the block of 10 kg is released, the kinetic energy of the system is zero. We are required to find the speed of the released block, when the block in contact with horizontal surface is about to be lifted. It means that there is no change in kinetic or potential energy for the block of 5 kg.

In order to assess the energy state of the block of 10 kg, let us analyze the process. As the block of 10 kg moves down, it stretches the spring. As such, it acquires the speed (kinetic energy), while spring acquires elastic potential

energy. However, at the same time, the gravitational potential energy of the block decreases as it comes down. Let the extension in the spring be “x” and let the speed of the mass of 10 kg be “v”, when block of 5 kg is about to be lifted.

Considering the system isolated, the change in kinetic energy is :

$$\varDelta K = \frac{1}{2}mv^2 = \frac{1}{2}X10Xv^2 = 5v^2$$

The change in potential energy is :

$$\varDelta U = \frac{1}{2}kx^2 - mgx$$

We need to know “x” to evaluate the expression of potential energy. When the block of 5kg is about to be lifted, then force across the spring is equal to the difference of weights. Applying Hook’e law,

$$Kx = (10 - 5)g = 5X10 = 50$$

$$\Rightarrow x = \frac{50}{50} = 1 \quad m$$

Now, applying conservation of energy,

$$\varDelta U + \varDelta K = 0$$

$$\Rightarrow \frac{1}{2}kx^2 - mgx + 5v^2 = 0$$

$$\Rightarrow v = \frac{10X10X1 - \frac{1}{2}X50X1^2}{5}$$

$$\Rightarrow v = 15 \quad m/s$$

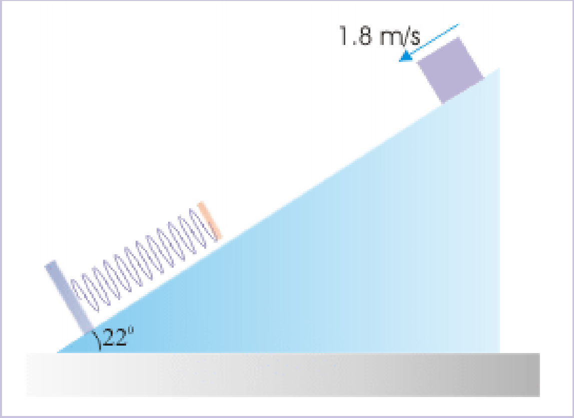
Incline – block - spring system

Problem 5: We are designing a delivery ramp for crates containing exercise equipment. The 1470-N crates will move at 1.8 m/s at the top of a ramp that slopes downward at 22 degrees. The ramp exerts a 550-N kinetic friction force on each crate, and the maximum static friction also has this value. Each crate

will compress a spring at the bottom of the ramp and will come to rest after traveling a total distance of 8.0 meters along the ramp. Once stopped, a crate must not rebound back up the ramp. Calculate the maximum force constant of the spring which will meet the design criteria. Take sin22° = 0.374 and g = 10 m/s^2.

Note: This problem is adapted from the question mailed by a student.

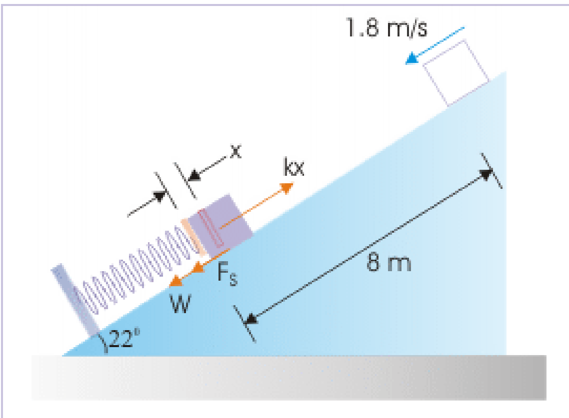
Incline – block - spring system



The block is moving with an initial speed down the incline.

Solution : Here the critical part of the question is to understand that the force required to return the crate up is equal to the sum of the component of its weight down the incline and the maximum static friction so that block can be initiated up the incline. Hence, spring force for this condition is :

Incline – block - spring system



Forces on the crate.

$$F_S = mg\sin\theta + F_K = mg\sin 22^0 + 550 = 1470X0.374 + 550 = 1100 \quad N$$

Let “x” be the compression in the spring. Then, applying Hooke’ law, we get the compression in the spring as :

$$F_S = Kx = 1100$$

The system here is an isolated system with both conservative and non-conservative force. Hence,

$$W_F = \Delta K + \Delta U$$

Here,

$$\Rightarrow W_F = -550x8 = -4400 \quad J$$

$$\Rightarrow \Delta K = K_f - K_i = 0 - \frac{1}{2}X147X1.82 = -238.14 \quad J$$

$$\Rightarrow \Delta U = \frac{1}{2}kx^2 - mgh = \frac{1}{2}kx^2 - 1470X8X\sin 22^0$$

$$\Rightarrow \Delta U = \frac{1}{2}kx^2 - 1470X8X\sin 22^0 = \frac{1}{2}kx^2 - 4398$$

Putting values in the equation, we have :

$$\Rightarrow W_F = \varDelta K + \varDelta U$$

$$\Rightarrow -4400 = -238 + \frac{1}{2}kx^2 - 4398$$

$$\Rightarrow \frac{1}{2}kx^2 = 238$$

$$\Rightarrow kx^2 = 476$$

Combining with Hooke's equation,

$$\Rightarrow K = \frac{476K^2}{1100^2}$$

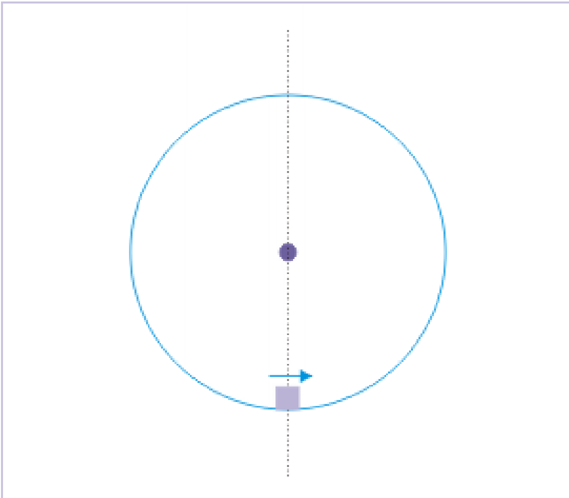
$$\Rightarrow K = \frac{1100^2}{476} = 2542 \quad N/m$$

Note: We can see that a spring having spring constant less than this value will also not return the crate. Hence, this spring constant represents the maximum value of the spring constant, which meets the required design criterion.

Vertical circular motion

Problem 5: An object inside a spherical shell is projected horizontally at the lowest point such that it just completes the circular motion in a vertical plane. If the radius of the shell is 2 m, then find the acceleration of the object when its velocity is vertical.

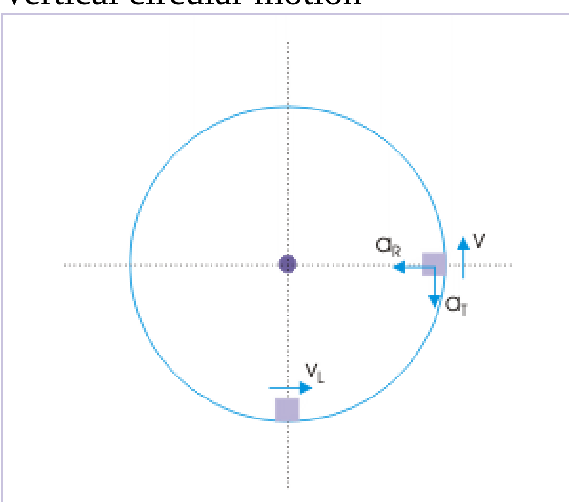
Vertical circular motion



An object inside a spherical shell is projected horizontally at the lowest point.

Solution : The object attains velocity in vertical direction, when it reaches half of the total height. The acceleration of the object is resultant of tangential and centripetal accelerations at that point :

Vertical circular motion



The acceleration of the object is resultant of tangential and centripetal accelerations.

$$a = \sqrt{a_T^2 + a_R^2}$$

Here,

$$a_T = g$$

and

$$a_r = \frac{v^2}{r}$$

We need to know the speed of the object when it has reached half the height. Now, the minimum speed at the lowest point to enable to complete the revolution is :

$$v_L = \sqrt{5gr}$$

From conservation of energy at lowest and middle levels,

$$K_L + U_L = K_M + U_M$$

Considering zero reference potential at the lowest point,

$$U_L = 0$$

$$U_M = mgr$$

Putting these values in energy equation, we have :

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{5gr}\right)^2 - mgr$$

$$\Rightarrow v^2 = \left(\sqrt{5gr}\right)^2 - 2gr = 3gr$$

and

$$\Rightarrow a_r = \frac{v^2}{r} = \frac{3gr}{r} = 3g$$

Thus,

$$\Rightarrow a = \sqrt{\left\{g^2 + (3g)^2\right\}} = \sqrt{10}g = 10\sqrt{10} \quad m/s^2$$

Note : It is interesting to note that acceleration is independent of the radius of the shell.

Center of mass
There is a characteristic geometric point of the three dimensional body in motion, which behaves yet as a particle.

Our study of motion has been limited up to this point. We referred particle, object and body in one and same way. We considered that actual three dimensional rigid body moved such that all constituent particles had same motion i.e. same trajectory, velocity and acceleration.

An actual body, however, can move differently to this simplified paradigm. Consider a ball rolling down an incline plane or consider a stick thrown in air. Different parts of the body have different motions. While translating in the air, the stick rotates about a moving axis. In the nutshell, this means that these bodies may not behave like a particle as assumed earlier.

Stick thrown in air

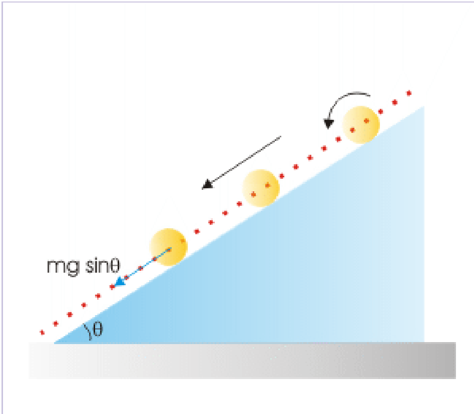
The stick does not behave like a particle.

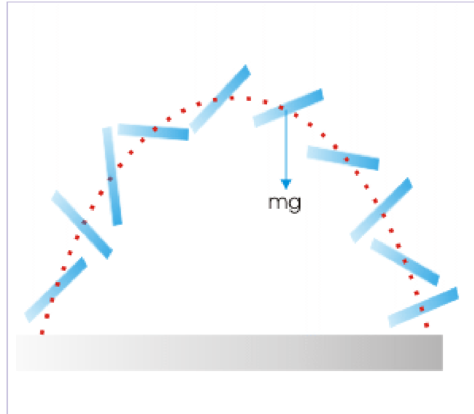
Apparently, it would be quite complicated to describe motions of parts or particles, having different motions, in an integrated manner. There is but one surprising simplifying characteristic of these motions. There is a characteristic geometric point of the three dimensional body in motion, which behaves yet as a particle. This point is known as "center of mass" in short "COM". It has following two characterizing aspects :

- COM appears to carry the whole mass of the body.
- All external forces appear to apply at COM.

Rolling ball on an incline and Stick thrown in air

The force appears to operate on the COM is “mgsinθ”.	The force appears to operate on the COM is “mg”.





Significantly, the center of ball, which is COM of rolling ball, follows a straight linear path; whereas the COM of the stick follows a parabolic path as shown in the figure. Secondly, the forces appear to operate on the COMs in two cases are “mgsinθ” and “mg” as if they were indeed particle like objects. This concept of COM, therefore, relieves us of the complexities that otherwise we would have faced describing motions of rigid bodies.

We must here note the word "appear" in describing characteristics of COM. In reality, COM is a geometric point. It need not be even the part of the body. Think of a hollow sphere, whose center is its COM. Is there any material there? Also, consider the motion of the stick again. Each particle constituting the stick is pulled by gravitational force. What it means that external forces, in actuality, may not act on COM alone as they appear in equivalent term.

Further the concept of COM is equally valid to a system of particles, which may not be in contact. This follows from the fact that a rigid body, after all, is an arrangement of particles, where inter-particle distances are extremely small.

In words, we can, therefore, define COM as :

Center of mass (COM)

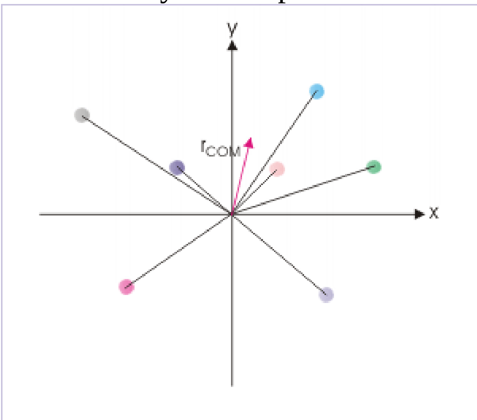
> COM of a system of particles is a geometric point, which assumes all the mass and external force(s) during motion.

Mathematical expression of COM for a system of particles

Let us now look at a baseball bat closely. It has a peculiar shape. Where should the COM lie? By experience, we can say that it should lie on the heavier side. This perception comes from the realization that heavier part has most of the mass. It gives us the clue about COM that it lies on the heavier side of an irregularly shaped body. Now, let us consider the motion of spherical sphere of uniform density. Here, mass is evenly distributed. Where should the COM lie? Obviously, COM coincides with the center of spherical ball.

It emerges from the discussion that COM is a statement of spatial arrangement of mass i.e. distribution of mass within the system. The position of COM is given a mathematical formulation which involves distribution of mass in a volumetric space, assuming as if all mass is concentrated there :

COM for a system of particles



COM assumes as if mass of all particles is concentrated there.

This relation is read as "position of COM is mass weighted average of the positions of particles".

Further, COM is a geometric point in three-dimensional volume and it involves distribution of mass with respect to some reference system. This expression of the position of COM is description of a position in three dimensional volume. Now, substituting with appropriate symbols, we have :

It is important to note here that this formulation is not arbitrary. We shall find subsequently that this mathematical expression of COM, as a matter of fact, leads us to formulation of Newton's second law for a system of particles, while keeping its form intact.

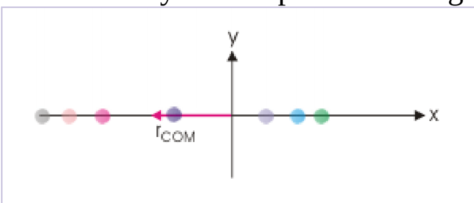
We can express the position of COM in scalar component form as :

We know that there are only two directions involved with each of the axes. As such, we assign positive value for distances in reference direction and negative in opposite direction.

Special cases

(a) Particle system along a straight line :

COM of particle system along a straight line is given by the scalar expression in one dimension:
COM for a system of particles along a straight line



Scalar COM expression in one dimension

Evidently, COM of particles lie on the line containing particles.

(b) Two particles system

The expression of COM reduces for two particles as :

As only one coordinate is involved, it is imperative that COM lies on the line joining two particles. It can also be seen that center of mass lies in between the masses on the line connecting two particles. There are two additional simplified cases for two particles system as :

(i) Origin of coordinate coincides with position of one of the particles :

The expression of COM is simplified when position of one of the particles is the origin. For two particles system,

where "x" is the linear distance between two particles. Since " " is a fraction, it follows that center of mass two particles system is indeed a point which lies between the positions of two particles.

(ii) The mass of the particles are equal :

In this case, (say)

When origin of coordinate coincides with position of one of the particles, :

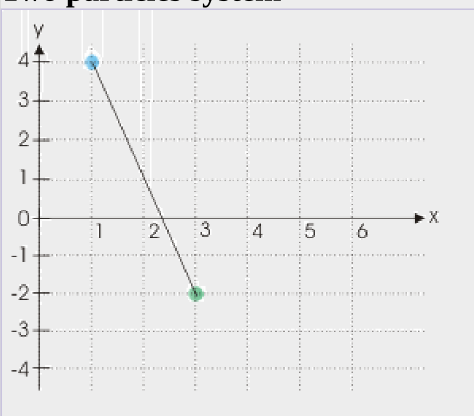
This result is on expected line. Center of mass (COM) of two particles of equal masses is midway between the particles.

Example:

Problem : Two particles of 2 kg and 4 kg are placed at (1,4) and (3,-2) in x and y plane. If the coordinates are in meters, then find the position of COM.

Solution : Using expression in scalar component form :

Two particles system



Positions of the particles are shown on "xy" coordinate system.

and

Thus, position of COM lies on x - axis at x = 7/3 m.

COM of rigid body

Rigid body is composed of very large numbers of particles. Mass of rigid body is distributed closely. Thus, the distribution of mass can be treated as continuous. The mathematical expression for rigid body, therefore, is modified involving integration. The integral expressions of the components of position of COM in three mutually perpendicular directions are :

— đ

— đ

— đ

Note that the term in the numerator of the expression is nothing but the product of the mass of particle like small volumetric element and its distance from the origin along the axis. Evidently, this terms when integrated is equal to sum of all such products of mass elements constituting the rigid body.

Evaluation of above integrals is simplified, if the density of the rigid body is uniform. In that case,

— đ/đ

đ/ đ/

Substituting,

— đ

— đ

— đ

We must understand here that once we determine COM of a rigid body, the same can be treated as a particle at COM with all the mass assigned to that particle. This concept helps to find COM of a system of rigid bodies, comprising of many rigid bodies. Similarly, when a portion is removed from a rigid body, the COM of the rigid body can be obtained by treating the "portion removed" and the "remaining body" as particles. We shall see the working of this concept in the example given in the next section.

Symmetry and COM of rigid body

Evaluation of the integrals for determining COM is very difficult for irregularly shaped bodies. On the other hand, symmetry plays important role in determining COM of a regularly shaped rigid body. There are certain simplifying facts about symmetry and COM :

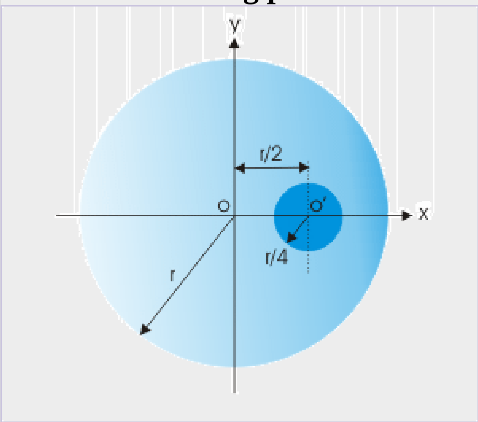
1. If symmetry is about a point, then COM lies on that point. For example, COM of a spherical ball of uniform density is its center.

2. If symmetry is about a line, then COM lies on that line. For example, COM of a cone of uniform density lies on cone axis.
3. If symmetry is about a plane, then COM lies on that plane. For example, COM of a cricket bat lies on the central plane.

The test of symmetry about a straight line or a plane is that the body on one side is replicated on the other side.

Example:
Problem : A small circular portion of radius r/4 is taken out from a circular disc of uniform thickness having radius "r" and mass "m" as shown in the figure. Determine the COM of the remaining portion of the uniform disc.

COM of remaining portion of circular disc

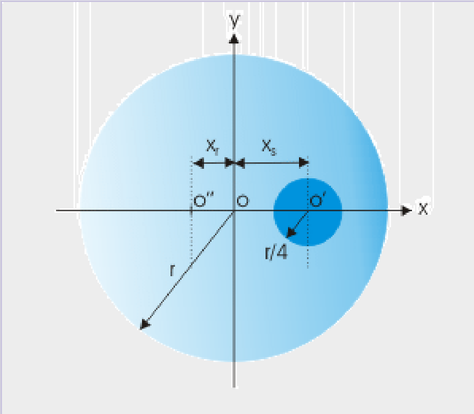


A small circular portion of radius r/4 is taken out.

Solution : If we start from integral to determine COM of the remaining disc portion, then it would be a really complex proposition. Here, we shall make use of the connection between symmetry and COM. We note that the COM of the given disc is "O" and COM of the smaller disc removed is "O'". How can we use these fact to find center of mass of the remaining portion ?
The main idea here is that we can treat regular bodies with known COM as particles, which are separated by a known distance. Then, we shall employ the expression of COM for two particles to determine the COM of the remaining portion. We must realize that when a portion is removed from the bigger disc on the right side, the COM of the remaining portion shifts towards left side (heavier side).
The test of symmetry about a straight line or a plane is that the body on one side is replicated on the other side. We see here that the remaining portion of the disc is not symmetric about y-axis, but is symmetric about x-axis. It means that the COM of the remaining portion lies on the x-axis on the left side of the center of original disc. It also means that we need to employ the expression of COM for one dimension only.

While employing expression of COM, we use the logic as explained here. The original disc (with known COM) is equivalent to two particles system comprising of (i) remaining portion (with unknown COM) and (ii) smaller disc (with known COM). Now, x-component of the COM of original disc is :

COM of remaining portion of circular disc



A small circular portion of radius r/4 is taken out.

But, x-component of the COM of original disc coincides with origin of the coordinate system. Further, let us denote the remaining disc by subscript "r" and the smaller circular disk removed by subscript "s".

As evident from figure, . We, now, need to find mass of smaller disc, , and mass of remaining portion, , using the fact that the density is uniform. The density of the material is :

where "t" is thickness of disc.

Putting these values in the expression of the position of COM, we have :

Acknowledgment

Author wishes to thank Aladesanmi Oladele A for making suggestion to remove error in the module.

Center of mass and rigid bodies
Under certain condition, COM of rigid bodies is same as geometric center.

Rigid bodies are composed of very small particles which interact with each other via electromagnetic force. They form a continuous distribution of mass. As such, expressions of COM in three coordinate directions involve evaluation of integrals as described in earlier module. This evaluation, however, is rendered difficult on two counts :

- Mass distribution may not be uniform.
- The body shape may be irregular.

The geometry of regularly shaped bodies are defined by mathematical equations. Such is not the case with irregular bodies. However, there is a good thing about center of mass (COM) that it represents the point where external force equivalently applies. This fact allows us to experimentally determine COM of even irregularly shaped bodies. We can balance a body on a pointed wedge. The COM of the body falls on the line of balance. In order to know the COM (a point), however, we need to balance the body with different orientation to get another line of balance. The point of intersection of the two lines of balance is the COM of the body.

COM of regular bodies with uniform density

We are saved from any mathematical calculation in cases of certain regularly shaped bodies with uniform density, which are symmetric to all the axes of the coordinate system involved. In all such cases, COM is same as geometric center. COMs of a sphere, spherical shell, ring, disc, cylinder, cone, rod, square plate etc. fall under this category where COM is simply the geometric center of the bodies.

However, there are cases of regular shaped bodies which are not symmetric to three (for three dimensional bodies) or two axes (for planar bodies). For example, consider the case of hemispherical body or the case of semi-circular wire. These bodies are not symmetric to all the axes involved. Here, geometric center and, thus, COM are not obvious. In this section, we shall

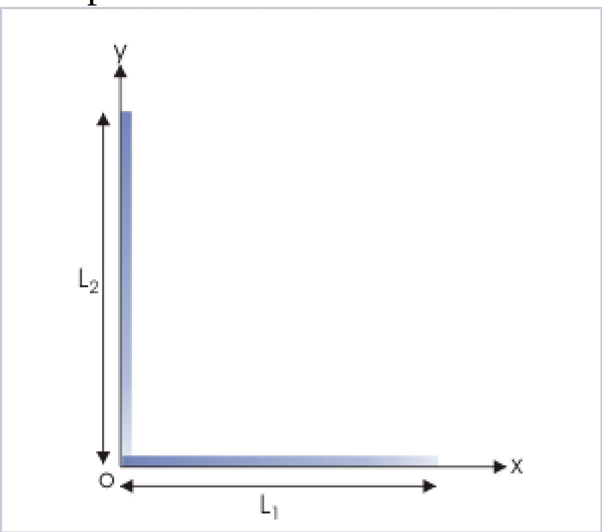
evaluate COM of such regular shaped bodies, which are not symmetric about all axes of the coordinate system.

Regular bodies allow us to evaluate integrals as geometry is defined. Evaluation of integral is simplified if the mass is evenly distributed. Unless otherwise indicated, we shall consider rigid bodies of uniform density only. Further, some rigid bodies are combination of other regular bodies, whose COMs are known. In that case, we would employ the formula of COM for the system of particles.

L shaped rod or wire

In this case, we need not resort to integration as L-shaped rod is simply a combination of two rods with known COMs, which can be treated as combination of particles.

L shaped rod



L-shaped rod is a combination of two rods.

Let the origin of the planar coordinate system coincides with the corner of the L-shaped rod. COM of a uniform rod is the middle point of the rod.

From the figure, the COM of rod along x-axis is ($\frac{l_1}{2}$, 0). On the other hand, COM of rod along y-axis is (0, $\frac{l_2}{2}$). These rods, therefore, can be considered as particles at these positions. Thus, the L-shaped rod system reduces to the case of two particles system separated by a distance. Let m_1 and m_2 be the mass of two rods respectively. Now, the linear mass density is :

$$\gamma = \frac{m_1}{l_1} = \frac{m_2}{l_2}$$

The COM of the L-shaped rod is :

$$x_{\text{COM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$
$$\Rightarrow x_{\text{COM}} = \frac{\gamma l_1 \text{x} \frac{l_1}{2}}{\gamma(l_1 + l_2)}$$
$$\Rightarrow x_{\text{COM}} = \frac{{l_1}^2}{2(l_1 + l_2)}$$

and similarly,

$$y_{\text{COM}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$
$$\Rightarrow x_{\text{COM}} = \frac{\gamma l_2 \text{x} \frac{l_2}{2}}{\gamma(l_1 + l_2)}$$
$$\Rightarrow x_{\text{COM}} = \frac{{l_2}^2}{2(l_1 + l_2)}$$

It is important to note that the expressions of COM are independent of mass term.

Example:
Problem : A thin rod of length 3 m is bent at right angles at a distance 1 m from one end. If 1 m segment lies horizontally, determine the COM of the resulting L-shaped wire structure.
Solution : Here l_1 = 1m and l_2 = 2 m. The COM of the L - shaped rod is :

$$\Rightarrow x_{\text{COM}} = \frac{{l_1}^2}{2(l_1+l_2)}$$
$$\Rightarrow x_{\text{COM}} = \frac{1}{2(1+2)} = \frac{1}{6}\ m$$

and

$$\Rightarrow y_{\text{COM}} = \frac{{l_2}^2}{2(l_1+l_2)}$$
$$\Rightarrow x_{\text{COM}} = \frac{4}{2(1+2)} = \frac{4}{6} = \frac{2}{3}\ m$$

Semicircular uniform wire

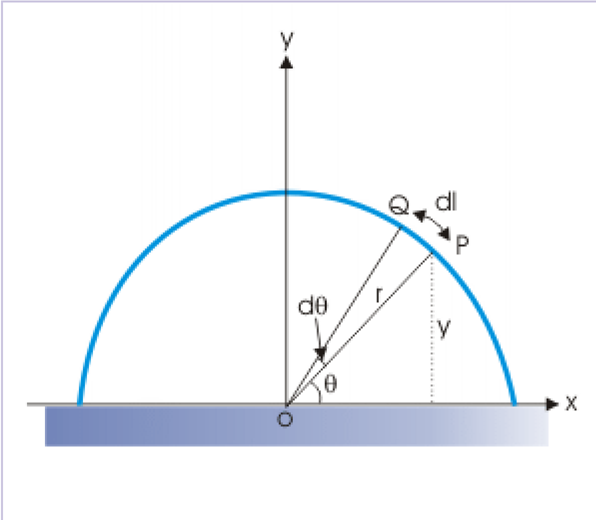
A semicircular wire is not symmetric to all the axes of the coordinates. The wire is a planar body involving two dimensions (x and y). However, we note that the wire is symmetric about y-axis, but not about x-axis. This has two important implications :

- COM lies on y - axis (i.e on the axis of symmetry).
- We only need to evaluate "y" coordinate of COM as x_{COM} = 0.

Thus, we need to evalute the expression :

$$y_{\text{COM}} = \frac{1}{M} \int y \mathbb{d}m$$

Evaluation of an integral is based on identifying the differential element. The semicircular wire is composed of large numbers of particle like elements, each of mass "$\mathbb{d}$m". The idea of this particle like element of mass "dm" is that its center of mass i.e COM lies at "y" distance from the origin in y-direction. This element is linked to the mass "m" of the wire as :
Semicircular uniform wire



The semicircular wire is composed of large numbers of particles.

$$dm = \gamma dl = \left(\frac{M}{\pi R}\right) dl$$

Now,

$$dl = R d\theta$$
$$y = R\sin\theta$$
$$\Rightarrow dm = \left(\frac{M}{\pi R}\right) R d\theta = \left(\frac{M}{\pi}\right) d\theta$$

Putting in the integral,

$$\Rightarrow y_{\text{COM}} = \frac{1}{M}\int R\sin\theta\left(\frac{M}{\pi}\right) d\theta$$
$$\Rightarrow y_{\text{COM}} = \frac{R}{\pi}\int \sin\theta d\theta$$

Integrating between θ = 0 and θ = π,

$$\Rightarrow y_{\text{COM}} = \frac{R}{\pi}\left[-\cos\theta\right]_0^{\pi}$$
$$\Rightarrow y_{\text{COM}} = \frac{R}{\pi}\left[-\cos\pi + \cos 0\right] = \frac{2R}{\pi}$$

The COM of the semicircular wire is (0, $\frac{2R}{\pi}$).

Semicircular unifrom thin disc

A semicircular disc is not symmetric to all the axes of the coordinates. The thin disc is a planar body involving two dimensions (x and y). However, we note that the disc is symmetric about y-axis, but not about x-axis. This has two important implications :

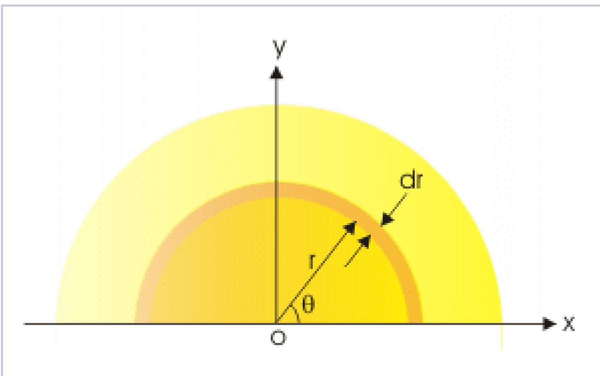
- COM lies on y - axis (i.e on the axis of symmetry).
- We only need to evaluate "y" coordinate of COM as x_{COM} = 0.

Thus, we need to evaluate the expression :

$$y_{\text{COM}} = \frac{1}{M} \int y \mathbb{d} m$$

Evaluation of an integral is based on identifying the differential element. The semicircular disc is composed of large numbers of semicircular thin wires. We must be careful in evaluating this integral. The elemental mass "dm" corresponds to a semicirular wire - not a particle as the case before. The term "y" in the integral denotes the distance of the COM of this elemental semicircular wire. The COM of this semi-circular thin wire element is at a distance 2r/Π (as determined earlier)in y-direction.

Semicircular uniform disc



The semicircular disc is composed of large numbers of

semicircular wires.

Now,

$$y = \frac{2r}{\pi}$$
$$dm = \sigma dA = (\pi r dr)\sigma$$

where "σ" denoted the surface density. It is given by :

$$\sigma = \frac{M}{\frac{\pi R^2}{2}} = \frac{2M}{\pi R^2}$$

Putting in the integral,

$$\Rightarrow y_{\mathrm{COM}} = \frac{1}{M} \int \left(\frac{2r}{\pi}\right) \sigma \pi r dr$$
$$\Rightarrow y_{\mathrm{COM}} = \frac{2\sigma}{M} \int r^2 dr$$

Integrating between r = 0 and r = R (radius of the semicircular disc),

$$\Rightarrow y_{\mathrm{COM}} = \frac{2\sigma}{M} \left[\frac{r^3}{3}\right]_0^R$$
$$\Rightarrow y_{\mathrm{COM}} = \frac{2\sigma R^3}{3M}$$
$$\Rightarrow y_{\mathrm{COM}} = \frac{4\mathrm{M}R^3}{3M\pi R^2} = \frac{4R}{3\pi}$$

The COM of the semicircular disc is (0, $\frac{4R}{3\pi}$).

COM rigid bodies with non-uniform density

Evaluation of integrals to determine COM of non-uniform rigid bodies may be cumbersome. There are two possibilities :

1. The variation in density is distinctly marked.
2. The variation in density is defined in mathematical form.

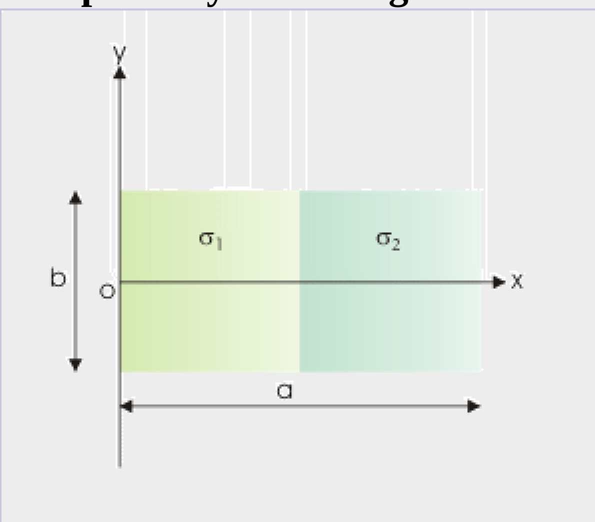
Variation in density is distinctly marked

If the variation in density is distinctly marked, then it is possible to determine COMs of individual uniform parts. Once COMs of the individual uniform parts are known, we can treat them as particle with their mass at their respective COMs. Thus, non-uniform rigid body is reduced to a system of particles, which renders easily for determination of its COM. This technique is illustrated in the example given here :

Example:

Problem : A thin plate (axb) is made up of two equal portions of different surface density of σ_1 and σ_1 as shown in the figure. Determine COM of the composite plate in the coordinate system shown.

Composite system of rigid bodies



Thin plate is made up of two equal portions of different surface density.

Solution : COM of the first half is at $\frac{a}{4}$, whereas COM of the second half is at $\frac{3a}{4}$ from the origin of the coordinate system in x-direction. As x-axis

lies at mid point of the height of the plate, y-component of COM is zero. Now two portions are like two particles having masses :

$$m_1 = \frac{ab}{2}\sigma_1$$

and

$$m_2 = \frac{ab}{2}\sigma_2$$

Also, the x-coordinates of the positions of these particles are $x_1 = \frac{a}{4}$ and $x_2 = \frac{3a}{4}$. The x-component of the COM of two particles system is :

$$x_{\text{COM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

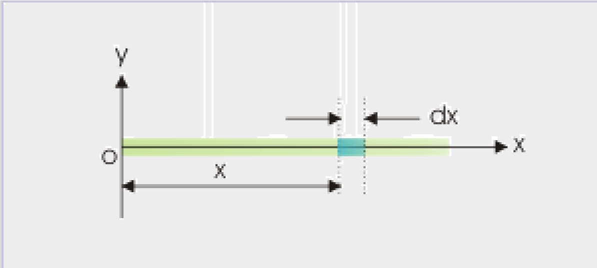
$$\Rightarrow x_{\text{COM}} = \frac{\frac{ab}{2}\sigma_1 \frac{a}{4} + \frac{ab}{2}\sigma_2 \frac{3a}{4}}{\frac{ab}{2}\sigma_1 + \frac{ab}{2}\sigma_2}$$

$$\Rightarrow x_{\text{COM}} = \frac{(\sigma_1 + 3\sigma_2)a}{4(\sigma_1 + \sigma_2)}$$

Variation in density is defined in mathematical form

If the variation in density is well defined, then it is possible to evaluate integral to determine the COM for non-uniform rigid body. The technique is illustrated in the example given here :

Example:
Problem : Linear density of a rod varies with length as γ = 2x + 3. If the length of the rod is "L", then find the COM of the rod.
Solution : The COM of the rod in one dimension is given as :
COM of a rod with varying density



Linear density of a rod varies with length as γ = 2x + 3.

$$x_{\text{COM}} = \frac{1}{M}\int x dm$$

Here,

$$dm = \gamma dx = (2x+3)dx$$

We can use this relation to determine COM as :

$$\Rightarrow x_{\text{COM}} = \frac{\int x dm}{\int dm}$$

$$\Rightarrow x_{\text{COM}} = \frac{\int x(2x+3)dx}{\int (2x+3)dx}$$

$$\Rightarrow x_{\text{COM}} = \frac{2\int x^2 dx + 3\int x dx}{2\int x dx + 3\int dx}$$

Evaluating between x = 0 and x = L, we have :

$$\Rightarrow x_{\text{COM}} = \frac{\frac{2L^3}{3}+\frac{3L^2}{2}}{\frac{2L^2}{2}+3L}$$

$$\Rightarrow x_{\text{COM}} = \frac{4L^2+9L}{6(L+3)}$$

Center of mass (check your understnading)
Objective questions, contained in this module with hidden solutions, help improve understanding of the topics covered in the modules "Center of mass" and "Center of mass and rigid bodies".

The questions have been selected to enhance understanding of the topics covered in the modules titled "Center of mass" and "Center of mass and rigid bodies". All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

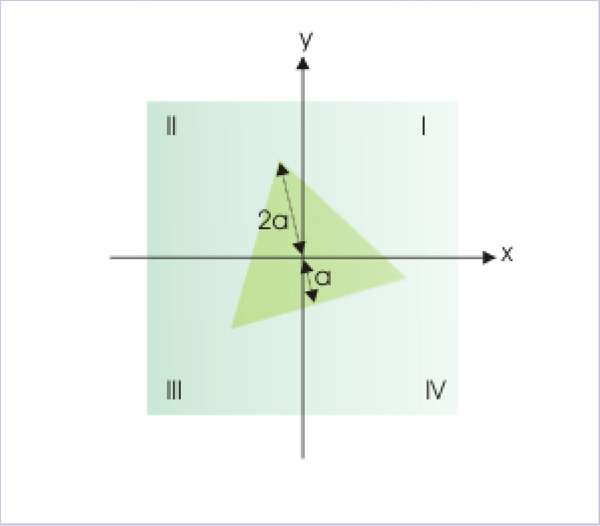
Understanding level (Center of mass)

Exercise:

Problem:

An equilateral triangular portion is removed from a uniform square plate as shown in the figure. The center of the mass of the plate :

COM of remaining plate



An equilateral triangular portion is removed from a uniform square plate .

(a) shifts to quadrant I (b) shifts to quadrant II (c) shifts to quadrant III (d) does not shift

---

Solution:

The COM of the portion removed is the geometric center of the triangle. Note here that the geometric center of the triangle coincides with the geometric center of the square - even though the triangle is asymmetrically oriented. Thus, removal of the triangular portion does not alter the COM. The COM of the remaining part of the square plate still lies at the center of the square.

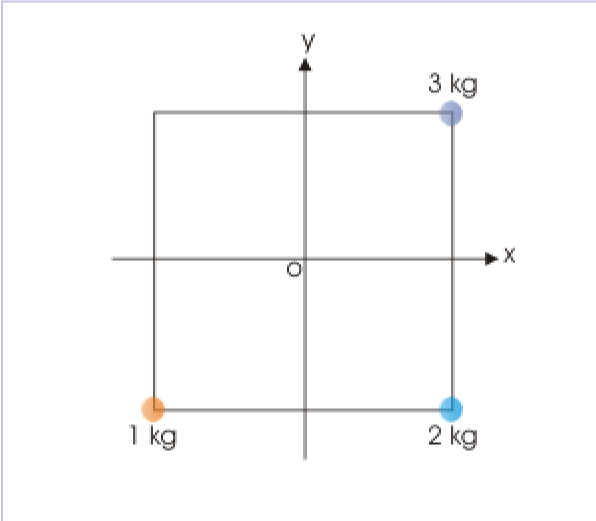
Hence, option (d) is correct.

Exercise:

Problem:

Three particles are placed at the three corners of a square as shown in the figure. What should be the mass of the particle at the remaining corner so that the COM of the system of particles lies at the center of square ?

System of particles



Particles are placed at the corners of a square .

(a) 4 kg or 8 kg (b) 4 kg (c) 6 kg (d) Center of the square can not be COM

Solution:

In order that the COM of the system of particles lies at the center, the masses of the particles on opposite corners should be equal such that masses are symmetrically distributed about the planner axes. Thus, for the given set of particle masses, the COM does not lie at the center of the square.

Hence, option (d) is correct.

Exercise:

Problem: The center of mass of a rigid body :

(a) coincides with geometric center.

(b) is a geometric point.

(c) lies always inside the rigid body.

(d) lies always outside the rigid body.

Solution:

The center of mass coincides with geometric center when density of the rigid body is uniform. However, a rigid body can have non-uniform density as well. In that case, center of mass does not coincide with geometric center. As discussed in the text in the modules on the topic, only choice (b) is correct.

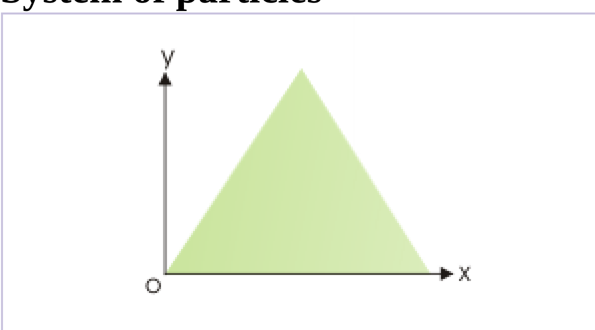
Hence, option (b) is correct.

Exercise:

Problem:

Three particles of 1 kg, 2 kg and 3 kg are positioned at the vertices of an equilateral triangle of side 1 m as shown in the figure. The center of mass of the particle system (in meters) is :

System of particles



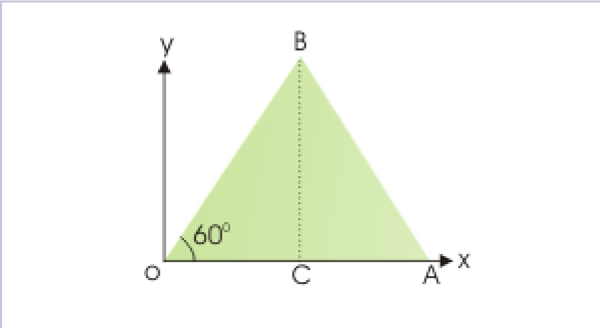
Three particles are positioned at the vertices of an equilateral triangle.

(a) 1, 0.0866 (b) 0.75, 0.43 (c) 1, 1.172 (d) 0.75, 1

Solution:

The height of the triangle is :

System of particles



Three particles are positioned at the vertices of an equilateral triangle.

$$\mathrm{AC} = \mathrm{OB} \sin 60^0 = 1 \text{ x } \frac{\sqrt{3}}{2}$$
$$\Rightarrow \mathrm{AC} = \frac{\sqrt{3}}{2} = 0.866\, m$$

The COM of the system of particles is :

$$x_{\mathrm{COM}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$
$$\Rightarrow x_{\mathrm{COM}} = \frac{1\text{x}0 + 2\text{x}1 + 3\text{x}0.5}{6} = 0.75\, m$$

and

$$y_{\mathrm{COM}} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$
$$\Rightarrow x_{\mathrm{COM}} = \frac{1\text{x}0 + 2\text{x}0 + 3\text{x}0.866}{6} = 0.43\, m$$

Hence, option (b) is correct.

Exercise:

Problem:

If all the particles are situated at a distance "r" from the origin of a three dimensional coordinate system, then COM of the system of particles is :

$$\text{(a)} \quad \le r \;\; \text{(b)} \quad = r \;\; \text{(c)} \quad > r \;\; \text{(d)} \quad \ge r$$

Solution:

If the particles are uniformly distributed, then COM lies at the center. On the other extreme, if there is only one particle in the system, then COM lies at a linear distance "r" from the center. For other possibilities, COM should lie between these two extremes.

Hence, option (a) is correct.

Exercise:

Problem:

The density of a rod is not constant. In which of the following situation COM can not lie at the geometric center ?

(a) Density increases from left to right for the first half and decreases from right to left for the second half

(b) Density increases from left to right

(c) Density decreases from left to right

(d) Density decreases from left to right for the first half and increases from right to left for the second half

---

Solution:

When density increases or decreases continuously from one end to another, the COM should lie on heavier side. In these conditions, there

is no possibility that the rod is balanced at the geometric center.

Hence, options (b) and (c) are correct.

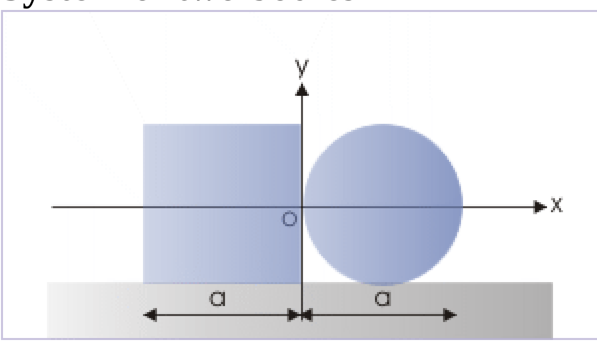
Application level (Center of mass)

Exercise:

Problem:

A circular and a square plate are placed in contact as shown in the figure. If the material and thickness of the two plates are same, then COM of the system of bodies as measured from the point of contact is :

System of two bodies



A circular and a square plate are placed in contact.

$$(a)\ \frac{a}{2},\ 0.05a\ \ (b)\ 0.05a,\ -\frac{a}{2}\ \ (c)\ -0.05a,\ 0\ \ (d)\ -0.06a,\ 0$$

Solution:

The COMs of each body is its geometric center. They form two particles system separated by a distance "a" in x - direction. Let subscripts "s" and "c" denote square and circular plates respectively. Let "σ" be the area surface density. Then the masses are :

$$m_s = a^2\sigma$$

and

$$m_c = \left(\frac{\pi a^2\sigma}{4}\right)$$

We see that the system of bodies is symmetric about x-axis, but not about y-axis. Thus, COM lies on x - axis. Now, the x-component of COM is :

$$x_{\text{COM}} = \frac{m_s x_s + m_c x_c}{m_s + m_c}$$

$$\Rightarrow x_{\text{COM}} = \frac{\left(a^2\sigma\right)\left(-\frac{a}{2}\right)+\left(\frac{\pi a^2\sigma}{4}\right)\left(\frac{a}{2}\right)}{\left(a^2\sigma\right)+\left(\frac{\pi a^2\sigma}{4}\right)}$$

$$\Rightarrow x_{\text{COM}} = \frac{-\left(1-\frac{\pi}{4}\right)}{2\left(1-\frac{\pi}{4}\right)} = -0.06a$$

Hence, option (d) is correct.

Answers

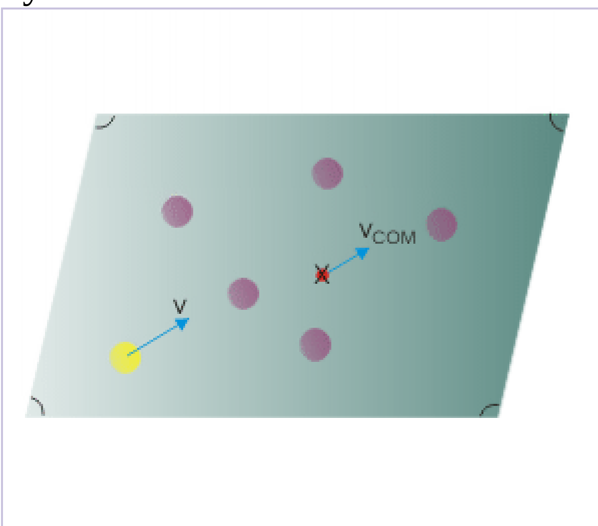
`1. (d) 2. (d) 3. (b) 4. (b) 5. (a) 6. (b) and (c) 7. (d)`

Laws of motion and system of particles

The point of application of a single external force can not be associated with more than one particle. In this sense, the consideration to extend law of motion to many particles system may appear to be meaningless. However, we find that the dynamics of a bunch of even unassociated particles can be studied collectively with startling accuracy and relevance.

Let us begin with a collection of six billiard balls, positioned arbitrarily on a smooth billiard table. Imagine we have initiated one of the ball with a constant velocity.

System of billiard balls



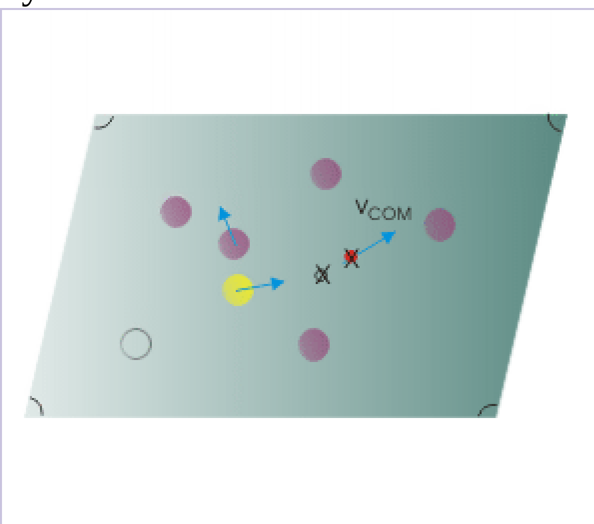
Six billiard balls on a smooth billiard table.

Since other balls are placed unconnected, they are not affected by the motion of one of the six billiard balls. It may be surprising, but we will see that "center of mass (COM)" is, as a matter of fact, governed by Newton's second law of motion! As no external force is acting on the collection of balls (neglect forces like weight and normal force, which are perpendicular to the motion), their COM is moving with a constant velocity in accordance with Newton's second law.

$$F_{\text{Ext.}} = 0; \;\; a_{\text{COM}} = 0; \;\; v_{\text{COM}} = \text{a constant}.$$

Not only this, if the moving ball collides with other ball, then also the motion of COM of the system of particles remains same. The magnitude and direction of velocity of COM remain what they were before collision. The contact forces, being the internal forces, do not affect motion of the COM. Whatever be the sequence of subsequent hits among the balls, COM moves with constant velocity. This underlines the connection between Newton's second law and the COM of the system of particles.

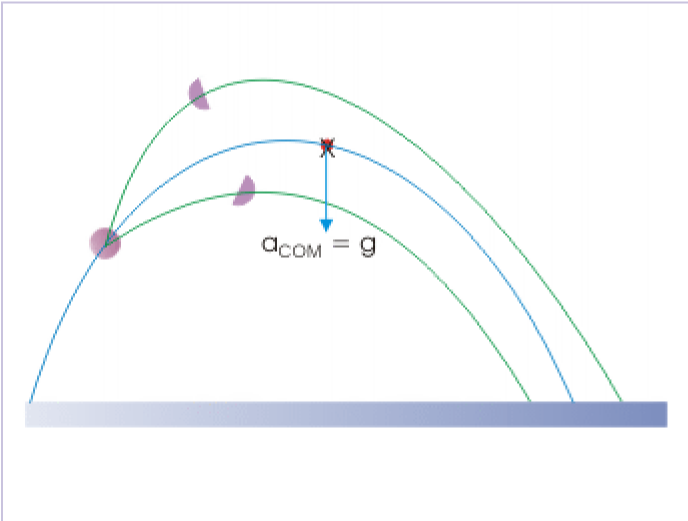
System of billiard balls



The magnitude and direction of velocity remain what they were before collision.

What if the external force is not zero. Let us take another example where a ball is projected at an angle in air. The ball is split in mid air, say, due to implanted cleavage (plane of weakness). The two fragments move down towards the surface after the split. The ball is acted upon by the gravity (force of gravitation) before being split and follows a parabolic path. Since splitting along the cleavage had been an internal process, the system of two parts is acted by the same external force of gravity. Therefore, the COM of split parts also follows exactly the same parabolic path (neglecting air resistance) as the ball would have transversed, if it were not split! Here,

Projectile motion



The ball is split in mid air.

$$F_{\text{Ext.}} = mg;\ \ a_{\text{COM}} = g;\ \ v_{\text{COM}} = \text{a variable.}$$

The external force on the system of two parts is :

$$\mathbf{F}_{\text{Ext.}} = M\mathbf{a}_{\text{COM}}$$

where "M" is the total mass of the ball and " $\mathbf{a}_{\text{COM}}$ " is acceleration of the COM.

In the nutshell, we conclude :

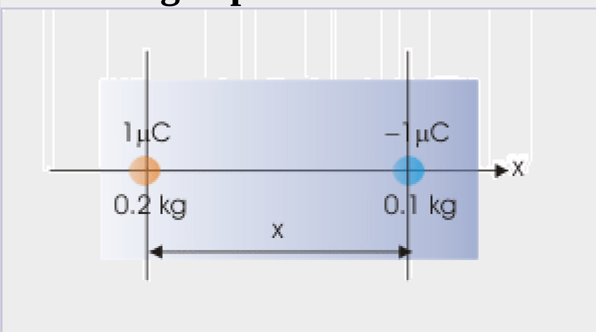
- COM represents an effective point, whose motion is governed by Newton's law of motion.
- The law of motion for system of particles is stated for the acceleration of COM - not for the individual particle. This equation does not give any information about the acceleration of individual particles.
- The law of motion for system of particles is stated assuming that all the mass of the system is concentrated at COM.
- The law of motion for system of particles is stated for the external forces acting on the particles of the system - not the internal forces.

- The law of motion for system of particles is stated for a closed system in which mass is constant i.e. there is no exchange of material "from" or "to" the system of particles.

Example:

Problem : Two spherical charged particles of 0.2 kg and 0.1 kg are placed on a horizontal smooth plane at a distance "x" apart. The particles are oppositely charged to a magnitude of 1μC. At what distance would they collide as measured from the initial position of the particle of 0.2 kg?

Two charged particles



The particles are oppositely charged to a magnitude of 1μC.

Solution : This is two particles system, which are acted by the electrostatic force of attraction. As the surface is smooth, we can neglect friction. We can also neglect weight of the particles and normal force as they are perpendicular to motion.

The important aspect to consider here is that electrostatic force of attraction between particles is internal force and, therefore, does not enter into our consideration. The position of collision, as we will see, is independent of the force of attraction and hence independent of the charge on the particles.

Note: Data on charge in the question is superfluous and is given only to highlight the independence of COM from internal forces.

Now, as the external force on the particles is zero, the motion of the COM of the two particles system will not change. Since the particles were at rest in the beginning, the COM of the particle system will remain where it was in the beginning. At the time of the collision, when the particles have same position (approximately), the COM of particles is also the point of collision. It means that point of collision coincides with the COM. Now, COM of the particle system as measured from the first particle is :

$$\Rightarrow x_{\mathrm{COM}} = \frac{m_2 x}{m_1 + m_2}$$

$$\Rightarrow x_{\mathrm{COM}} = \frac{0.1x}{0.3} = \frac{x}{3}$$

Thus, charged particles collide at a distance "x/3" from the initial position of the particle of mass 0.2 kg.

Newton's second law of motion

Here, we set out to write Newton's second law of motion for the system of particles by differentiating the expression of COM for a system of particles. First differentiation leads to expression of velocity for the COM of the system. Second differentiation leads to expression of acceleration for the system and, in turn, the expression of Newton's second law for the system of particles.

Velocity of COM

We consider a number of particles which are individually acted by different forces. Now, according to the definition of COM, we have :

$$\mathbf{r}_{\text{COM}} = \frac{\sum m_i \mathbf{r}_i}{M}$$

$$\Rightarrow M\mathbf{r}_{\text{COM}} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n$$

Differentiating with respect to time, we have :

$$\Rightarrow M\mathbf{v}_{\text{COM}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + ... + m_n\mathbf{v}_n$$

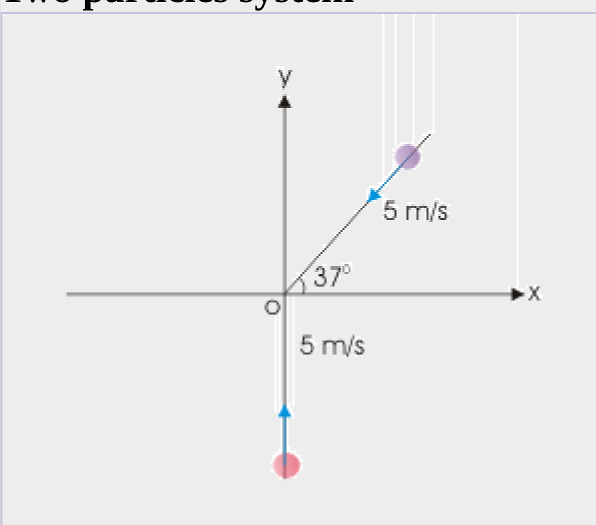
$$\mathbf{v}_{\text{COM}} = \frac{\sum m_i \mathbf{v}_i}{M}$$

This equation relates velocities of individual particles to that of COM of the system of particles.

Example:

Problem : Two particles of 0.2 kg and 0.3 kg, each of which moves with initial velocity of 5 m/s, collide at the origin of planar coordinate system. Find the velocity of their COM before and after the collision.

Two particles system



Two particles collide at the origin .

Solution : The components of velocities of COM are given as :

$$\Rightarrow v_{\text{COMx}} = \frac{m_1 v_{1\text{x}} + m_2 v_{2\text{x}}}{m_1 + m_2}$$

$$\Rightarrow v_{\text{COMx}} = \frac{0.2 \text{ X } \left(-5 \cos 37^0\right) + 0.3 \text{ X } 0}{0.5}$$

$$\Rightarrow v_{\text{COMx}} = \frac{0.2 \text{ X } \left(-5 \text{ X } \frac{4}{5}\right)}{0.5} = -1.6\, m/s$$

and

$$\Rightarrow v_{\text{COMy}} = \frac{m_1 v_{1\text{y}} + m_2 v_{2\text{y}}}{m_1 + m_2}$$

$$\Rightarrow v_{\text{COMy}} = \frac{0.2 \text{ X } \left(-5 \sin 37^0\right) + 0.3 \text{ X } 5}{0.5}$$

$$\Rightarrow v_{\text{COMy}} = \frac{0.2 \text{ X } \left(-5 \text{ X } \frac{3}{5}\right)) + 1.5}{0.5} = 1.8\, m/s$$

Thus, velocity of COM is :

$$\Rightarrow \mathbf{v}_{\text{COM}} = -1.6\mathbf{i} + 1.8\mathbf{j}$$

Since there is no external force, the COM of the two particles move with same velocity before and after the collision.

Acceleration of COM

Differentiating the relation for velocity with respect to time, we have :

$$\Rightarrow M\mathbf{a}_{\text{COM}} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \text{..........} + m_n\mathbf{a}_n$$

$$\mathbf{a}_{\text{COM}} = \frac{\sum m_i \mathbf{a}_i}{M}$$

In terms of force,

$$\Rightarrow M\mathbf{a}_{\text{COM}} = \mathbf{F}_1 + \mathbf{F}_2 + \text{..........} + \mathbf{F}_n$$

$$\Rightarrow M\mathbf{a}_{\text{COM}} = \sum \mathbf{F}_i$$

This relation relating accelerations of individual particles to the acceleration of COM needs a closer look. It brings about an important revelation about internal and external forces. Individual particles are indeed acted by both internal and external forces. Many of the forces listed on the right hand of the above equation, therefore, may be internal forces.

The internal forces are equal and opposite forces and as such cancel out in the above sum. The resultant force on the right hand side is, thus, external forces only.

$$\Rightarrow \mathbf{F}_{\text{Ext.}} = M\mathbf{a}_{\text{COM}}$$

This is the expression of Newton's second law for a system of particles. This relation also underlines that COM of the system of particles represents a point where the resultant external force appears to act. This is additional to the assumptions earlier that COM is the point where all the mass of the system appears to be concentrated. Importantly, we must always remember that this is a point where resultant external force "appears" to act and not the point where forces actually act.

Example:

Problem : A log of wood of length "L" and mass "M" is floating on the surface of still water. One end of the log touches the bank. A person of mass "m" standing at the far end of the log starts moving toward the bank. Determine the displacement of the log when the person reaches the nearer end of the log?

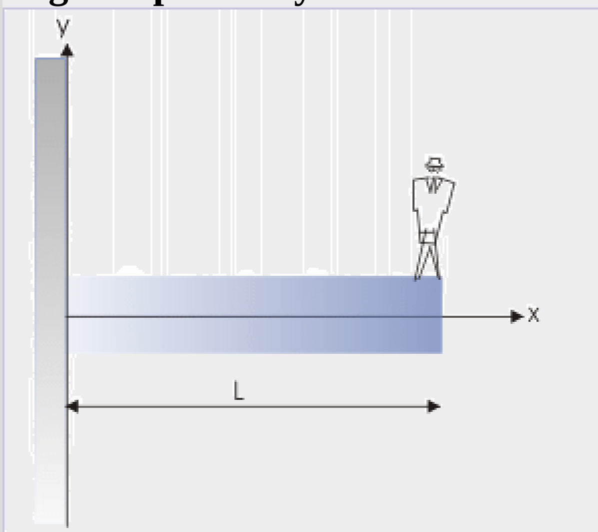
Solution : AIn this case, no external force is involved in the direction of motion. We neglect weight and normal force perpendicular to the surface of log as they do not affect the motion in x-direction. Now, considering law of motion for the man and log system :

$$\mathbf{F}_{\text{Ext.}} = (m + M)\mathbf{a}_{\text{COM}}$$
$$\Rightarrow \mathbf{a}_{\text{COM}} = 0$$

Thus, COM is not accelerated. Also as the system was at rest in the beginning, COM of the system at the end should remain where it was in

the beginning. Now, considering measurement of COM in horizontal direction (x - direction) as measured from bank for the initial condition, we have :

Log and person system



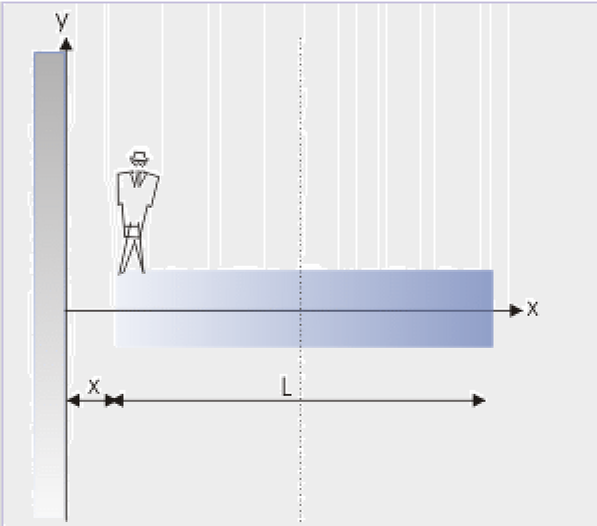
The system is at rest in the beginning.

$$x_{\text{COM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\Rightarrow x_{\text{COMi}} = \frac{mL + M \text{ X } \frac{L}{2}}{m + M}$$

Let the log has moved off "x" distance when the person reaches the nearer end. The COM in horizontal direction (x - direction) as measured from the bank for the final condition is :

Log and person system



The system has moved away by a distance "x".

$$\Rightarrow x_{\text{COMf}} = \frac{mx + M \text{ X } \left(\frac{L}{2} + x\right)}{m+M}$$

But,

$$x_{\text{COMi}} = x_{\text{COMf}}$$

$$\Rightarrow mL + M \text{ X } \frac{L}{2} = mx + M \text{ X } \left(\frac{L}{2} + x\right)$$

$$\Rightarrow x = \frac{mL}{m+M}$$

Laws of motion and system of particles (check your understanding)

The questions have been selected to enhance understanding of the topics covered in the module titled " Laws of motion and system of particles ". All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

Understanding level (Laws of motion and system of particles)

Exercise:

Problem:

A spherical object placed on a smooth horizontal table is at rest. It suddenly breaks in two unequal parts as a result of chemical reaction taking place inside the object. Then, the parts of the object move :

(a) in same direction

(b) in different directions

(c) in opposite directions with equal speeds

(d) in opposite directions with unequal speeds

Solution:

The particle is at rest before breaking into parts due to internal force. It means that the COM of the parts should remain at the initial position. In other words, its velocity should be zero even after breaking. Now, the speed of the COM is :

$$\mathbf{v}_{\mathrm{COM}} = \frac{m_1\mathbf{v}_1+m_2\mathbf{v}_2}{m_1+m_2} = 0$$

$$\Rightarrow m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0$$

$$\mathbf{v}_1 = -\left(\frac{m_2}{m_1}\right)\mathbf{v}_2$$

Clearly, the velocities of the parts of the sphere should move in opposite directions. As the parts are unequal, they should move with different speeds as well.

Hence, option (d) is correct.

Exercise:

Problem:

In a system of two particles of 0.1 kg and 0.2 kg, The particle of 0.1 kg is at rest, whereas other particle moves with an acceleration "a". The magnitude of acceleration of the COM of the system of two particles is :

$$\text{(a)}\ \frac{a}{3}\ \ \text{(b)}\ \frac{2a}{3}\ \ \text{(c)}\ \frac{3a}{2}\ \ \text{(d)}\ \frac{a}{3}$$

Solution:

The acceleration of the system of two particles is given by :

$$\mathbf{a}_{\mathrm{COM}} = \frac{m_1\mathbf{a}_1+m_2\mathbf{a}_2}{m_1+m_2}$$
$$\mathbf{a}_{\mathrm{COM}} = \frac{0.1\text{x}0+0.2\text{x}\mathbf{a}}{0.3} = \frac{2\mathbf{a}}{3}$$

Hence, option (b) is correct.

Exercise:

Problem:

A sphere of radius "r" with uniform density is placed on a smooth horizontal surface. A horizontal force "F" is applied on the sphere. The acceleration of the center of sphere :

(a) is maximum when force is applied near the bottom edge of the sphere.

(b) is maximum when force is applied near the top edge of the sphere.

(c) is maximum when force is applied at middle of the sphere.

(d) is independent of position of application.

Solution:

Sphere is composed by a system of particle. As the density of sphere is uniform, its COM coincides with the center of sphere. Now, the acceleration of the "center of mass" is dependent on the net external force on the system. It does not depend on the point of application. Recall that we explained that COM represents a point where external force "appears" to apply - not the point where net external force acts.

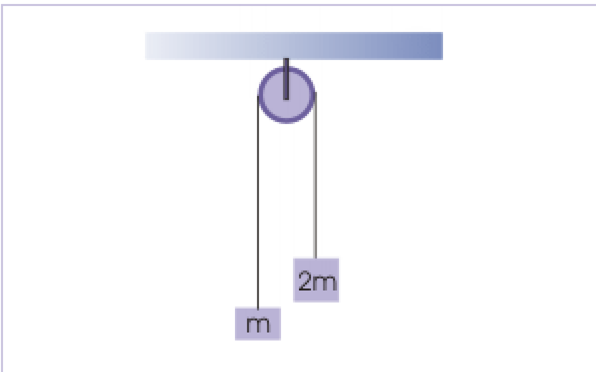
Hence, option (d) is correct.

Exercise:

Problem:

Two blocks of masses "m" and "2m" are released from rest as shown in the figure. The displacement of the COM of the block system (in meters) after 1 second is (neglect friction and masses of string and pulley) :

Blocks and pulley system



(a) 0.51 (b) 0.61 (c) 0.71 (d) 0.81

Solution:

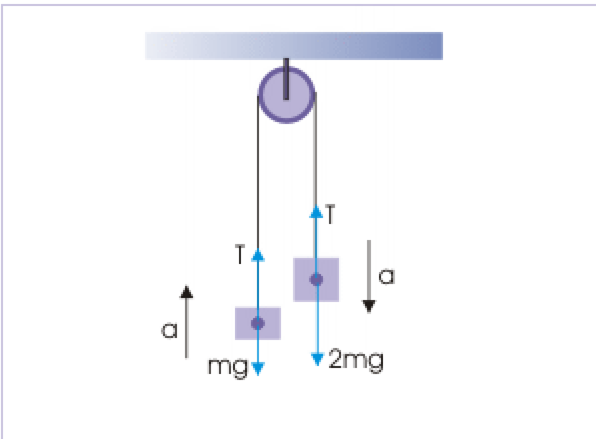
In the beginning when the blocks are released from rest, the COM of the two blocks system is also at rest. Once the blocks are released, they are acted upon by force due to gravity. We consider tensions in the string as the internal force. It means that COM is accelerated downward. In order to find the displacement of the COM, we need to know its vertical acceleration. The acceleration of the COM is obtained by :

$$\mathbf{a}_{\mathrm{COM}} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}$$

$$\Rightarrow \mathbf{a}_{\mathrm{COM}} = \frac{\mathbf{a}_1 + 2\mathbf{a}_2}{3}$$

Thus, we need to know the accelerations of individual block. As the blocks are connected with a taut string, the magnitude of acceleration of each block is same, but they are directed opposite to each other. From force analysis on individual block,

Free body diagram



For first block :

$$\Rightarrow \sum F_y = T - mg = ma$$

For second block :

$$\Rightarrow \sum F_y = 2mg - T = 2ma$$

Solving two equations, we have :

$$\Rightarrow a = \frac{g}{3}$$

Putting in the equation of acceleration of COM in y - direction with appropriate sign (downward direction as positive):

$$\Rightarrow a_{\text{COM}} = \frac{a_1 + 2a_2}{3} = \frac{-\frac{g}{3} + \frac{2g}{3}}{3}$$

$$\Rightarrow a_{\text{COM}} = \frac{g}{9} = \frac{10}{9}$$

Now, let "y" be the displacement of COM in 1 second,

$$\Rightarrow y = \frac{1}{2}at^2$$

$$\Rightarrow y = \frac{1}{2} \text{ x } \frac{10}{9} \text{ x } 1 = 0.51\, m$$

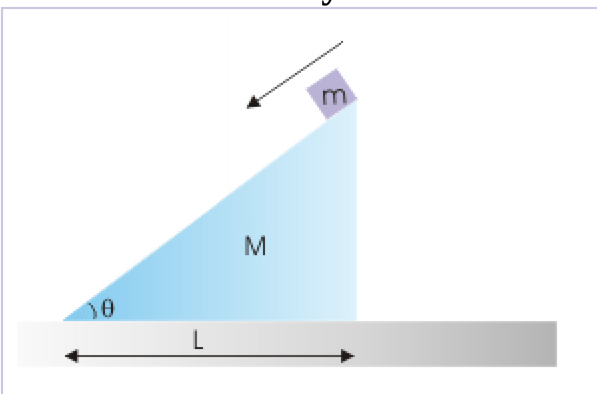
Hence, option (a) is correct.

Exercise:

Problem:

A block of mass "m" is released from the top of an incline of mass "M" as shown in the figure, where all the surfaces in contact are frictionless. The displacement of the incline, when the block has reached bottom is :

Block and incline system



$$(a)\ \frac{mL}{m+M}\quad (b)\ \frac{mML}{m+M}\quad (c)\ \frac{mL}{M-m}\quad (d)\ \frac{mML}{M-m}$$

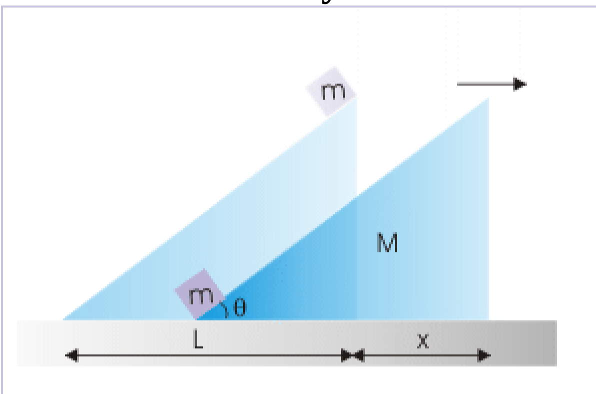
Solution:

This question makes use of the concept of COM to a great effect. Answer to this question involving force analysis would have been difficult. Here, we note that the block and incline is at rest in the beginning. This means that COM of the system is at rest before the block is released. When the block is released, the system is subjected to external force due to gravity. This means that COM should move along the direction of applied external force. As the direction of external force is vertically downward, the COM will also be accelerated downward. Importantly, COM will have no displacement in horizontal direction. We shall make use of this fact in solving this problem.

We can estimate here that the block slides down towards left. In order that the COM has no horizontal displacement, we can conclude that

incline moves towards right. Let us consider that the incline moves to right by a distance "x".

Block and incline system



The horizintal component of displacement of block on reaching bottom is given as "L" with respect to incline. This component of displacement with respect to ground,"d", is (note that in the meantime incline has also moved in opposite direction and the block is carried along with the incline) :

$$\Rightarrow d = L - x$$

As COM of the "block and incline" system has no displacement in horizontal direction,

$$x_{\text{COM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = 0$$

$$m_1 x_1 + m_2 x_2 = 0$$
$$\Rightarrow m(L - x) - Mx = 0$$

$$\Rightarrow x = \frac{mL}{m + M}$$

Hence, option (a) is correct.

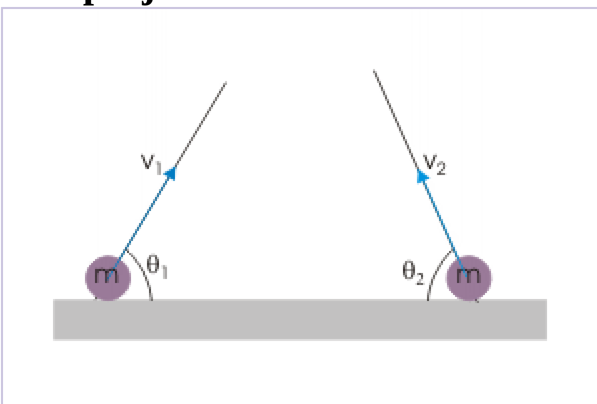
Application level (Laws of motion and system of particles)

Exercise:

Problem:

Two particles are projected with speeds v_1 and v_2 making angles θ_1 and θ_2 with horizontal at the same instant as shown in the figure. Neglecting air resistance, the trajectory of the COM of the two particles :

Two projectiles



(a) can be a vertical straight line

(b) can be a horizontal straight line

(c) is a straight line

(d) is a parabola

Solution:

To answer this question, we need to analyze the speeds of COM of two particles in horizontal (x) and vertical (y) directions. Here,

$$v_{\text{COMx}} = \frac{m_1 v_{1\text{x}} + m_2 v_{2\text{x}}}{m_1 + m_2} \implies v_{\text{COMx}} = \frac{v_{1\text{x}} + v_{2\text{x}}}{2}$$

$$\Rightarrow v_{\text{COMx}} = \frac{v_1 \cos\theta_1 - v_2 \cos\theta_2}{2} = \text{a constant}$$

and

$$v_{\text{COMy}} = \frac{m_1 v_{1\text{y}} + m_2 v_{2\text{y}}}{m_1 + m_2} \implies v_{\text{COMy}} = \frac{v_{1\text{y}} + v_{2\text{y}}}{2}$$

$$\Rightarrow v_{\text{COMy}} = \frac{v_1 \cos\theta_1 - gt + v_2 \cos\theta_2 - gt}{2}$$

$$\Rightarrow v_{\text{COMy}} = \frac{v_1 \cos\theta_1 + v_2 \cos\theta_2 - 2gt}{2}$$

On inspection of the speed of COM in x-direction, we see that it can be zero if :

$$v_1 \cos\theta_1 = v_2 \cos\theta_2$$

For this condition, COM has only speed in vertical direction and varies with time. For all other values of attributes of motion, which do not satisfy the equality, the COM has both components of velocity and, therefore, moves in parabolic trajectory.

Note: This is a case of two particles system, which is acted by the force due to gravity. If the initial directions of velocities and that of force are same, then COM would move in the direction of force. Here, situation is different, initial velocities and the force have different directions to begin with. As such the motion of COM is parabolic except for a condition as derived above.

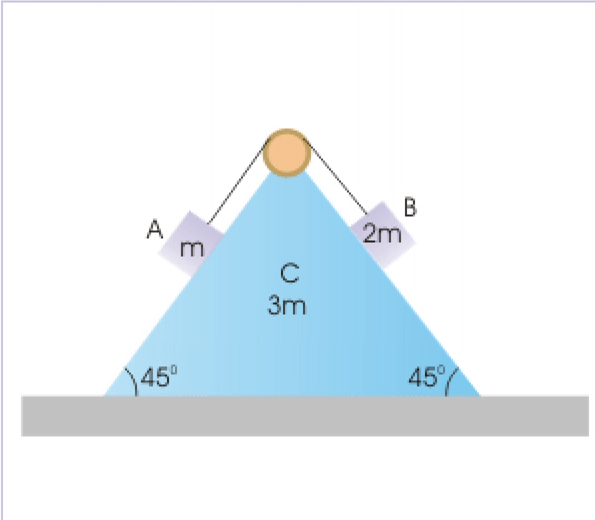
Hence, option (a) is correct.

Exercise:

Problem:

The wedge "C" is placed on a smooth horizontal plane. Two blocks "A" and "B" connected with "mass-less" string is released over a double sided wedge "C". The masses of blocks "A","B" and "C" are m, 2m and 3m respectively. After the release of the blocks, the block "B" moves on the incline by 10 cm, before it is stopped. Neglecting friction between all surfaces, the displacement of wedge "C", in the meantime, is (in cm):

Blocks and wedge system

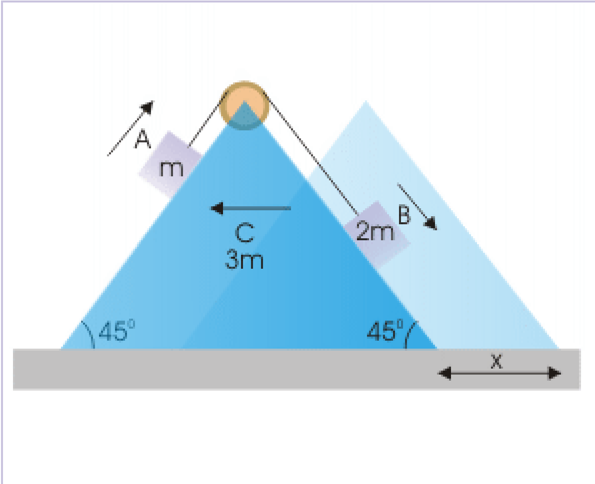


(a) 5 (b) $\frac{5}{\sqrt{2}}$ (c) $\frac{5}{2\sqrt{2}}$ (d) 10

Solution:

This question makes use of the concept of COM to a great effect. Answer to this question involving force analysis would have been difficult. Here, we note that The system of blocks and wedge is at rest in the beginning. This means that COM of the system is at rest before the block is released. When the blocks are released, the system is subjected to external force due to gravity. This means that COM should move along the direction of applied external force. As the direction of external force is vertically downward, the COM will also be accelerated downward. Importantly, COM will have no displacement in horizontal direction. We shall make use of this fact in solving this problem.

We can estimate here that the heavier block "B" slides down, whereas lighter block "A" slides up the incline. It means that block "B" moves towards right. In order that the COM has no horizontal displacement, we can conclude that wedge "C" moves towards left. Let us consider that the wedge "C" moves to left by a distance "x".
Free body diagram



Now, there is yet another twist in the question. The displacement of block "B" is given with respect to wedge "C". The horizontal component of this displacement with respect to wedge ,d, is :

$$d = \cos 45^0 = 5\sqrt{2}$$

The horizontal component of displacement of the block "B" with respect to ground, d', is :

$$\text{d'} = d - x = 5\sqrt{2} - x$$

As COM has no displacement in horizontal direction,

$$x_{\text{COM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = 0$$

$$m_1 x_1 + m_2 x_2 = 0$$
$$\Rightarrow m(5\sqrt{2} - x) - 3mx = 0$$

$$\Rightarrow x = \frac{5}{2\sqrt{2}}$$

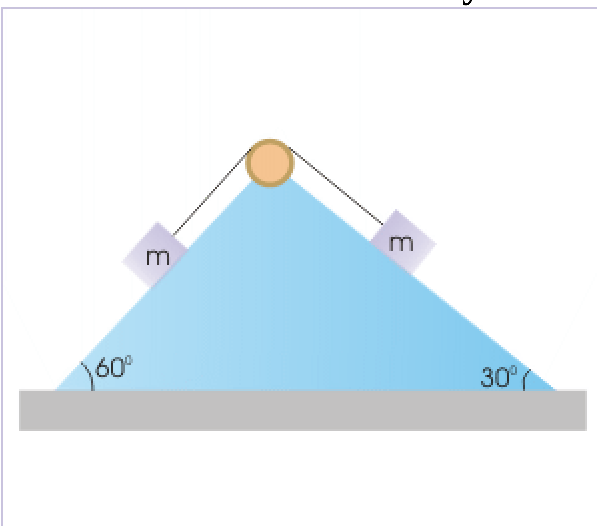
Hence, option (c) is correct.

Exercise:

Problem:

Two blocks of equal mass are released over a fixed double incline as shown in the figure. Neglecting friction between incline and blocks, the acceleration of the COM of the blocks and double incline system is :

Blocks and double incline system



$$(a)\ \frac{\sqrt{2}-1}{2\sqrt{3}}g\ (b)\ \frac{\sqrt{2}+1}{4\sqrt{3}}g\ (c)\ \frac{\sqrt{3}-1}{4\sqrt{3}}g\ (d)\ \frac{\sqrt{3}+1}{2\sqrt{3}}g$$

Solution:

Each of the block behaves as particle. Thus, the acceleration of the COM of two blocks connected with "mass-less" string is given by :

$$\mathbf{a}_{\text{COM}} = \frac{m_1\mathbf{a}_1+m_2\mathbf{a}_2}{m_1+m_2}$$

The two blocks have same mass. Thus,

$$\mathbf{a}_{\text{COM}} = \frac{(\mathbf{a}_1+\mathbf{a}_2)}{2}$$

The two blocks are connected by a taut string having acceleration of same magnitude, say, "a". Now, as individual accelerations are at right angles, the magnitude of resultant acceleration is :

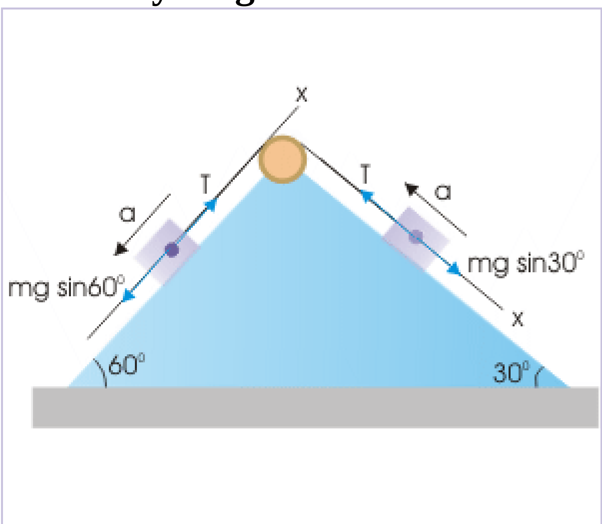
$$\left|\mathbf{a}_1 + \mathbf{a}_2\right| = \sqrt{(}a^2 + a^2 = a\sqrt{2}$$

Thus, magnitude of acceleration of the COM is :

$$a_{\text{COM}} = \frac{a\sqrt{2}}{2} = \frac{a}{\sqrt{2}}$$

We need to find the common acceleration of the individual blocks by force analysis :

Free body diagram



For first block :

$$\Rightarrow \sum F_x = mg \sin 60^0 - T = ma$$

For second block :

$$\Rightarrow \sum F_x = T - mg \sin 30^0 = ma$$

Solving two equations, we have :

$$a_{\text{COM}} = \frac{g\left(\sin 60^0 - \sin 30^0\right)}{2}$$

$$a = \frac{\sqrt{3} - 1}{4}$$

Substituting this value in the equation of acceleration of COM, we have :

$$a = \frac{\sqrt{3}-1}{4\sqrt{3}} g$$

Hence, option (c) is correct.

Answers

`1. (d) 2. (b) 3. (d) 4. (a) 5. (a) 6. (a) 7. (c) 8. (c)`

Conservation of linear momentum
The linear momentum can not change in a closed system of particles.

We have briefly defined linear momentum, while describing Newton's second law of motion. The law defines force as the time rate of linear momentum of a particle. It directly provides a measurable basis for the measurement of force in terms of mass and acceleration of a single particle. As such, the concept of linear momentum is not elaborated or emphasized for a single particle. However, we shall see in this module that linear momentum becomes a convenient tool to analyze motion of a system of particles - particularly with reference to internal forces acting inside the system.

It will soon emerge that Newton's second law of motion is more suited for the analysis of the motion of a particle like objects, whereas concept of linear momentum is more suited when we deal with the dynamics of a system of particles. Nevertheless, we must understand that these two approaches are interlinked and equivalent. Preference to a particular approach is basically a question of suitability to analysis situation.

Let us now recapitulate main points about linear momentum as described earlier :

(i) It is defined for a particle as a vector in terms of the product of mass and velocity.

$$\mathbf{p} = m\mathbf{v}$$

The small "**p**" is used to denote linear momentum of a particle and capital "**P**" is used for linear momentum of the system of particles. Further, these symbols distinguish linear momentum from angular momentum (**L**) as applicable in the case of rotational motion. By convention, a simple reference to "momentum" means "linear momentum".

(ii) Since mass is a positive scalar quantity, the directions of linear momentum and velocity are same.

(iii) In physical sense, linear momentum is said to signify the "quantum or quantity of motion". It is so because a particle with higher momentum generates greater impact, when stopped.

(iv) The first differentiation of linear momentum with respect to time is equal to external force on the single particle.

$$\mathbf{F}_{\text{Ext.}} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$$

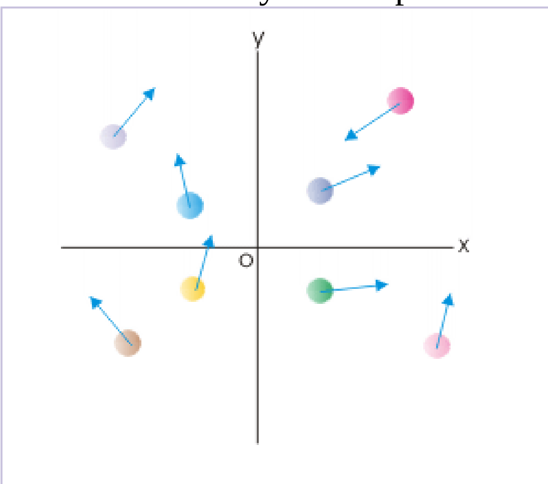
Momentum of a system of particles

The concept of linear momentum for a particle is extended to a system of particles by summing the momentum of individual particles. However, this sum is a vector sum of momentums. We need to either employ vector addition or equivalent component summation with appropriate sign convention as discussed earlier. Linear momentum of a system of particles is, thus, defined as :

Momentum of a system of particles

> The linear momentum of a system of particles is the vector sum of linear momentums of individual particles.

Momentum of a system of particles



Particles moving with diferent velocities.

$$\mathbf{p} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \text{ .... } + m_\text{n}\mathbf{v}_\text{n}$$

$$\Rightarrow \mathbf{p} = \sum m_i\mathbf{v}_i$$

From the concept of "center of mass", we know that :

$$M\mathbf{v}_{\text{COM}} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \text{ .... } + m_\text{n}\mathbf{v}_\text{n}$$

Comparing two equations,

$$\mathbf{P} = M\mathbf{v}_{\text{COM}}$$

The linear momentum of a system of momentum is, therefore, equal to the product of total mass and the velocity of the COM of the system of particles.

External force in terms of momentum of the system

Just like the case for a single particle, the first differentiation of the total linear momentum gives the external force on the system of particles :

$$\mathbf{F}_{\text{Ext.}} = \frac{d\mathbf{P}}{dt} = M\mathbf{a}_{\text{COM}}$$

This is the same result that we had obtained using the concept of center of mass (COM) of the system of particles. The application of the concept of linear momentum to a system of particles, however, is useful in the expanded form, which reveals the important aspects of "internal" and "external" forces :

$$\mathbf{F}_{\text{Ext.}} = \frac{d\mathbf{P}}{dt} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \text{................} + m_{\text{n}}\mathbf{a}_{\text{n}}$$

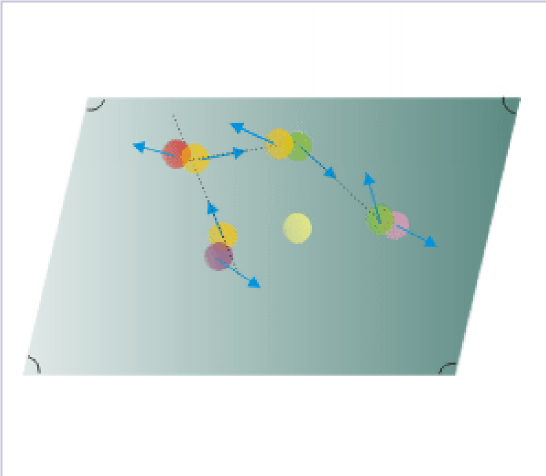
The right hand expression represents the vector sum of all forces on individual particles of the system.

$$\mathbf{F}_{\text{Ext.}} = \frac{d\mathbf{P}}{dt} = \mathbf{F}_1 + \mathbf{F}_2 + \text{................} + \mathbf{F}_{\text{n}}$$

This relation is slightly ambiguous. Left hand side symbol, " $\mathbf{F}_{\text{Ext.}}$ " represents net external force on the system of particles. But, the individual forces on the right hand side represent all forces i.e. both internal and external forces. This means that :

$$\mathbf{F}_{\text{Ext.}} = \frac{d\mathbf{P}}{dt} = \mathbf{F}_1 + \mathbf{F}_2 + \text{................} + \mathbf{F}_{\text{n}} = \mathbf{F}_{\text{Ext.}} + \mathbf{F}_{\text{Int.}}$$

It is not difficult to resolve this apparent contradiction. Consider the example of six billiard balls, on of which is moving with certain velocity. The moving ball may collide with another ball. The two balls after collision, in turn, may collide with other balls and so on. The point is that the motions (i.e. velocity and acceleration) of the balls in this illustration are determined by the "internal" contact forces.



Particles moving with diferent velocities.

It happens (law of nature) that the motion of the "center of mass" of the system of particles depend only on the external force - even though the motion of the constituent particles depend on both "internal" and "external" forces. This is the important distinction as to the roles of external and internal forces. Internal force is not responsible for the motion of the center of mass. However, motions of the particles of the system are caused by both internal and external forces.

We again look at the process involved in the example of billiard balls. The forces arising from the collision is always a pair of forces. Actually all force exists in pair. This is the fundamental nature of force. Even the external force like force due to gravity on a projectile is one of the pair of forces. When we study projectile motion, we consider force due to gravity as the external force. We do not consider the force that the projectile applies on Earth. We think projectile as a separate system. In nutshell, we consider a single external force with respect to certain object or system and its motion.

However, the motion within a system is all inclusive i.e both pair forces are considered. It means that internal forces always appear in equal and opposite pair. The net internal force, therefore, is always zero within a system.

$$\mathbf{F}_{\text{Int.}} = 0$$

This is how the ambiguity in the relation above is resolved.

Conservation of linear momentum

If no external force is involved, then change in linear momentum of the system is zero :

$$\mathbf{F}_{\text{Ext.}} = \frac{d\mathbf{P}}{dt} = 0$$
$$\Rightarrow d\mathbf{P} = 0$$

The relation that has resulted is for infinitesimally small change in linear momentum. By extension, a finite change in the linear momentum of the system is also zero :

$$\Delta\mathbf{P} = 0$$

This is what is called "conservation of linear momentum". We can state this conservation principle in following two different ways.

Conservation of linear momentum
: When external force is zero, the change in the linear momentum of a system of particles between any two states (intervals) is zero.

Mathematically,

$$\Delta\mathbf{P} = 0$$

We can also state the law as :

Conservation of linear momentum
: When external force on a system of particles is zero, the linear momentum of a system of particles can not change.

Mathematically,

$$\mathbf{P}_i = \mathbf{P}_f$$

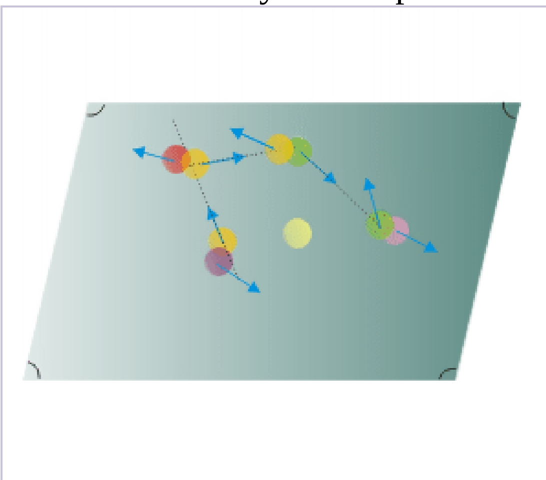
In expanded form,

$$m_1\mathbf{v}_{1\text{i}} + m_2\mathbf{v}_{2\text{i}} + \text{................} + m_{\text{n}}\mathbf{v}_{\text{ni}} = m_1\mathbf{v}_{1\text{f}} + m_2\mathbf{v}_{2\text{f}} + \text{................} + m_{\text{n}}\mathbf{v}_{\text{nf}}$$

We can interpret conservation of linear momentum in terms of the discussion of billiards balls. The system of billiard balls has certain linear momentum, say, "Pi", as one of the balls is moving. The ball, then, may collide with other balls. This

process may repeat as well. The contact forces between the balls may change the velocities of individual balls. But, these changes in velocities take place in magnitude and direction such that the linear momentum of the system of particles remains equal to "Pi".

Momentum of a system of particles



Particles moving with diferent velocities.

This situation continues till one of the billiard balls strikes the side of the table. In that case, the force by the side of the table constitutes external to the system of six billiard balls. Also, we must note that when one of the ball goes into the corner hole, the linear momentum of the system changes. This fact adds to the condition for the conservation of linear momentum. Conservation of linear momentum of a system of particles is, thus, subjected to two conditions :

1. No external force is applied on the system of particles.
2. There is no exchange of mass "to" or "from" the system of particles.

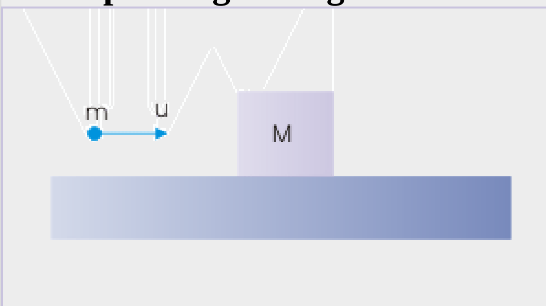
In short, it means that the system of particles is a closed system. The definition of the conservation principle is, thus, modified as :

Conservation of linear momentum
: The linear momentum can not change in a closed system of particles.

Example:

Problem : A bullet of mass "m" , moving with a velocity "u", hits a wooden block of mass "M", placed on a smooth horizontal surface. The bullet passes the wooden block and emerges out with a velocity "v". Find the velocity of bullet with respect to wooden block.

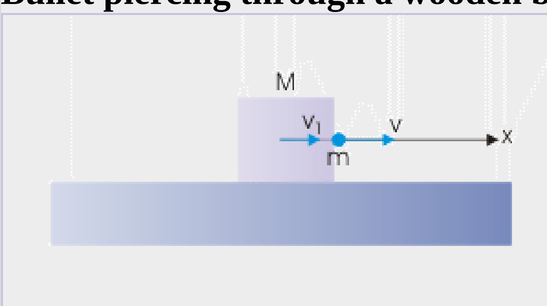
Bullet piercing through a wooden block



A bullet of mass "m" , moving with a velocity "u", hits a wooden block of mass "M".

Solution : To determine relative velocity of the bullet with respect to wooden block, we need to find the velocity of the wooden block. Let the velocity of block be " v_1 ", then relative velocity of the bullet is :

Bullet piercing through a wooden block



The bullet passes the wooden block and emerges out with a velocity "v".

$$v_{\text{rel}} = v - v_1$$

The important aspect of the application of conservation of linear momentum is that we should ensure that the system of particles/ bodies is not subjected to net

external force (or component of net external force) in the direction of motion.
We note here that there is no external force in horizontal direction on the system comprising of bullet and wooden block. We can, therefore, find the velocity of the wooden block, using conservation of linear momentum.

$$\Delta P = mv + mv_1 - mu = 0$$

$$\Rightarrow v_1 = \frac{m(u-v)}{M}$$

and

$$\Rightarrow v_{\text{rel}} = v - \frac{m(u-v)}{M}$$

$$\Rightarrow v_{\text{rel}} = \frac{Mv - m(u-v)}{M}$$

Conservation of linear momentum in component form

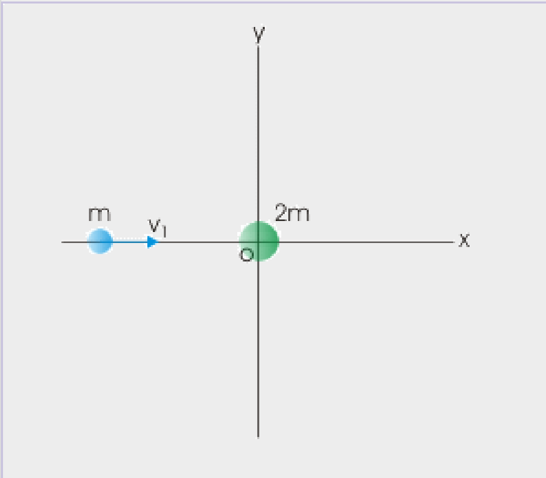
Linear momentum is a vector quantity. It, then, follows that we can formulate conservation principle in three linear direction. In three mutually perpendicular directions of a rectangular coordinate system :

$$P_{\text{xi}} = P_{\text{xf}}$$
$$P_{\text{yi}} = P_{\text{yf}}$$
$$P_{\text{zi}} = P_{\text{zf}}$$

These are important result. In most of the situations, we are required to analyze motion in linear direction i.e. one direction. In that case, we are required to analyze force, momentum etc in that direction only. This is a great simplification of analysis paradigm as we shall find out in solving problems.

Example:
Problem : A ball of mass "m", which is moving with a speed " v_1 " in x-direction, strikes another ball of mass "2m", placed at the origin of horizontal planar coordinate system. The lighter ball comes to rest after the collision, whereas the heavier ball breaks in two equal parts. One part moves along y-axis with a speed " v_2 ". Find the direction of the motion of other part.

Collision of balls



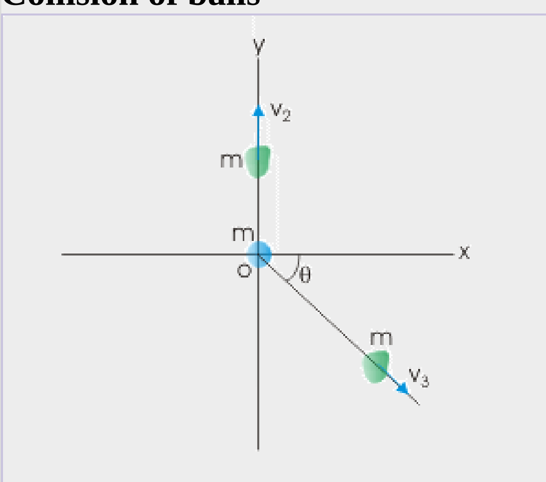
A ball collides with another ball at the origin of coordinate system.

Solution : Answer to this question makes use of component form of conservation law of momentum. Linear momentum before strike is :

$$P_{\mathrm{xi}} = mv_1$$
$$P_{\mathrm{yi}} = 0$$

Let the second part of the heavier ball moves with a speed " v_3 " at an angle "θ" as shown in the figure. Linear momentum in two directions after collision :

Collision of balls



The ball at the origin breaks up
in two parts.

$$P_{\text{xf}} = mv_3 \cos\theta$$
$$P_{\text{yf}} = mv_2 - mv_3 \sin\theta$$

Applying conservation of linear momentum :

$$\Rightarrow mv_3 \sin\theta = mv_1$$

and

$$mv_2 - mv_3 \sin\theta = 0$$

$$\Rightarrow mv_3 \sin\theta = mv_2$$

Taking ratio, we have :

$$\tan\theta = \frac{v_2}{v_1}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{v_2}{v_1}\right)$$

Force and invariant mass

Newton’s second law is stated in terms of either time rate of change of momentum or as a product of mass and acceleration.

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt}$$

and

$$\mathbf{F} = m\mathbf{a}$$

There is a bit of debate (even difference of opinion) about the forms (momentum .vs. acceleration) in which Newton’s second law is presented – particularly with respect to consideration of mass as invariant or otherwise. In this module, we shall discuss to resolve this issue.

As far as classical mechanics is concerned, we shall see that mass is always considered invariant. The reference to variable mass is actually about redistribution of mass - not the change in the mass of the matter. Such is the case with a leaking balloon or with a rocket. Matter (gas exhaust) is simply let out in the surrounding. So long, mass is not redistributed during the application of external force, we can safely take out mass from the differential of first equation. In that case, expression of force in two forms are same and equal.

However, the two forms of force defining relations are not equivalent in relativistic mechanics. Here, the definition of force in terms of linear momentum stands, but not the form involving mass and acceleration. We shall refer the actual relation in the end of this module.

Mass invariance

The mass invariance in classical mechanics enables us to take the mass out of the differential equation given by Newton’s second law :

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt}$$

This is important from the perspective that external force can be expressed in terms of acceleration :

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$$

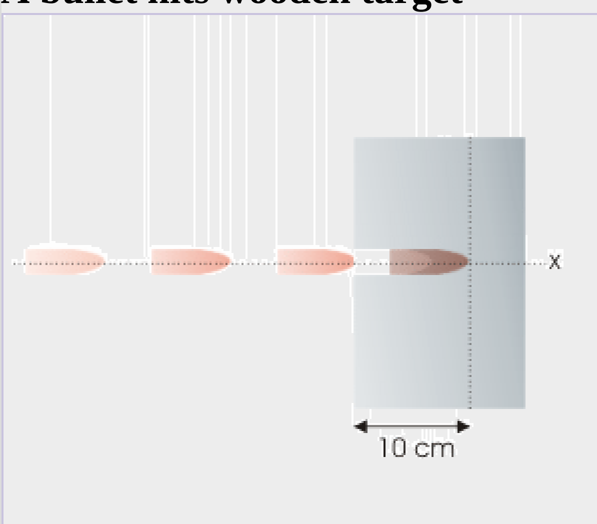
A situation, where measurement of force is required, assumes that mass is invariant. We proceed to measure force either in terms of acceleration or change in momentum in equivalent manner.

Force as product of mass and acceleration

Example:

Problem : A bullet of 10 gram enters into a wooden target with a speed of 200 m/s and comes to stop with constant deceleration. If linear penetration into the wood is 10 cm, then find the force on the bullet.

A bullet hits wooden target



Solution : The motion of the bullet is resisted by wood, which applies a force in a direction opposite to that of the bullet. We assume wood to be uniform. The resistance offered by it, therefore, is constant. Now, as force is constant, resulting deceleration is also constant. Applying equation of motion,

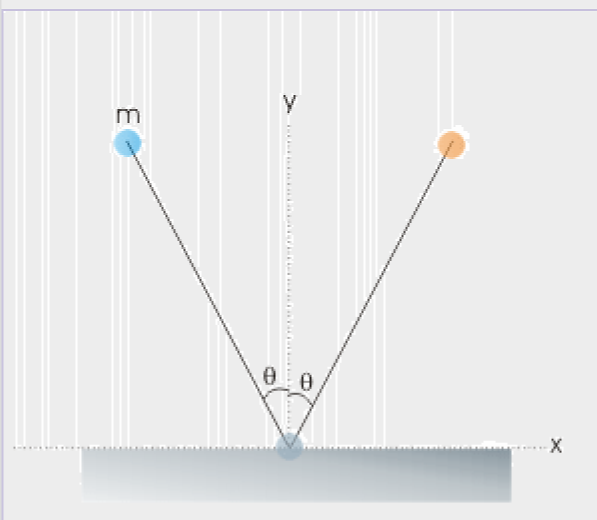
u = 500 m/s ; v = 0; x = 10/100 = 0.1 m.

$$v^2 = u^2 + 2ax$$
$$\Rightarrow 0 = 200^2 + 2a \text{ x } 0.1$$
$$\Rightarrow a = -\frac{40000}{0.2} = -200000\, m/s^2$$

Force on the bullet,
F = ma = - 0.01 x 200000 = -2000 N
Note that bullet, in turn, applies 2000 N of force on the wooden target. This explains why bullet hits are so fatal for animals. Further note that we measured force as product of mass and acceleration.

Force as a time rate of change in linear momentum

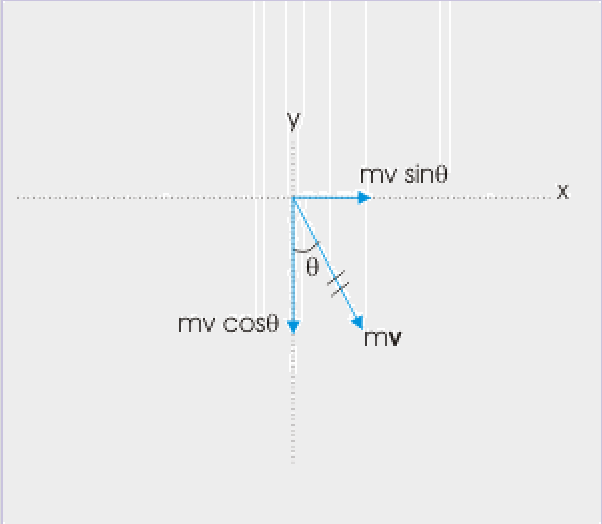
Example:
Problem : A ball of mass "m" with a speed "v" hits a hard surface as shown in the figure. The ball rebounds with the same speed and at the same angle, θ, with vertical as before. Find average force acting on the ball.
A ball strikes the hard surface



Solution : We can determine force either (i) by measuring acceleration or (ii) by measuring change in linear momentum during the motion. Here, we take the second approach.

The linear momentum (remember that it is a vector) before hitting surface is :

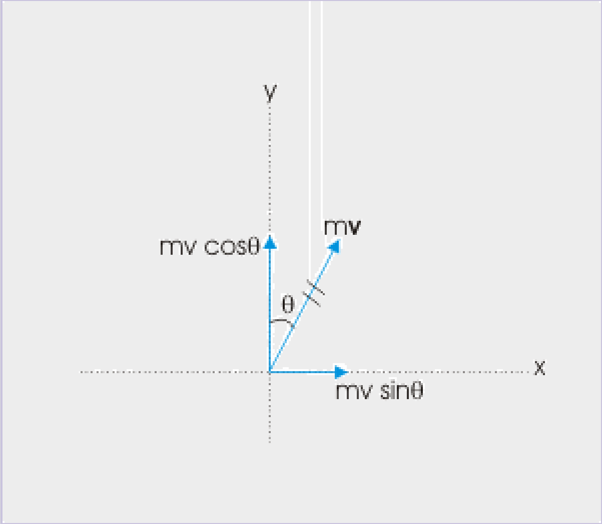
Components of linear momentum before hitting surface



$$\mathbf{p}_i = mv\sin\theta\mathbf{i} - mv\cos\theta\mathbf{j}$$

The linear momentum (remember that it is a vector) after hitting surface is :

Components of linear momentum after hitting surface



$$\mathbf{p}_f = mv\sin\theta\mathbf{i} + mv\cos\theta\mathbf{j}$$

The change in linear during contact with surface, Δp,

$$\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = mv\sin\theta\mathbf{i} + mv\cos\theta\mathbf{j} - mv\sin\theta\mathbf{i} + mv\cos\theta\mathbf{j}$$
$$\Rightarrow \Delta \mathbf{p} = 2mv\cos\theta\mathbf{j}$$

The average force is :

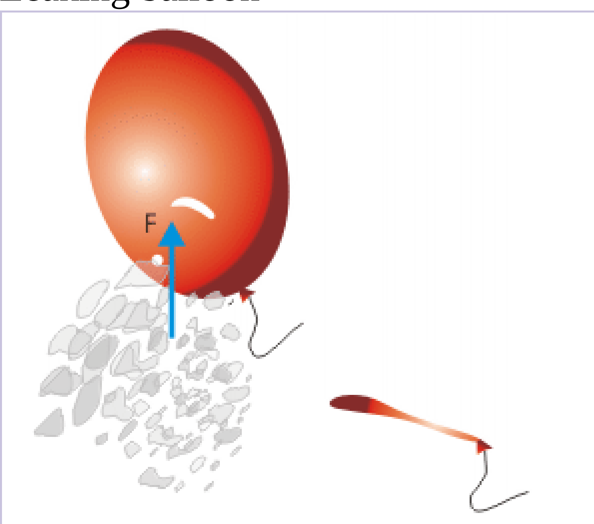
$$\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t} = \frac{2mv\cos\theta\mathbf{j}}{\Delta t}$$

Note that force is acting in vertical upward direction.

Mass redistribution and Newton's second law

There are certain system in which mass is redistributed with the surrounding. The change in mass of the system results as there is exchange of mass between the system and its surrounding. The motion of a leaking balloon, as shown, is an example of changing mass system.

Leaking balloon



The air gushing out of the balloon generates an external force on balloon.

Similar is the situation in the case of a rocket. Now, does this exchange of mass with surrounding have any bearing on the form of Newton’s second law or about the meaning of force as product of mass and acceleration? This is the question that we need to answer. To investigate this question, let us consider the motion of the rocket. It acquires very high speed quickly as a result of high speed gas escaping in the opposite direction.

Applying Newton’s second law, considering that the mass of the system is changing :

$$F = \frac{d(mv)}{dt}$$
$$\Rightarrow F = m\frac{dv}{dt} + v\frac{dm}{dt}$$
$$\Rightarrow F = ma + v\frac{dm}{dt}$$

This apparent mass variant form of Newton’s second law appears to destroy the well founded meaning of force as "product of invariant mass and acceleration". As a matter of fact, it is not so.

Let us recall that the relation, F = ma, essentially underlines relation between force (cause) and acceleration (effect). Now look closely at the additional term resulting from change in mass in the context of a real time case like of a rocket. Rearranging, we have :

$$\Rightarrow F - v\frac{dm}{dt} = ma$$

Here, we need to be cautious in interpreting this expression. The first question that we need to answer is : what does "v" represent in the expression? Is it the velocity of rocket or escaping gas or is it the relative velocity of rocket with respect to escaping gas? The second question, then, is what does "a" represent?

We see that "v" should represent the velocity of the rocket in the ground inertial reference. This is the way we interpret Newton's second law for a constant mass body? The situation here, however, is different in that a part of the body is continuously being transferred to other system, which itself is moving. In order to simplify the analysis, we consider that rocket is ejecting

gas at constant rate and with constant relative velocity. We can then say that ejected gas is applying force on the rocket in the reference of ejected gas, which is moving at uniform velocity. As such moving mass of gas constitute an inertial frame in which force is applied. The velocity, "v", therefore, represents the velocity of the rocket with respect to ejected mass of gas. We, then, rewrite the equation as :

$$\Rightarrow F - v_r \frac{dm}{dt} = ma$$

where " v_r " represents the velocity of rocket with respect to gas being ejected.

Now, we turn to answer second question about acceleration. We know that acceleration is time rate of change of velocity. But, as we discussed, we measure velocity of the rocket with reference to ejected gas. Therefore, it follows that rate of change should also be associated with relative velocity - not the absolute velocity with respect to ground. However, there is an important difference between velocity and acceleration, as measured in two inertial frames viz. ground and ejected gas. Though, measurement of velocities are different, but measurement of "change" in velocity remains same in all inertial frames. Hence, we can interpret "a" as acceleration measured in either of two references without any distinction.

Now, we are in position to correctly interpret the additional term " $-v_r \frac{dm}{dt}$ " by answering the following question : "How could a high speed ejection of gas make the rocket accelerate in the absence of any other external force? We should remember that gravitational pull and air resistance, as a matter of fact, retards the motion of rocket. The answer lies in the fact that the additional term " $-v_r \frac{dm}{dt}$ " actually represents a force called "thrust" on the rocket. For rocket to accelerate, this thrust is an external force on the rocket. If we represent thrust by symbol "T", then :

$$\Rightarrow T = -v_r \frac{dm}{dt}$$

Substituting in the equation of Newton's law,

$$\Rightarrow F+T=ma$$

The additional term, therefore, is not an “effect” term like “ma”, but a "cause" term representing “external force”, which results from the exchange of mass with the surrounding.

Clearly, there is no other external force other than thrust that gives such a great acceleration to a rocket. Gravitational and air resistance actually retards the motion of a rocket moving vertically upward. Thus, we can safely say that thrust is the only force that imparts acceleration in the direction of the motion of the rocket. In case rocket is fired from a region where other external forces like that of gravity and air resistance can be neglected, then we can put , F = 0. Then,

$$\Rightarrow T=ma$$

In the nutshell, we see that a varying mass does not change the fundamental aspect of Newton’s second law of motion. It only modifies the external force on the body. The net external force is still equal to the product of mass and acceleration.

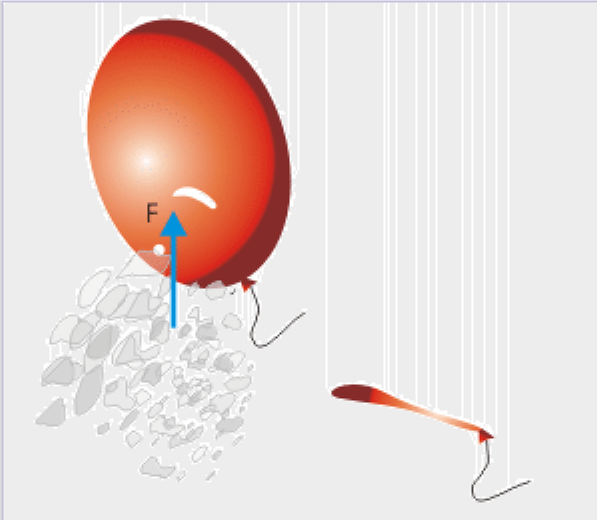
We shall reinforce these concepts with another approach that makes use of conservation of linear momentum in a separate module on rocket.

Example:

Problem : Air from a leaking balloon of mass 15 gm gushes out at a constant speed of 10 m/s. The balloon shrinks completely in 10 seconds and reduces to a mass of 5 gm. Find the average force on the balloon.

Solution : There is no external force on balloon initially. However, gushing air constitutes an exchange of mass between balloon and its surrounding. This generates an external force on the balloon given by :

Leaking balloon



The air gushing out of the balloon generates an external force on balloon.

$$F_{\text{avg}} = -v_r \text{ x } \frac{\Delta m}{\Delta t}$$
$$\Rightarrow F_{\text{avg}} = -10 \text{ x } \frac{0.005-0.015}{10}$$
$$\Rightarrow F_{\text{avg}} = 0.01 \text{ N}$$

We summarize the discussion held, in the context of classical mechanics, so far as :

1. Given mass is invariant.
2. In some instances, the the variance of the mass of body system observed is actually a redistribution of mass from one body system to more than one body systems.
3. Irrespective of the nature of mass of a body system (varying or constant), external force is equal to time rate change of momentum.
4. If there is redistribution of mass, then it results into an external force on the original body, modifying the external force on it.

5. Irrespective of the nature of mass of a body system (varying or constant), the net external force is equal to the product of mass and acceleration.
6. Irrespective of the nature of mass of a body system (varying or constant), momentum and acceleration forms of Newton's second law are equivalent and consistent to each other.

Mass variance in relativistic mechanics

The detailed discussion of this topic is not part of the course. For information sake, we shall only outline the features of force law in relativistic mechanics. Here, momentum and acceleration forms are not equivalent. As a matter of fact, the momentum form is valid even in relativistic mechanics i.e.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

However, the acceleration form is not valid. For relativistic mechanics, this form is written, in terms of rest mass, m_0 , as :

$$\mathbf{F} = \frac{m_0\mathbf{a}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} + \left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}} m_0 \frac{\mathbf{v.a}}{c^2}\mathbf{v}$$

Yet another important aspect of force here is that force and acceleration vectors need not be parallel or in the same direction.

We summarize the discussion held, in the context of relativistic mechanics, so far as :

1. Given mass is not invariant.
2. The momentum and acceleration forms of Newton's second law are not equivalent.
3. Momentum form of Newton's second law is valid.
4. Acceleration form of Newton's second law is not valid.
5. Force and acceleration need not be in the same direction.

Rocket

Space investigation revolves around this complicated machine. Our consideration in this module, however, is limited to the governing laws that broadly guide this piece of engineering marvel to destinations that we could not have imagined. We shall, therefore, study motion of a rocket from the perspective of physical laws only – not the sophisticated technology otherwise associated with it.

A rocket is known to gain quick acceleration and speed. It achieves outstanding speed good enough to propel itself beyond the influence of gravity and into the space that is being actively probed since its advent.

The rocket attains speed by ejecting high speed gas through exhaust nozzles. This results in a force termed as "thrust" (we studied about this force in previous module). Along the way, the rocket keeps loosing mass, while it is accelerated against air resistance and gravitational pull. These opposing forces gradually vanish as it reaches sufficient distance away from Earth.

It is apparent from the description of the motion so far that a good part of rocket mass is fuel. If the time rate of gas ejection is constant, then the same thrust would propel rocket with increasing acceleration (because of reduced mass) with the progress of motion through the space.

In this module, we would establish working relations to determine speed, thrust and time rate of ejection. We shall, though, restrict ourselves to linear motion of rocket in one dimension to keep the description simple.

Motion of a rocket

Conservation of linear momentum provides appropriate analysis framework for analyzing motion of a rocket.

The important difference in the approach here vis-a-vis application of Newton's second is that we can consider "rocket" and "ejected gas mass" as components of a closed, isolated system. Though, the mass of the rocket is

varying with time, but the mass of the “rocket – gas mass” system is constant. This fact eliminates the complexity resulting from varying mass.

It is easy to visualize that “thrust” on the rocket should be greater than forces like gravity and air resistance, which are opposing its motion. Since rocket acquires great enough velocity and goes beyond the influence of opposing forces quickly, it is intuitive to study motion in ideal condition, when no external force other than “thrust” operates on it. This enables us to draw a base case, which can be appropriately modified by taking other external forces into account, if so required.

Thus, system is closed and isolated. The net external force is zero. As such, the linear momentum of the system is conserved.

For the analysis of the motion of a rocket, we shall make few simplifying assumptions :

- There is no external force like gravity and air resistance.
- The motion is taking place in one dimension.
- Fuel is consumed at constant rate.
- Gas is ejected at constant relative velocity with respect to rocket.

Reference to different velocities

Conventionally, we represent velocity of rocket with respect to ground by symbol “v”, velocity of escaping gas with respect to ground by “u” and relative velocity of rocket with respect to escaping gas by “ v_r ”. Also, we consider the direction of motion of rocket as the reference direction (x – direction).

The absolute velocities of the rocket and ejected gas mass are shown in the figure :

Absolute velocities

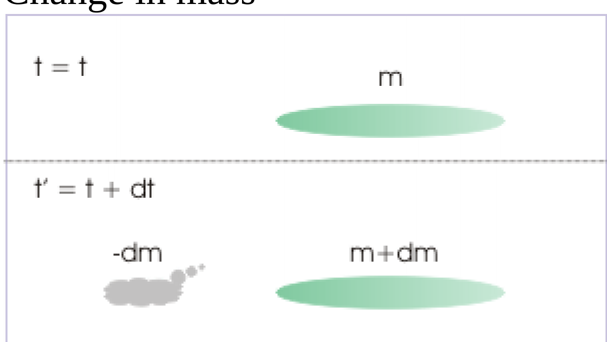
The velocities of rocket and ejected gas mass are shown with respect to ground.

The relative velocity of rocket with respect to escaping gas is equal to difference of absolute velocities with respect to ground. It may also be emphasized here that relative velocity of rocket with respect to escaping gas and relative velocity of escaping gas with respect to rocket are equal and opposite to each other.

Time rate of change in mass

The expression " $\frac{dm}{dt}$ " is time rate of change in the mass of the rocket. Since final mass of the rocket is less than initial mass during a small time interval "dt", the time rate of change is essentially a negative quantity. We should note that mass is an unsigned scalar, which can not be negative. However, change in mass can be negative. Clearly, the mass of rocket, after a small time interval, is "m+dm" - not "m-dm". The masses of two components of system are shown in the figure.

Change in mass



The change in the mass of the rocket with time is negative.

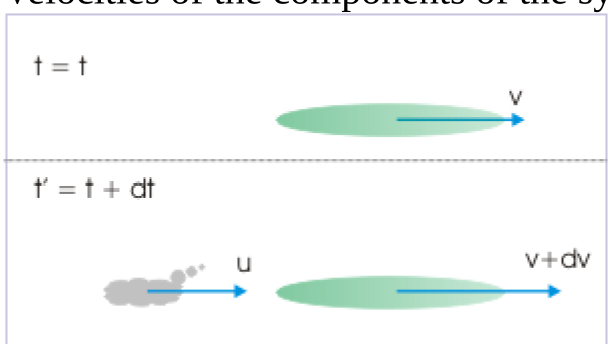
Conservation of linear momentum

We analyze motion of “rocket and ejected mass” system in the inertial frame of reference of ground. There is no difficulty in the interpretation as system is a single entity without any variation in mass.

At a given instant, t = t, let “m” be the mass of the rocket and “v” be the velocity of the rocket.

We, now, consider the situation at a time instant t’ = t + dt, after a small time interval, “dt”. Let “m+dm” be the mass of the rocket, v+dv be the absolute velocity of the rocket with respect to ground, “-dm” be the mass of the ejected gas and “u” be the absolute velocity of the gas mass with respect to ground.

Velocities of the components of the system



Velocity of rocket increases with time.

Now, applying conservation of linear momentum, we have :

$$mv = (m + dm)(v + dv) + (-dm)u$$

This form of equation, however, is not very useful as it consists of absolute velocity of escaping gas, which is variable and is difficult to be measured. We need to convert this velocity in terms of relative velocity of rocket with respect to ejected mass (v_r) and the velocity of the rocket (v).

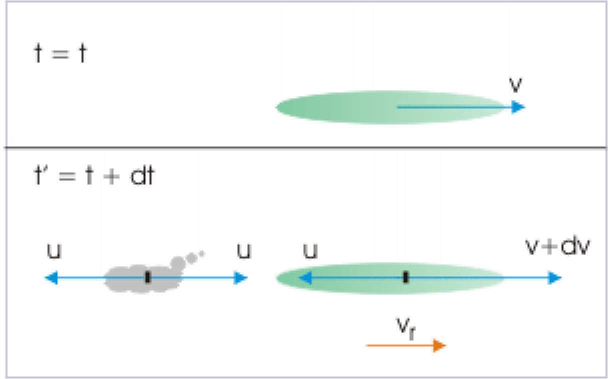
The relative velocity of rocket with respect to ejected gas is equal to the magnitude of relative velocity of ejected mass with respect to rocket. This later velocity i.e. relative velocity of ejected mass with respect to rocket is actually the velocity that can be calibrated for different time rates of mass ejection at the ground.

From consideration of relative motion, we know that

$$v_r = v + dv - u$$

In the figure, negative of the velocity of gas mass is applied to both components of the system to obtain relative velocity of the rocket with respect to ejected gas mass.

Relative velocity of the rocket



Relative velocity is obtained by subtracting velocities.

Rearranging, we have :

$$\Rightarrow u = v + dv - v_r$$

Substituting this expression for “u” in the equation of conservation of linear momentum, we have :

$$\Rightarrow mv = (m + dm)(v + dv) + (-dm)(v + dv - v_r)$$

$$\Rightarrow mv = mv + vdm + mdv + dmdv - vdm - dmdv + v_r dm$$

$$\Rightarrow mdv + v_r dm = 0$$

$$\Rightarrow mdv = -v_r dm$$

Dividing both sides by “dt”, we have :

$$m\frac{dv}{dt} = -v_r\frac{dm}{dt}$$

$$\Rightarrow ma = -v_r\frac{dm}{dt}$$

We have discussed the expression on the right hand side of the equation in the module named “Force and invariant mass”. This term was found to be the “cause” element in Newton’s second law of motion known as “thrust”. The thrust, “T”, is a force that results from exchange of mass between rocket and its surrounding. It acts on the rocket in the direction opposite to the direction in which gas escapes from the rocket i.e. in the direction of motion of the rocket.

$$\Rightarrow T = ma = -v_r\frac{dm}{dt}$$

This equation is the governing expression for the motion of a rocket in the absence of other external forces.

Rocket parameters

A rocket is usually designed for different uniform time rates of change in mass ($\frac{dm}{dt}$). by This rate, in turn, determines (i) fuel consumption rate (ii)

velocity with which gas is ejected for given nozzle size (iii) thrust on the rocket and (iv) velocity of the rocket.

Fuel consumption rate

We have seen that time rate of change in mass is a negative quantity. On the other hand, we report fuel consumption with a positive number like 1 kg/s or so. Negative number, as we can see, does not make sense in reporting fuel consumption rate. It is, therefore, imperative that we define fuel consumption rate, "r", as negative of time rate of change in mass i.e.

$$r = -\frac{dm}{dt}$$

The equation of rocket can be written in terms of fuel consumption rate as :

$$\Rightarrow T = ma = rv_r$$

Velocity of a rocket

We have established relation for thrust in terms of relative velocity of rocket and time rate of change in mass. Yet another interesting aspect to know about rocket would be the velocity of rocket for a given set of design parameters. In order to find the same, we need to have another close look at the governing expression of the rocket,

$$ma = -v_r \frac{dm}{dt}$$

The important thing to emphasize here is that acceleration "a" is the acceleration of the rocket and as such is equal to the time rate of change of velocity of the rocket – not the time rate of change of relative velocity. Rewriting the equation,

$$\Rightarrow m\frac{dv}{dt} = -v_r\frac{dm}{dt}$$

Rearranging,

$$\Rightarrow dv = -v_r\frac{dm}{m}$$

Integrating both sides of the equation,

$$\Rightarrow \quad dv = - \quad v_r\frac{dm}{m}$$

As pointed out, rocket design considers uniform ejection rate and relative velocity. Hence, we can take " v_r " term out of the integral. Now, integrating both sides between initial and final values, we have :

$$\Rightarrow v_f - v_i = -v_r \;{}_{m_f}^{m_i}\; \frac{dm}{m}$$

$$\Rightarrow v_f - v_i = -v_r \ln\frac{m_f}{m_i}$$

$$\Rightarrow v_f - v_i = v_r \ln\frac{m_i}{m_f}$$

Converting natural logarithm to the base "10", we have :

$$\Rightarrow v_f - v_i = 2.303 v_r \log\frac{m_i}{m_f}$$

If initial velocity is zero and final velocity, $v_f = v$, then velocity of rocket, "v", is given as :

$$\Rightarrow v = 2.303 v_r \log\frac{m_i}{m_f}$$

We should note here that $m_f < m_i$. Hence, ratio " $\frac{m_i}{mf}$ " is greater than "1". Its logarithm is positive.

Multistage rocket system

The expression of velocity of a rocket as derived is :

$$v = 2.303 v_r \log \frac{m_i}{m_f}$$

This expression is indicative of one important aspect about the motion of rocket. The motion of the rocket can be increased in two ways (i) by increasing gas ejection velocity, " v_r " and (ii) by reducing mass of the rocket, " m_f ". The first factor is the primary cause of motion. We can not impart acceleration and velocity to a standstill rocket by removing a part of it. However, if the rocket has acquired certain velocity, then it is possible to increase velocity of the rocket by merely dislodging unnecessary part (mass), therefore reducing final mass at any time.

This is what is achieved in the multi – stage rocket system. Fuel load is divided in different segments. Once the fuel is exhausted, then that segment is dislodged successively.

Other external force(s)

We are now equipped to analyze motion of a rocket with external force other than thrust. Analyzing air resistance is complicated as it depends on many factors including velocity of the rocket. This aspect is beyond the scope of this module. The effect of force due to gravity, in the range in which acceleration due to gravity is considered constant, is relatively easy to assess.

As pointed out earlier in the module named "Force and invariant mass",

$$F + T = ma$$

Gravity acts downward – opposite to the direction of motion of a vertically moving rocket. Hence,

$$\Rightarrow ma = T - mg$$

$$\Rightarrow a = \frac{T}{m} - g$$

We should be careful in using this relation. This relation is valid for a given instant, when mass of the rocket is “m”. Generally, we know the initial mass and as such this relation can be used to determine acceleration of the rocket in the beginning of motion of rocket fired from the ground or from near the ground.

Rocket (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to rocket. The questions are categorized in terms of the characterizing features of the subject matter :

- Initial acceleration
- Fuel consumption rate
- Projection from space
- Motion under gravity

Initial acceleration

Problem 1 : A rocket of total mass 500 kg is fired from the surface of Earth in vertical direction. The rocket consumes fuel at 1.5 kg/s and ejects gas at 2000 m/s. Find initial acceleration of the rocket. Neglect air resistance.

Solution : We consider thrust on the rocket assuming no other external force and then apply force due to gravity in the opposite direction in order to get the initial force on the rocket.

The equation of rocket in terms of fuel consumption rate is :

$$T = rv_r$$

Putting values,

$$\Rightarrow T = 1.5X2000 = 3000 \quad N$$

Now, the magnitude of force due to gravity on the rocket is equal to its initial weight,

$$\Rightarrow F_g = mg = 500X10 = 5000 \quad N$$

Thus, there is net force of 5000 – 3000 = 2000 N on the rocket in the downward direction. The rocket, therefore, is unable lift itself. Its initial acceleration is zero.

$$\Rightarrow a = 0$$

Problem 2 : A rocket of total mass 500 kg is fired from the surface of Earth in vertical direction. The rocket consumes fuel at 3 kg/s and ejects gas at 2500 m/s. Find initial acceleration of the rocket. Neglect air resistance.

Solution : The initial acceleration of the rocket, which includes effect of gravity, is given by :

$$a = \frac{T + F_g}{m}$$

Here,

$$\Rightarrow T = rv_r = 3X2500 = 7500_N$$

$$\Rightarrow F_g = -mg = -500X10 = -5000 \quad N$$

Putting values,

$$\Rightarrow a = \frac{7500 - 5000}{500} = 5 \quad m/s^2$$

Fuel consumption rate

Problem 3 : A rocket of total mass 500 kg is fired from the surface of Earth in vertical direction. The rocket ejects gas at 2500 m/s. Find fuel consumption rate to just lift the rocket.

Solution : For the given situation, the upward direction is equal to the weight of the rocket in downward direction.

$$\Rightarrow T = mg$$

$$\Rightarrow rv_r = mg$$

$$\Rightarrow r = \frac{mg}{v_r}$$

Putting values,

$$\Rightarrow r = \frac{500X10}{2500} = 2 \quad kg/s$$

Problem 4 : A rocket of total mass 500 kg is fired from the surface of Earth in vertical direction. The rocket ejects gas at 2500 m/s. Find fuel consumption rate so that its initial acceleration is twice that of acceleration due to gravity.

Solution : The required acceleration is twice to that of acceleration due to gravity :

$$a = 2g = 2X10 = 20 \quad m/s^2$$

The equation of rocket is :

$$a = \frac{T + F_g}{m} = \frac{rv_r - mg}{m}$$

Putting values,

$$\Rightarrow 20 = \frac{rX2500 - 500X10}{500}$$

$$\Rightarrow rX2500 = 20X500 + 500X10 = 10000 + 5000 = 15000$$

$$\Rightarrow r = \frac{15000}{2500} = 6 \quad kg/s$$

Projection from space

Problem 5 : A rocket of total mass 500 kg is fired from the surface of a permanent space station. The rocket consumes fuel at 1.5 kg/s and ejects gas at 2000 m/s. Find its velocity at the time its mass reduces to 50 kg.

Solution : We have seen in earlier example that the rocket was unable to lift itself from the surface of Earth for the given design parameters. However, the same can be projected from a space station where gravity is zero. Assuming that space station is massive enough to remain in its position, the velocity of the rocket is given by :

$$v = 2.303 v_r \log \frac{m_i}{m_f}$$

Putting values,

$$\Rightarrow v = 2.303X2000X \log\left(\frac{500}{50}\right)$$

$$\Rightarrow v = 2.303X2000X1 = 4606 \quad m/s$$

Motion under gravity

Problem 5 : A rocket of total mass 500 kg, carrying 450 kg of fuel, is fired from the surface of Earth in vertical direction. The rocket consumes fuel at 3 kg/s and ejects gas at 2500 m/s. If acceleration due to gravity is assumed constant during its flight, find its velocity, when all fuel has been consumed. Neglect air resistance.

Solution : This problem assumes that acceleration due to gravity is constant in the region of flight. We see here that gravity decelerates rocket at a constant rate irrespective of the mass of rocket. Recall that acceleration due

to gravity does not depend on mass, as “mass” appears both (i) as cause of gravitational force and (ii) as object subjected to the gravitation force.

Further, it can be seen that rocket is able to lift itself for the given design parameters. Now, applying deceleration due to gravity to the equation of velocity of rocket, we have :

$$v = 2.303 v_r \log \frac{m_i}{m_f} - gt$$

Here fuel consumption rate is given. Hence, we can find the total time for the consumption of total fuel,

$$\Rightarrow t = \frac{450}{3} = 150 \quad s$$

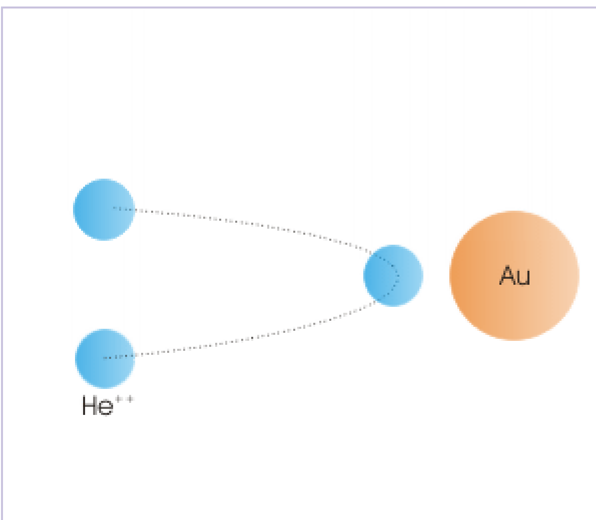
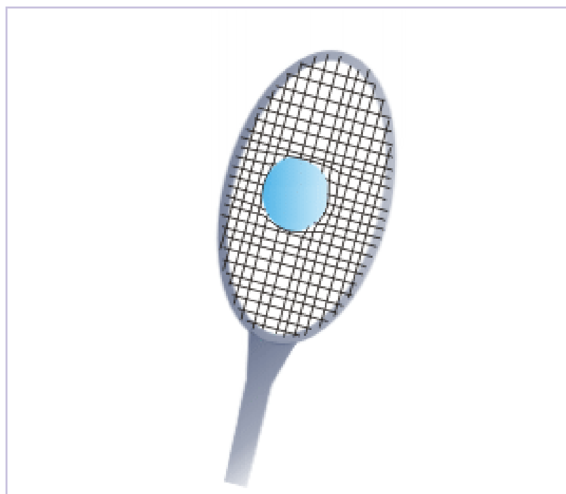
Putting values in the expression of velocity of the rocket, we have :

$$\Rightarrow v = 2.303 X 2500 X \log \frac{500}{(500 - 50)} - 10 X 150$$

$$\Rightarrow v = 4606 - 1500 = 4106 \quad m/s$$

Collision
Forces in collision are internal to the system of colliding bodies.

Collision of objects is a brief phenomenon of interaction with force of relatively high magnitude. It is a common occurrence in physical world. Atoms, molecules and basic particles frequently collide with each other. An alpha particle, for example, is said to collide when the nucleus of gold atom repels incoming alpha particle with a strong electrostatic force. The "contact" is an assessment of the event involving force of high magnitude for small period . Two objects of macroscopic dimensions are said to collide making physical contact like, when a tennis ball is hit hard to rebound.
Collision

Alpha particle and nucleus of gold atom come very close to each other.	Tennis racket and ball collide making contact with each other.
	

Two stationary objects in a given reference can not collide. Therefore, one of the basic requirements of collision is that at least one of the objects must be moving. Since motion is involved, the system of colliding objects has certain momentum and kinetic energy to begin with. Our objective in this module is to investigate these physical quantities during different states associated with a collision process. A collision is typically studied in terms of three states in "time" reference : (i) before (ii) during and (iii) after.

The state of contact - actual or otherwise - spans a very brief period. The total duration of contact between tennis racket and ball may hardly sum up to 30 seconds or so in a set of a tennis match, which might involve hundreds of hits ! Typically the contact period during a single collision is milli-seconds only. This poses a serious problem as to the measurement of physical quantities during collision. We shall see that study of collision is basically about making assessment of happenings in this brief period in terms of quantities that can be measured before and after the collision.

Further, a collision involves forces of relatively high magnitude. To complicate the matter, the force during collision period is a variable force. It makes no sense to attempt investigate the nature of this force quantitatively as it may not be possible to correlate the force with the motions of colliding objects using force law for this very small period. There is yet another reason why we would avoid force analysis during collision. We must understand that a collision need not end up with change in motion only. It is entirely possible that a collision may result in the change in shape (deformation) or even decomposition of the colliding bodies. Such changes resulting from force is not the domain of analysis of motion here.

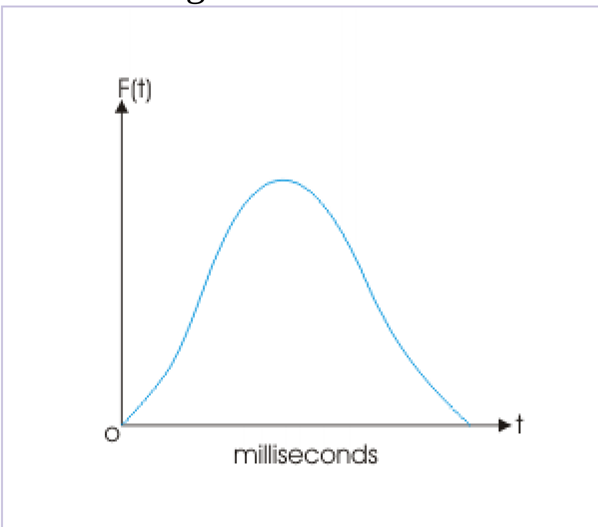
It is, therefore, imperative that we take a broader approach involving momentum and kinetic energy to study the behavior of objects, when they collide. However, before we proceed to develop the theoretical framework for the collision, it would be helpful to enumerate the characterizing aspects of a collision as :

- A collision involves interaction of an object with force.
- A collision involves forces of relatively high magnitude.
- Collision takes place for a very small period usually in milliseconds or less.
- A collision involves certain momentum and kinetic energy to begin with.
- The study of collision is not suited for force analysis, using laws of motion.

Forces during collision

Collision usually occurs in the absence of any external force or comparatively negligible external force with respect to collision force. Two bodies collide because a body comes in the line of motion of other body. The forces during collision between colliding bodies are essentially internal forces to the system of colliding bodies. The internal force is a variable force of relatively high magnitude.

Force during collision

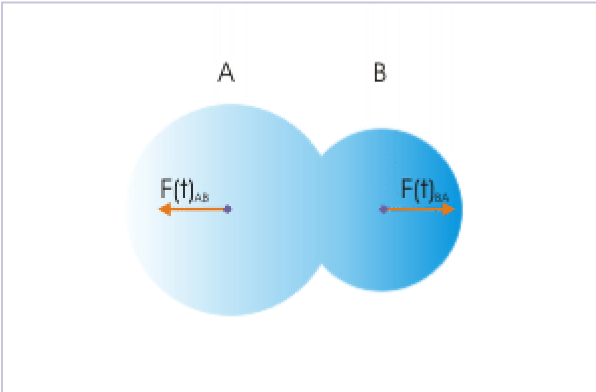


The internal force is a variable force of relatively high magnitude, which operates for a very small period.

The figure here captures the attributes of the collision force. It is important to note that the plot has time in milli-seconds.

The collision forces between colliding objects appear in pair and follow Newton's third law. The force pairs are equal and opposite. This has important implications. Irrespective of the masses or sizes of the colliding bodies, the pair (two) forces are always equal at any time during the collision. What it means that force-time plot of either pair force is exact replica of force-time plot of the other collision force.

Forces during collision



The force pairs are equal and opposite.

Impulse

As pointed out earlier, it is not very useful studying contact force as it is a variable force and secondly, we can not measure the same with any accuracy for the short interval it operates. We can, however, relate the collision force to the change in linear momentum. Here is a word of caution. If the collision forces are internal forces, then how can there be a change in linear momentum? We are actually referring change in linear momentum of one of the colliding bodies, which changes in such a manner so that linear momentum of the system of colliding bodies remain constant. Now, using Newton's second law of motion, the collision force operating on one of the colliding objects is :

$$\mathbf{F}\left(t\right) = \frac{đ\mathbf{p}}{đt}$$

$$\Rightarrow đ\mathbf{p} = \mathbf{F}(t)đt$$

Integrating for the period of collision,

$$\Rightarrow \int đ\mathbf{p} = \int \mathbf{F}\left(t\right)đt$$

$$\Rightarrow \Delta\mathbf{p} = \int \mathbf{F}\left(t\right)đt$$

This is an important result. This enables us to measure the product of two immeasurable quantities (i) variable force of relatively high magnitude, F(t) and (ii) very small time, dt, during the collision in terms of measurable change in momentum, Δ**p**, before and after the collision. The integral of the product of force and time equals change in momentum and is known as impulse denoted by the symbol "J" :

Equation:

$$\Rightarrow \mathbf{J} = \mathbf{\Delta p} = \int \mathbf{F}\left(t\right) đt$$

Like linear momentum, impulse is a vector quantity. Evidently, it has the same dimension and unit as that of linear momentum. It may be noted here that impulse is a measure of the product of force and time of operation, but not either of them individually. Further, it must also be understood that impulse is rarely evaluated using its defining integral; rather it is evaluated as the change in linear momentum - most of the time.

We shall learn that a collision may be elastic (where no loss of kinetic energy of the system takes place) or inelastic (where certain loss of kinetic energy of the system is involved).

The collision forces are normal to the tangent through the surfaces in contact. In real situation, the force may not be passing through a contact point, but over an area. In order to keep the analysis simple, however, we consider that contact is a point and the collision forces are normal to the tangent. This simplifying assumption has important implications as we shall see while studying inelastic collision is a separate module.

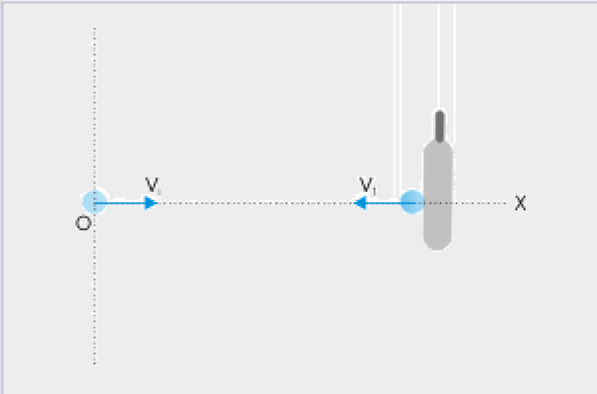
Example:

Problem : A ball of 0.1 kg in horizontal flight moves with a speed 50 m/s and hits a bat held vertically. The ball reverses its direction and moves with the same speed as before. Find the impulse that acts on the ball during its contact with the bat.

Solution : We calculate the impulse on the ball by measuring the change in linear momentum. Here, vectors involved are one dimensional. Therefore,

we can use scalar representation of quantities with appropriate sign. Let us consider that the ball is moving in positive x-direction. Then,

Collision



The ball reverses its direction with same speed after its hits the bat.

$$J = \Delta p = p_f - p_i = mv_f - mv_i$$

$$\Rightarrow J = m\left(v_f - v_i\right) = 0.1 \text{ x } \left(-50 - 50\right) = -10 \, \frac{\text{kg x}\, m}{s}$$

Negative sign means that the impulse is acting in the "-x" direction, in which the ball finally moves.

Component form of impulse

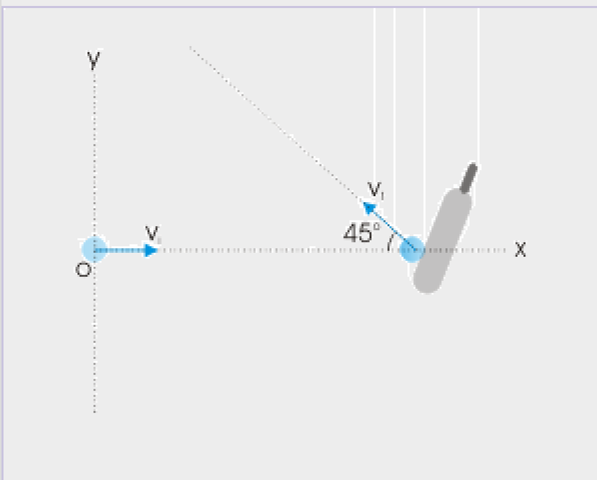
Like linear momentum, impulse is a vector quantity. Impulse vector on a body is directed in the direction of Δ**p**. This direction may not be the direction of either initial or final linear momentum of the body under investigation. Like stated before in the course, the direction of a vector and that of "change in vector" may not be same. The component of impulse vector along any of the coordinate direction is equal to the difference of the components of linear momentum in that direction. Mathematically,

Equation:

$$J_x = \Delta p_x = p_{\mathrm{fx}} - p_{\mathrm{ix}}$$
$$J_y = \Delta p_y = p_{\mathrm{fy}} - p_{\mathrm{iy}}$$
$$J_z = \Delta p_z = p_{\mathrm{fz}} - p_{\mathrm{iz}}$$

Example:
Problem : A ball of 0.1 kg in horizontal flight moves with a speed 50 m/s and is deflected by a bat as shown in the figure at an angle with horizontal. As a result, the ball is reflected in a direction, making 45° with the horizontal. Find the impulse that acts on the ball during its contact with the bat.

Collision



The ball is deflected at an angle with horizontal.

Solution : We calculate the impulse on the ball by measuring the change in linear momentum in horizontal (x-direction) and vertical (y-direction) directions. Let us consider that the ball is initially moving in positive x-direction. Then,

$$J_x = \Delta p_x = p_{\mathrm{fx}} - p_{\mathrm{ix}} = m(v_{\mathrm{fx}} - v_{\mathrm{ix}})$$
$$\Rightarrow J_x = 0.1 \ \text{x} \ \left(-50\cos 45^0 - 50\right) = -8.54\,\mathrm{kg} \ \text{x} \ \frac{m}{s}$$

Similarly, component of impulse in y-direction is :

$$J_y = \Delta p_y = p_{\mathrm{fy}} - p_{\mathrm{iy}} = m(v_{\mathrm{fy}} - v_{\mathrm{iy}})$$
$$\Rightarrow J_y = 0.1 \ \text{x} \ \left(50\sin 45^0 - 0\right) = 3.54\,\frac{\mathrm{kg\,x}\,m}{s}$$

The impulse on the ball is :

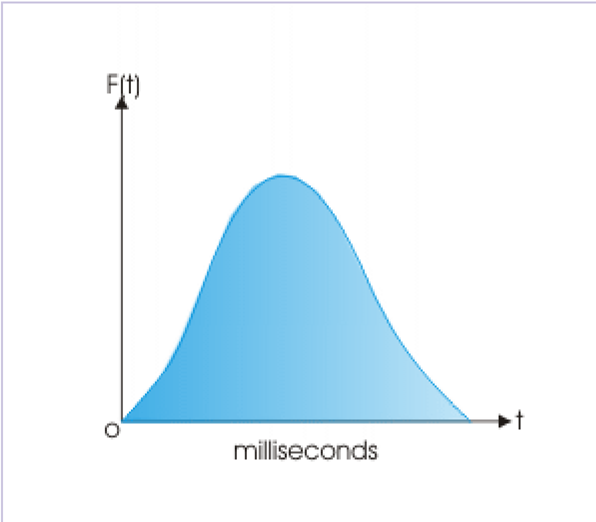
$$\mathbf{J} = J_x\mathbf{i} + J_y\mathbf{j}$$
$$\Rightarrow \mathbf{J} = \left(-8.54\mathbf{i} + 3.54\mathbf{j}\right)\frac{\mathrm{kg\,x}\,m}{s}$$

This example and the one given earlier illustrate that the impulse and its evaluation in terms of linear momentum is independent of whether the collision is head on or not. No doubt the result is different in two cases. The impulse force is greater for head on collision than when the colliding object glances the target at an angle. It is expected also. The change in linear momentum when the striking object reverses it motion is maximum and as such impulse imparted is also maximum when collision is head on.

Average force

We may not be able to relate impulse with the instantaneous force during the collision, but we can estimate the average force if we have accurate estimate of the time involved during collision. For the sake of simplicity, we consider one-dimensional contact force. Looking at the expression of impulse i.e the integral of force over the time of collision is graphically equal to the area under the curve.

Force - time plot during collision

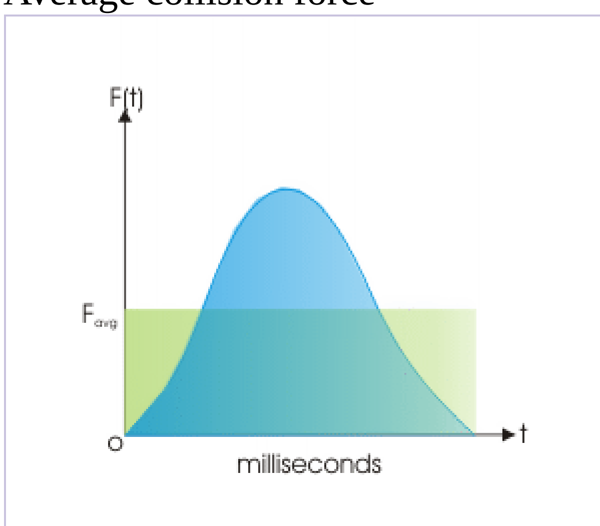


The integral of force over the time of collision is graphically equal to the area under the curve.

$$J = \int F(t)\,dt = \text{Area}$$

We can draw a rectangle such its area is equal to the integral. In that case ordinate on the force axis represents the average force during the collision period.

Average collision force



Average force is equal to the ordinate of the rectangle of area equal to that of area under the force - time actual curve.

Mathematically,

Equation:

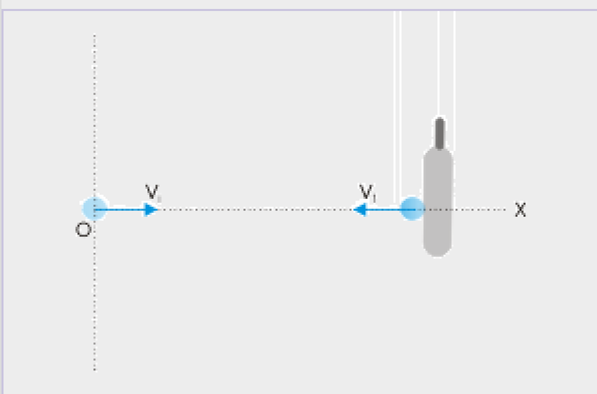
$$F_{\text{avg}} = \frac{\int F(t)\,\text{đ}t}{\Delta t} = \frac{J}{\Delta t} = \frac{\Delta p}{\Delta t}$$

Example:

Problem : A ball of 0.1 kg in horizontal flight moves with a speed 50 m/s and it hits a bat held vertically. The ball reverses its direction and moves with the same speed as before. If the time of contact is 1 milli-second, then find the average force that acts on the ball during its contact with the bat.

Solution : We calculate the impulse on the ball by measuring the change in its linear momentum. Let us consider that the ball is moving in positive x-direction. Then,

Collision



The ball reverses its direction with same speed after it hits the bat.

$$J = \Delta p = p_f - p_i = mv_f - mv_i$$

$$\Rightarrow J = m\left(v_f - v_i\right) = 0.1 \ \text{ x } \ \left(-50 - 50\right) = -10 \ \frac{\text{kg x } m}{s}$$

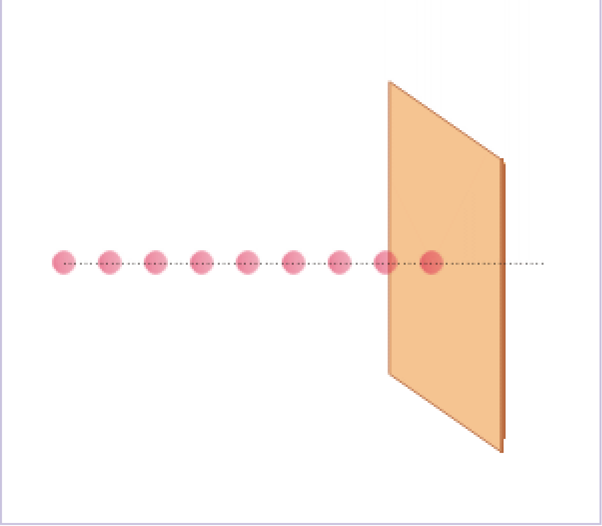
The magnitude of average force is :

$$F_{\text{avg}} = \frac{10}{1\text{x}10^{-3}} = 10000 \ N$$

A collision may take place when external force is operating like when a cricket ball strikes the racket in the presence of gravitational force. However, gravitational force is relatively small (mg = 0.1 x 10 = 1.5 N). This force is negligible in comparison to the collision force of 10000 N! It is, therefore, realistic to assume that collision takes place in the absence of external force.

This concept of average force allows us to estimate the average force on a target, which is bombarded with successive colliding masses like a stream of bullets or balls as shown in the figure below. Here, the average force is given as :

Series Collision



The ball hits the target in quick succession.

$$F_{\text{avg}} = \frac{J}{\Delta t} = \frac{\sum \Delta p}{\Delta t}$$

If the projectiles have equal mass "m" and there are "n" numbers of the projectiles hitting the target, then

$$\Rightarrow \sum \Delta p = nm\Delta v$$

and

Equation:

$$F_{\text{avg}} = \frac{nm\Delta v}{\Delta t}$$

Linear momentum during collision

We have noted in the examples given here that we can neglect external force for all practical purpose as far as the collision is concerned. Collision force is too large in comparison to forces like gravitational force. For this reason, we shall neglect external forces unless stated otherwise. As such, linear momentum of the system of colliding objects is conserved. This means that linear momentum of the system before and after the collision is same.

Equation:

$$\mathbf{P}_{\text{before}} = \mathbf{P}_{\text{after}}$$

This fact provides the basic framework for analyzing collision. It is valid irrespective of the type of collision : elastic or inelastic. We shall make use of this fact in subsequent modules, while analyzing various scenarios of collision.

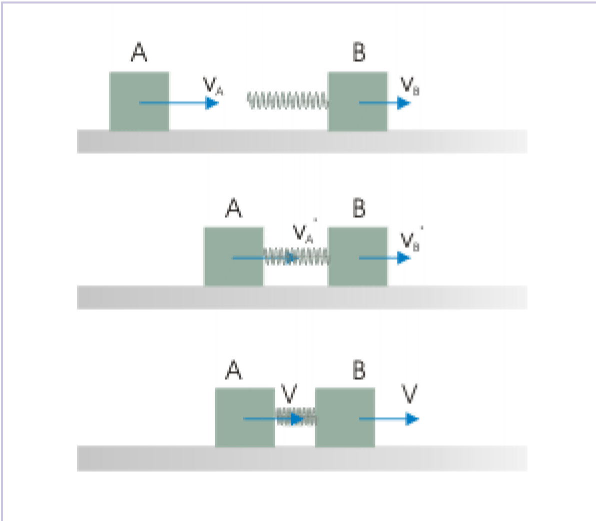
Kinetic energy of colliding bodies

There are two types of collision - elastic and inelastic. In the elastic collision, the colliding bodies is restored to the normal shape and size when the bodies separate. The deformation of colliding bodies is temporary and is limited to the period of collision. This means that kinetic energy of the system is simply stored as elastic energy during collision and is released to the system when bodies regain their normal condition. On the other hand, there is loss of kinetic energy during inelastic collision as kinetic energy is converted to some other form of energy like sound energy or heat energy, which can not be released to the system of colliding bodies as kinetic energy subsequent to the collision.

In reality most of the macroscopic collisions involving physical contact are inelastic as kinetic energy of the system is irrevocably converted to other form of energy. For example, when a tennis ball is released from a height, it does not bounce to the same height on the return journey. However, if the ground is plane and hard, then the ball may transverse the greater part of the height and we may say that the collision is approximately elastic.

There is an interesting set up, which explains various aspects of elastic collision. We will study this set up, which comprises of two blocks and a spring to get an insight into the collision process. We imagine two identical blocks "A" and "B" moving on a smooth floor as shown in the figure below. The block "A" moves right with speed v_A towards block "B". On the other hand block "B" moves with speed v_B such that $v_A > v_B$. We also imagine that a "mass-less" spring is attached to the block "B" as shown in the figure. Let us also consider that the spring follows Hook's law for elongation (F = -kx).

Collision



The spring is compressed.

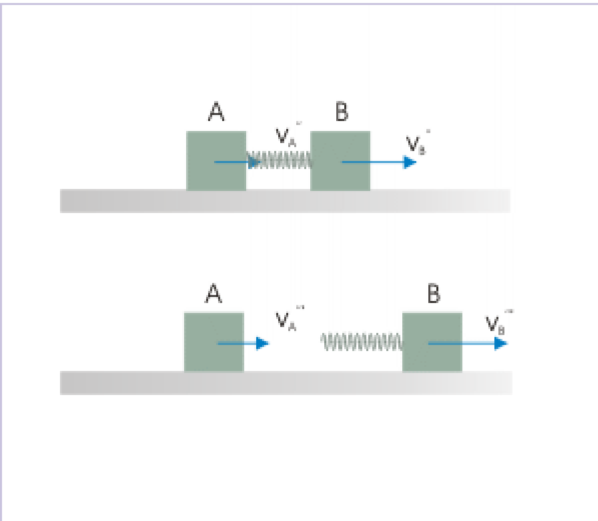
At the time when block "A" hits the spring, the end of the spring in contact with block "A" acquires the speed of the block "A" (v_A). The end of the spring in contact with block "B", however, moves with speed v_B . As the block "A" moves with relative speed towards "B", the spring is compressed. The spring force for any intermediate compression "x" is given by :

$$F(t) = -kx$$

The spring forces act on both the blocks but in opposite directions. The spring force is equivalent to the internal collision force. The spring force pushes the block "A" towards left and hence decelerates it. Let its speed at a given instant after hitting the spring be v_A' . On the other hand, spring force pushes the block "B" towards right and hence accelerates it. As such, block "B" acquires certain speed, say v_B' . If $v_A' > v_B'$, then the spring is further compressed. The speed of block "A" further decreases and that of block "B" further increases. The spring force during this period keeps increasing. This process continues till the speeds of blocks become equal. Let us consider that common speed of each of the blocks be V.This situation corresponds to the maximum compression and maximum spring force i.e the maximum collision force.

The maximum spring force continues to decelerate block "A" and accelerates block "B". As such, block "B" (v_B'') begins to move faster than the block "A" (v_A'') . It results in elongation of spring. The spring force during this period keeps decreasing. The spring force, however, continues to decelerates block "A" and accelerates block "B". This process continues till the spring attains its normal length. At this moment, the block "A" looses contact with the spring. The block "B" moves with greater speed (v_B''') than block "A" (v_A'''). Therefore, the separation between two blocks keeps increasing with time as they move with different speeds.

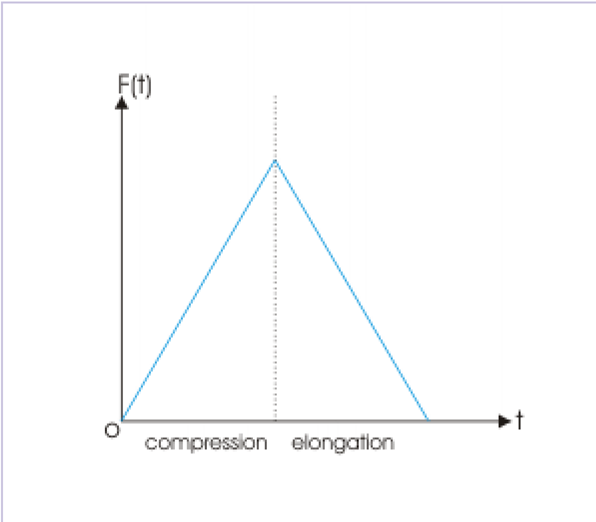
Collision



The spring is elongated.

The spring force - time plot during the contact with spring approximates the force curve during a collision :

Collision force



The force - time plot.

Since the spring is mass-less and elastic (follows Hooke's law), the kinetic energy of the system before the collision is temporarily converted to elastic potential energy i.e spring energy during the compression of the spring. The stored elastic potential energy is then released as kinetic energy during elongation of the spring. At the end of elongation when spring attains its normal length, the kinetic energy of the system is restored as before. In the nutshell,

Equation:

$$K_{\text{before}} = K_{\text{after}}$$

During collision, however, total energy has component of elastic potential energy as well and is equal to the kinetic energy of the system before and after the collsion :

$$E = K + U = K_{\text{before}} = K_{\text{after}}$$

In the same fashion, we can visualize inelastic collision. In this case, some of the energy is converted to a form of energy, which can not be regained as

kinetic energy like when a rubber ball hits a ground and is unable to regain the height. Here,
Equation:

$$K_{\text{before}} > K_{\text{after}}$$

Summary

1: Collision of objects is a brief phenomenon of interaction with force of relatively high magnitude.

2: The integral of the product of force and time equals change in momentum and is known as impulse denoted by the symbol "**J**" :

$$\mathbf{J} = \Delta\mathbf{p} = \int \mathbf{F}\left(t\right)\text{đ}t$$

3: The component of impulse vector along any of the coordinate direction is equal to the difference of the components of linear momentum in that direction. Mathematically,

$$J_x = \Delta p_x = p_{\text{fx}} - p_{\text{ix}}$$
$$J_y = \Delta p_y = p_{\text{fy}} - p_{\text{iy}}$$
$$J_z = \Delta p_z = p_{\text{fz}} - p_{\text{iz}}$$

4: The average force during collision is by the ratio of impulse and time interval as :

$$F_{\text{avg}} = \frac{\int F\left(t\right)\text{đ}t}{\Delta t} = \frac{J}{\Delta t} = \frac{\Delta p}{\Delta t}$$

5: Elastics collision :

$$\mathbf{P}_{\text{before}} = \mathbf{P}_{\text{after}}$$

and

$$K_{\text{before}} = K_{\text{after}}$$

6: Inelastic collision :

$$\mathbf{P}_{\text{before}} = \mathbf{P}_{\text{after}}$$

and

$$K_{\text{before}} > K_{\text{after}}$$

Acknowledgment

Author wishes to thank Manuel Widmer for making suggestions to remove error in the module.

Laws governing collision

Collision, as we know, is an event of very short time interval. It is imperative that we look for some mechanism that enables us to predict the result subsequent to collision without knowing what happens during collision. Fortunately, there are simple underlying principles that govern collision process. In this module, we shall discuss governing laws for different situation pertaining to collision.

Conservation of linear momentum

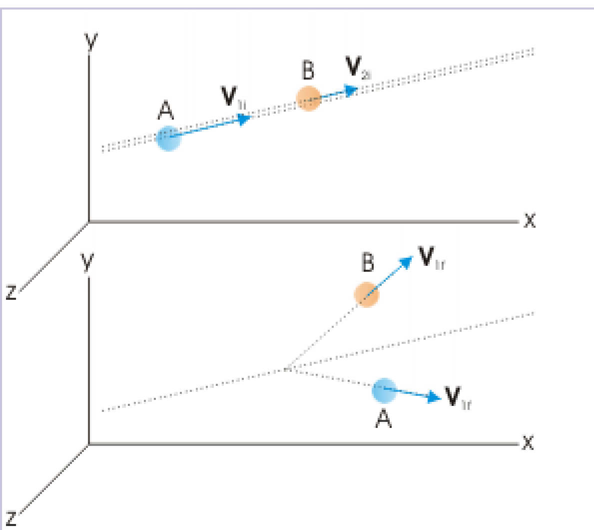
Foremost among the laws governing collision is the conservation of linear momentum as there is no external force involved or the external force is small enough with respect to collision force and can be neglected without affecting the result in any appreciable manner.

For the system of colliding bodies, linear momentum of the system is same before and after collision :

$$\mathbf{P}_i = \mathbf{P}_f$$

For two colliding bodies, the above conservation law can be written as :

Conservation of linear momentum



The bodies collide to galnce off
in different direction.

$$m_1\mathbf{v}_{1\text{i}} + m_2\mathbf{v}_{2\text{i}} = m_1\mathbf{v}_{1\text{f}} + m_2\mathbf{v}_{2\text{f}}$$

Suffix "1" and "2" refer to the two colliding bodies, whereas suffix "i" and "f" refer to the "initial" and "final" states (before and after) of the collision respectively. It is important to visualize that two bodies will collide when their trajectory is such that the physical dimension of the bodies overlap. Further, one of the bodies approaches second body with greater speed ($v_{1\text{i}} > v_{2\text{i}}$) for collision to occur. In the figure above, two bodies are shown to have parallel trajectories so that body "A" does not strike "B" head on, but glances to move in different directions after collision.

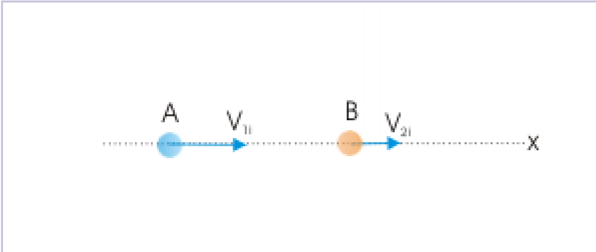
Generally, the body with greater velocity that collides with another body is conventionally termed as "projectile" and the body which is hit is termed as "target".

We must understand that the law of conservation of linear momentum is valid for a closed system, wherein there is no exchange of mass between the system and its surrounding. Also, we must note that above mathematical construct is a vector equation. This can be expressed in component form as :

$$m_1 v_{1\text{ix}} + m_2 v_{2\text{ix}} = m_1 v_{1\text{fx}} + m_2 v_{2\text{fx}}$$
$$m_1 v_{1\text{iy}} + m_2 v_{2\text{iy}} = m_1 v_{1\text{fy}} + m_2 v_{2\text{fy}}$$
$$m_1 v_{1\text{iz}} + m_2 v_{2\text{iz}} = m_1 v_{1\text{fz}} + m_2 v_{2\text{fz}}$$

When bodies collide head-on, they move along a straight line. Such collision in one dimension can be handled with any of the component equations. For convenience, we can drop the component suffix and write the conservation law without suffix :

Collision in one dimension



Head on collision results motion in one diemsion.

$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 v_{1\mathrm{f}} + m_2 v_{2\mathrm{f}}$$

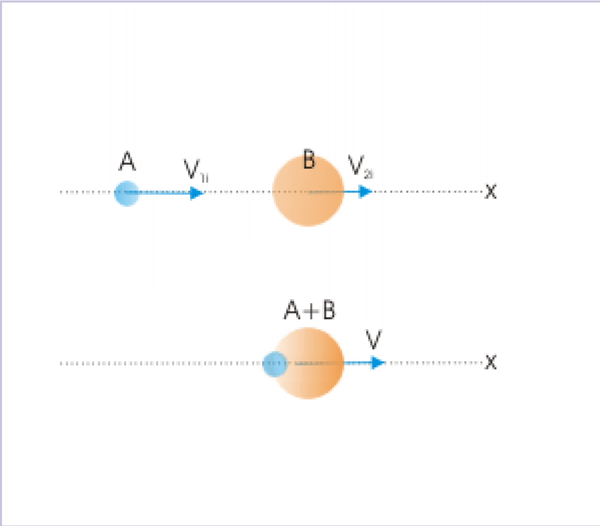
As a reminder, we need to emphasize here that consideration of conservation of linear momentum is independent of following important aspects of collision :

- Collision force or time of collision
- Nature of collision : elastic or inelastic
- Dimension of collision : one, two or three

Conservation of linear momentum in completely inelastic collision

Completely inelastic collision is a special case of inelastic collision. In this case, the colliding body is completely embedded into target after collision. The two colliding bodies thereafter move with same velocity. Let the common speed be V such that :

Completely inelastic collision



The colliding move with same velocity.

$$\mathbf{v}_{1\mathrm{f}} = \mathbf{v}_{2\mathrm{f}} = \mathbf{V}$$

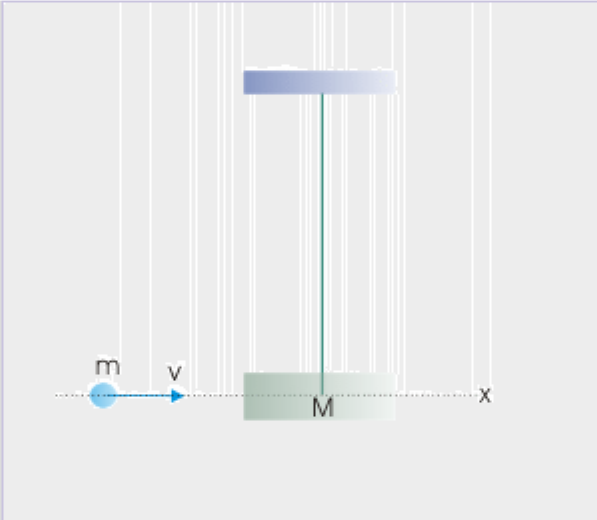
Putting this in to the equation of linear momentum, we have :

$$m_1\mathbf{v}_{1\mathrm{i}} + m_2\mathbf{v}_{2\mathrm{i}} = m_1\mathbf{V} + m_2\mathbf{V} = (m_1 + m_2)\mathbf{V}$$

If the collision is one dimensional also, then we can drop vector notation :

$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 V + m_2 V = (m_1 + m_2)V$$

Example:
Problem : A ballistic pendulum comprising of wooden block of 10 kg is hit by a bullet of 10 gm in horizontal direction. The bullet quickly comes to rest as it is embedded into the wooden block. If the wooden block and embedded bullet moves with a speed of 1.414 m/s after collision, then find the speed of bullet.
Ballistic pendulum



The bullet and block undergo completely inelastic collision.

Solution : The collision lasts for short period. During collision, the combined mass of the block and bullet is subjected to gravitational and tension forces, which balance each other. As such, the system of colliding bodies is subjected to zero external force. The governing principle, in this case, is conservation of linear momentum. Now, as the bullet sticks to the wooden block, the collision is completely inelastic.

Let m and M be the masses of bullet and block respectively. Let us also consider that bullet strikes the block with a velocity "v" and the block and embedded bullet together move with the velocity "V" after the collision. We assume that bullet hits the block head on and the combined mass of block and bullet moves in the horizontal direction just after the collision. Hence, applying conservation of linear momentum in one dimension, we have :

$$(m+M)V = mv$$

$$\Rightarrow v = \frac{m+M}{m} V$$

$$\Rightarrow v = \frac{0.01+10}{0.01} \text{ x } 1.414 = 1415\, m/s$$

Center of mass and collision

The fact that there is no external force during collision has important bearing on the motion of center of mass. The motion of the center of mass can not change in the absence of external force; internal forces appearing during the collision can not change the velocity of center of mass. As such, motion of the center of mass remains unaffected after collision. In other words, acceleration of center of mass is zero. Mathematically :

$$\mathbf{v}_{\text{com}} = \text{constant}$$
$$\mathbf{a}_{\text{com}} = 0$$

We can combine this fact with the conservation law to obtain the expression of center of mass. The linear momentum of the system of two colliding bodies in terms of center of mass is :

$$M\mathbf{v}_{\text{com}} = m_1\mathbf{v}_{1\text{i}} + m_2\mathbf{v}_{2\text{i}} = m_1\mathbf{v}_{1\text{f}} + m_2\mathbf{v}_{2\text{f}}$$

$$\Rightarrow \mathbf{v}_{\text{com}} = \frac{m_1\mathbf{v}_{1\text{i}}+m_2\mathbf{v}_{2\text{i}}}{m_1+m_2} = \frac{m_1\mathbf{v}_{1\text{f}}+m_2\mathbf{v}_{2\text{f}}}{m_1+m_2}$$

This vector equation can be equivalently written into three scalar component equations. For one dimensional case, however, we drop the vector notation altogether as in the case of linear momentum :

$$\Rightarrow v_{\text{com}} = \frac{m_1 v_{1\text{i}}+m_2 v_{2\text{i}}}{m_1+m_2} = \frac{m_1 v_{1\text{f}}+m_2 v_{2\text{f}}}{m_1+m_2}$$

It must be understood that these relations are general relation like that of linear momentum. Again as a reminder, we need to emphasize here that consideration of center of mass for collision is independent of following important aspects of collision :

- Collision force or time of collision
- Nature of collision : elastic or inelastic
- Dimension of collision : one, two or three

Center of mass and completely inelastic collision

As in the case of conservation of linear momentum in one dimension, the completely inelastic collision provides the unique condition when,

$$v_{1\mathrm{f}} = v_{2\mathrm{f}} = v$$

$$\Rightarrow v_{\mathrm{com}} = \frac{m_1 V + m_2 V}{m_1 + m_2} = V$$

This result is expected also as two bodies is converted into one. The center of mass, therefore, has the same velocity as that of the combined (or stuck) mass.

Kinetic energy and collision

Unlike linear momentum, kinetic energy is not conserved in all collisions. This depends on the type of collision i.e. whether the collision is elastic or inelastic. Here, we shall discuss these cases.

Note: Like kinetic energy, the potential energy of the system before and after collision is not considered for analyzing collision. It is so because, we concentrate on the velocities just before and after the collision, which occurs in very short period. There may not be any change in the elevation of colliding bodies before and after collision. For this reason, we analyze collision even along an incline with out considering potential energy of the colliding bodies as far as immediate states "before" and "after" are concerned.

Kinetic energy in elastic collision

In elastic collision, the colliding bodies get deformed to different degree depending on the nature of material composing them. The kinetic energy of the system is temporarily transferred into elastic potential energy, which is regained subsequently. The nature of bodies, however, determine the time for which colliding bodies are in contact. Harder the bodies, smaller is the

time of contact and vice-versa. The collision between two hard bodies is likely to produce elastic collision. As kinetic energy is regained at the end of collision, kinetic energy is conserved in elastic collision. Thus,

$$\mathbf{K}_i = \mathbf{K}_f$$

For two colliding bodies, the above conservation law can be written as :

$$\frac{1}{2}m_1 v_{1\mathrm{i}}^2 + \frac{1}{2}m_2 v_{2\mathrm{i}}^2 = \frac{1}{2}m_1 v_{1\mathrm{f}}^2 + \frac{1}{2}m_2 v_{2\mathrm{f}}^2$$

It is important to realize that this relation for elastic collision is an additional relation, which can be used in conjunction with the relation for conservation of linear momentum. We must also note that kinetic energy is a scalar quantity unlike linear momentum. This has one important implication. Irrespective of the dimension of motion, there is no component form of relation for kinetic energy. In all case, there is only one relation of kinetic energy as given above (compare this with three component equation for conservation of momentum).

Kinetic energy in inelastic collision

In inelastic collision, kinetic energy of the system is not conserved. Some of the kinetic energy is irrevocably transferred to the other forms of energy like sound or heat energy. As a result,

$$\mathbf{K}_i > \mathbf{K}_f$$

For two colliding bodies, the relation for kinetic energy can be written as :

$$\frac{1}{2}m_1 v_{1\mathrm{i}}^2 + \frac{1}{2}m_2 v_{2\mathrm{i}}^2 > \frac{1}{2}m_1 v_{1\mathrm{f}}^2 + \frac{1}{2}m_2 v_{2\mathrm{f}}^2$$

It is evident that this relationship of inequality does not help to evaluate motion of bodies after collision as the same can not be used in conjunction with momentum equation.

The loss of energy in the collision is estimated taking the difference of kinetic energy before and after the collision.

$$\Delta K = K_i - K_f = \frac{1}{2}\left(m_1 v_{1\mathrm{i}}^2 + m_2 v_{2\mathrm{i}}^2 - m_1 v_{1\mathrm{f}}^2 - m_2 v_{2\mathrm{f}}^2\right)$$

If loss of kinetic energy (ΔK) can be determined experimentally, then it is possible to render inequality into an equality as :

$$K_i = K_f + \Delta K$$

For two colliding bodies, the conservation of kinetic energy can be written as :

$$\frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2 = \frac{1}{2} m_1 v_{1\mathrm{f}}^2 + \frac{1}{2} m_2 v_{2\mathrm{f}}^2 + \Delta K$$

Now, we can use this equation in conjunction with the equations of linear momentum to determine the unknown attributes of collision.

Example:

Problem : A block of mass "m" moving at speed "v" collides with a stationary block of mass "2m" head-on. If the lighter block comes to a stop immediately after the collision, then determine (i) whether the collision is elastic or inelastic and (ii) loss of kinetic energy to the surrounding, if any.

Solution : It is a one dimensional collision. Applying conservation of linear momentum in one dimension,

$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 v_{1\mathrm{f}} + m_2 v_{2\mathrm{f}}$$

Here, $m_1 = m, m_2 = 2\mathrm{m}, v_{1\mathrm{i}} = v, v_{2\mathrm{i}} = 0$. Let the heavier block moves with a velocity "V" after the collision. Then,

$$mv + 2\mathrm{m \ x \ }0 = m\mathrm{\ x\ }0 + 2\mathrm{mV}$$
$$\Rightarrow V = \frac{v}{2}$$

Now, kinetic energy of the system of two colliding bodies before collision is :

$$K_i = \frac{1}{2} m v^2$$

and the kinetic energy of the system of two colliding bodies after collision is :

$$K_f = \frac{1}{2} m V^2 = \frac{1}{2} m \left(\frac{v}{2}\right)^2 = \frac{m v^2}{8}$$

Thus, $K_i > K_f$. Hence, collision is inelastic. The loss of kinetic energy to the surrounding is :

$$\Delta K = K_i - K_f = \frac{m v^2}{2} - \frac{m v^2}{8} = \frac{3 m v^2}{8}$$

Kinetic energy in completely inelastic collision

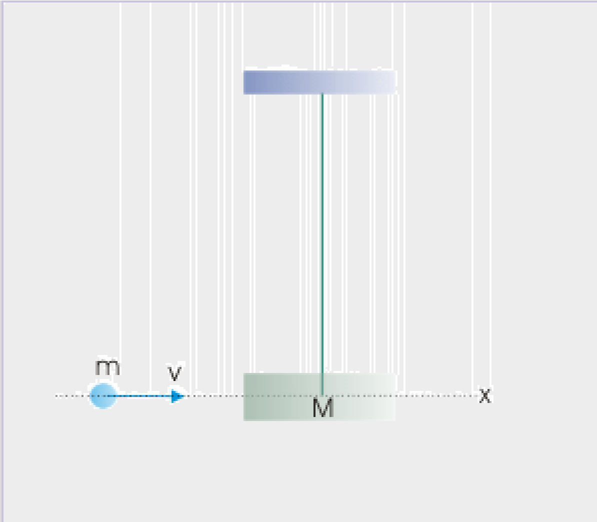
The completely inelastic collision is a subset of inelastic collision. It provides the unique condition when final kinetic energy pertains to the combined mass. Let us consider that collision takes place in one dimension and the common velocity of the embedded body is V. Then,

$$\mathbf{v}_{1\mathrm{f}} = \mathbf{v}_{2\mathrm{f}} = \mathbf{V}$$

Kinetic energy inequality is written as :

$$\frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2 > \frac{1}{2} \left(m_1 + m_2\right) V^2$$

Example:
Problem : A ballistic pendulum comprising of wooden block of 10 kg is hit by a bullet of 10 gm in horizontal direction. The bullet quickly comes to rest as it is embedded into the wooden block. If the wooden block rises by a height 10 cm as it swings up, then find the speed of the bullet.
Ballistic pendulum



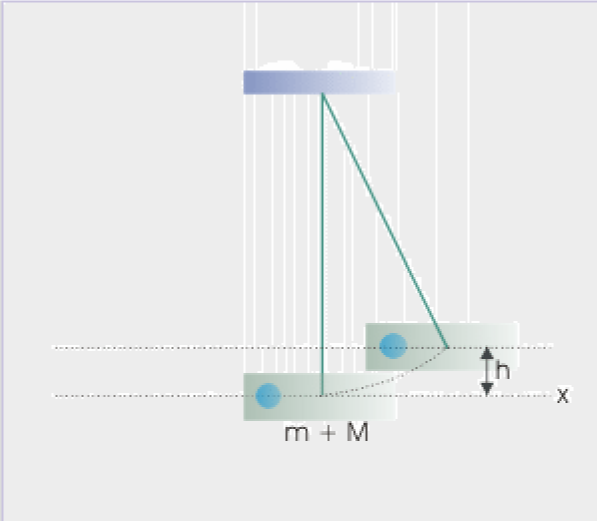
The bullet and block undergo completely inelastic collision.

Solution : We find the relation of velocities using conservation of linear momentum as discussed in the first example of this module.

$$(m+M)V = mv$$
$$\Rightarrow V = \frac{mv}{m+M}$$

The combined mass of the bullet and block rises to a vertical height "h" due to swing. In this process, the kinetic energy of the block with embedded bullet is converted into the change in potential energy of the system. Hence,

Ballistic pendulum



Kinetic energy is converted into gravitational potential energy as the bullet and block system rises as .

$$\frac{1}{2}\left(m+M\right)V^2 = \left(m+M\right)gh$$
$$\Rightarrow \frac{1}{2}V^2 = gh$$

Substituting for "V" from the first equation,

$$\frac{m^2v^2}{2\left(m+M\right)^2} = gh$$
$$\Rightarrow v = \frac{m+M}{m}\sqrt{\left(2gh\right)}$$

Putting values, we have :

$$v = \frac{0.01+10}{0.01} \text{ x } \sqrt{\left(2 \text{ x } 10 \text{ x } 0.1\right)} = 1415\,m/s$$

Analyzing collision
Analyzing collision situation involves application of basic conservation principles in a closed system.

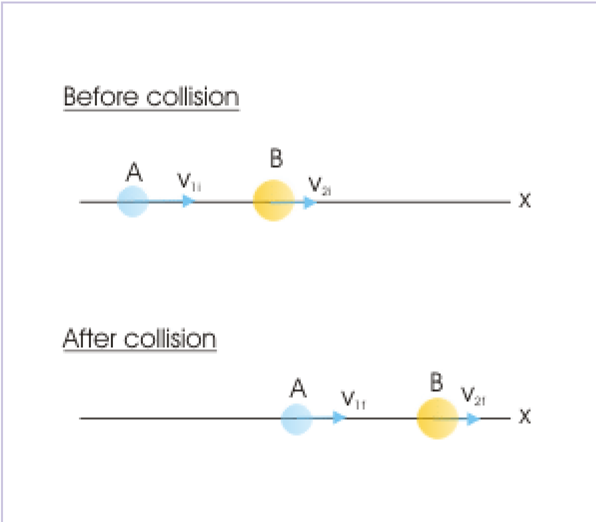
We have discussed different physical laws pertaining to collision. In this module, we shall combine these concepts to analyze different types of collision. The basic classification of collision is based on the nature of collision i.e. elastic and inelastic collisions. Further, we shall derive expressions for attributes like final velocities of the colliding bodies in both one and two dimensions.

Elastic collision in one dimension

Elastic collision in one dimension involves collision of two bodies, which move along the same straight line. In order that collision is one dimensional, it should be "head - on" collision so that bodies after collision follow the same straight line. This restriction is implied for one dimensional collision.

In this case, there are two governing equations with respect to conservation of linear momentum and kinetic energy. Since collision is in one dimension, we can use scalar form of linear momentum equation with a sign convention. The velocity in the reference direction of associated coordinate system is considered positive.

To write the governing equations, let us consider that m_1 and m_2 be the masses of the projectile body and target body respectively. Let us also consider that they move with velocity $v_{1\mathrm{i}}$ and $v_{2\mathrm{i}}$ before collision; and $v_{1\mathrm{f}}$ and $v_{2\mathrm{f}}$ after collision. For the collision, it is imperative that velocity of the projectile ($v_{1\mathrm{i}}$) is greater than that of target ($v_{2\mathrm{i}}$). Now the momentum equations is :
Elastic collision in one dimension



Projectile body collides "head - on" with the target.

Equation:

$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 v_{1\mathrm{f}} + m_2 v_{2\mathrm{f}}$$

We can rearrange this equation by clubbing attributes of projectile and target on either side of the equation as :

Equation:

$$m_1 (v_{1\mathrm{i}} - v_{1\mathrm{f}}) = m_2 (v_{2\mathrm{f}} - v_{2\mathrm{i}})$$

From kinetic energy consideration, we have :

Equation:

$$\frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2 = \frac{1}{2} m_1 v_{1\mathrm{f}}^2 + \frac{1}{2} m_2 v_{2\mathrm{f}}^2$$

Again rearranging attributes of bodies on either side of equation :

Equation:

$$m_1\left(v_{1\mathrm{i}}^2 - v_{1\mathrm{f}}^2\right) = m_2\left(v_{2\mathrm{f}}^2 - v_{2\mathrm{i}}^2\right)$$

Dividing equation - 4 by equation - 2, we have :
Equation:

$$v_{1\mathrm{i}} + v_{1\mathrm{f}} = v_{2\mathrm{i}} + v_{2\mathrm{f}}$$
$$v_{1\mathrm{i}} - v_{2\mathrm{i}} = v_{2\mathrm{f}} - v_{1\mathrm{f}}$$

This is an important equation. This equation can be interpreted in terms of relative velocity with which projectile and target move with respect to each other. The left hand side of the expression gives the velocity of approach i.e. the relative velocity with which projectile moves towards the target for the collision to occur. On the other hand, right hand side of the equation gives the velocity of separation with which target is separated from the projectile.

$$\text{Velocity of approach} = \text{Velocity of separation}$$

We can summarize the discussion so far about elastic collision in one dimensions with following characterizing aspects :

- The bodies move along same straight line before and after collision.
- The collision is "head - on".
- There is no loss of kinetic energy of the system during collision.
- Velocity of approach (of projectile) is equal to velocity of separation (of target).

Velocities after collision

We can use following two equations to find expression for final velocities of two bodies after collision :
Equation:

$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 v_{1\mathrm{f}} + m_2 v_{2\mathrm{f}}$$

Equation:

$$v_{1\mathrm{i}} - v_{2\mathrm{i}} = v_{2\mathrm{f}} - v_{1\mathrm{f}}$$

To find final velocity of projectile, $v_{1\mathrm{f}}$, we multiply equation - 5 by m_2 :
Equation:

$$m_2 v_{1\mathrm{i}} - m_2 v_{2\mathrm{i}} = m_2 v_{2\mathrm{f}} - m_2 v_{1\mathrm{f}}$$

Subtracting equation - 6 from equation - 1, we have :

$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} - m_2 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 v_{1\mathrm{f}} + m_2 v_{2\mathrm{f}} - m_2 v_{2\mathrm{f}} + m_2 v_{1\mathrm{f}}$$

$$(m_1 - m_2) v_{1\mathrm{i}} + 2 m_2 v_{2\mathrm{i}} = (m_1 + m_2) v_{1\mathrm{f}}$$

Equation:

$$v_{1\mathrm{f}} = \frac{(m_1 - m_2)}{(m_1 + m_2)} v_{1\mathrm{i}} + \frac{2 m_2}{(m_1 + m_2)} v_{2\mathrm{i}}$$

Similarly, we can find the final velocity of the target by multiplying equation - 5 by m_1 and solving the resulting equations,
Equation:

$$v_{2\mathrm{f}} = \frac{2 m_1}{(m_1 + m_2)} v_{1\mathrm{i}} - \frac{(m_1 - m_2)}{(m_1 + m_2)} v_{2\mathrm{i}}$$

Special cases

Here, we discuss some special cases of elastic collision in one dimension :

1: Collision between equal masses :

In this case, $m_1 = m_2 = m$ (say). Putting this value in the equations of final velocities (equation 7 and 8), we have :

$$v_{1f} = v_{2i}$$

and

$$v_{2f} = v_{1i}$$

This is an interesting result. When bodies of two equal masses collide elastically and "head - on", the velocities of projectile and target are exchanged. The projectile acquires the initial velocity of target and the target acquires the initial velocity of the projectile. If the target is initially at rest, then projectile is rendered to come to a dead stop as it acquires the initial velocity of the target. We can probably experience such thing if we are able to hit an identical billiard ball or coin "head - on" with identical billiard ball or coin. If the hit is "head - on", then the striking ball or coin should come to a stop.

2: Collision between light projectile and heavy target :

From our daily experience, we can visualize such collision. What happens when a hard ball (light projectile) hits a wall or hard surface (heavy target). We know that wall remains where it stands, whereas the ball reverses its trajectory without any change in the magnitude of velocity. Let us see whether this common experience is supported by the analysis. Here, $m_2 >> m_1$. As such, we can neglect m_1 in comparison to m_2 in the expressions of final velocities. Various expressions as contained in the equations of final velocities can be approximated as :

$$\frac{(m_1 - m_2)}{(m_1 + m_2)} \approx \frac{-m_2}{(m_1 + m_2)} \approx \frac{-m_2}{m_2} = -1$$

$$\frac{2m_2}{(m_1 + m_2)} \approx \frac{2m_2}{m_2} = 2$$

$$\frac{2m_1}{(m_1 + m_2)} \approx \frac{2m_1}{m_2} \approx 0$$

Putting these approximate values in the equations of final velocities (equations 7 and 8), we have :

$$v_{1f} = -1 \text{ x } v_{1i} + 2 \text{ x } v_{2i}$$

and

$$v_{2\mathrm{f}} = v_{2\mathrm{i}}$$

The second result reveals that the heavier target maintains its initial velocity. If the target is initially at rest , then it remains at rest.

On the other hand, velocity of the lighter projectile varies and depends on the combination of initial velocities of both projectile and target. If the target is initially at rest, then

$$v_{1\mathrm{f}} = -v_{1\mathrm{i}}$$

This means that the projectile reverses its motion.

3: Collision between heavy projectile and light target :

Could we think of such collision and the consequent result? It is likely that the motion of heavy projectile is not affected at all, whereas it is likely that the lighter target flies off with a higher velocity. It is what our common sense indicates. Let us test our common sense.

Here, $m_1 >> m_2$. As such, we can neglect m_2 in comparison to m_1 in the expressions of final velocities. Various expressions as contained in the equations can be approximated as :

$$\frac{(m_1 - m_2)}{(m_1 + m_2)} \approx \frac{m_1}{(m_1 + m_2)} \approx \frac{m_2}{m_1} = 1$$

$$\frac{2m_2}{(m_1 + m_2)} \approx \frac{2m_2}{m_1} \approx 0$$

and

$$\frac{2m_1}{(m_1 + m_2)} \approx \frac{2m_1}{m_1} = 2$$

Putting these approximate values in the equations of final velocities (equations 7 and 8), we have :

$$v_{1\mathrm{f}} = 1 \text{ x } v_{1\mathrm{i}} + 2 \text{ x } 0 = v_{1\mathrm{i}}$$

and

$$v_{2\mathrm{f}} = 2v_{1\mathrm{i}} - v_{2\mathrm{i}}$$

The first result reveals that the heavier projectile indeed maintains its initial velocity.

On the other hand, velocity of the lighter target varies and depends on the combination of initial velocities of both projectile and target. If the target is initially at rest, then

$$v_{2\mathrm{f}} = 2v_{1\mathrm{i}}$$

This means that the light target flies off with double the initial velocity of the projectile.

We can summarize these results for elastic collision in one dimension as :

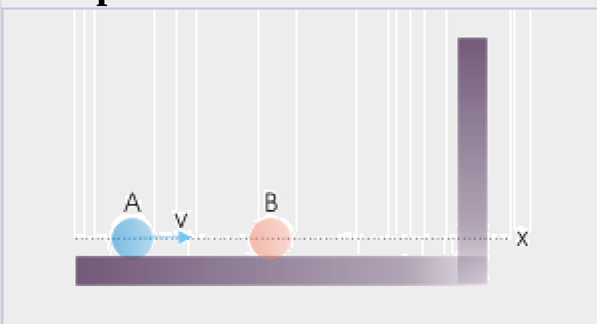
- Velocities are exchanged when bodies have equal masses.
- If the heavier target is at rest, then the motion of target remains same, whereas projectile reverses its direction without any change in magnitude.
- If the projectile is heavier and target at rest, the motion of projectile remains same, whereas target acquires double the initial velocity of the projectile.

We can generally conclude from above results that velocity of the heavier (projectile or target) is unaltered by collision. Most of the time, it is not possible to memorize the expressions of final velocities for different cases. It is generally more convenient to apply the governing laws for the given situation and solve the equation independently. However, the approximated results for the special cases as enumerated above helps save time for working with problems of elastic collision in one dimension. It is, therefore, convenient to remember the results of special cases as enumerated above.

Example:

Problem : Two spheres "A" and "B" as shown in the figure are identical in shape, size and mass. The sphere "A" moves right with a velocity "v", whereas sphere "B" is at rest. If the wall on the right is fixed, find the velocities of the spheres after all possible collisions have taken place. Assume that collisions are elastic collisions in one dimension and no friction is involved between surfaces.

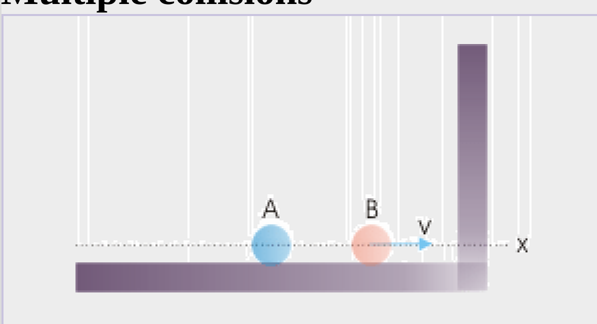
Multiple collisions



Solution : Since collisions take place in one dimension, it is imperative that all collisions are "head on" collisions. In order to know the final states of the motion, we proceed from the first collision till the possibility of collision exhausts.

The sphere "A" collides "head - on" with sphere "B". It is one dimensional elastic collision involving equal masses. Therefore, the velocities of the colliding spheres are exchanged after collision. It means that sphere "A" acquires the velocity of "B" i.e it comes to rest. On the other hand, sphere "B" acquires the velocity of sphere "A" and moves with velocity "v" towards the fixed wall. The situation is shown in the figure below.

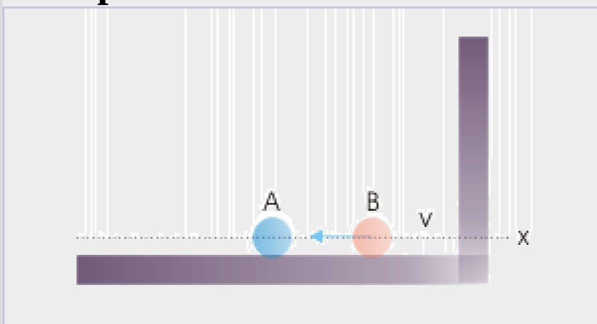
Multiple collisions



The sphere "B" moving with velocity "v" collides with fixed wall. This is elastic collision of a lighter projectile with a heavy target. As such, the velocity of the colliding sphere "B" is simply reversed after collision. It means that sphere "B" moves towards sphere "A" with velocity "v" after

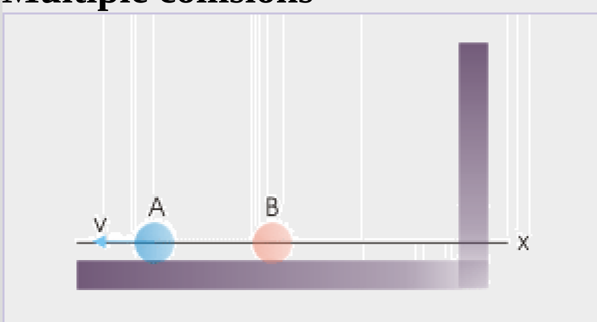
reversing its motion subsequent to collision with fixed wall. This situation is depicted in the figure.

Multiple collisions



The sphere "B" collides with stationary sphere "A". Again the collision being one dimensional elastic collision between spheres of equal masses, velocities are exchanged. The sphere “B” come to a stop, whereas sphere "A" starts moving left with velocity "v". This constitutes the last collision as sphere "A" moves away from the "B". The final situation is shown in the figure. Evidently, final situation is same as initial situation with only difference that sphere is moving towards left as against its motion towards left in the beginning.

Multiple collisions



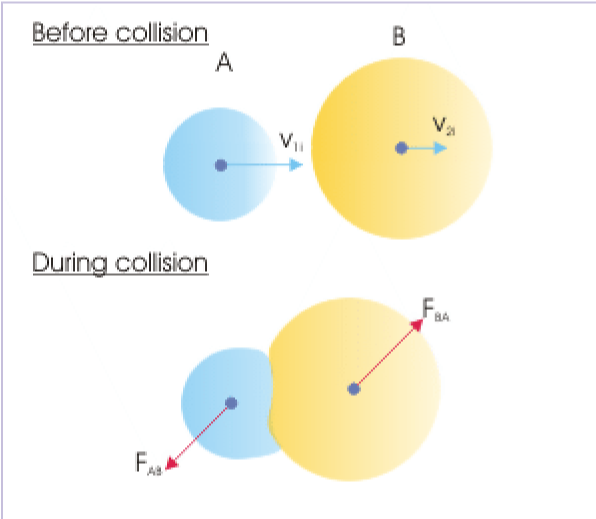
This example illustrates how we can analyze elastic collision situations in one dimension with the help of results arrived for special cases. Notably, solution completely avoids any mathematical analysis.

Elastic collision in two dimensions

Subsequent to collision, the motion of bodies is determined or affected by the impulse on the bodies. When the collision is "head - on", the impulse is directed along the line of strike. As such, motion of the bodies after

collision follows the same line of motion as before collision. Evidently, such is not the case when collision is not "head - on".

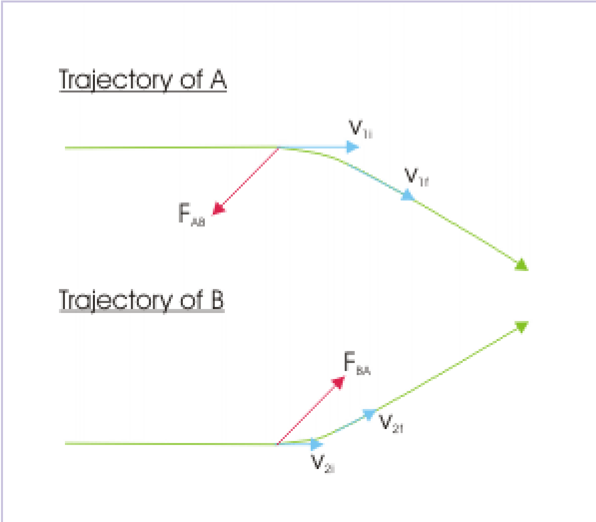
Elastic collision in two dimensions



Bodies move in different directions after collision.

The collision force pair during collision is equal and opposite. The collision force modifies the motion of the bodies as shown in the figure below. The direction of force with respect to the direction of velocity determines what would be the final velocity and its direction. It must be realized that collision force is a variable force. As such, change in velocity takes place during the period of collision and the bodies acquire the final velocity at the end of the collision.

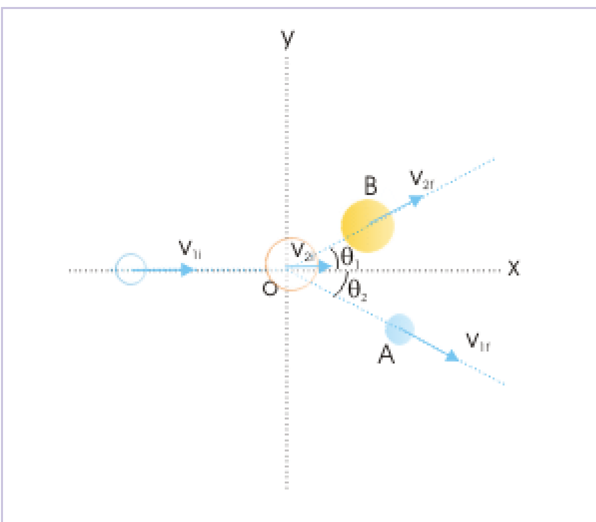
Elastic collision in two dimensions



Trajectories of the colliding bodies after collision.

The figure below shows the situation after collision. The two bodies move in different directions to the one before collision. The motion, therefore, is described in two dimensions x and y.

Elastic collision in two dimensions



Velocities of colliding bodies after collision.

The underlying principles governing collision are a pair of component linear equations and one equation of kinetic energy. Thus, three defining equations are :

1: Momentum equations :

$$m_1 v_{1\mathrm{ix}} + m_2 v_{2\mathrm{ix}} = m_1 v_{1\mathrm{fx}} + m_2 v_{2\mathrm{fx}}$$
$$m_1 v_{1\mathrm{iy}} + m_2 v_{2\mathrm{iy}} = m_1 v_{1\mathrm{fy}} + m_2 v_{2\mathrm{fy}}$$

2: Kinetic energy equation :

$$\tfrac{1}{2} m_1 v_{1\mathrm{i}}^2 + \tfrac{1}{2} m_2 v_{2\mathrm{i}}^2 = \tfrac{1}{2} m_1 v_{1\mathrm{f}}^2 + \tfrac{1}{2} m_2 v_{2\mathrm{f}}^2$$

For the set up shown in the figure, we note here that initial velocities have no component in y-direction. Thus, momentum equation in y-direction is modified as :

$$0 = m_1 v_{1\mathrm{fy}} + m_2 v_{2\mathrm{fy}}$$

Writing the component momentum equations in terms of angles,

$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 v_{1\mathrm{f}} \cos\theta_1 + m_2 v_{2\mathrm{f}} \cos\theta_2$$

and

$$0 = -m_1 v_{1\mathrm{f}} \sin\theta_1 + m_2 v_{2\mathrm{f}} \sin\theta_2$$

There are four unknowns for given bodies with given initial velocities. They are $v_{1\mathrm{f}}$, $v_{2\mathrm{f}}$, θ_1, and θ_2 . However, there are only three equations. Evidently, we need to have some experimentally determined value to solve equations for the remaining unknowns.

Elastic and plastic collisions (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to elastic collision. The questions are categorized in terms of the characterizing features of the subject matter :

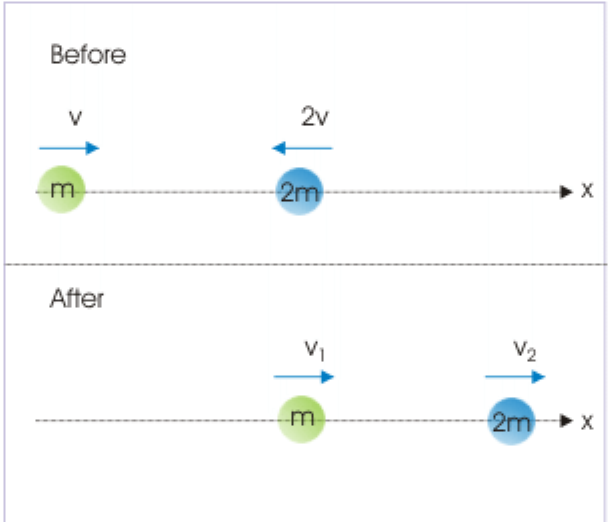
- Collision in one dimension
- Collision in two dimensions
- Elastic potential energy in collision
- Oblique collision
- Loss in kinetic energy

Collision in one dimension

Problem 1 : Two particles of mass “m” and “2m” coming from opposite sides collide “head-on” elastically. If velocity of particle of mass “2m” is twice that of the particle of mass “m”, then find their velocities after collision.

Solution : We see here that collision and motions are in one dimension. Let us denote particles of mass “m” and “2m” as “1” and “2” respectively. Let us consider that particle “1” moves in the positive x-direction and particle “2” moves in opposite direction to the reference.

Let the velocities of particle of mass “m” and “2m” after collision are “ v_1 ” and “ v_2 ” respectively. Hence,
Elastic collision



Velocities before and after collision

$$v_{1i} = v$$

$$v_{2i} = -2v$$

and

$$v_{1f} = v_1$$

$$v_{2f} = v_2$$

Applying equation of conservation of linear momentum,

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Putting values,

$$\Rightarrow mv - 2mX2v = mv_1 + 2mv_2$$

$$\Rightarrow v_1 + 2v_2 = -3v$$

For elastic collision, velocity of approach equals velocity of separation. Hence,

$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

Putting values,

$$\Rightarrow v - (-2v) = v_2 - v_1$$

$$\Rightarrow v_2 - v_1 = 3v$$

Adding two resulting equations,

$$\Rightarrow 2v_2 + v_2 = 0$$

$$\Rightarrow v_2 = 0$$

Putting this value in the equation of conservation of linear momentum,

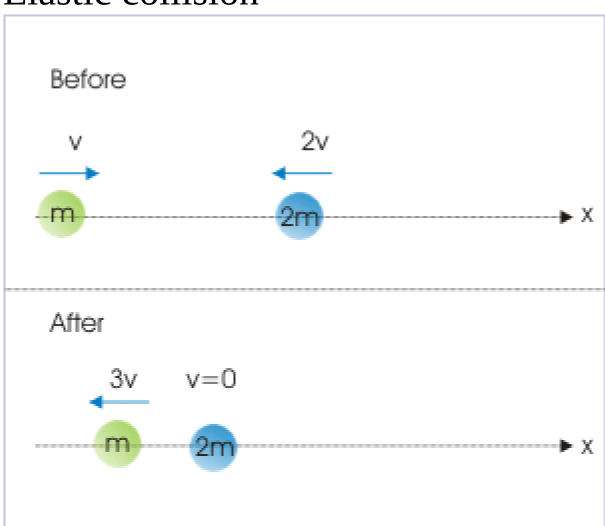
$$\Rightarrow v_1 + 2v_2 = -3v$$

$$\Rightarrow v_1 + 0 = -3v$$

$$\Rightarrow v_1 = -3v$$

It means that heavier particle of mass “2m” comes to rest, but the lighter particle of mass “m” moves with thrice its original speed in opposite direction. The situation after the collision is shown in the figure below :

Elastic collision

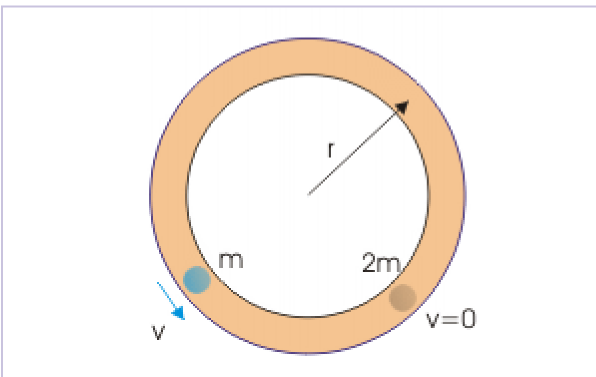


Velocities before and after collision

Collision in one dimension

Problem 2 : A particle of mass "m", moving with velocity "v" in a horizontal circular tube of radius "r", hits another particle of mass "2m" at rest. If the inside surface of the circular tube is smooth and collision is perfectly elastic, find the time interval after first collision, when they collide again.

Elastic collision



The particles collide in the circular tube.

Solution : The collision along circular path can be dealt in the same fashion as that of collision in one dimension, because particle is constrained to move along a fixed path. The particles will collide after covering a distance equal to perimeter of the circular path at a relative speed with which they separate after the collision.

For elastic collision, we know that relative speed of separation is equal to relative speed of approach.

Here,

Relative speed of approach = v = relative speed of separation.

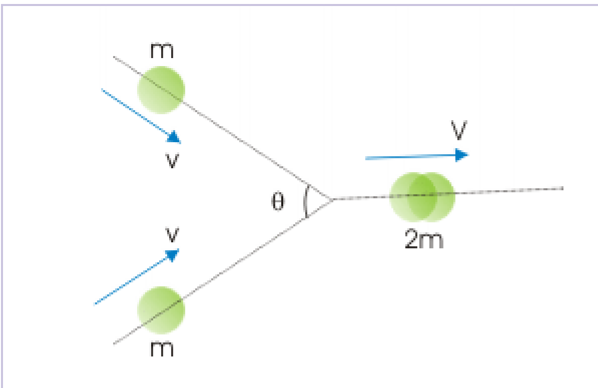
Hence, time taken to collide again, “t”, is :

$$t = \frac{2\pi r}{v}$$

Collision in two dimensions

Problem 3 : Two identical particles moving with the same speed along two different straight lines collide at the point where their trajectories meet. The particles, after a perfectly inelastic collision, move with half the initial speed. Find the angle between the paths of two particles before collision.

Elastic collision



Motion in two dimensions

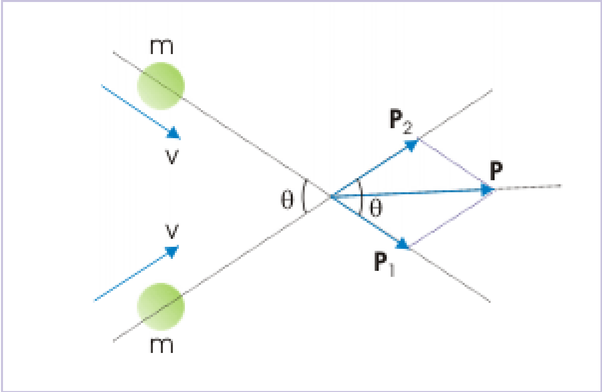
Solution : This is a question involving motion in two dimensions. However, we can not proceed, using component conservation equations. It is because we do not know the direction of final common velocity. If we proceed with component equations, we shall have more unknowns than equations.

We, therefore, make use of vector equation for conservation of linear momentum. We can find resultant linear momentum of the particles before collision. According to conservation of linear momentum, the magnitude of

resultant linear momentum should be equal to the magnitude of final linear momentum.

Let " P_1 " and " P_2 " be the linear momentums of two particles, making an angle "θ" with each other. Let each has mass "m" and speed "v". Then, applying theorem of parallelogram for vector addition, the magnitude of resultant of linear momentums (P) is related as :

Elastic collision



Vector sum of momentums

$$P^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\theta$$

$$\Rightarrow P^2 = m^2v^2 + m^2v^2 + 2mvmv\cos\theta$$

$$\Rightarrow P^2 = 2m^2v^2 + 2m^2v^2\cos\theta$$

The common velocity of the combined mass after collision for perfectly inelastic collision between identical bodies is half the initial velocity i.e. "v/2". On the other hand, magnitude of linear momentum after collision is :

$$P^2 = \left\{2m\left(\frac{v}{2}\right)\right\}^2 = m^2v^2$$

Combining two equations, we have :

$$\Rightarrow m^2v^2 = 2m^2v^2 + 2m^2v^2\cos\theta$$

$$\Rightarrow \cos\theta = -\frac{1}{2} = \cos 120^0$$

$$\Rightarrow \theta = 120^0$$

Elastic potential energy in collision

Problem 4 : A particle with kinetic energy collides head on with an identical particle at rest. If the collision is elastic, then find the maximum elastic energy possessed by the system during collision.

Solution : Recall our discussion on drawing parallel of collision process with that of motions of two block, one of which has a spring tail. We had discussed that when projectile block hits the spring with greater velocity than that of the target block, the spring compresses till velocities of two blocks equal. This situation represents the maximum elastic potential energy of the colliding system.

Let the projectile block has velocity “v”, whereas the target block is stationary in the beginning as given in the question. Let “V” be the common velocity of colliding particles during the collision, corresponding to situation when system has maximum elastic potential energy.

Applying conservation of linear momentum to the system,

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Putting values,

$$\Rightarrow mv - mX0 = mV + mV$$

$$\Rightarrow V = \frac{v}{2}$$

Kinetic energy before collision is equal to total mechanical energy of the system.

$$\Rightarrow K_i = K = \frac{1}{2} mv^2$$

The kinetic energy when two particles move with common velocity is given as :

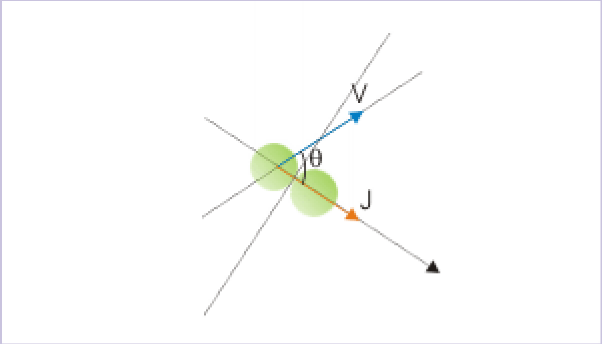
$$\Rightarrow Kc = \frac{1}{2} X2mXV^2 = mX\left(\frac{v}{2}\right)^2 = mv^2/4 = \frac{K}{2}$$

Oblique collision

Problem 5 : A ball of mass “m” collides elastically with an identical ball at rest obliquely. Find the angle between velocities after collision.

Solution : We can answer this question using equations available for collision in two dimensions. However, there is an elegant solution to this problem that we can attempt using the nature of collision. We know that collision normal force is perpendicular to the tangent drawn at the contact point. There is no component of collision force in the tangential direction.

Oblique elastic collision



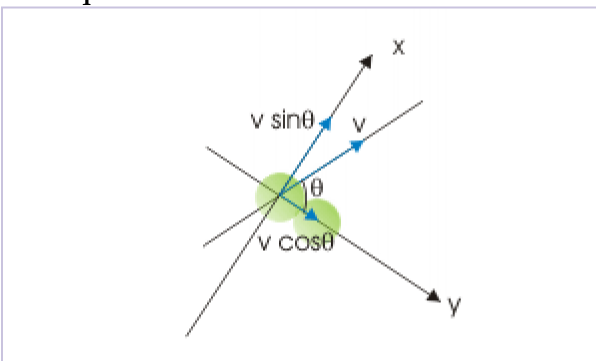
The projectile makes an angle with normal collision force.

Thus, if we stick to the analysis in these two perpendicular directions, then we can get the answer quickly. Let the velocity of projectile ball (“1”) makes an angle “θ” with the direction of normal force as shown in the figure.

Analyzing motion in y – direction, we see that the projectile hits the target (“2”) “head – on” with a component of velocity “v cos θ”. For this situation

involving head on collision between identical balls, velocities are simply exchanged after collision. It means that projectile's velocity becomes zero in this direction, whereas target gains a velocity "v cos θ" in this direction.

Oblique elastic collision

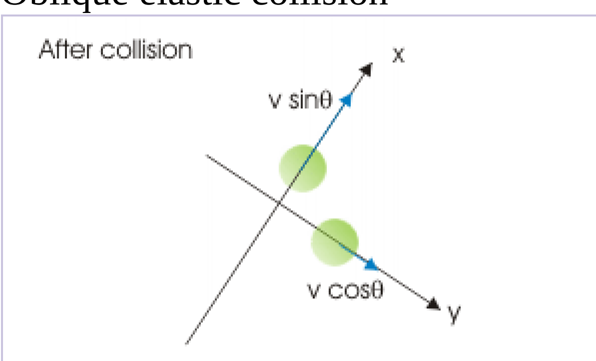


Analysis in mutually perpendicular directions.

Analyzing motion in x-direction, we see that there is no component of collision normal force in this direction. This means that component of velocity in this direction remains constant. As such, projectile has velocity "v sin θ" in this direction, whereas target has no component of velocity in this direction.

In the nutshell, projectile moves with resultant velocity "v sin θ" in x-direction, whereas target moves resultant velocity "v cos θ" in y-direction.

Oblique elastic collision



Velocities are mutually

perpendicular to each other after
collision.

Hence, two identical balls, after collision, move in directions, which are at right angle with each other. This is a very important result as it says that resulting motion in elastic collision of two identical bodies are at right angle to each other irrespective of the angle at which they collide!

Loss in kinetic energy

Problem 6 : A particle of mass 1 kg with velocity “v” collides “head on” with another identical particle at rest. If kinetic energy of the system of particles after collision is 2 J, then find minimum and maximum values of initial velocity “v” of the projectile.

Solution : There is no (minimum) loss of kinetic energy in perfectly elastic collision. On the other hand, there is maximum loss in kinetic energy in perfectly inelastic collision.

The velocity of projectile particle is minimum for perfectly elastic collision for a given kinetic energy after collision.

$$\Rightarrow K_i = K_f = \frac{1}{2}mv^2 = 2$$

$$\Rightarrow \frac{1}{2}X1Xv^2 = 2$$

$$\Rightarrow v^2 = 4$$

$$\Rightarrow v = 2 \quad m/s$$

The velocity of projectile particle is maximum for perfectly inelastic collision for a given kinetic energy after collision, which is 2 J. Let v’ be the velocity of the combined mass after plastic collision.

Then, applying conservation of linear momentum,

$$mv + mX0 = 2mv\prime$$

$$v\prime = \frac{v}{2}$$

$$K_f = \frac{1}{2}2m(v\prime)^2 = \frac{1}{2}2m\left(\frac{v}{2}\right)^2$$

$$\Rightarrow \frac{v^2}{4} = 2$$

$$\Rightarrow v = 2\sqrt{2} \quad m/s$$

Inelastic collision

Inelastic collision differs to elastic collision in one important respect that there is certain loss of kinetic energy during collision in the form of sound, heat or in other energy forms. As kinetic energy is not conserved in inelastic collision, we use the concept of "coefficient of restitution" to supplement the analysis of inelastic collision.

One of the dictionary meanings of “restitution” is “refund”. As a matter of fact, it has similar meaning in the context of collision also. Sometimes, we want to measure how much of the speed (magnitude of velocity) of the colliding bodies is retained after collision. In other words, what is refunded (speed) after the collision?

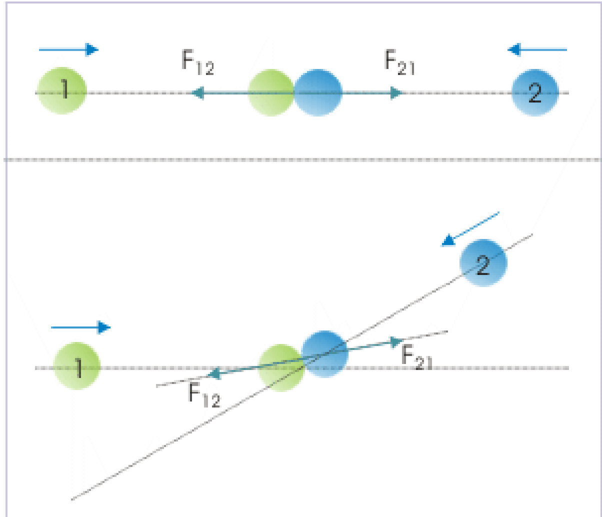
For example, we would like to have a tennis racket that returns the ball with the most of kinetic energy (speed). Actually, the golf club with thin head is designed to punch the ball with maximum velocity. To save the sanctity of the game from technology, it is, however, restricted that no golf club should have coefficient of restitution greater than a specific value.

Coefficient of restitution

Coefficient of restitution is a measure of two colliding bodies. It is specific to the given pair of colliding bodies, which can be measured with suitable arrangement.

We, however, need to understand the context of collision in order to define “coefficient of restitution”. There is a difficulty here that two bodies can collide “head – on” or obliquely. As there can be infinite numbers of angles involved between the pair of colliding bodies, it is apparent that we need to specify the nature of collision so that this (coefficient of restitution) property is a unique value property for the pair of colliding bodies in question.

Collision



Head-on and oblique collision

For this reason, coefficient of restitution is defined for “head-on” collision. The next question, however, is that : what is “head – on” collision? In simple words, the paths of bodies and direction of normal force during collision are along the same straight line. Now, the stage is set to define the term as :

The coefficient of restitution (e) of two bodies for “head-on” collision is a constant and is equal to the ratio of velocity of separation and velocity of approach.

$$e = \frac{\text{velocity of separation}}{\text{velocity of approach}}$$

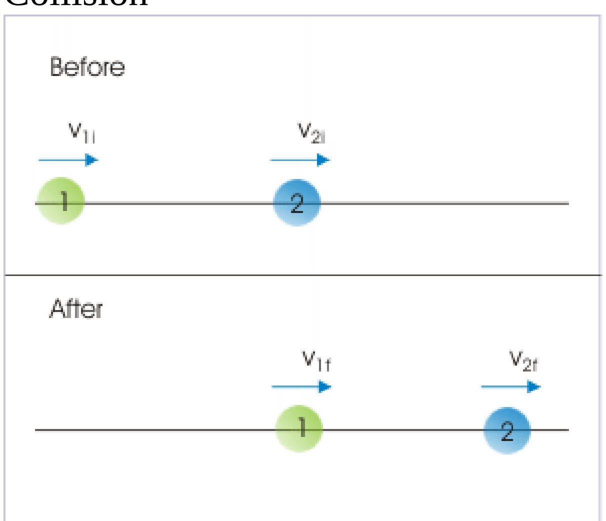
The very fact that “head – on” collision and motion of bodies take place along a straight line has important bearing on the evaluation of this constant. If we look closely at the above ratio, then it is very easy to understand following aspects of this quantity :

1. It is possible to take the ratio of two vectors (division of one vector by another) because both velocity terms are along a straight line. Had they been directed differently, this ratio can not be evaluated in the first place.

2. The coefficient of restitution is a positive constant. This fact is very helpful in working problems based on coefficient of restitution. If this ratio evaluates to negative values, then we should be certain that there is something amiss in assigning signs of terms involved in the ratio.
3. The coefficient of restitution (e) is a positive number, whose value falls in the range 0≤ e ≤1. We know that velocity of separation and approach are equal in the case of elastic collision. The value of "e" is 1 in this case. On the other hand, velocity of separation is zero (0) for completely inelastic (also called plastic) collision. The value of "e" is 0 in this case. Thus, elastic and plastic collisions represent the bounding values of the coefficient of restitution (e). The value of "e" falls between these bounding values for other inelastic collision. The coefficient of restitution is a fraction (final kinetic energy of the colliding system can not greater than initial) for inelastic colision.
4. Since motions are along a straight line, we can use scalar representation of velocity with appropriate sign convention with respect to reference direction.

In order to understand the terms involved in the ratio, we consider an example. One block designated as “1” approaches another block designated as “2” on a smooth surface. For collision to occur, it is clear that $v_{1i} > v_{2i}$. And the velocity of approach, is :

Collision



"Head - on" Collision

$$v_{\text{approach}} = v_{1i} - v_{2i}$$

After collision, the bodies move with different velocities and move away from each other. Therefore, the velocity of separation is given by :

$$v_{\text{separation}} = v_{2f} - v_{1f}$$

The coefficient of restitution is :

$$e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}}$$

In order to avoid confusion on account of subscripts, it is helpful to follow certain conventions as given here :

1. Decide a reference direction i.e an axis.
2. Consider one of the bodies as “projectile”, which is subscripted with “1” and consider other as “target”, which is subscripted with “2”.
3. Write velocity of separation as relative velocity of “target” with respect to “projectile” and
4. Write velocity of approach as relative velocity of “projectile” with respect to “target”.

If we stick with this scheme, then we can write the expression of coefficient of restitution as :

$$\Rightarrow e = \frac{v_{21f}}{v_{12i}}$$

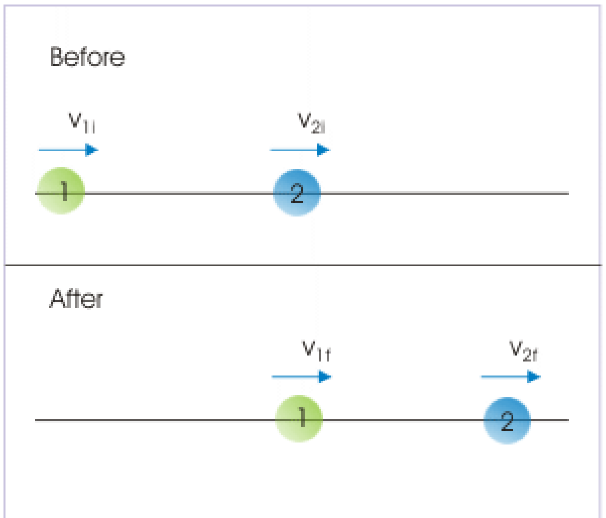
Analysis of collision

The analysis of collision between two bodies involves many unknowns – two masses and four velocities. On the other, we have only one equation of conservation of linear momentum in the case of inelastic collision. Since kinetic energy is not conserved in inelastic collision, we can not use equation of kinetic energy as in the case of elastic collision.

The measurement of coefficient of restitution is helpful to obtain another equation that can be used in conjunction with conservation of linear momentum.

Continuing with the earlier example, involving two blocks, the conservation of linear momentum is written as :

Collision



Head-on collision

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

The coefficient of restitution equation is :

$$e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}}$$

$$\Rightarrow v_{2f} - v_{1f} = e(v_{1i} - v_{2i})$$

Our objective here is to write final velocities in terms of initial velocities for two blocks of given masses. We can eliminate “ $v_{2\mathrm{f}}$ ” from equation of conservation of linear momentum, using its expression from equation of restitution,

$$\Rightarrow m_1 v_{i_1} + m_2 v_{2i} = m_1 v_{1f} + m_2 \{e(v_{1i} - v_{2i}) + v_{1f}\}$$

$$\Rightarrow v_{1f}(m_1+m_2) = m_1 v_{1i} + m_2 v_{2i} - e m_2 v_{1i} + e m_2 v_{2i}$$

$$\Rightarrow v_{1f}(m_1+m_2) = v_{1i}(m_1 - e m_2) + v_{2i}(m_2 + e m_2)$$

$$\Rightarrow v_{1f} = \frac{(m_1 - e m_2)}{(m_1+m_2)} v_{1i} + \frac{(m_2 + e m_2)}{(m_1+m_2)} v_{2i}$$

Similarly, we can eliminate " v_{1f} " from equation of conservation of linear momentum, using equation of restitution,

$$\Rightarrow m_1 v_{1i} + m_2 v_{2i} = m_1\{v_{2f} - e(v_{1i} - v_{2i})\} + m_2 v_{2f}$$

$$\Rightarrow v_{2f}(m_1+m_2) = m_1 v_{1i} + m_2 v_{2i} + m_1 e v_{1i} - e m_1 v_{2i}$$

$$\Rightarrow v_{2f}(m_1+m_2) = v_{2i}(m_2 - e m_1) + v_{i_1}(m_1 + e m_1)$$

$$\Rightarrow v_{2f} = \frac{(m_2 - e m_1)}{(m_1+m_2)} v_{2i} + \frac{(m_1 + e m_1)}{(m_1+m_2)} v_{1i}$$

Collision between identical bodies

Here, $m_1 = m_2 = m$ (say). If second body is also at rest in the beginning, then $v_{2i} = 0$. The final velocities in this case are :

$$\Rightarrow v_{1f} = \frac{(1-e)}{2} v_{1i}$$

$$\Rightarrow v_{2f} = \frac{(1+e)}{2} v_{1i}$$

These results are intuitive about the role of coefficient of restitution in the analysis of inelastic collision. We see here that final speeds are some fraction of initial speed. In turn, the fraction is a function of coefficient of restitution (e).

Since kinetic energy consists of term of speed squared, we can conclude that final kinetic energy of the colliding bodies are fraction of initial kinetic

energy and this fraction is a function of coefficient of restitution. Thus, coefficient of restitution enables us to measure kinetic energy of the colliding bodies after collision. For this reason, we can say that using coefficient of restitution is an equivalent analysis method for specifying loss of kinetic energy during inelastic collision.

Example

Problem : A block of mass 1 kg moving with a velocity 1 m/s collides "head - on" with another identical block at rest. If the coefficient of restitution is 2/5, find the loss of kinetic energy during collision.

Solution : This is an inelastic "head - on" collision. We note here that calculation of kinetic energy will require determination of final velocities. We can use the momentum and coefficient of restitution equations to find the final velocities of the blocks after collision. Here,

$$m_1 = m_2 = 1 \text{ kg}, v_{1\text{i}} = 1\, m/s, v_{2\text{i}} = 0$$

The momentum equation in one dimension is :

$$\begin{aligned} & m_1 v_{1\text{i}} + m_2 v_{2\text{i}} = m_1 v_{1\text{f}} + m_2 v_{2\text{f}} \\ & \Rightarrow 1 \text{ X } 1 + 1 \text{ X } 0 = 1 \text{ X } v_{1\text{f}} + 1 \text{ X } v_{2\text{f}} \\ & \Rightarrow v_{1\text{f}} + v_{2\text{f}} = 1 \end{aligned}$$

The coefficient of restitution equation is :

$$\begin{aligned} v_{2\text{f}} - v_{1\text{f}} &= e(v_{1\text{i}} - v_{2\text{i}}) \\ v_{2\text{f}} - v_{1\text{f}} &= \tfrac{2}{5}\left(1 - 0\right) = 0.4 \end{aligned}$$

Adding two equations and solving for $v_{2\text{f}}$, we have :

$$\begin{aligned} & \Rightarrow 2v_{2\text{f}} = 1.4 \\ & \Rightarrow v_{2\text{f}} = 0.7\, m/s \end{aligned}$$

and

$$\Rightarrow v_{1\mathrm{f}} = 0.3\, m/s$$

Once final velocities are known, we can calculate initial and final kinetic energies:

$$\mathbf{K}_i = \frac{1}{2} m_1 v_{1\mathrm{i}}^2 + \frac{1}{2} m_2 v_{2\mathrm{i}}^2 = \frac{1}{2} \text{ x } 1 \text{ x } 1^2 = 0.5\, J$$

and

$$\mathbf{K}_f = \frac{1}{2} m_1 v_{1\mathrm{f}}^2 + \frac{1}{2} m_2 v_{2\mathrm{f}}^2 = \frac{1}{2} \text{ X } 1 \text{ X } 0.3^2 + \frac{1}{2} \text{ X } 1 \text{ X } 0.7^2 = 0.29\, J$$

The loss of kinetic energy is :

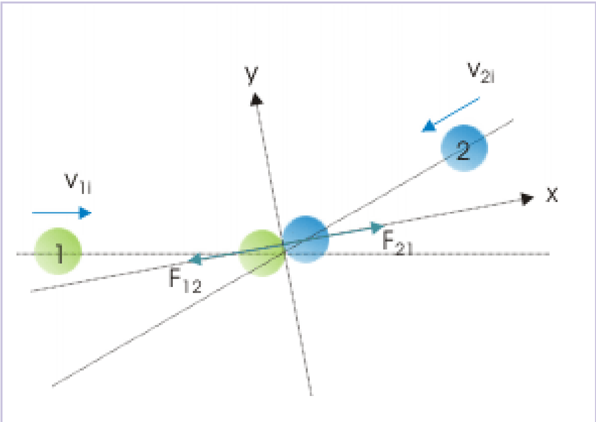
$$\Delta K = K_i - K_f = 0.5 - 0.29 = 0.21\, J$$

Oblique collision

The real time collision, however, is not “head – on” collision most of the times. In this section, we shall apply the concept of coefficient of restitution to oblique collision.

The normal force during collision is along a straight line even in an oblique collision. However, motions of the colliding bodies are not along the direction in which collision normal force acts. In this case, we analyze motion with coordinate system having axes parallel and perpendicular to the direction of normal force. This set up facilitates application of the concept of the coefficient of restitution in the direction of normal force.

Collision



Oblique collision

We consider an oblique collision in which normal force acts in “y” direction as shown in the figure above. We can analyze collision using following aspects of oblique collision,

1: We use equation of restitution in the direction of normal force . Recall that coefficient of restitution is defined for "head - on" collision. This implies that it is defined for motion in the direction of normal force acting during collision.

2: The component of normal force during collision is zero in the perpendicular direction. This means that the components of velocity of each body are unchanged before and after collision in the direction perpendicular to normal force.

3: Since net external force on the system of two particles is zero (collision normal forces are internal forces for the system of particles), we can use equation of conservation of linear momentum in component form in directions parallel and perpendicular to normal force.

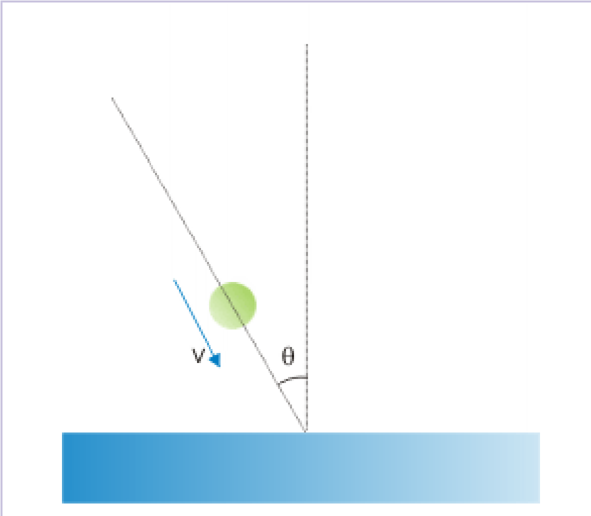
$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Example

Problem : A ball of mass “m” hits a floor with a speed “v” making an angle “θ”. If “e” be the coefficient of restitution, then find the angle of reflection of the ball.

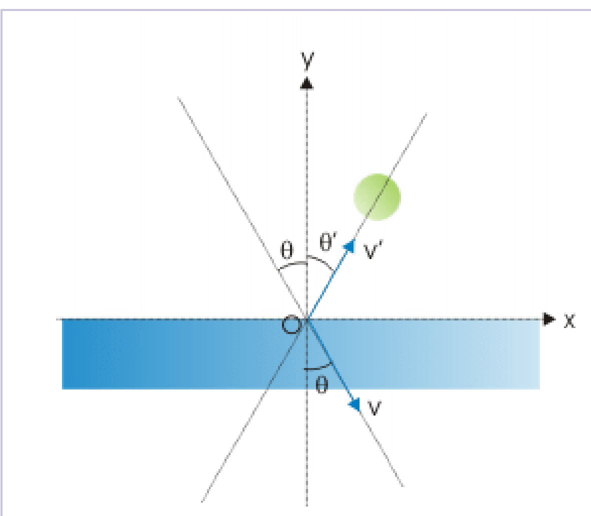
Collision



A ball of mass “m” hits a floor with a speed “v” making an angle “θ”.

Solution : Force of collision acts normal to the floor at the point, where ball strikes it. There is no component of collision force parallel to the surface. As such, component of velocity parallel to the surface before and after remains same.

Collision



A ball of mass “m” hits a floor with a speed “v” making an

angle “θ”.

Let “θ’” be the angle of reflection. Then, components of velocity before and after are equal,

$$v \sin\theta = v' \sin\theta'$$

We, now, consider motion in the normal direction. Here, the coefficient of restitution is given by :

$$e = \frac{v_{21f}}{v_{12i}}$$

$$e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} = \frac{0 - v_{1fy}}{v_{1iy} - 0}$$

$$\Rightarrow v_{1fy} = -ev_{1iy} = -ev\cos\theta$$

and

$$\tan\theta' = \frac{v_{1fx}}{v_{1fy}} = \frac{v\sin\theta}{ev\cos\theta} = \frac{\tan\theta}{e}$$

Repetitive collisions

In this section, we consider a case of tennis ball falling from a height and undergoing repetitive collisions with a horizontal hard surface. The ball looses some kinetic energy (hence speed) after each successive strike due to inelastic collision between ball and horizontal surface. As a consequence, the height attained by the ball diminishes as the number of collision increases.

Our objective here is to find an expression for the speed of the ball and the height attained subsequent to nth collisions.

Let coefficient of friction between ball and the surface be “e”. Also, let the initial height of the ball be “ h_0 ” and speed before the first strike be “ v_0 ”. For the analysis of motion, we consider upward vertical direction as positive. Now, applying equation of motion in vertical downward direction, we have :

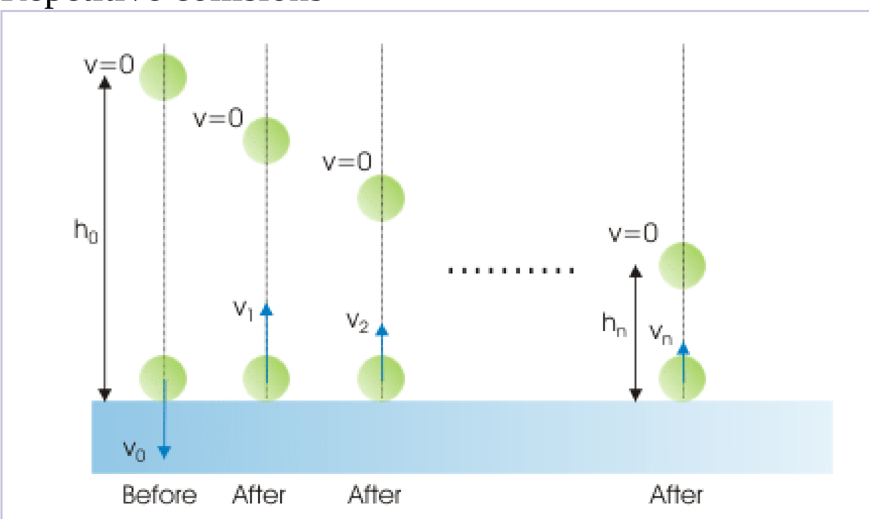
$$v_0^2 = 0 + 2(-g)(-h_0)$$

$$\Rightarrow v_0 = \sqrt{(2gh_0)}$$

and

$$\Rightarrow h_0 = \frac{v_0^2}{2g}$$

Repetitive collisions



The ball looses speed and height with successive collision.

Ist strike

Let “ v_1 ” be the velocity of the ball after first strike. The velocity after 1st collision is obtained from consideration of equation of restitution,

$$v_{2f} - v_{1f} = v_{1i} - v_{2i}$$

Putting values, we have :

$$\Rightarrow 0 - v_1 = e(-v_0)$$

$$\Rightarrow v_1 = ev_0$$

Height attained with this velocity is given as :

$$h_1 = \frac{v_1^2}{2g} = \frac{e^2 v_0^2}{2g} = e^2 h_0$$

2nd strike

Let " v_2 " be the velocity of the ball after second strike. The velocity after 2nd collision is obtained from consideration of equation of restitution,

$$v_{2f} - v_{1f} = v_{1i} - v_{2i}$$

Putting values, we have :

$$0 - v_2 = eXv_1$$

$$\Rightarrow v_2 = ev_1 = eXev_0 = e^2 v_0$$

Height attained with this velocity is given as :

$$\Rightarrow h_2 = \frac{v_2^2}{2g} = \frac{e^4 v_0^2}{2g} = e^4 h_0$$

nth strike

Continuing in the same fashion, we can get expressions for nth strike. Let " v_n " be the velocity of the ball after nth strike. Then,

$$\Rightarrow v_n = e^n v_0$$

Let " h_n " be the height attained. Then,

$$h_n = e^{2n} h_0$$

Inelastic collision (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

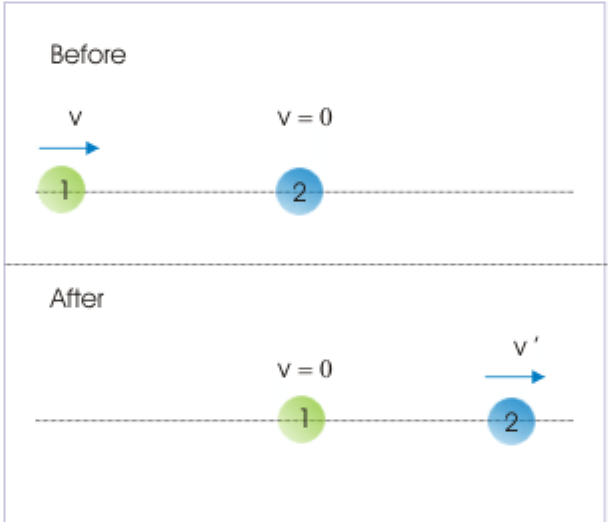
We discuss problems, which highlight certain aspects of the study leading to inelastic collision. The questions are categorized in terms of the characterizing features of the subject matter :

- Coefficient of restitution
- Successive collisions
- Kinetic energy and coefficient of restitution
- Motion subsequent to collision

Coefficient of restitution

Problem 1 : A block of mass 1 kg, moving at a speed “v”, collides with another block of mass 10 kg at rest. The lighter block comes to rest after the collision. Find the coefficient of restitution.

Solution : Let us consider that lighter block is moving along x-axis. The initial velocities are :
Inelastic collision



Velocities before and after collision

$$v_{1i} = v$$

$$v_{2i} = 0$$

The final velocities are :

$$v_{1f} = 0$$

$$v_{2f} = v\prime$$

From conservation of linear momentum,

$$1Xv + 10X0 = 1X0 + 10Xv\prime$$

$$\Rightarrow v\prime = \frac{v}{10}$$

The coefficient of restitution is :

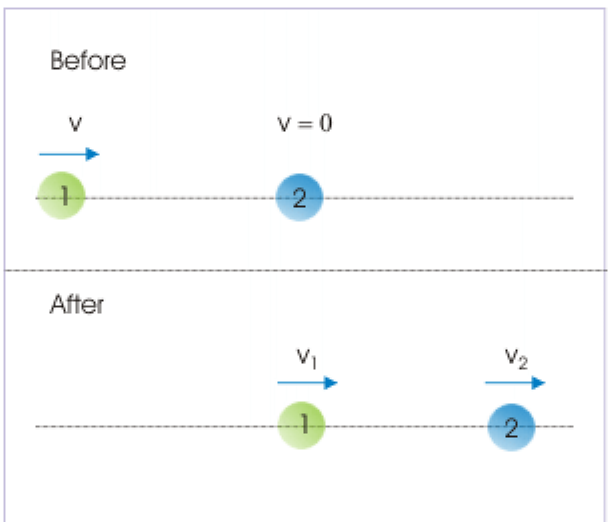
$$e = \frac{\text{velocity of separation}}{\text{velocity of appraoch}}$$

$$\Rightarrow e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} = \frac{v\prime - 0}{v} = \frac{\frac{v}{10}}{v} = \frac{1}{10} = 0.1$$

Problem 2 : A spherical ball hits "head – on" another identical ball, which is stationary. The velocity of second ball, immediately after the collision, is twice that of first ball. Find coefficient of restitution.

Solution : We solve this problem using equation of conservation of linear momentum and equation of restitution. Let velocity of first ball before collision be "v", velocity of first ball after collision be " v_1 " and velocity of second ball after collision be " v_2 ".

Inelastic collision



Velocities before and after collision

Applying equation of conservation of linear momentum, we have :

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Putting values, we have :

$$\Rightarrow mv + mX0 = mv_1 + mv_2$$

$$\Rightarrow v_1 + v_2 = v$$

It is given that :

$$\Rightarrow v_2 = 2v_1$$

Substituting in the equation,

$$\Rightarrow v_1 + 2v_1 = 3v_1 = v$$

$$v_1 = \frac{v}{3}$$

Putting this value of " v_1 " in the equation of conservation of linear momentum, we get " v_2 " as :

$$\Rightarrow v_2 = v - v_1 = v - \frac{v}{3} = \frac{2v}{3}$$

Now, applying equation of restitution, we have :

$$e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} = \frac{v_2 - v_1}{v - 0}$$

$$\Rightarrow e = \frac{\frac{2v}{3} - \frac{v}{3}}{v}$$

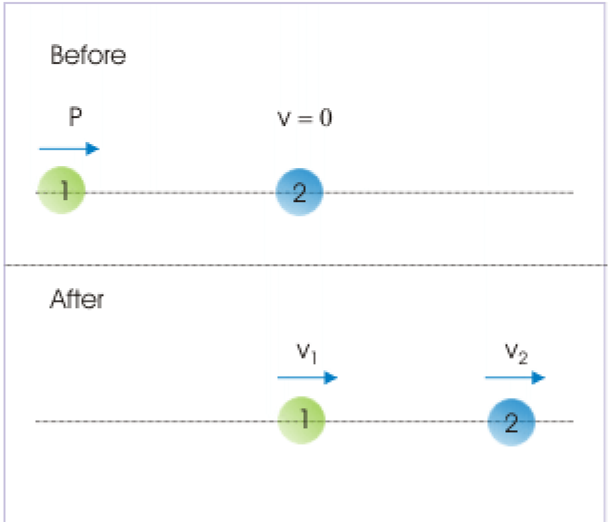
$$\Rightarrow e = \frac{1}{3}$$

Problem 3 : In a "head – on" inelastic collision between two identical particles, the linear momentum of the projectile is "P", whereas the target particle is stationary. If the target applies an impulse of magnitude "J" on the projectile, find coefficient of restitution.

Solution : In order to solve this problem, we need to evaluate the defining expression of coefficient of restitution. This means that we need to know final velocities.

We note here that we need to consider motion and collision in one dimension as the collision is “head – on”.

Let the mass of each particle is “m”. Let velocity of first ball before collision be “v”, velocity of first ball after collision be “ v_1 ” and velocity of second ball after collision be “ v_2 ”.
Inelastic collision



Velocities after collision

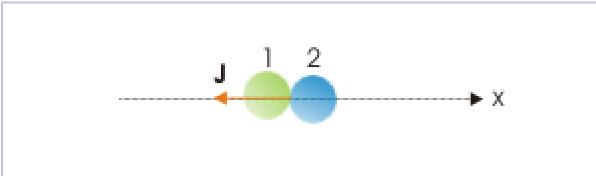
According to question,

$$v = \frac{P}{m}$$

Now, we know that impulse is equal to change in linear momentum :

$$J = \varDelta P = P_{1f} - P_{1i} = m(v_{1f} - v_{1i}) = m(v_1 - v)$$

However, $v_1 > v$. It means that impulse acts in opposite direction to that of reference direction of motion. Hence,
Inelastic collision



Direction of impulse

$$\Rightarrow J = -m(v_1 - v) = -mv_1 + mv$$

$$\Rightarrow v_1 = -\frac{J}{m} + v$$

Putting value of “v”,

$$\Rightarrow v_1 = -\frac{J}{m} + \frac{P}{m}$$

Now, applying conservation of linear momentum,

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\Rightarrow mv + mX0 = mv_1 + mv_2$$

$$\Rightarrow v = v_1 + v_2$$

Substituting for “ v_1 ”, we have :

$$\Rightarrow v = -\frac{J}{m} + v + v_2$$

$$\Rightarrow v_2 = \frac{J}{m}$$

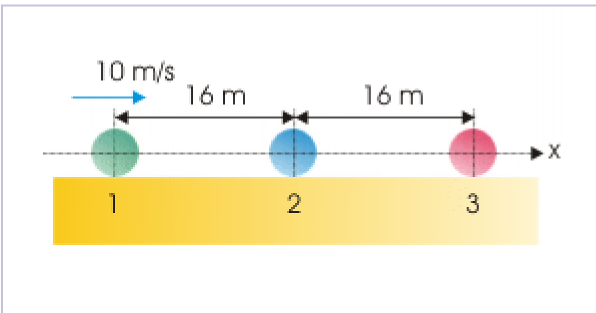
The coefficient of restitution is :

$$e = \frac{v_2 - v_1}{v} = \frac{\frac{J}{m} + \frac{J}{m} - \frac{P}{m}}{P/m} = \frac{2J}{P} - 1$$

Successive collisions

Problem 4 : Three identical balls designated “1”, “2” and “3” are placed 16 meters apart on a smooth horizontal surface. The ball “1” is given an initial velocity 10 m/s towards “2”. The collision between “1” and “2” is inelastic with coefficient of restitution 0.6, whereas collision between “2” and “3” is perfectly inelastic. Find the time elapsed between two consecutive collisions between “1” and “2”.

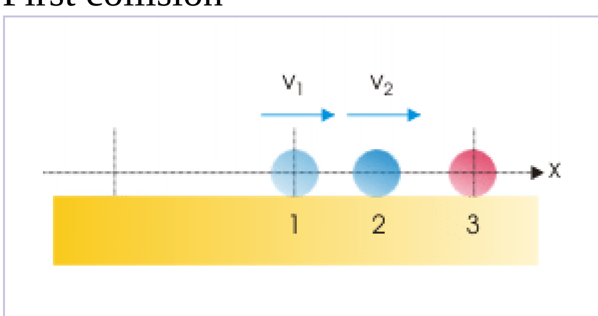
Successive collisions



Three identical balls are placed on a smooth horizontal surface.

Solution : To answer this question, we need to understand the collision sequence. The ball “1” moving with velocity 10 m/s strikes second ball in - elastically. As a consequence, depending on coefficient of restitution two balls move with different velocities. This collision constitutes the first collision.

First collision



First collision is inelastic.

Let the mass of each particle is “m”. Let velocity of first ball before collision be “v”, velocity of first ball after collision be “ v_1 ” and velocity of second ball after collision be “ v_2 ”. Applying conservation of linear momentum,

$$mv = mv_1 + mv_2$$

$$\Rightarrow v = v_1 + v_2$$

$$\Rightarrow v_1 + v_2 = 10$$

Applying equation of restitution,

$$\Rightarrow v_2 - v_1 = ev = 0.6X10 = 6$$

Adding two resulting equations,

$$\Rightarrow 2v_2 = 16$$

$$\Rightarrow v_2 = 8 \quad m/s$$

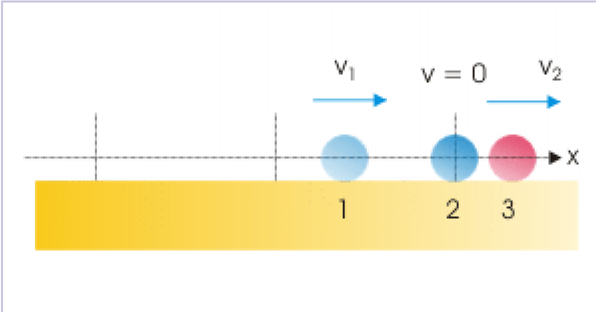
Substituting this value in the equation of conservation of linear momentum,

$$\Rightarrow v_1 = 10 - v_2 = 10 - 8 = 2 \quad m/s$$

Clearly, second ball covers the horizontal distance of 16 m, subsequent to collision at 8 m/s and strikes the third ball.

The second ball strikes third stationary identical ball elastically. As a result, the balls exchange their velocities. It means second ball becomes stationary and third ball moves off towards right with velocity at 8 m/s. First ball, however, keeps moving towards right at 2 m/s and covers the distance of 16 m to strike the second ball again, which has come to a standstill. The time taken by the first ball to hit the second ball again is :

Second collision



Second collision is perfectly elastic.

$$t = \frac{16}{2} = 8 \quad s$$

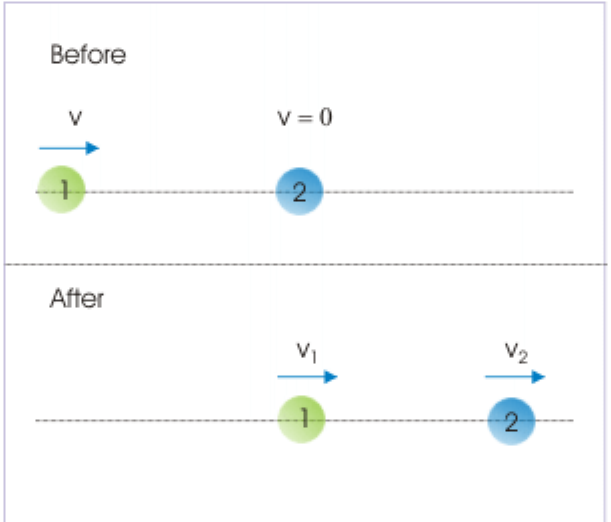
This is the time interval between required two consecutive collisions.

Kinetic energy and coefficient of restitution

Problem 5 : A spherical ball of mass “m” moving at a speed “v” makes a head on collision with an identical ball at rest. The kinetic energy of the balls after collision is 3/4 th of the value before collision. Find coefficient of restitution.

Solution : The spherical balls have equal mass and initial velocity of “target” ball is zero.

Let the mass of each particle is “m”. Let velocity of first ball before collision be “v”, velocity of first ball after collision be “ v_1 ” and velocity of second ball after collision be “ v_2 ”.
Inelastic collision



Velocities before and after collision

For identical mass and stationary target, we have :

$$v_1 = \frac{(1-e)}{2} v$$

and

$$v_2 = \frac{(1+e)}{2} v$$

Note: Instead of remembering these relations, we can alternatively solve simultaneous equations of conservation of linear momentum and restitution to obtain these relations.

According to question,

$$K_f = \frac{3}{4} K_i$$

$$\Rightarrow \frac{1}{2} m\left(v_1^2 + v_2^2\right) = \frac{3}{4} X \frac{1}{2} m v^2$$

$$\Rightarrow v_1^2 + v_2^2 = \frac{3}{4} m v^2$$

Substituting for final velocities, we have :

$$\Rightarrow \frac{(1-e)^2}{4} v^2 + \frac{(1+e)^2}{4} v^2 = \frac{3}{4} m v^2$$

$$\Rightarrow (1-e)^2 + (1+e)^2 = 3$$

$$\Rightarrow 1 + e^2 - 2e + 1 + e^2 + 2e = 3$$

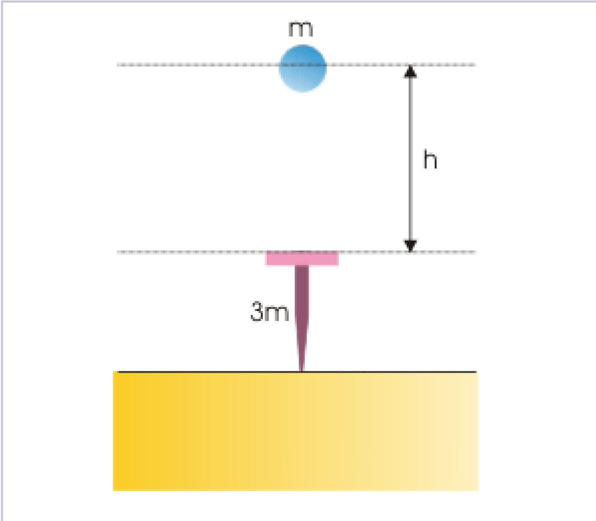
$$\Rightarrow e^2 = \frac{1}{2}$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

Motion subsequent to collision

Problem 6 : The ram of a pile driver has a mass m and is released from rest at a height h above the top of the pile which has the mass 3m. To what height does the ram rebound if the coefficient of restitution between the ram and the pile is e and e>(1/3)? If the pile meets a constant resistance F from the material in which it is lodged, calculate how far into the material it penetrates before coming to rest.

Inelastic collision



The ram of a pile driver is released from rest at a height "h".

Note: This question is as submitted by Baline Patrick through e-mail.

Solution : We need to know the velocities of ram and pile immediately after the inelastic collision. Let the velocity of the ram just before strike be "v", velocity of the ram after the strike be " v_1 " velocity of the pile after strike be " v_2 ".

Let the downward vertical y-direction be positive. We can determine velocity of the ram just before it hits the pile. Applying equation of motion for constant acceleration,

$$v^2 = 0 + 2gh = 2gh$$

In order to determine velocities immediately after collision, we shall apply conservation of linear momentum in vertical direction (downward direction

as positive)

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Putting values, we have :

$$\Rightarrow mv + 0x3m = mv_1 + 3mv_2$$

$$\Rightarrow v = v_1 + 3v_2$$

Now, applying equation of restitution, we have :

$$e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} = \frac{v_2 - v_1}{v - 0}$$

$$\Rightarrow ev = v_2 - v_1$$

Thus, we have two unknowns “ v_1 ” and “ v_2 ”. Also we have two equations. Adding two equations,

$$\Rightarrow (1 + e)v = 4v_2$$

The velocity of pile, therefore, is :

$$\Rightarrow v_2 = \frac{(1 + e)v}{4}$$

Putting this value in the first equation, we get the velocity of the ram after collision,

$$\Rightarrow v_1 = v - 3v_2 = \frac{(1 - 3e)v}{4}$$

Putting expression of “v”, as obtained earlier, in above two expressions, we have :

$$\Rightarrow v_2 = \frac{(1 + e)v}{4}$$

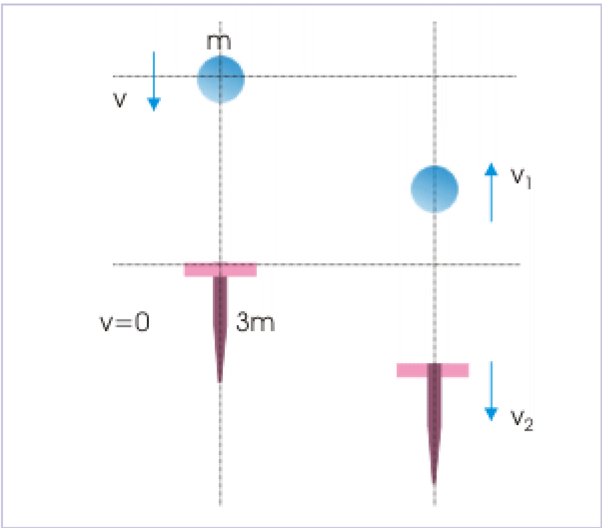
$$\Rightarrow v_2 = \frac{(1+e)\sqrt{(2gh)}}{4}$$

and

$$\Rightarrow v_1 = \frac{(1-3e)\sqrt{(2gh)}}{4}$$

Note that 1 - 3e < 0 as e > 1/3. It means that the velocity of ram is opposite to the assumed positive reference direction. This further means that it rebounds after the strike in upward direction.

Inelastic collision



Velocities of colliding bodies.

$$\Rightarrow v_1 = \frac{(3e-1)\sqrt{(2gh)}}{4}$$

Let h' be the height attained by the ram after the strike. For the upward motion of ram,

$$0 = v_1^2 - 2gh\prime$$

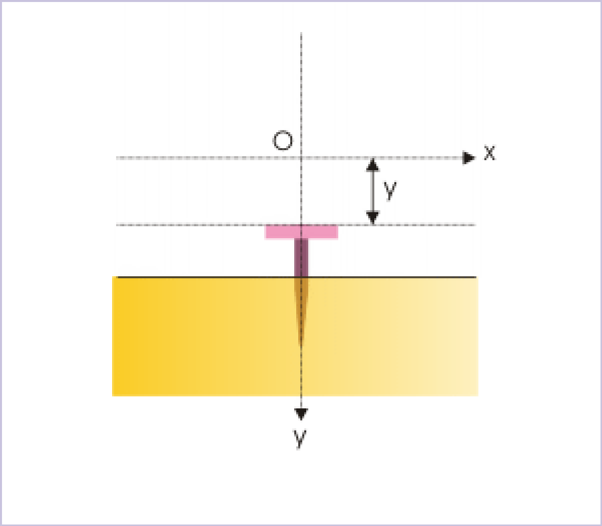
$$\Rightarrow h\prime = \frac{v_1^2}{2g} = \frac{(3e-1)^2 Xgh}{2gX16} = \frac{(3e-1)^2 Xh}{16}$$

Now, we consider the motion of pile into the material. There are two forces acting on the pile (i) constant resistance “F” and (ii) force due to gravity “3mg". As given, the net force acts up to decelerate the pile. The deceleration is :

$$a = \frac{F - 3mg}{3m}$$

Let "y" be the vertical distance moved by the pile. Applying equation of motion,

Motion of pile



The pile is decelerated due to resistance offered by the material.

$$0 = v_2^2 - 2X\frac{(F - 3mg)}{3m}Xy$$

$$\Rightarrow y = \frac{v_2^2 X 3m}{2(F - 3mg)}$$

$$\Rightarrow y = \frac{(1+e)^2 X 6mgh}{2(F - 3mg)X16} = \frac{(1+e)^2 X 3mgh}{16(F - 3mg)}$$

Angular velocity
Angular quantities are not limited to rotation about fixed axis only.

Angular quantities like angular displacement, velocity, acceleration and torque etc. have been discussed in earlier modules. We were, however, restricted in interpreting and applying these quantities to circular motion or pure rotational motion. Essentially, these physical quantities have been visualized in reference to an axis of rotation and a circular path.

In this module, we shall expand the meaning and application of angular quantities in very general terms, capable of representing pure as well as impure rotation and translation. We shall find, in this module, that pure translation and rotation are, as a matter of fact, special cases.

General interpretation of angular quantities

Here, we shall define and interpret angular quantities very generally with respect to a "point" in the reference system - rather than an axis. This change in reference of measurement allows us to extend application of angular quantities beyond the context of rotational motion. We can actually associate all angular quantities even with a straight line motion i.e. pure translational motion. For example, we can calculate torque on a particle, which is moving along a straight line. Similarly, we can determine angular displacement and velocity for a projectile motion, which we have studied strictly from the point of view of translation. We shall work out appropriate examples to illustrate extension of angular concepts to these motions.

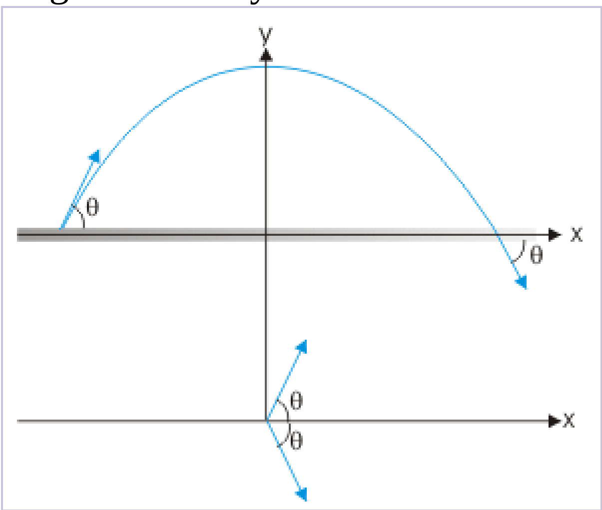
We must understand here that the broadening the concept of angular quantities is not without purpose. We shall find out in the subsequent modules that the de-linking of angular concepts like torque and angular momentum from an axis, lets us derive very powerful law known as conservation of angular momentum.

The example given below calculates average angular velocity of a projectile to highlight the generality of angular quantity.

Example

Problem 1 : A particle is projected with velocity "v" at an angle of "θ" with the horizontal. Find the average angular speed of the particle between point of projection and point of impact.

Solution : The average angular speed is given by :
Angular velocity



Average angular speed during the flight of a projectile.

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t}$$

From the figure, magnitude of the total angular displacement is :

$$\Delta\theta = 2\theta$$

On the other hand, time of flight is given by :

$$\Delta t = \frac{2v\sin\theta}{g}$$

Putting these values in the expression of angular velocity, we have :

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t} = \frac{2\theta g}{2v\sin\theta}$$
$$\omega_{\text{avg}} = \frac{\theta g}{v\sin\theta}\ \text{rad}/s$$

From this example, we see that we can indeed associate angular quantity like angular speed with motion like that of projectile, which is not strictly rotational.

Angular velocity

As a particle moves, the line joining a fixed point and particle, moves through angular displacement. Important thing to note here is that the particle may not follow a circular path - it can describe any curve even a straight line. We, then, define average angular velocity as :

Average angular velocity
: Average angular velocity of a particle about a point is equal to the ratio of change in angular displacement about that point and time.

Equation:

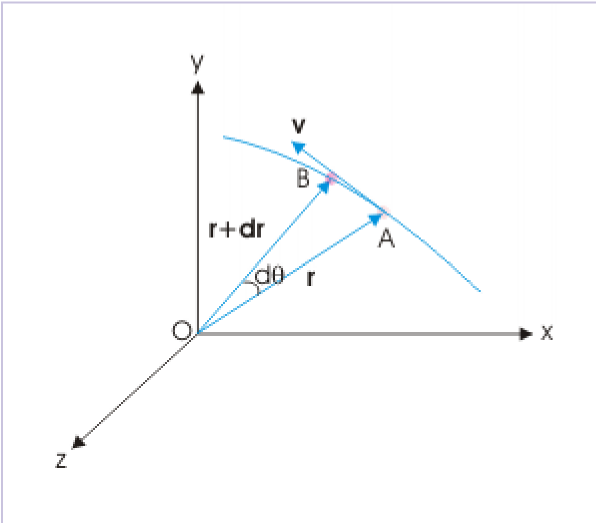
$$\boldsymbol{\omega}_{\text{avg}} = \frac{\boldsymbol{\Delta\theta}}{\boldsymbol{\Delta t}}$$

Instantaneous angular velocity is obtained by taking the limit when time interval tends to become zero. In other words, the instantaneous angular velocity (simply referred as angular velocity) is equal to the first differential of angular displacement with respect to time :

Equation:

$$\Rightarrow \boldsymbol{\omega} = \frac{\mathbf{d\theta}}{\mathbf{dt}}$$

Angular velocity



The angular displacement in a small time interval as measured with respect to a point.

It is convenient to consider the point as the origin of the coordinate system. In the figure above, origin serves as the point about which angular displacement and velocity are defined. Now since angle is measured with respect to origin, the measurement of angular velocity, in turn, will depend on the coordinate system chosen for the measurement. Further note that the linear distance of the particle from the origin is not constant like in circular motion.

Angular and linear velocity

In the case of circular motion about an axis, we have seen that linear velocity of a particle is related to angular velocity by the vector relation given by :

Equation:

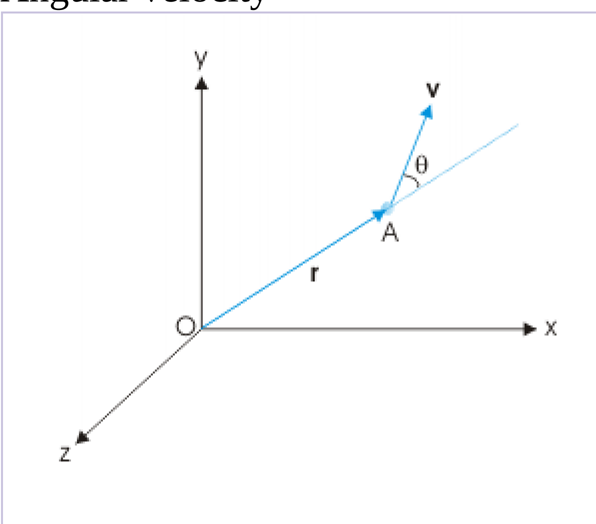
$$\mathbf{v} = \boldsymbol{\omega} \text{ x } \mathbf{r}$$

In the case of rotation about a fixed axis, the plane of velocity and position vector is same. This is not necessary for motion of a particle about any arbitrary point. The position vector and velocity can be oriented in any direction. It means that plane of motion can be any plane in three dimensional coordinate system and is not constrained in any manner.

Interpretation of the vector equation as given above is difficult for the motion of a particle along a path, which is not circular. We observe that angular velocity is an operand of the cross product. What we mean to say that it is not the vector, which is expressed in terms of other vectors. By looking at the vector relation, we can say that linear velocity is perpendicular to the plane formed by angular velocity and position vector. From the point of view of angular velocity, however, we can only say that it is perpendicular to linear velocity, but it can have any orientation with respect to position vector. This presents difficulty in interpreting this relation for angular velocity about a point.

Here, we consider a situation as shown in the figure. We see that velocity of the particle and the position vector form an angle, "θ" between them. For convenience, we have considered the point about which angular velocity is defined as the origin of coordinate system.

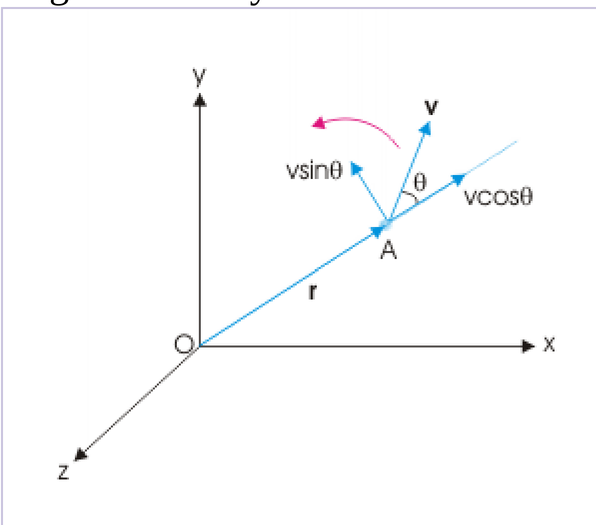
Angular velocity



The angular displacement in a small time interval as measured with respect to a point.

To analyze the situation, let us consider components of velocity in the radial and tangential directions.

Angular velocity



The angular displacement in a small time interval as measured with respect to a point.

The radial component of velocity is not a component for rotation. Its direction is along the line passing through the origin. The rotation is accounted by the component of velocity perpendicular to position vector. Now, tangential velocity is perpendicular to position vector. Thus, we can use the fact that instantaneous angular velocity is equal to ratio of tangential velocity and linear distance from the origin (as defined for circular motion about an axis). Hence, the magnitude of angular velocity is given as :

Equation:

$$\Rightarrow \omega = \frac{v \sin \theta}{r}$$

Equation:

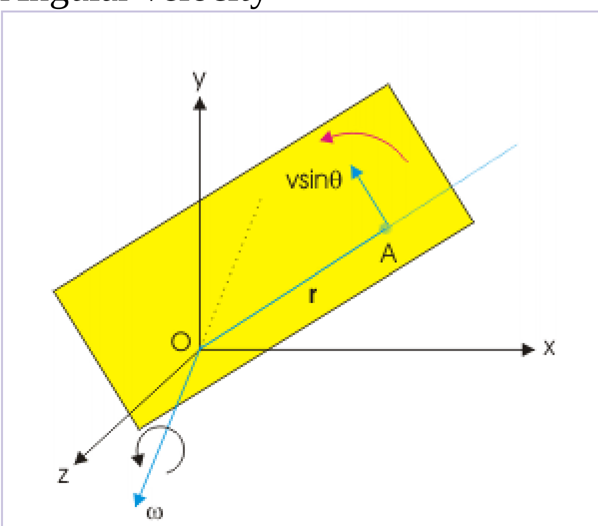
$$\Rightarrow \omega = \frac{v_{\perp}}{r}$$

We can use the expression as defined above to calculate the magnitude of angular velocity. The evaluation of the first expression will require us to determine angle between two vectors. The second expression, which uses tangential velocity, gives easier option, when this component of velocity is known.

Direction of angular velocity

As the particle moves along a curved path, position of the particle changes with respect to point about which angular velocity is measured. In order to determine the direction of angular velocity, we need to know the plane of rotation at a particular position. We define plane of rotation as the plane, which contains the point (origin of the coordinate system in our case) and the tangential velocity. The direction of angular velocity is perpendicular to this plane.

Angular velocity



The angular displacement in a small time interval as measured with respect to a point.

By convention, we consider anticlockwise rotation as positive. Looking at the figure, we can see that this direction is same that of the direction of vector product of "position vector" and "velocity".

Following the convention, we can write vector equation for angular velocity as :

Equation:

$$\Rightarrow \boldsymbol{\omega} = \frac{\mathbf{rxv}}{r^2}$$

It can be easily visualized that plane of rotation can change as particle moves in three dimensional space. Accordingly, direction of angular velocity can also change.

Example

Problem : The velocity of a particle confined to xy - plane is (8**i** – 6**j**) m/s at an instant, when its position is (2 m, 4m). Find the angular velocity of the particle about the origin at that instant.

Solution : Angular velocity is related to linear velocity by the following relation,

$$\mathbf{v} = \boldsymbol{\omega} \mathbf{\ x\ r}$$

From question, we can see that motion (including rotation) takes place in xy-plane. It, then, follows that angular velocity is perpendicular to xy-plane i.e. it is directed along z-axis. Let angular velocity be represented as :

$$\boldsymbol{\omega} = a\mathbf{k}$$

Putting the values, we have :

$$8\mathbf{i} - 6\mathbf{j} = a \ \mathbf{kx} \ (2\mathbf{i} + 4\mathbf{j})$$

$$\Rightarrow 8\mathbf{i} - 6\mathbf{j} = 2a\ \mathbf{j}\ -4a\mathbf{i})$$

Comparing the coefficients of unit vectors on either side of the equation, we have :

$$2a = -6$$
$$\Rightarrow a = -3$$

and

$$-4a = 8$$
$$\Rightarrow a = -2$$

We see here that we do not get unique value of angular velocity. As a matter of fact, we have included this example to highlight this aspect of the vector relation used here. It was pointed out that angular velocity is one of the operands of the vector product. From the relation, we can only conclude that angular velocity is perpendicular to linear velocity. We can see that this condition is fulfilled for any direction of velocity in the plane of motion. As such, we do not get the unique value of angular velocity as required.

Let us now apply the other vector relation that we developed to define angular velocity :

$$\Rightarrow \boldsymbol{\omega} = \frac{\mathbf{rxv}}{r^2}$$

Putting values, we have :

$$\Rightarrow \boldsymbol{\omega} = \frac{(2\mathbf{i}+4\mathbf{j})\mathbf{x}(8\mathbf{i}-6\mathbf{j})}{\sqrt{(2^2+4^2)}^2}$$

$$\Rightarrow \boldsymbol{\omega} = \frac{(-12\mathbf{k}-32\mathbf{k})}{20}$$

$$\Rightarrow \boldsymbol{\omega} = -1.71\ \mathbf{k}\ \text{rad}\,/s$$

Instantaneous axis of rotation

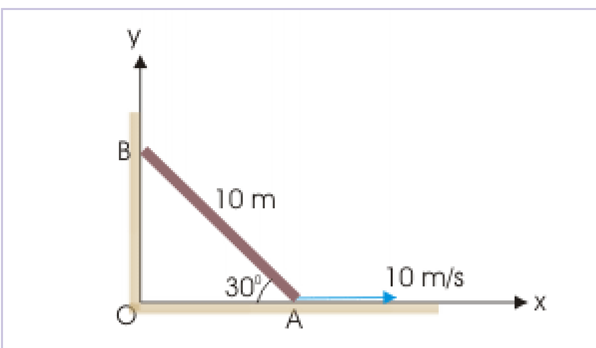
The notion of rotation with the motion of a particle is visualized in terms of moving axis. We have discussed that the direction of angular velocity is perpendicular to the plane formed by tangential velocity and the point about which angular velocity is measured. We can infer this direction to be the axial direction at this point for the angular velocity at that instant.

In case, these instantaneous axes are parallel to each other, then it is possible to assign unique angular velocity of the particle about any one of these instantaneous axes. We shall work out an example here to illustrate determining angular velocity about an instantaneous axis.

Example 3

Problem 3 : A rod of length 10 m lying against a vertical wall starts moving. At an instant, the rod makes an angle of 30° as shown in the figure. If the velocity of end “A” at that instant, is 10 m/s, then find the angular velocity of the rod about “A”.

Motion of a rod



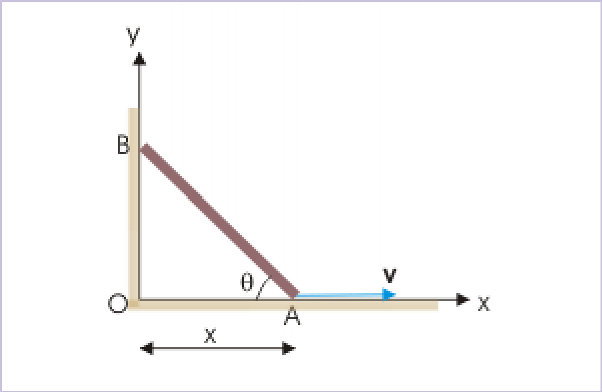
The rod is constrained to move between a vertical wall and a horizontal surface.

Solution : In this case, the point about which angular velocity is to be determined is itself moving towards right with a velocity of 10 m/s. However, the instantaneous axis of rotation is parallel to each other. As

such, it is possible to assign unique angular velocity for the motion under consideration.

We shall examine the situation with the geometric relation of the length of rod with respect to the position of its end “A”. This is a logical approach as the relation for the position of end “A” shall let us determine both linear and angular velocity.

Motion of a rod



The rod is constrained to move between a vertical wall and a horizontal surface.

$$x = \text{AB} \cos \theta$$

Note here that length of rod is constant. An inspection of the equation reveals that if we differentiate the equation with respect to time, then we shall be able to relate linear velocity with angular velocity.

$$\frac{đx}{đt} = \text{AB} \frac{đ}{đt} \left(\cos \theta \right) = - \text{AB} \sin \theta \text{ x } \frac{đ\theta}{đt}$$

$$\Rightarrow v = - \text{AB}\, \omega \sin \theta$$

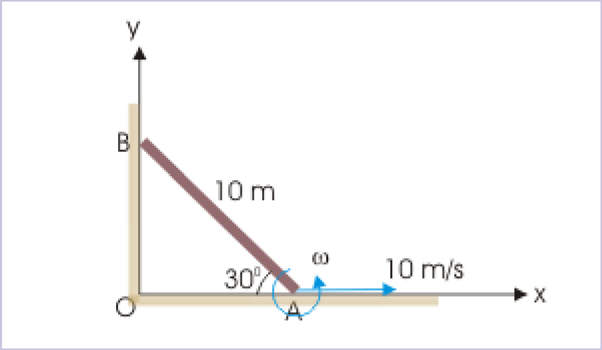
$$\Rightarrow \omega = - \frac{v}{\text{AB} \sin \theta}$$

Putting values,

$$\Rightarrow \omega = -\frac{10}{10 \sin 30^\circ} = -2 \text{ rad}/s$$

Negative sign here needs some clarification. We can visualize that rod is rotating about instantaneous axis in anticlockwise direction. As such, the sign of angular velocity should have been positive - not negative. But, note that we are measuring angle from the negative x-direction as against positive x-direction. Therefore, we conclude that the negative sign in this instant case represents rotation of rod about the instantaneous axis in anticlockwise direction.

Motion of a rod



The rod is constrained to move between a vertical wall and a horizontal surface.

Summary

1: Angular quantities are general quantities, which can be defined and interpreted for any motion types – pure/impure translation or rotation.

2: Angular velocity is defined as the time rate of change of angular displacement with respect to a point :

$$\omega = \frac{d\theta}{dt}$$

3: If the point, about which angular velocity is determined, is the origin of the reference system, then the position of the particle is represented by

position vector.

4: The measurement and physical interpretation of angular velocity are different than that in the case of circular (rotational) motion about an axis.

5: The relation between linear and angular velocity has the following form,

$$\mathbf{v} = \boldsymbol{\omega} \ \mathbf{x} \ \mathbf{r}$$

where “**r**” determines the position of the particle with respect to the point. The magnitude of angular velocity can be determined, using any of the following two relations :

$$\omega = \frac{v \sin \theta}{r}$$

and

$$\omega = \frac{v_{\perp}}{r}$$

where " $v_{\perp}$ " denotes tangential component of the velocity vector, which is perpendicular to the position vector.

7: Direction of angular velocity is perpendicular to the plane of motion. The plane of motion, however, may not be fixed like in the case of rotation. We define plane of motion, which contains origin (or point) and tangential component of velocity.

Torque
Torque is the cause of rotation.

Torque about a point is a concept that denotes the tendency of force to turn or rotate an object in motion. This tendency is measured in general about a point. It is also termed as "moment of force". The torque in angular motion corresponds to force in translation. It is the "cause" whose effect is either angular acceleration or angular deceleration of a particle in general motion. Quantitatively, it is defined as a vector given by :

Equation:

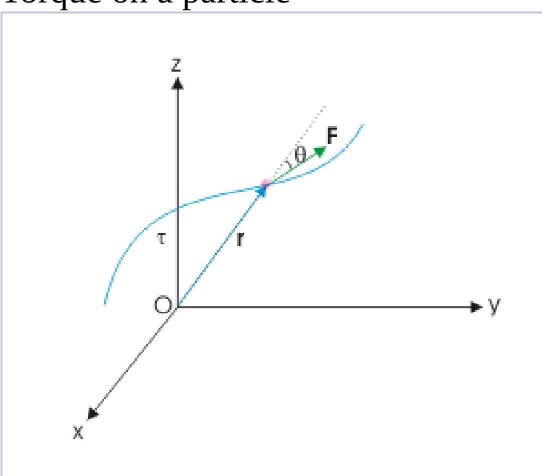
$$\boldsymbol{\tau} = \mathbf{r} \ \mathbf{x} \ \mathbf{F}$$

Rotation is a special case of angular motion. In the case of rotation, torque is defined with respect to an axis such that vector "**r**" is constrained to be perpendicular to the axis of rotation. In other words, the plane of motion is perpendicular to the axis of rotation. Clearly, the torque in rotation corresponds to force in translation.

Torque about a point

An external force on a particle constitutes a torque with respect to a point. Only condition is that the point, about which torque is defined or measured, does not lie on the line of force, in which case torque is zero.

Torque on a particle



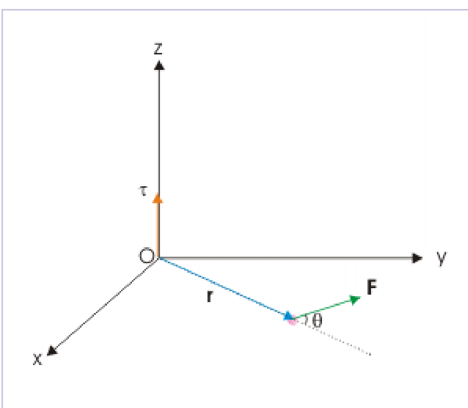
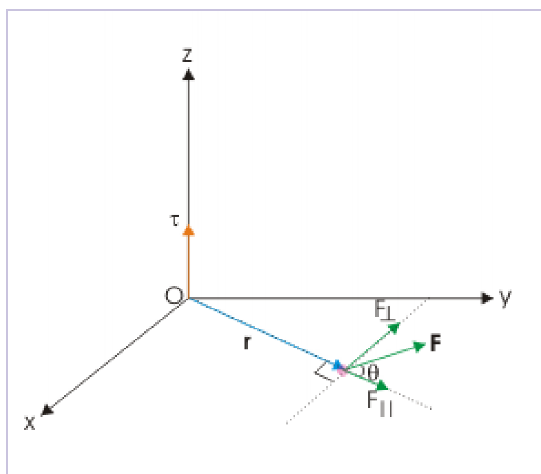
The particle moves in three dimensional coordinate space.

We can make note of the fact that it is convenient to construct reference system in such a manner so that the point (about which torque is measured) coincides with the origin. In that case, the vector (**r**) denoting the position of the particle with respect to point, becomes the position vector, which is measured from the origin of reference system.

Magnitude of torque

With the reference of origin for measuring torque, we can find the magnitude of torque, using any of the following relations given below. Here, we have purposely considered force in xy - plane for illustration and visualization purpose as it provides clear directional relationship of torque " **τ** " with the operand vectors " **r** " and " **F** ".

Torque on a particle

Torque in terms of angle enclosed.	Torque in terms of force, perpendicular to position vector.
	

1: Torque in terms of angle enclosed

Equation:

$$\tau = rF\sin\theta$$

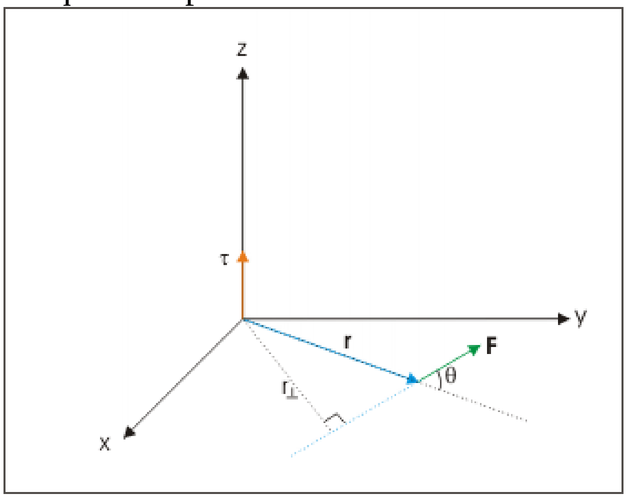
2: Torque in terms of force perpendicular to position vector
Equation:

$$\tau = r(F\sin\theta) = rF_{\perp}$$

3: Torque in terms of moment arm
Equation:

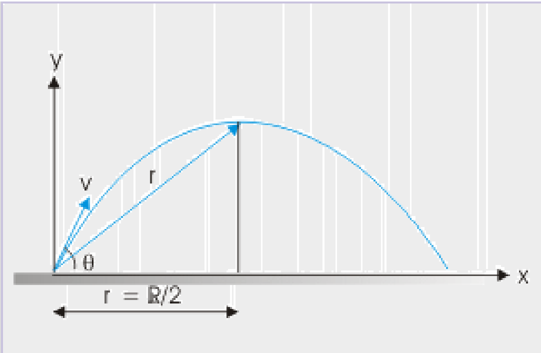
$$\tau = (r\sin\theta)F = r_{\perp}F$$

Torque on a particle



Torque in terms of moment arm

Example:
Problem : A projectile of mass "m" is projected with a speed "v" at an angle "θ" with the horizontal. Calculate the torque on the particle at the maximum height in relation to the point of projection.
Solution : The magnitude of the torque is given by :
Projectile motion



Torque on a projectile.

$$\tau = r_{\perp}F = \text{product of moment arm and magnitude of force}$$

where $r\perp$ is moment arm. In this case, the moment arm is equal to half of the horizontal range of the flight,

$$r_{\perp} = \frac{R}{2} = \frac{v^2 \sin 2\theta}{2g}$$

Now, the force on the projectile of mass "m" is due to the force of gravity :

$$F = mg$$

Putting these expressions of moment arm and force in the expression of torque, we have :

$$\Rightarrow \tau = \frac{mgv^2 \sin 2\theta}{2}$$

Application of right hand rule indicates that torque is clockwise and is directed in to the page.

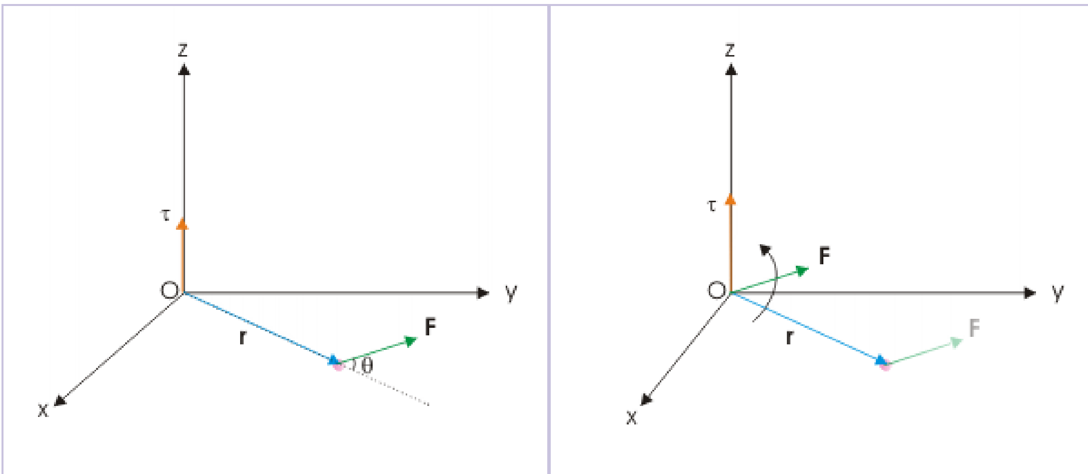
Direction of torque

The determination of torque's direction is relatively easier than that of angular velocity. The reason is simple. The torque itself is equal to vector product of two vectors, unlike angular velocity which is one of the two operands of the vector product. Clearly, if we know the directions of two operands here, the direction of torque can easily be interpreted.

By the definition of vector product, the torque is perpendicular to the plane formed by the position vector and force. Besides, it is also perpendicular to each of the two vectors individually. However, the vector relation by itself does not tell which side of the plane formed by operands is the direction of torque. In order to decide the orientation of the torque, we employ right hand vector product rule.

For this, we need to shift one of the operand vectors such that their tails meet at a point. It is convenient to shift the force vector, because application of right hand vector multiplication rule at the point (origin of the coordinate system) gives us the sense of angular direction of the torque.
Vector product

Right hand rule. Application of Right hand rule.



In the figure, we shift the second vector (**F**) so that tails of two operand vectors meet at the point (origin) about which torque is calculated. The two operand vectors define a plane ("xy" - plane in the figure). The torque (**τ**) is, then, acting along a line perpendicular to this plane and passing through the meeting point (origin).

The orientation of the torque vector in either of two direction is determined by applying right hand rule. For this, we sweep the closed fingers of the right hand from position vector (first vector) to force vector (second vector). The outstretched thumb, then, indicates the orientation of torque. In the case shown, the direction of torque is positive z-direction.

While interpreting the vector product, we must be careful about the sequence of operand. The vector product " **F x r** " is negative or opposite in sign to that of " **r x F** ".

Illustration

The problem involving torque is about evaluating the vector product. We have the options of using any of the three methods described above for calculating the magnitude of torque. In this section, we shall work with two important aspects of calculating torque :

- More than one force operating simultaneously on the particle
- Relative orientation of the plane of operand vectors and planes of coordinate system

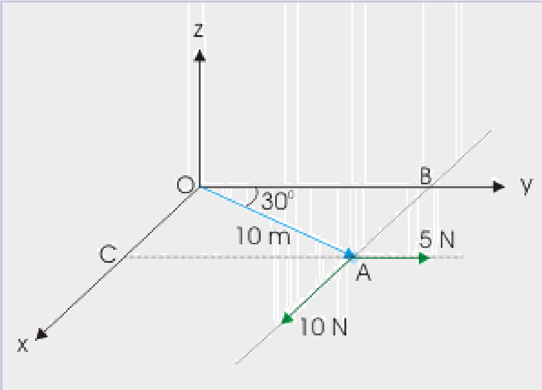
More than one force operating simultaneously on the particle

Here, we have two choices. Either, we evaluate torques for each force and then find the resultant torque or we evaluate the resultant (net) force first and then find the torque. The choice depends on the situation in hand.

In particular, if we find that the resulting torques are along the same direction (say one of the coordinate direction), then it is always better to calculate torque individually. This is because torques along a direction can be dealt as scalars with appropriate sign. The resultant torque is simply the algebraic sum. Example below highlights the this aspect.

The bottom line in making choice is that we should keep non-linear vector summation - whether that of force or torque - to a minimum. Otherwise, analytical technique for vector addition will be need to be employed.

Example:
Problem : At an instant, a particle in xy plane is acted by two forces 5 N and 10 N in xy plane as shown in the figure. Find the net torque on the particle at that instant.
Torque on a particle



Two forces are acting simultaneously on the particle.

Solution : An inspection of the figure reveals that forces are parallel to axes and perpendicular linear distances of the lines of forces can be known. In this situation, calculation of the magnitude of torque suits to the form of expression that uses moment arm. Here, moment arms for two forces are :
For 5 N force, the moment arm "AB" is :

$$\text{AB} = \text{OA} \sin 30^0 = 10 \text{ x } 0.5 = 5\, m$$

For 10 N force, the moment arm "AC" is :

$$\text{AC} = \text{OA} \cos 30^0 = 10 \text{ x } 0.866 = 8.66\, m$$

The torques due to 5 N is :

$$\Rightarrow \tau_1 = 5 \text{ x } 5 = 25 \;\; Nm \;\; \text{(positive being anti-clockwise)}$$

The torques due to 10 N is :

$$\Rightarrow \tau_2 = -8.66 \text{ x } 10 = -86.6 \;\; Nm \;\; \text{(negative being clockwise)}$$

Net torque is :

$$\Rightarrow \tau_{\text{net}} = \tau_1 + \tau_2 = 25 - 86.6 = -61.6 \;\; Nm \;\; \text{(negative being clockwise)}$$

Note: We can also first calculate the resultant force and then apply the definition of torque to obtain net torque. But, then this would involve non-linear vector

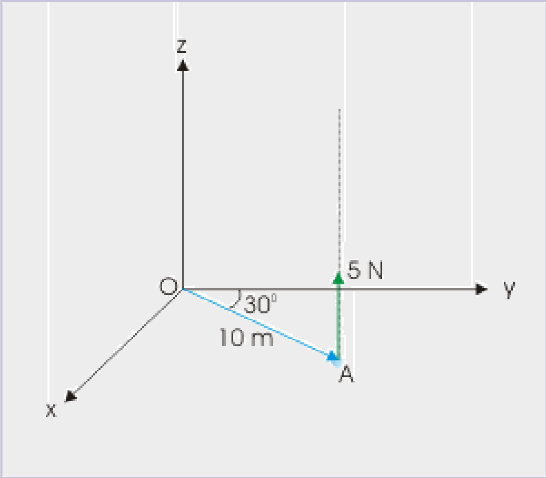
addition.

Relative orientation of the plane of operand vectors

There are two possibilities here. The plane of operands are same as that of the plane of coordinate system. In simple word, the plane of velocity and position vector is same as one of three planes formed by coordinate system. If this is so, then problem analysis is greatly simplified. Clearly, the torque will be along the remaining third axis. We have only to find the orientation (in the positive or negative direction of the axis) with the help of right hand rule. It is evident that we should always strive to orient our coordinate system, if possible, so that the plane formed by position vector and force align with one of the coordinate planes.

In case, the two planes are different, then we need to be careful (i) in first identifying the plane of operands (ii) finding relation of the operands plane with coordinate plane and (iii) applying right hand rule to determine orientation (which side of the plane). Importantly, we can not treat torque as scalar with appropriate sign, but have to find direction of torque with respect to one of the coordinates.

Example:
Problem : A particle in xy plane is acted by a force 5 N in z-direction as shown in the figure. Find the net torque on the particle.
Torque on a particle



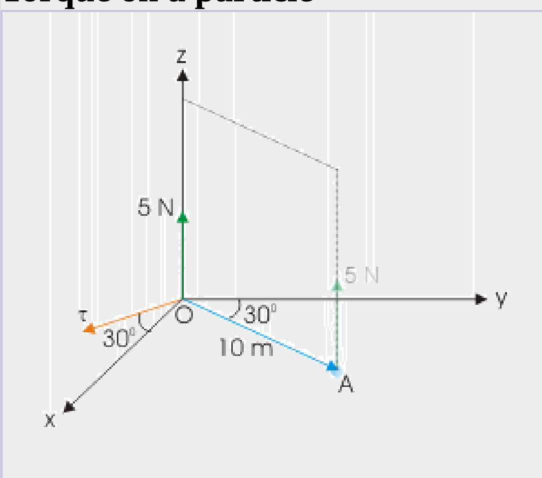
Two force is acting parallel to z-axis.

Solution : In this case, moment arm is equal to the magnitude of position vector. Hence, torque is :

$$\tau = 10 \text{ x } 5 = 50 \ Nm$$

However, finding its direction is not as straight forward. The situation here differs to earlier examples in one important respect. The plane containing position vector and force vector is a plane defined by zOA. Also, this plane is perpendicular to xy - plane. Now, torque is passing through the origin and is perpendicular to the plane zoA. We know a theorem of geometry that angles between two lines and their perpendiculars are same. Applying the same and right hand rule, we conclude that the torque is shifted by 30 degree from the y-axis and lies in xy-plane as shown in the figure here.

Torque on a particle



The torque makes an angle of 30 with y-axis.

Torque in component form

Torque, being a vector, can be evaluated in component form with the help of unit vectors along the coordinate axes. The various expressions involved in the vector algebraic analysis are as given here :

$$\boldsymbol{\tau} = \mathbf{r} \text{ x } \mathbf{F}$$

$$\boldsymbol{\tau} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \text{ x } (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})$$

`τ = | i j k | | x y z | | Fx Fy Fz|`

Equation:

$$\boldsymbol{\tau} = (yF_z - zF_y)\mathbf{i} + (zF_x - xF_z)\mathbf{j} + (xF_y - yF_x)\mathbf{k}$$

Example:

Problem : A force **F** = 2**i** + **j** – 2**k** Newton acts on a particle at **i** + 2**j** –**k** meters at a given time. Find the torque about the origin of coordinate system.

Solution : Hence, torque is :

`τ = | i j k | | 1 2 -1 | | 2 1 -2 |`

$$\Rightarrow \boldsymbol{\tau} = [(2 \text{ x } -2) - (1 \text{ x } -1)]\mathbf{i} + [(-1 \text{ x } 2) - (1 \text{ x } -2)]\mathbf{j} + [(1 \text{ x } 1) - (2 \text{ x } 2)]\mathbf{k}$$

$$\Rightarrow \boldsymbol{\tau} = -3\mathbf{i} - 3\mathbf{k} \;\; Nm$$

Summary

1: Torque in general angular motion is defined with respect to a point as against axis in rotation. The expression of torque in two cases, however, is same :

$$\boldsymbol{\tau} = \mathbf{r} \text{ x } \mathbf{F}$$

2: When the point (about which torque is defined) coincides with the origin of coordinate system, the vector "**r**" appearing in the expression of torque is position vector.

3: Magnitude of torque is given by either of the following relations,

(i) Torque in terms of angle enclosed

$$\tau = rF\sin\theta$$

(ii) Torque in terms of force perpendicular to position vector

$$\tau = rF_{\perp}$$

(iii) Torque in terms of moment arm

$$\tau = r_{\perp}F$$

4: Direction of torque

The torque is perpendicular to the plane formed by velocity vector and force and also individually to either of them. We get to know the orientation of the torque vector by applying right hand rule.

5: When more than one force acts, then we should determine the resultant of forces or resultant of torques, depending on which of the approach will minimize or avoid non-linear vector summation to obtain the result.

6: When the plane of operands is same as one of the coordinate plane, then torque acts along the remaining axis.

7: Torque is expressed in component form as :

`τ = | i j k | | x y z | | Fx Fy Fz|`

Torque about a point (check your understanding)
Objective questions, contained in this module with hidden solutions, help improve understanding of the topics covered under the module "Torque about a point".

The questions have been selected to enhance understanding of the topics covered in the module titled " Torque ". All questions are multiple choice questions with one or more correct answers.

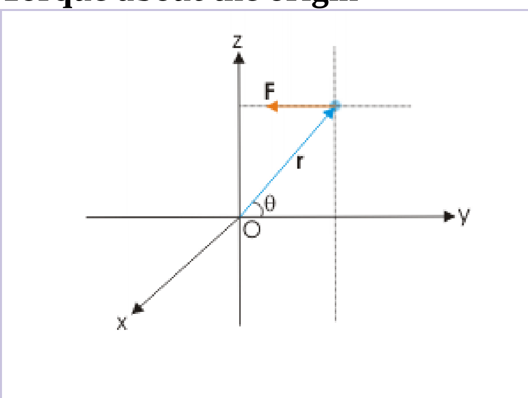
Understanding level (Torque about a point)

Exercise:

Problem:

A force, “F”, acts on a particle at a linear distance “r” from the origin of a coordinate system. If force acts in coordinate plane “yz”, then the toque on the particle is :

Torque about the origin



The force acts in “yz” plane.

(a) makes an angle “θ” with “–z” axis
(b) makes an angle “θ” with x axis
(c) acts along x axis
(d) acts along “-x” axis

Solution:

In this case, vectors “**r**” and “**F**” are in yz – plane. As torque is perpendicular to this plane, it is directed either along “x” or “-x” axis. Applying right hand rule by shifting force vector at the origin, we see that the torque acts in x-direction.

Hence, option (c) is correct.

Exercise:

Problem:

The torque on a particle at a position on x-axis (other than origin) is zero. If the force applied is not zero, then force is acting :

$$\begin{array}{l}\text{(a) either in “y” or “z” direction}\\ \text{(b) in x direction}\\ \text{(c) in} \pm \text{x direction}\\ \text{(d) - z direction}\end{array}$$

Solution:

The torque on the particle is given as :

$$\boldsymbol{\tau} = \mathbf{r} \text{ } \mathbf{x} \text{ } \mathbf{F}$$

Its magnitude is given by :

$$\tau = rF \sin\theta$$

The torque can be zero for following conditions (i) force is zero (ii) particle is at the origin and (iii) sine of enclosed angle is zero. Since "r" and "F" are non-zero, it follows that :

$$\theta = 0^\circ \text{ or } 180^\circ$$

Thus, the force on the particle is acting ± x direction.

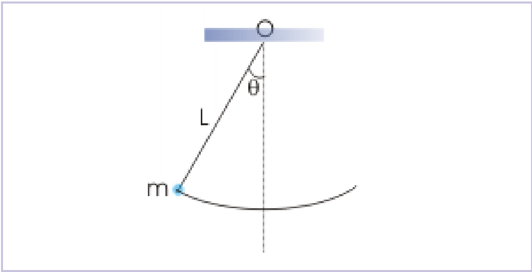
Hence, option (c) is correct.

Exercise:

Problem:

The bob of a pendulum of length "L" is raised to one side and released to oscillate about the mid point. The torque about the point of suspension, at an instant when the bob makes an angle “θ” with respect to vertical, is :

Pendulum



The pendulum bob makes an angle “θ” with the vertical.

$$(a)\ mgL\sin\theta\ \ (b)\ mgL\cos\theta\ \ (c)\ mgL\tan\theta\ \ (d)\ mgL\cot\theta$$

Solution:

There are two forces that operate on the pendulum bob (see the left figure):

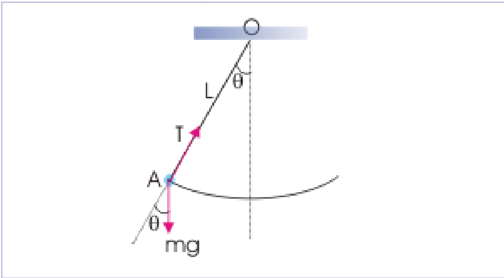
- Weight of the bob (mg)
- Tension in the string (T)

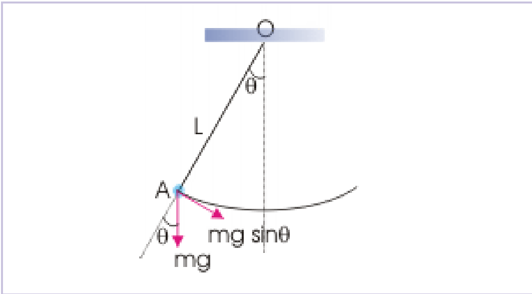
The line of action of tension in the string of the pendulum is along the string itself, which passes through center of mass. It means that torque is applied by only the weight of the bob. Since the linear distance of the weight acting on the bob from the point of suspension is given, it would be easier to find magnitude of torque as product of linear distance and component of force perpendicular to it.

Pendulum

Two forces act on the pendulum bob.

The component of weight perpendicular to string





The component of weight perpendicular to the string is (see the right figure) :

$$F_{\perp} = mg \sin\theta$$

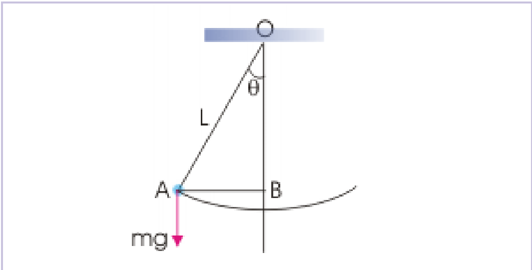
The torque due to the weight is :

$$\tau = LF_{\perp} = mgL \sin\theta$$

Hence, option (a) is correct.

Note : The torque about the suspension point can also be determined, using moment arm. In this case moment arm i.e. perpendicular distance between suspension point and line of action of force is "AB" :

Pendulum



The pendulum bob makes an angle "θ" with the vertical.

$$\tau = L_{\perp}F = \text{AB x } F = L \sin\theta \text{ x } mg = mgL \sin\theta$$

Exercise:

Problem:

A force **F** = (2**i** + 2**j** – 3**k**) Newton acts on a particle, placed at a point given by **r** = (**i** + **j** – **k**) meters. If the particle is constrained to rotate about x-axis along a circular path, then the magnitude of torque about the axis is :

$$\text{(a) 1 (b) 2 (c) 3 (d) 4}$$

Solution:

The torque about the origin of coordinate system is given by :

$$\boldsymbol{\tau} = \mathbf{r\ x\ F}$$

`τ = | i j k | | 1 1 -1 | | 2 2 -3 |`

$$\Rightarrow \boldsymbol{\tau} = [(1 \text{ x } -3) - (2 \text{ x } -1)]\mathbf{i} + [(-1 \text{ x } 2) - (1 \text{ x } -3)]\mathbf{j} + [(1 \text{ x } 2\} - (1 \text{ x } 2)]\mathbf{k}$$
$$\Rightarrow \boldsymbol{\tau} = (-3 + 2)\mathbf{i} + (-2 + 3)\mathbf{j}$$
$$\Rightarrow \boldsymbol{\tau} = -\mathbf{i} + \mathbf{j}$$

The particle, here, moves about the axis of rotation in x-direction. In this case, the particle is restrained to rotate or move about any other axis. Thus, torque in rotation about x-axis is equal to the x- component of torque about the origin. Now, the vector x-component of torque is :

$$\boldsymbol{\tau}_x = -\mathbf{i}$$

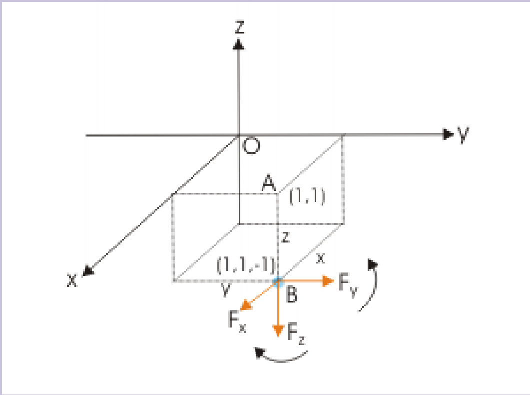
The magnitude of torque about x-axis for rotation is :

$$\tau_x = 1\ Nm$$

Hence, option (a) is correct.

Note : We can check the result, considering individual force components. We first identify A (1,1) in xy-plane, then we find the position of the particle, B(1,1,-1) by moving “-1” in negative z-direction as in the figure below.

Torque on the particle



The position of particle along
with components of force

The component in x-direction does not constitute a torque about x-axis as the included angle is zero.

The component of force in y-direction is Fy = 2 N at a perpendicular distance z = 1 m. For determining torque about the axis, we multiply perpendicular distance with force. We apply appropriate sign, depending on the sense of rotation about the axis. The torque about x-axis due to component of force in y - direction is anticlockwise and hence is positive :

$$\tau_{\text{xy}} = zF_y = 1 \text{ x } 2 = 2 \ Nm$$

The component of force in z-direction is Fz = 3 N at a perpendicular distance y = 1 m. The torque about x-axis due to the component of force in z - direction is clockwise and hence is negative :

$$\tau_{\text{xz}} = -yF_z = -1 \text{ x } 3 = -3 \ Nm$$

Since both torques are acting along same axis i.e. x-axis, we can obtain net torque about x-axis by arithmetic sum :

$$\Rightarrow \tau_x = \tau_{\text{xy}} + \tau_{\text{xz}} = 2 - 3 = -1 \ Nm$$

The magnitude of net torque about x-axis is :

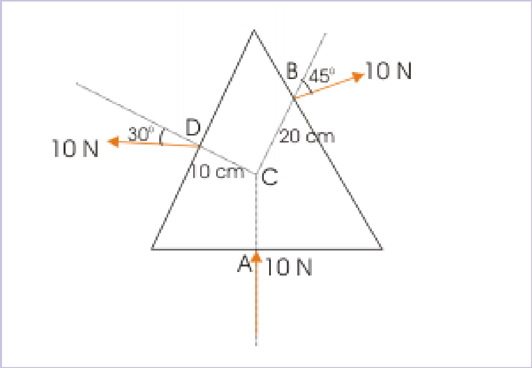
$$\Rightarrow \tau_x == 1 \ Nm$$

Exercise:

Problem:

In the figure, three forces of 10 N each act on a triangular plate as shown in the figure. If “C” be the center of mass of the plate, then torque, in N-m, about it is :

Torque about center of mass



Three forces are acting.

$$(a)\ 0.61\ \ (b)\ \ -1.33\ \ (c)\ \ -0.91\ \ (d)\ \ -1.11$$

Solution:

We note here that line of action of the force acting on the horizontal face passes through center of mass. Thus, it does not constitute a torque about the center of mass.

Further, we see that the linear distances of the point of action of the forces are given. Therefore, it would be easy to apply the formula for the magnitude of torque, which makes use of perpendicular force component.

Now, the perpendicular force component at “B” is :

$$F_{\mathrm{B}\perp} = 10 \sin 45^\circ = \frac{10}{\sqrt{2}}\ \ N$$

Similarly, the perpendicular force component at “D” is :

$$F_{\mathrm{D}\perp} = 10 \sin 30^\circ = 5\ \ N$$

The torque due to force at “B” about center of mass, “C”, is clockwise and hence is negative :

$$\tau_{\text{CB}} = -0.2 \text{ x } \frac{10}{\sqrt{2}} = -\sqrt{2}\ Nm$$

Torque due to force at “D” about center of mass, “C”, is anticlockwise and hence is positive :

$$\Rightarrow \tau_{\text{CD}} = 0.1 \text{ x } 5 = 0.5\ Nm$$

The torques are along the same direction i.e. perpendicular to the surface of the plate. Therefore, we can find the net torque as algebraic sum :

$$\Rightarrow \tau_C = -\sqrt{2} + 0.5 = -0.91\ Nm$$

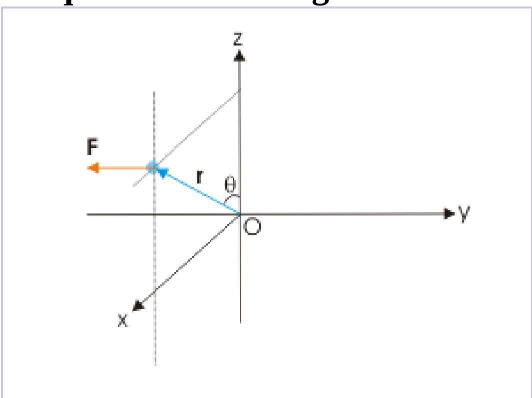
Hence, option (c) is correct.

Exercise:

Problem:

A force, “F”, acts on a particle lying in “xz” plane at a linear distance “r” from the origin of a coordinate system. If the direction of force "F" is parallel to y-axis, then the toque on the particle :

Torque about the origin



The particle lies in “xz” plane.

(a) makes an angle “θ” with “–z” axis
(b) makes an angle “θ” with x - axis
(c) acts along x – axis
(d) acts along “-x” axis

Solution:

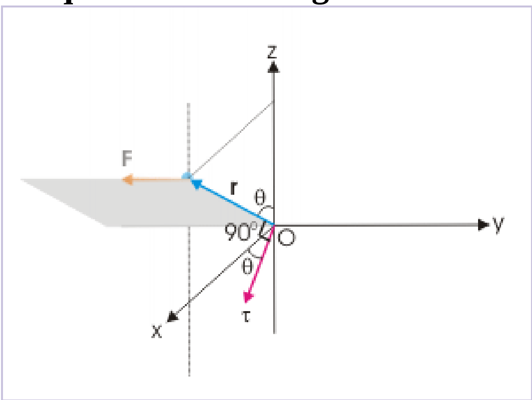
First thing that we note here is that the plane formed by operands is not same as any of the coordinate planes. Now, the torque on the particle is given as :

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

From the relation, we come to know that the vector product "**τ**" is perpendicular to the plane formed by the vectors "**r**" and "**F**".

Let us imagine force vector is moved to origin keeping the direction same. Applying Right hand cross product rule, we move from vector "**r**" from its position to that of "**F**" such that the curl of fingers is along the direction of movement. The direction of thumb downward tells us that torque is acting perpendicular to plane and below the surface i.e. making an acute angle with x-axis.

Torque about the origin



Direction of torque

We observe here that torque "τ" and position vector "r" are perpendicular to each other. On the other hand, "z" and "x" axes are perpendicular to each other. Since the angle between perpendiculars on two lines is same, we conclude that torque acts in xz plane, making an angle "θ" with the x-axis as shown in the figure above.

Hence, option (b) is correct.

Answers

`1. (c) 2. (c) 3. (a) 4. (a) 5. (c) 6. (b)`

Rotation
Each particle of a rigid body follows a circular path about a fixed axis in pure rotation.

The study of rotational motion is an important step towards the study of real time motion. Rotational motion is essentially a circular motion about a point or an axis (for three dimensional bodies). The rotation of a rigid body about an axis passing through the body itself is termed "spin motion" or simply "spin". On the other hand, rotation about an external axis is termed "orbital motion". We shall treat both these types of rotation in the same manner as treatment of either rotation types is same from the point of view of governing laws of motion.

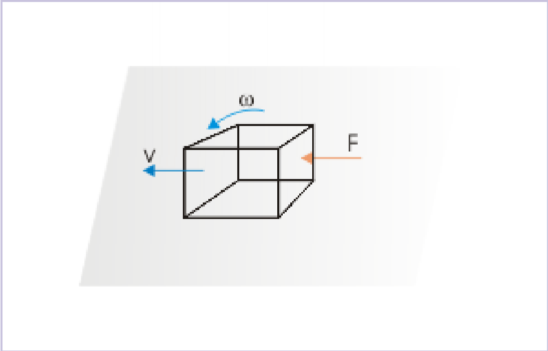
Rotation is one of two basic components of general motion of rigid body. The motion of a rigid body is either translation or rotation or combination of two. In translation, each of the particles constituting the rigid body has same linear velocity and acceleration. On the same footing, we say that each of the particles constituting the rigid body has same angular velocity and acceleration in rotation. A consequence of this analogy is that each particle constituting the rigid body moves in a circular motion such that their centers lie on a straight line called axis of rotation.

Pure translation refers to a straight line motion. Similarly, pure rotation refers to rotation about a fixed axis. Importantly, pure translation excludes rotation whereas pure rotation excludes translation.

Torque in rotation

A rigid body in rotation about a fixed axis should continue to rotate with the given angular velocity indefinitely unless obstructed externally. The angular velocity of rotation is changed by external cause in the same manner as in translational motion. In the case of translation motion, the external cause is "force". We have to investigate what is the equivalent "cause" in the case of rotational motion?

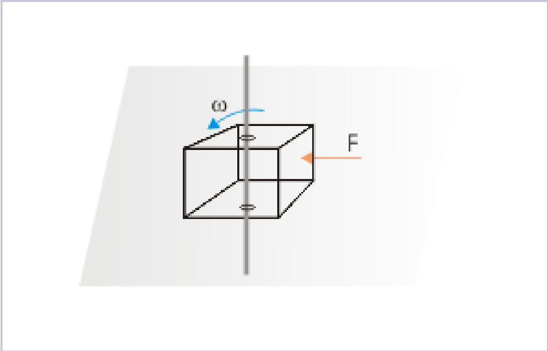
Let us consider a rigid block placed over a smooth horizontal surface as shown in the figure, which is subjected to a force across one of its face. What is expected? The center of mass will move with linear acceleration following Newton's second law. However, the force is not passing through the center of mass. As such, the face of the block will also have turning tendency in the direction shown.
The motion of block



The block has turning tendency.

Let us now imagine if the body is pierced through in the middle by a vertical bar with some clearance. The vertical bar will inhibit translation and the block will only rotate about the vertical bar. The question is what caused the block to turn around? Indeed it is the force that caused the angular motion. However, it is not only the force that determined the outcome (magnitude and direction of angular velocity and acceleration). There are other considerations as well.

Rotational motion



The vertical bar inhibits translation and the block only rotates about the vertical bar. .

We all have the experience of opening and closing a hinged door in our house. It takes lesser effort (force) to open the door, when force is applied farther from the hinge. We can further reduce the effort by applying force normal to the plane of the door. On the other hand, we would require greater effort (force), if we push or pull the door from a point closer to the hinge. If we apply force in the radial direction

(in the plane of door) towards the hinge, then the door does not rotate a bit - whatever be the magnitude of force. In the nutshell, rotation of the door depends on :

- Magnitude of force
- Point of application of force with respect to hinge (axis of rotation)
- Angle between force and perpendicular line from the axis of rotation

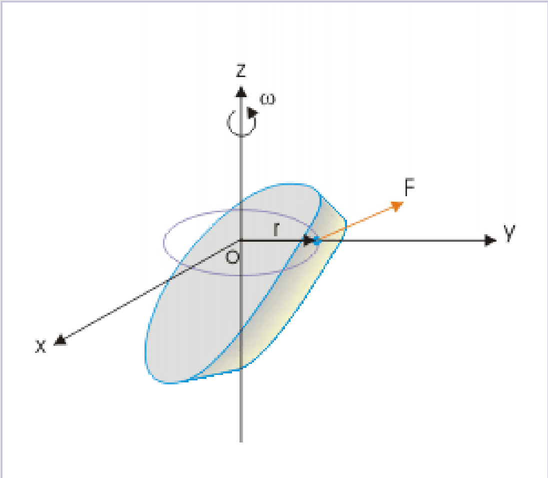
These factors, which cause rotation, are captured by a quantity known as torque, which is defined as :

Equation:

$$\boldsymbol{\tau} = \mathbf{r} \ \mathbf{x} \ \mathbf{F}$$

where "**r**" is the position vector. There is a slight difficulty in interpreting this vector equation as applicable to rotation. The position vector,"**r**", as we can recall, is measured from a point. Now, the question is what is that point in the case of rotation about a fixed axis? Here, we observe that there is a unique point of application of the force. By the nature of motion, this point rotates in a plane perpendicular to the axis of rotation. We can, therefore, uniquely define the position of the point of application of force by measuring it from the point on the axis in that plane of rotation.

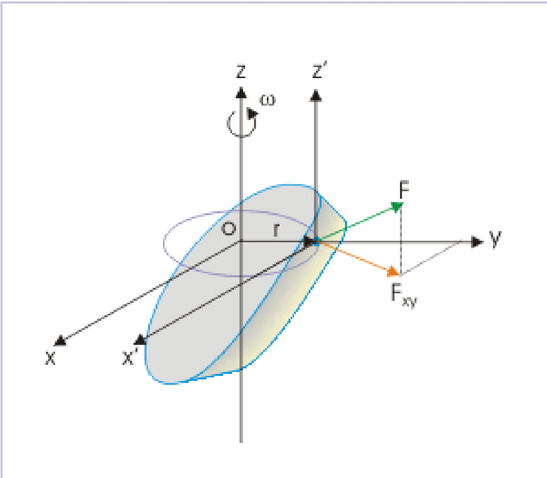
Rotational motion



Rotation about z-axis. .

The force may be directed in any possible direction. However, the body rotates about a fixed axis. It is not allowed to rotate or move about any other axes. What it means that we are limited only to the component of torque along the axis of rotation. It, then, means that we are only interested in components of force that lie in the plane of rotation, which is perpendicular to the axis of rotation. This we achieve by ensuring to only consider the components of force which are perpendicular to the axis of rotation i.e. in the plane of rotation. In the figure below, we have tried to capture this aspect. We consider only the component of force in xy-plane while evaluating the vector torque equation given above.

Rotational motion



Rotation about z-axis.

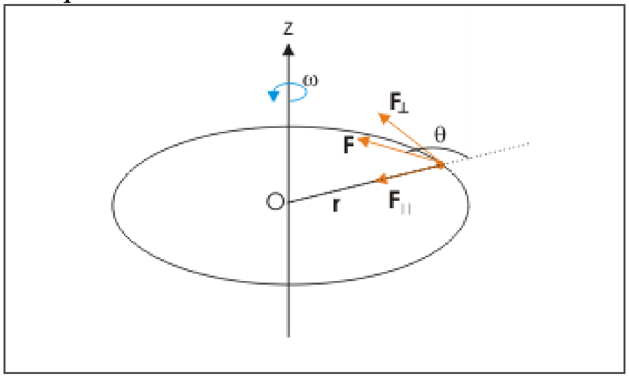
In the nutshell, we interpret the vector product with two specific conditions :

- The position vector (**r**) is measured from a point on the axis, which is in the plane of rotation.
- The force vector (**F**) is component of force in the plane of rotation.

The rotation of the rigid body about an axis passing through the body itself is composition of very large numbers of circular motions by particles composing it. Each of the particles undergo circular motion about the axis in a plane which is perpendicular to it.

Magnitude of torque

With the understandings as deliberated above, let us determine the magnitude of torque. Here, let us consider that a force is applied at a point which is at a perpendicular distance "r" from the axis and in the plane of rotation. As far as application of force is concerned, it is a force applied at a point in the plane of rotation. In order to keep the context simplified, we have not drawn the rigid body assuming that the point of application rotates in a circle whose plane is perpendicular to the axis of rotation. Now, the magnitude of torque is given as :
Torque



Torque depends on the enclosed angle.

Equation:

$$\tau = rF\sin\theta$$

where "r","F" and "θ" are quantities measured in the plane perpendicular to the axis of rotation.

Equivalently, we can determine the magnitude of torque as :
Equation:

$$\tau = r(F\sin\theta) = rF_{\perp}$$

The magnitude of torque is equal to the product of the magnitude of position vector as drawn from the center of the circle and component of force perpendicular to the position vector. This is an important interpretation as it highlights that it is the tangential or perpendicular component (perpendicular to radial direction) of force, which is capable to produce change in the rotation. The component of force in the

radial direction (F cosθ) does not affect rotation. We can also rearrange the expression of magnitude as :

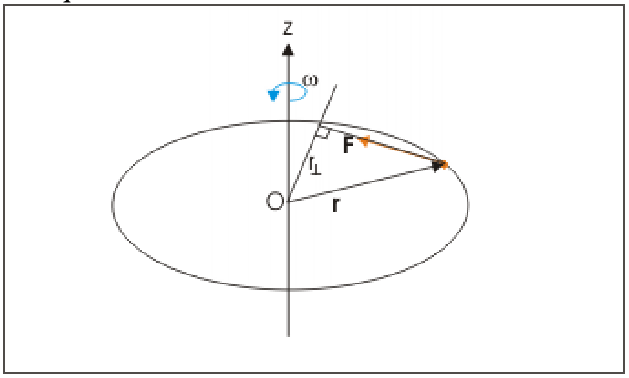
Also,

Equation:

$$\tau = (r \sin\theta)F = r_{\perp}F$$

The magnitude is equal to the product of the perpendicular distance and magnitude of the force. Perpendicular distance is obtained by drawing a perpendicular on the extended line of application of force as shown in the figure below. We may note here that this line is perpendicular to both the axis of rotation and force. This perpendicular distance is also known as "moment arm" of the force and is denoted as " $r_{\perp}$ ".

Torque



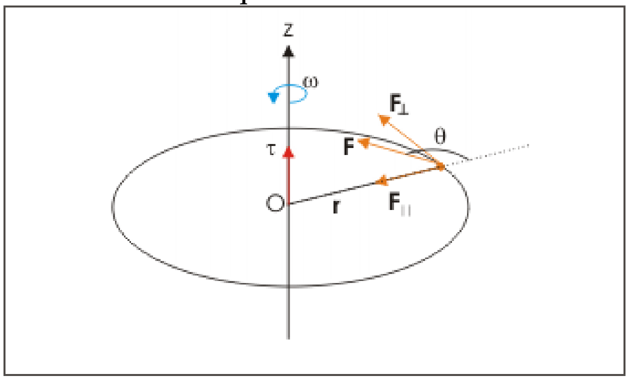
The magnitude is equal to the product of moment arm and magnitude of the force.

Direction of torque

The nature of cross (or vector) product of two vectors, conveys a great deal about the direction of cross product i.e. torque where both position and force vectors are in the plane of rotation. It tells us that (i) torque vector is perpendicular to the plane formed by operand vectors i.e. "**r**" and "**F**" and (ii) torque vector is individually perpendicular to each of the operand vectors. Applying this explanation to the case in hand, we realize here that torque is perpendicular to the plane formed by radius

and force vectors i.e z-axis. However, we do not know which side of the plane i.e +z or -z direction, the torque is directed.

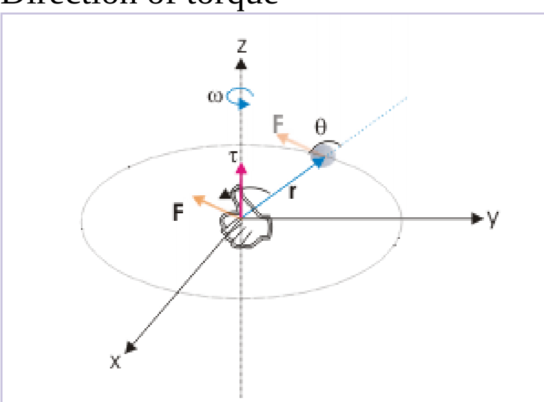
Direction of torque



Direction of torque is perpendicular to the plane of rotation.

We apply right hand rule to determine the remaining piece of information, regarding direction of torque. We have two options here. Either we can shift radius vector such that tails of two vectors meet at the position of particle or we can shift the force vector (parallel shifting) so that tails of two vectors meet at the axis. Second approach has the advantage that direction of torque vector along the axis also gives the sense of rotation about that axis. Thus, following the second approach, we shift the force vector to the origin, while keeping the magnitude and direction same as shown here.

Direction of torque



Force vector is shifted to origin in order to apply right hand rule.

.

Now, the direction of rotation is obtained by applying rule of vector cross product. We place right hand with closed fingers such that the curl of fingers point in the direction as we transverse from the direction of position vector (first vector) to the force vector (second vector). Then, the direction of extended thumb points in the direction of torque. Alternatively, we see that a counter clock-wise torque is positive, whereas clock-wise torque is negative. In this case, torque is counter-clockwise and is positive. Therefore, we conclude that torque is acting in +z-direction.

Pure rotation about a fixed axis gives us an incredible advantage in determining torque. We work with only two directions (positive and negative). In the case of torque about a point, however, we consider other directions of torque as well. It is also noteworthy to see that torque follows the superposition principle. Mathematically, we can add torque vectors algebraically as there are only two possible directions to obtain net or resultant torque. In words, it means that if a rigid body is subjected to more than one torque, then we can represent the torques by a single torque, which has the same effect on rotation.

Torque has the unit of Newton - meter.

Rotational torque .vs. torque about a point

We have seen that rotational torque is calculated with the component of force which lies in the plane of rotation. This is required as the rigid body is free to move only about the fixed axis perpendicular to the plane of rotation. Accordingly, we evaluate vector expression of torque by measuring position vector from center of circle and using component of force in the plane of rotation. However, if we evaluate the vector expression of torque **r** X **F** with respect to the origin of reference lying on the axis of rotation as is the general case of torque about a point, then we get torque which needs not be along the axial direction.

Clearly, rotational torque is a subset of torque defined for a point on the axis of rotation. This fact gives an alternate technique to determine rotational torque. We can approach the problem of determining rotational torque as described in the module, wherein we first resolve the given force in the plane of rotation containing the point of application. Then, we find the moment arm or perpendicular force component and determine the rotational torque as already explained. Alternatively, we can determine the torque about a point on the axis using vector expression of

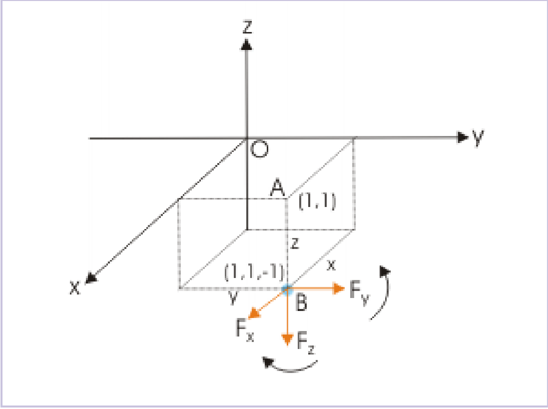
torque and then consider only the component of torque in the direction of axis of rotation.

In order to fully understand the two techniques, we shall work out an example here to illustrate the two approaches discussed here.

Let us consider a force **F** = (2**i** + 2**j** – 3**k**) Newton which acts on a rigid body at a point **r** = (**i** + **j** – **k**) meters as measured from the origin of reference. In order to determine the rotational torque say about x-axis as the axis of rotation, we first proceed by considering only the force in the plane of rotation. The figure below shows the components of force and their perpendicular distances from three axes.

We find the position of application of force by first identifying A (1,1) in xy-plane, then we find the position of the particle, B(1,1,-1) by moving "-1" in negative z-direction as in the figure below.

Torque on the particle



The position of particle along with components of force

Since the rigid body rotates about x-axis, the plane of rotation is yz plane. We, therefore, do not consider the component of force in x-direction. The components of force relevant here are (i) the component of force in y-direction Fy = 2 N at a perpendicular distance z = 1 m from axis of rotation and (ii) the component of force in z-direction Fz = 3 N at a perpendicular distance y = 1 m. For determining torque about x-axis, we multiply perpendicular distance with force. We apply appropriate sign, depending on the sense of rotation about the axis. The torque about x-axis due to component of force in y - direction is anticlockwise and hence is positive :

$$\tau_1 = zF_y = 1 \text{ x } 2 = 2 \ Nm$$

The torque about x-axis due to the component of force in z - direction is clockwise and hence is negative :

$$\tau_2 = -yF_z = -1 \text{ x } 3 = -3 \ Nm$$

Since both torques are acting along same axis i.e. x-axis, we can obtain net torque about x-axis by algebraic sum :

$$\Rightarrow \tau_x = \tau_1 + \tau_2 = 2 - 3 = -1 \ Nm$$

The net rotational torque is negative and hence clockwise in direction. The magnitude of net torque about x-axis is :

$$\Rightarrow \tau_x == 1 \ Nm$$

Now, we proceed to calculate torque about the origin of reference which lies on the axis of rotation. The torque about the origin of coordinate system is given by :

$$\boldsymbol{\tau} = \mathbf{r} \text{ x } \mathbf{F}$$

`τ = | i j k | | 1 1 -1 | | 2 2 -3 |`

$$\Rightarrow \boldsymbol{\tau} = [(1 \text{ x } -3) - (2 \text{ x } -1)]\mathbf{i} + [(-1 \text{ x } 2) - (1 \text{ x } -3)]\mathbf{j} + [(1 \text{ x } 2\} - (1 \text{ x } 2)]\mathbf{k}$$
$$\Rightarrow \boldsymbol{\tau} = (-3 + 2)\mathbf{i} + (-2 + 3)\mathbf{j}$$
$$\Rightarrow \boldsymbol{\tau} = -\mathbf{i} + \mathbf{j}$$

The particle, here, moves about the axis of rotation in x-direction. In this case, the particle is restrained not to rotate about any other axis. Thus, torque in rotation about x-axis is equal to the x- component of torque about the origin. Now, the vector x-component of torque is :

$$\boldsymbol{\tau}_x = -\mathbf{i}$$

The net rotational torque is negative and hence clockwise in direction. The magnitude of torque about x-axis for rotation is :

$$\tau_x = 1 \ Nm$$

Nature of motion

Nature displays varieties of motion. Most of the time, we encounter motion, which is composed of different basic forms of motion. The most important challenge in the study of motion is to establish a clear understanding of the components (types) of motion that ultimately manifest in the world around us. Translational motion is the basic form. It represents the basic or inherent property of natural objects. A particle moving in straight line keeps moving unless acted upon by external force. However, what we see around is not what is fundamental to the matter (straight line motion), but something which is grossly modified by the presence of force. This is the reason planets move around the Sun; electrons orbit about nucleus and aero-planes circle the Earth on inter-continental flight.

Variety of motion is one important aspect of the study of motion. Another important aspect is the restrictive paradigm of fundamental laws that needs to be expanded to real time bodies and motions. For example, Newton's laws of motion as postulated are restricted to particle or particle like bodies. It is required to be adapted to system of particles and rigid body systems with the help of concepts like center of mass. In real time, motions may be composed of even higher degree of complexities. We can think of motions which involve rotation while translating. Now, the big question is whether Newtonian dynamics is capable to describe such composite motions? The answer is yes. But, it needs further development that addresses issues like rotational dynamics of particles and rigid bodies. We shall consider this aspect of rotational motion in the next module titled Rotation of rigid body.

Summary

1. The particles in a rigid body are locked and are placed at fixed linear distances from others.

2. Each particle constituting a rigid body executes circular motion about a fixed axis in pure rotation.

3. The cause of change in angular velocity in rotation is torque. It is defined as :

$$\boldsymbol{\tau} = \mathbf{r} \ \mathbf{x} \ \mathbf{F}$$

where position vector,"**r**", and the force vector," (**F**)" are in the plane of rotation of the point of application of force, which is perpendicular to the axis of rotation.

4. The magnitude of torque

The magnitude of torque is determined in three equivalent ways :

(i) In terms of angle between position vector (**r**) and force (**F**)

$$\tau = rF\sin\theta$$

(ii) In terms of tangential component of force (F_T or $F_\perp$)

$$\tau = r(F\sin\theta) = rF_\perp$$

(iii) In terms of moment arm ($r_\perp$)

$$\tau = (r\sin\theta)F = r_\perp F$$

4. Direction of torque

The vector equation of torque reveals that torque is perpendicular to the plane formed by position and force vectors and is also perpendicular to each of them individually. In order to know the sign of torque, we apply right hand rule (positive for anticlockwise and negative for clockwise rotation).

Rotation of rigid body
Rotation of rigid body is governed by an equivalent relation called Newton's second law of rotation.

Rotation of a rigid body is characterized by same angular velocity and acceleration of particles comprising it. The situation is similar to the case of translation in which linear velocity and acceleration of all particles comprising rigid body are same. In the previous module titled Rotation, we discussed torque as the “cause” of rotation and ways to calculate torque. In this module, we seek to study the torque (cause) and angular acceleration (effect) relationship for the rotational motion of a rigid body. In other words, we seek to state Newton’s laws of motion for rotation in line with the one that exists for translation.

Rigid body is composed of particles, which are at fixed distance with respect to each other. In simple words, if a particle "A" is at a distance of 10 mm (say) from another particle "B" within a rigid body, then they continue to remain 10 mm apart during motion. This requirement is important in describing rotational motion of a rigid body. The distribution of mass about the axis affects rotational inertia of the body. As such, change in inter-particle distance shall amount to changing "rotational inertia" of the body.

Before, we proceed we need to distinguish between two separate force requirements for rotational motion. In the previous module, we have discussed the force requirement for the torque which produces angular acceleration or causes rotational motion. What about the centripetal force requirement for a particle of the rigid body to move in circular motion? This force requirement is met by the inter-molecular forces. The requirement of centripetal force is the inherent requirement for circular motion of a particle and thereby for the rotation of rigid body. While studying cause and effect relation for the rotation, it should be clearly understood that we are only concerned with the force requirement of torque for the angular acceleration of the rigid body in rotation.

Newton's first law of rotation

In translation, a particle or particle like rigid body has constant linear velocity unless there is an external force being applied on it. By conjecture, we can extend this law to rotation saying that a rigid body in rotation about a fixed axis has constant angular velocity unless it is subjected to external torque. This is exactly the Newton's first law of rotation.

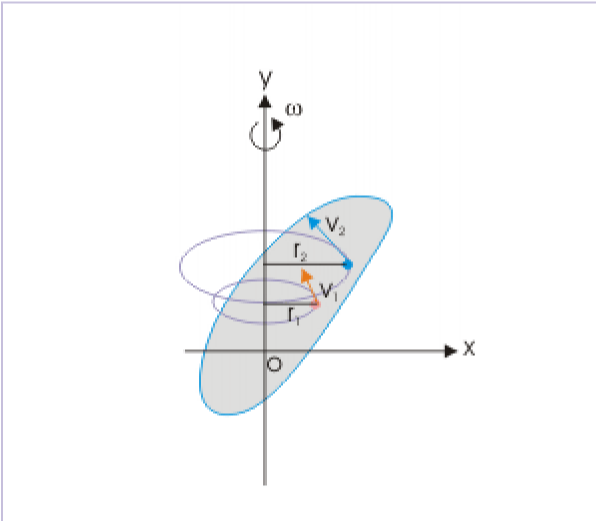
If the rigid body is at rest, then it will remain in rest. This is the exactly same assertion as for translation. On the other hand, if the rigid body is in rotation with a constant angular velocity, then it will continue to rotate with that angular velocity indefinitely. Of course, we do not realize the second assertion in our daily life because it is almost impossible to get rid of torques opposing rotational motion due to air resistance and resistance caused by the friction at the axis of rotation.

Newton's second law of rotation

Every particle of the rigid body in rotation undergoes circular motion irrespective of the shape of rigid body. The centers of the circular paths described by them lie on the axis of rotation. It should be noted that the different particles, constituting rigid body, have different linear velocities, but same angular velocity. It means that each particle traverses same angle in a given time. The linear velocity of a particle is related to angular velocity as :

$$v = \omega r$$

where "r" is the perpendicular distance of the particle's position from the axis of rotation in the plane of rotation. It is easy to visualize that particles constituting rigid body at different levels undergo rotation in different planes of rotation.
Pure rotational motion



Each particle of the body follows a circular path about axis in pure rotational motion.

It is also clear from the given relation for linear velocity that particle closer to the axis (smaller "r") will have lesser linear velocity than the one away from the axis (greater "r").

However, each of the particle of the body undergoes same angular displacement (θ) and has same angular velocity (ω) and angular acceleration (α). In other words, we can say that the angular attributes of motion of the rigid body in rotation are uniquely (single valued) defined at a particular instant.

This situation is analogous to pure translational motion of rigid body, in which each particle constituting the body has same linear velocity and acceleration at a particular instant. For this reason, the motion of a particle or a rigid body in pure rotational motion is governed by the same form of Newton's second law i.e. the relation that connects external torque to the angular acceleration has the same form as that of translational motion. In translational motion, this relationship is given as :

$$\mathbf{F} = m\mathbf{a}$$

Intuitively, we may conclude that relation between torque (cause) and angular acceleration (effect) might have the following form :
Equation:

$$\boldsymbol{\tau} = m\boldsymbol{\alpha}$$

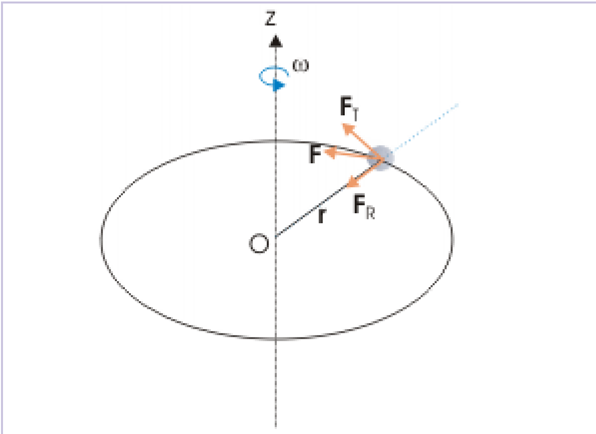
However, we find that the role of inertia is not represented by mass alone in rotation. As there are corresponding quantities for force and acceleration, there is also a separate corresponding term or quantity that represents inertia to rotation. This quantity is termed as "moment of inertia", represented by symbol, "I". The equivalent Newton's second law for rotation is, thus :
Equation:

$$\boldsymbol{\tau} = I\boldsymbol{\alpha}$$

However, we need to evaluate moment of inertia of the rigid body appropriately so that it represents the inertia of the body to external torque (cause). In the next section, we shall drive the basic expression of moment of inertia first for a single particle, then for a system of particle and then finally for the rigid body.

Moment of inertia of a particle

For the rotation of a particle, we simulate a situation in which a particle mass "m" is rotated by applying external force. We consider a particle of mass "m" is attached to one end of a mass-less (it is a mere theoretical consideration) rod of length "r". The other end of the rod is hinged at "O" such that the particle can be rotated in a horizontal plane about a vertical axis passing through center of circle, "O" .
Rotation



The path rotates following a circular path about axis of rotation. .

Now let us consider that a force "**F**" is applied to the particle in the plane of the circular path as shown in the figure. The radial component of force (F_R) does not produce any acceleration as the line of action of the radial force passes through the axis of rotation. The tangential component of force (F_T) produces tangential acceleration, a_T . The particle follows a circular path of radius "r" with its center at "O". From second law of translational motion,

$$F_T = ma_T$$

On the other hand, torque on the particle is :

$$\tau = rF_T = rma_T$$

But, we know that tangential acceleration is related to angular acceleration as :

$$a_T = \alpha r$$

Substituting in the expression of torque, we have :

$$\tau = rm\alpha r = \left(mr^2\right)\alpha$$

where $I = mr^2$. Its unit is $\text{kg} - m^2$.
Equation:

$$\tau = I\alpha$$

Example:
Problem : A particle of mass 0.1 kg is attached to one end of a light rod of length 1 m. The rod is hinged at the other end and rotates in a plane perpendicular to the axis of rotation. If the angular speed of the particle is 60 rpm, find the constant torque that can stop the rotation of the particle in one minute.
Solution : The idea here is to apply Newton's second law of rotation for a particle. For this, we need to find angular deceleration. For this, we first convert angular velocity in rad/s :

$$\omega_1 = 60\,\text{rpm} = 60 \text{ x } \frac{2\pi}{60} = 2\pi \text{ rad}/s$$

Since torque "τ" is constant as given in the question, this is obviously the case of uniform deceleration. As such, we can apply equation of rotational motion for constant acceleration,

$$\omega_1 = \omega_2 + \alpha t$$

Here,

$$\omega_1 = 2\pi \text{ rad}/s;\ \omega_2 = 0;\ t = 60\ s$$

Putting values in the equation of motion, we have :

$$\Rightarrow 0 = 2\pi + \alpha \text{ x } 60$$
$$\Rightarrow \alpha = -\frac{2\pi}{60} \text{ rad}/s^2$$

Now, applying Newton's second law of motion for rotation,

$$\tau = I\alpha$$

Here,

$$I = mr^2 = 0.1 \text{ x } 1^2 = 0.1\,\text{kg} - m^2$$

Dropping the sign of deceleration and putting values in the expression of torque, we have the magnitude of torque :

$$\tau = I\alpha = 0.1 \text{ x } \frac{2\pi}{60} = 0.01\,N - m$$

Moment of inertia of a system of particles

Moment of inertia of a particle about fixed axis is given by :
Equation:

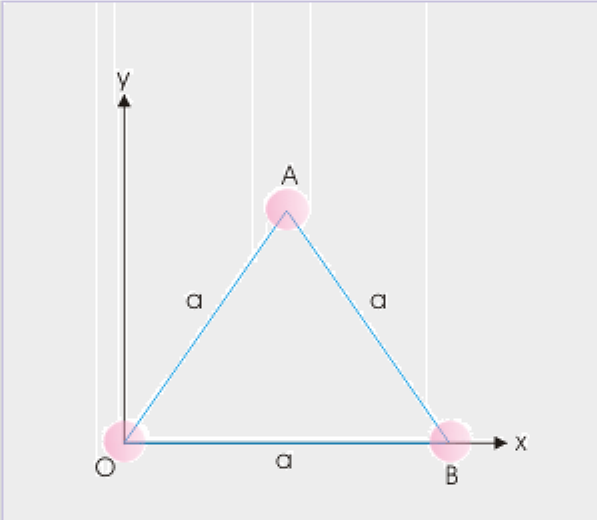
$$I = mr^2$$

The moment of inertia of the system of discrete particles is equal to the sum of moments of inertia of individual particles,
Equation:

$$I = \sum m_i {r_i}^2$$

where " m_i " is the mass of "i" th particle at perpendicular linear distance " r_i " from the axis of rotation.

Example:
Problem : Three particles each of mass "m" are situated at the vertices of an equilateral triangle OAB of length "a" as shown in the figure. Calculate moment of inertia (i) about an axis passing through "O" and perpendicular to the plane of triangle (ii) about axis Ox and (iii) about axis Oy.
Moment of inertia



Three particles each of mass "m" are situated at the vertices of an equilateral triangle.

Solution : We shall make use of the formulae of moment of inertia for discrete particles in each of the cases :
(i) about an axis passing through "O" and perpendicular to the plane of triangle
Here, distances of three particles from the axis are :

$$r_O = 0; r_A = a; r_B = a$$

The moment of inertia about an axis through "O" and perpendicular to the plane of triangle is :

$$I = \sum m_i {r_i}^2 = m_o {r_o}^2 + m_A {r_A}^2 + m_B {r_B}^2$$

$$\Rightarrow I = m \text{ x } 0 + ma^2 + ma^2 = 2ma^2$$

(ii) about axis Ox
Here distances of three particles from the axis of rotation are :

$$r_O = 0; r_A = \frac{\sqrt{3}a}{2}; r_B = 0$$

$$I=\sum m_i {r_i}^2 = m_o {r_o}^2 + m_A {r_A}^2 + m_B {r_B}^2$$

$$\Rightarrow I = m \text{ x } 0 + m\left(\frac{\sqrt{3}a}{2}\right)^2 + m \text{ x } 0 = \frac{3ma^2}{4}$$

(iii) about axis Oy
Here distances of three particles from the axes are :

$$r_O = 0, r_A = \frac{a}{2}, r_B = a$$

$$I=\sum m_i {r_i}^2 = m_o {r_o}^2 + m_A {r_A}^2 + m_B {r_B}^2$$

$$\Rightarrow I = m \text{ x } 0 + \frac{ma^2}{4} + m \text{ x } a^2 = \frac{5ma^2}{4}$$

This example illustrates how choice of axis changes moment of inertia i.e. the inertia of the system of particles to rotation. This is expected as change in the reference axis actually changes distribution of mass about axis of rotation.

Moment of inertia of rigid body

Rigid body is a continuous aggregation of particles. We, therefore, need to modify the summation in the expression of moment of inertia by integration as :

Equation:

$$I = \int r^2 đm$$

Evaluation of this integral for a given body is a separate task in itself. It is mathematically possible to evaluate this integral for bodies of regular shape. It would, however, be very difficult to evaluate the same for irregularly shaped rigid body. In such cases, it is pragmatic to resort to experimental methods to calculate moment of inertia. Mathematical evaluation of moment of inertia even for regularly shaped bodies would require specialized analysis and evaluation.

Two theorems, pertaining to moment of inertia, are of great help in the mathematical evaluation of moment of inertia of regularly shaped bodies. They are (i) parallel axes theorem and (ii) perpendicular axes theorem. These theorems extend the result of moment of inertia of basic geometric forms of rigid bodies to other axes. We shall cover these aspects and shall evaluate moment of inertia of certain important geometric rigid bodies in separate modules.

Kinetic energy of rigid body in rotation

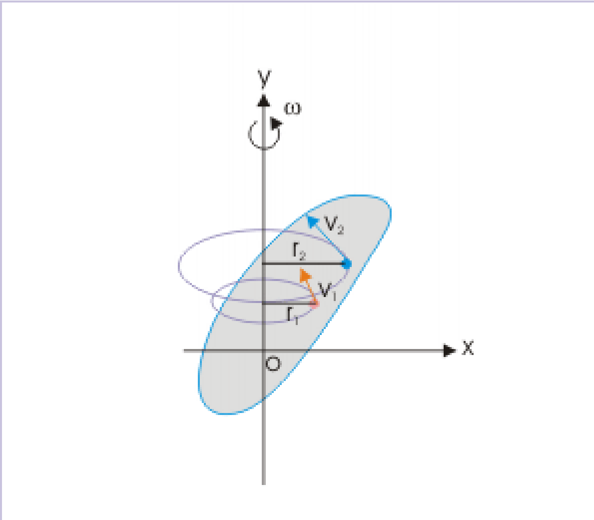
Kinetic energy of a particle or body represents the form of energy that arises from motion. We are aware that kinetic energy of a particle in translation is given by the expression :

Equation:

$$K = \frac{1}{2}mv^2$$

In pure rotation, however, the rigid body has no "over all" translation of the body. However, the body in rotation must have kinetic energy as it involves certain motion. A closer look on the rotation of rigid body reveals that though we may not be able to assign translation to the rigid body as a whole, but we can recognize translation of individual particles as each of them rotate about the axis in circular motion with different linear speeds. The speed of a particle is given by :

Pure rotational motion



Each particle of the body follows a circular path about axis in pure rotational motion.

$$v_i = \omega r_i$$

Thus, kinetic energy of an individual particle is :

$$K_i = \frac{1}{2} m_i {v_i}^2$$

where " K_i " is the kinetic energy of "i" th particle having a speed " v_i ". In terms of angular speed, the kinetic energy of an individual particle is :

$$K_i = \frac{1}{2} m_i (\omega r_i)^2$$

Now, the kinetic energy of the rigid body is sum of the kinetic energies of the particles constituting the rigid body :

$$K = \sum K_i = \sum \frac{1}{2} m_i \omega^2 {r_i}^2$$

We note here that angular speeds of all particles constituting the body are same. Hence, the constant "1/2" and " ω^2 " can be taken out of the summation sign :

$$K = \frac{1}{2}\omega^2 \sum m_i {r_i}^2$$

However, we know that :

$$I = \sum m_i {r_i}^2$$

Combining two equations, we have :
Equation:

$$K = \frac{1}{2} I \omega^2$$

This is the desired expression of kinetic energy of a rigid body rotating about a fixed axis i.e. in pure rotational motion. The form of expression of the kinetic energy here emphasizes the correspondence between linear and angular quantities. Comparing with the expression of kinetic energy for translational motion, we find that "moment of inertia (I)" corresponds to "mass (m)" and "linear speed (v)" corresponds to "angular speed (ω)".

We can also interpret the result obtained above from a different perspective. We could have directly inferred that expression of kinetic energy in rotation should have an equivalent form as :

$$\text{K (Kinetic energy)} = \frac{1}{2} \text{ x (inertia) x (speed)}^2$$

In rotation, inertia to the rotation is "moment of inertia (I)" and speed of the rigid body is "angular speed (ω)". Substituting for the quantities, we have the expression for kinetic energy of rigid body in rotation as :

$$K = \frac{1}{2} I \omega^2$$

Comparing this equation with the expression of the sum of kinetic energy of individual particles as derived earlier :

$$K = \frac{1}{2}\omega^2 \sum m_i {r_i}^2$$

Clearly,

$$I = \sum m_i {r_i}^2$$

This conclusion, thus, clearly establishes that the expression as given by $\sum m_i {r_i}^2$ indeed represents the inertia of the rigid body in rotation.

Note: Though it is clear, but we should emphasize that expressions of kinetic energy of rotating body either in terms of angular speed or linear speed are equivalent expressions i.e. two expressions measure the same quantity. Two expressions do not mean that the rotating body has two types of kinetic energy.

Summary

1. Moment of inertia is the inertia of an object against any change in its state of rotation. This quantity corresponds to “mass”, which determines inertia of an object in translational motion.

2. Moment of inertia is defined for rotation about a fixed axis for a particle, a system of particles and a rigid body.

3. Expressions of Moment of inertia

(i) For a particle

$$I = mr^2$$

(ii) For a system of particles

$$I = \sum m_i {r_i}^2$$

(iii) For a rigid body

$$I = \int r^2 dm$$

4. Moment of inertia, also referred in short as MI, is a scalar quantity. The directional positions (angular positions) of the particles/objects with respect to axis of rotation does not matter. The unit of MI is $\text{kg} - m^2$.

5. Objects in rotation have rotational kinetic energy of rotation due to rotational motion of individual particles, constituting the object.

6. Object in pure rotation has only rotational kinetic energy i.e. no translational kinetic energy is involved.

7. The expression of rotational kinetic energy is given by :

$$K = \frac{1}{2} I \omega^2$$

Rotation (check your understanding)

The questions have been selected to enhance understanding of the topics covered in the modules on " Rotation ". All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

Understanding level (Rotation)

Exercise:

Problem:

A force **F** = 2**i** + **j** + 3**k** Newton acts at a point of a rigid body, defined by the position vector **r** = **i** + **j** + **k** meters. If the body rotates about z-axis of the coordinate system, then the magnitude of torque (in N-m) accelerating the rigid body is :

$$\text{(a) } 1 \ \text{(b) } 2 \ \text{(c) } 3 \ \text{(d) } 4$$

Solution:

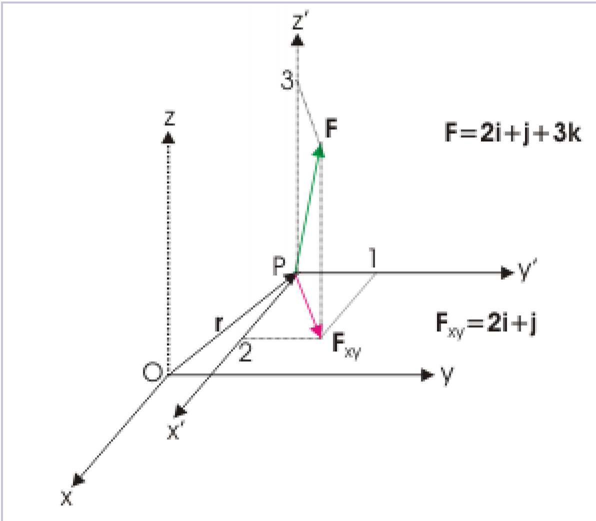
This question is included here with the purpose to highlight conceptual and visualization aspects of the calculation of torque. The torque about the axis of rotation i.e. z-axis is given as :

$$\boldsymbol{\tau} = \mathbf{r} \ \mathbf{x} \ \mathbf{F}$$

where “r” and “F” are in the plane perpendicular to the axis of rotation i.e. in “xy”-plane. Explicitly, we can write :

$$\boldsymbol{\tau}_z = \mathbf{r}_{\mathrm{xy}} \ \mathbf{x} \ \mathbf{F}_{\mathrm{xy}}$$

Rotation



Component of force in “xy”-plane. .

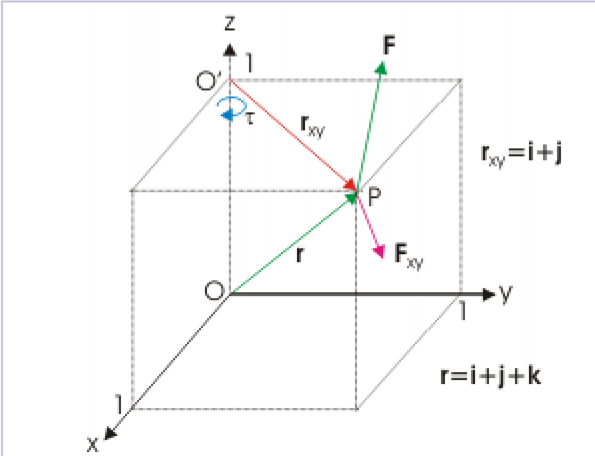
The figure above shows the force component in “xy”-plane. Have a careful look at the force component in relation to the point of application, "P". The rigid body is deliberately not shown so that we can visualize vector components without any congestion in the figure.

Note here that body is rotating about z-axis. Thus, we need to calculate torque about z-axis only. The z-component of force does not constitute a torque about z-axis. Hence,

$$\mathbf{F}_{\mathrm{xy}} = 2\mathbf{i} + \mathbf{j}$$

The component of radius vector in xy - plane is shown in the figure below. Note that it is measured in the plane of rotation of the point of application of force about the axis. Again, we drop z-component. The “xy” components of force and radius vectors are shown in the red colors in the figure.

Rotation



Component of position vector in "xy"-plane. .

$$\mathbf{r}_{\mathrm{xy}} = \mathbf{i} + \mathbf{j}$$

Therefore, torque about the axis of rotation is :

$$\boldsymbol{\tau}_z = \mathbf{r}_{\mathrm{xy}} \;\mathbf{x}\; \mathbf{F}_{\mathrm{xy}} = (\mathbf{i} + \mathbf{j}) \;\mathbf{x}\; (2\mathbf{i} + \mathbf{j}) = \mathbf{k} - 2\mathbf{k} = -\mathbf{k}$$

The magnitude of torque is :

$$\tau_z = 1\, N - m$$

Note that the torque is in the direction of "-z"-axis and the body is rotating clockwise.

Alternatively, we can also work this problem without restricting evaluation of vector product in any plane as :

$$\boldsymbol{\tau} = \mathbf{r} \;\mathbf{x}\; \mathbf{F}$$

`τ = | i j k | | 1 1 1 | | 2 1 3 |`

$$\boldsymbol{\tau} = (1 \;\mathrm{x}\; 3 - 1 \;\mathrm{x}\; 1)\,\mathbf{i} + (1 \;\mathrm{x}\; 2 - 1 \;\mathrm{x}\; 3)\,\mathbf{j} + (1 \;\mathrm{x}\; 1 - 2 \;\mathrm{x}\; 1)\,\mathbf{k}$$

$$\boldsymbol{\tau} = 2\,\mathbf{i} - \mathbf{j} - \mathbf{k}$$

The component of torque in z-direction is :

$$\boldsymbol{\tau}_z = -\,\mathbf{k}$$

The magnitude of torque is :

$$\tau_z = 1\,N - m$$

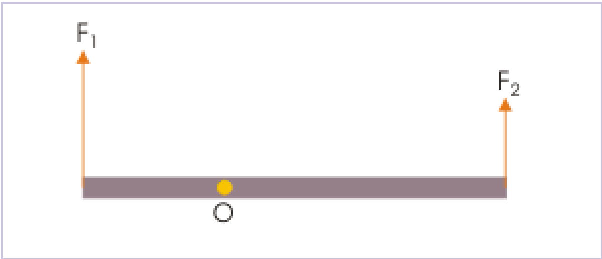
Hence, option (a) is correct.

Exercise:

Problem:

The overview of two horizontal forces acting at the ends of a rod, pivoted at the middle point, is shown in the figure. The rod is free to rotate in horizontal plane. However, the magnitudes of forces, acting perpendicular to the rod, are such that rod is stationary. If the angle of force F_1 with respect to rod is changed, then rod can be held stationary if :

Rotation



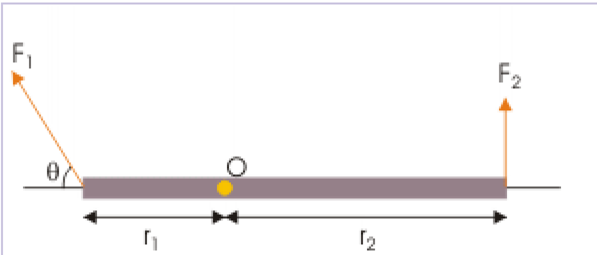
A rod is acted upon by a pair of forces in horizontal plane. .

(a) force $F_1$is increased
(b) force $F_2$is decreased
(c) point of application of force $F_1$is brought closure to pivot
(d) point of application of force $F_2$is brought closure to pivot

Solution:

The rod will not rotate when torques due to the forces are equal in magnitude and opposite in direction :

Rotation



A rod is acted upon by a pair of forces in horizontal plane. .

$$\tau_1 = \tau_2$$

$$F_1 \text{ x } r_1 = F_2 \text{ x } r_2$$

Since $r_1 < r_2$, we can conclude $F_1 > F_2$. The magnitude of torque due to force F_1 is given as :

$$\tau_1 = r_1 \text{ x } F_1 \sin\theta$$

Evidently, torque due to F_1 is reduced when angle of application of force with respect to rod changes. In order to maintain the balance, the force F_1 needs to be increased or the force F_2 is decreased or the force F_2 is brought closure to the pivot.

Hence, options (a), (b) and (d) are correct.

Exercise:

Problem:

A flywheel with moment of inertia of 2 $\text{ kg} - m^2$, is rotating at 10 rad/s about a perpendicular axis. The flywheel is brought to rest uniformly in 10 seconds due to friction at the axle. The magnitude of torque due to friction is (in N-m) :

$$\text{(a) } 0 \text{ (b) } 5 \text{ (c) } 10 \text{ (d) } 15$$

Solution:

The angular deceleration is uniform i.e. constant as given in the question. The torque due to friction at the axle is, therefore, also constant. Let “α” be the deceleration.

Applying, equation of motion for constant angular acceleration, we have :

$$\omega_f = \omega_i + \alpha t$$

$$\text{Here, } \omega_i = 10 \text{ rad } / s; \ \omega_f = 0; t = 2s,$$

$$\Rightarrow 0 = 10 + \alpha \text{ x } 2$$

$$\Rightarrow \alpha = -5 \text{ rad } / s^2$$

The magnitude of torque is :

$$\Rightarrow \tau = I\alpha = 2 \text{ x } 5 = 10 \, N - m$$

Hence, option (c) is correct.

Exercise:

Problem:

A circular disk of mass 2 kg and radius 1 m is free to rotate about its central axis. The disk is initially at rest. Then, the constant tangential force (in Newton) required to rotate the disk with angular velocity 10 rad/s in 5 second is :

$$\text{(a) } 2 \text{ (b) } 4 \text{ (c) } 8 \text{ (d) } 16$$

Solution:

This question is similar to earlier one except that we need to find the tangential force instead of torque. Since torque is equal to the product of tangential force and moment arm, we need to find torque first and then find the required tangential force.

A constant tangential force results in a torque along the axis of rotation. The torque, in turn, produces a constant angular acceleration "α". As such, we can apply equation of motion for constant angular acceleration,

$$\omega_f = \omega_i + \alpha t$$

$$\text{Here,} \omega_i = 0;\ \omega_f = 10 \text{ rad } / s; t = 2s,$$

$$\Rightarrow 10 = 0 + \alpha \text{ x } 2$$

$$\Rightarrow \alpha = -5 \text{ rad } / s^2$$

Applying Newton's second law for rotation, we have :

$$\tau = I\alpha$$

Here,

$$I = MR^2 = 2 \text{ x } 1^2 = 2\,\text{Kg} - m^2$$

Hence,

$$\tau = I\alpha = 2 \text{ x } 2 = 4\,N - m$$

Now, torque is also given as :

$$\tau = RF$$

$$F = \frac{\tau}{R} = \frac{4}{1} = 4\,N$$

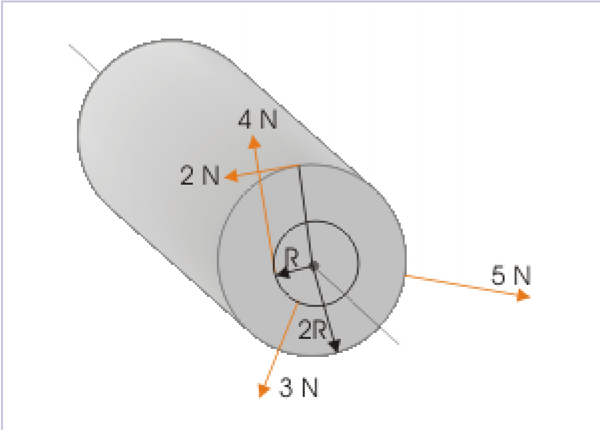
Hence, option (b) is correct.

Exercise:

Problem:

A wheel of 10 kg, mounted on its central axis, is acted upon by four forces. The lines of action of forces are in the plane perpendicular to the axis of rotation as shown in the figure. If the magnitudes of forces and their directions during rotation are same, then (considering, R = 10 m) :

Rotation



Rotation about central axis. .

(a) the magnitude of net torque is zero
(b) the magnitude of net acceleration is $2\,\text{rad}/s^2$
(c) the magnitude of net acceleration is $4\,\text{rad}/s^2$
(d) the direction of rotation is anti-clockwise

Solution:

The forces of 3N and 5N act through the axis of rotation. They do not constitute torque. The force of 4N constitute a clockwise torque of 4 x 10 = 40 N-m. On the other hand, the force of 2N constitute an anticlockwise torque of 2 x 20 = 40 N-m.

The cylinder, therefore, is acted by two equal, but opposite torques. Therefore, the net torque on the body is zero.

Hence, option (a) is correct.

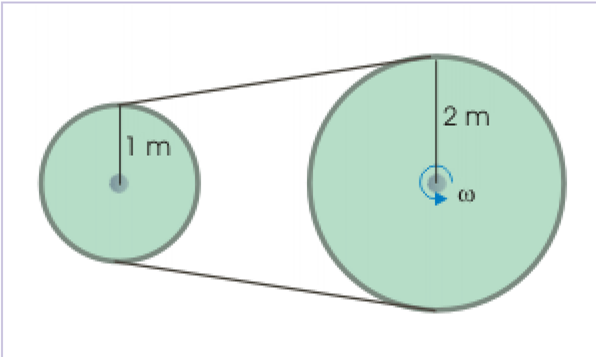
Application level (Rotation)

Exercise:

Problem:

Two flywheels of moments of inertia 1 and 2 $\text{kg}-m^2$ and radii 1 and 2 meters respectively are connected by a belt as shown in the figure. The angular speed of larger wheel is increased at a constant rate 2 rad/s^2 without any slippage between wheels and the belt. Then,

Connected flywheels



There is slippage between belt
and flywheels .

(a) Tension in the belt is same
(b) Linear velocity of the belt is same everywhere
(c) Torques on the wheels are same
(d) Angular accelerations of the wheels are same

Solution:

Since, no slippage between wheels and the belt is involved, the linear velocity of the belt is same through out.

The rotation of the larger wheel causes the smaller wheel to rotate as well. Since two wheels are accelerated from at constant rate, it means that they are acted upon by constant torques. On the other hand, a net tangential force on either of the wheels is required to constitute a torque. As such, tension in the belt needs to be different so that there is a net tangential force on the flywheel.

Let ω be the angular speed of the larger wheel. The corresponding linear speed of the belt is :

$$v = \omega_1 \text{ x } r_1 = \omega \text{ x } 2 = 2\omega$$

The angular speed of the smaller wheel is :

$$\omega_1 = \frac{v}{r_2} = \frac{2\omega}{1} = 2\omega$$

On the other hand, the linear acceleration of the belt is :

$$a = \alpha_1 \text{ x } r_1 = 2 \text{ x } 2 = 4\, m/s^2$$

The angular acceleration of the smaller wheel is :

$$\Rightarrow \alpha_2 = \frac{a}{r_2} = \frac{4}{1} = 4 \text{ rad}\,/s^2$$

Thus, angular accelerations of the wheels are not same. Now, the torque on the larger wheel is :

$$\Rightarrow \tau_1 = I_1 \text{ x } \alpha_1 = 2 \text{ x } 2 = 4\, N - m$$

The torque on the smaller wheel is :

$$\Rightarrow \tau_2 = I_2 \text{ x } \alpha_2 = 1 \text{ x } 4 = 4\, N - m$$

Thus, the torques on the wheels is same.

Hence, options (b) and (c) are correct.

Note: Torques need not be always equal. This result is specific for the set of values given in the question.

Answers

`1. (a) 2. (a), (b) and (d) 3. (c) 4. (b) 5. (a) 6. (b) and (c)`

Moments of inertia of rigid bodies
Moment of inertia of rigid body depends on the distribution of mass about the axis of rotation.

In the module titled Rotation of rigid body , we derived expressions of moments of inertia (MI) for different object forms as :

$$1.\ \text{For a particle : } \quad I = mr^2$$

$$2.\ \text{For a system of particles : } \quad I = \sum m_i {r_i}^2$$

$$3.\ \text{For a rigid body : } \quad I = \int r^2 đm$$

In this module, we shall evalaute MI of different regularly shaped rigid bodies.

Evaluation strategy

We evaluate right hand integral of the expression of moment of inertia for regularly shaped geometric bodies. The evaluation is basically an integration process, well suited to an axis of rotation for which mass distribution is symmetric. In other words, evaluation of the integral is easy in cases where mass of the body is evenly distributed about the axis. This axis of symmetry passes through "center of mass" of the regular body. Calculation of moment of inertia with respect to other axes is also possible, but then integration process becomes tedious.

There are two very useful theorems that enable us to calculate moment of inertia about certain other relevant axes as well. These theorems pertaining to calculation of moment of inertia with respect to other relevant axes are basically "short cuts" to avoid lengthy integration. We must, however, be aware that these theorems are valid for certain relevant axes only. If we are required to calculate moment of inertia about an axis which can not be addressed by these theorems, then we are left with no choice other than evaluating the integral or determining the same experimentally. As such, we limit ourselves in using integral method to cases, where moment of inertia is required to be calculated about the axis of symmetry.

In this module, we will discuss calculation of moment of inertia using basic integral method only, involving bodies having (i) regular geometric shape (ii) uniform mass distribution i.e uniform density and (iii) axis of rotation passing through center of mass (COM). Application of the theorems shall be discussed in a separate module titled " Theorems on moment of inertia ".

As far as integration method is concerned, it is always useful to have a well planned methodology to complete the evaluation. In general, we complete the integration in following steps :

1. Identify an infinitesimally small element of the body.
2. Identify applicable density type (linear, surface or volumetric). Calculate elemental mass "dm" in terms of appropriate density.
3. Write down the expression of moment of inertia (đI) for elemental mass.
4. Evaluate the integral of moment of inertia for an appropriate pair of limits and determine moment of inertia of the rigid body.

Identification of small element is crucial in the evaluation of the integral. We consider linear element in evaluating integral for a linear mass distribution as for a rod or a plate. On the other hand, we consider thin concentric ring as the element for a circular plate, because we can think circular plate being composed of infinite numbers of thin concentric rings. Similarly, we consider a spherical body, being composed of closely packed thin spherical shells.

Calculation of elemental mass "đm" makes use of appropriate density on the basis of the nature of mass distribution in the rigid body :

$$\text{mass} = \text{ appropriate density } \;\; \text{x} \;\; \text{ geometric dimension}$$

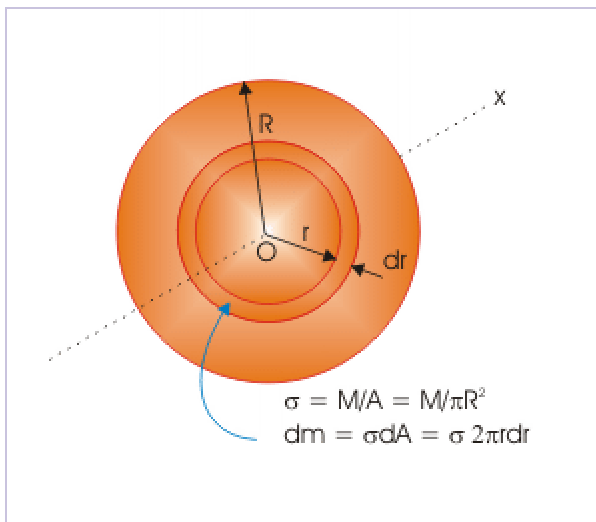
The choice of density depends on the nature of body under consideration. In case where element is considered along length like in the case of a rod, rectangular plate or ring, linear density (λ) is the appropriate choice. In cases, where surface area is involved like in the case of circular plate, hollow cylinder and hollow sphere, areal density (σ) is the appropriate choice. Finally, volumetric density (ρ) is suitable for solid three

dimensional bodies like cylinder and sphere. The elemental mass for different cases are :

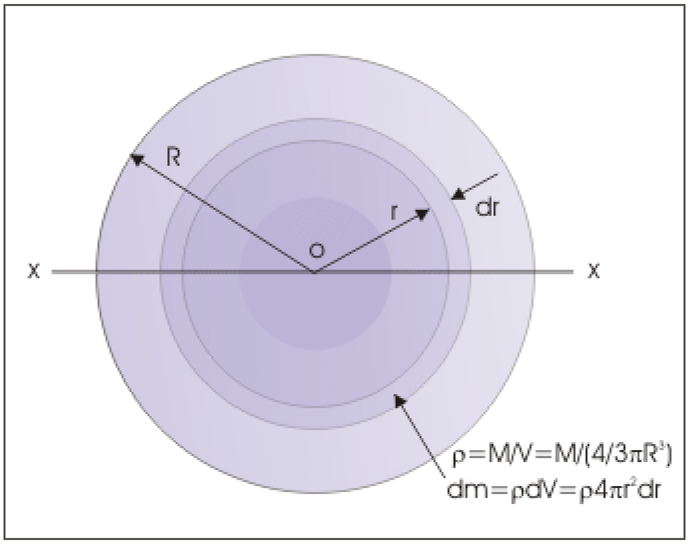
Equation:

$$đm = \lambda đx$$
$$đm = \sigma đA$$
$$đm = \rho đV$$

Elemental mass

Elemental mass of a rod for MI about perpendicular bisector	Elemental mass of a circular disc for MI about perpendicular axis through the center
[missing_resource: mi1b.gif]	

Elemental mass



Elemental mass of a sphere for MI about a diameter

The MI integral is then expressed by suitably replacing "dm" term by density term in the integral expression. This approach to integration using elemental mass assumes that mass distribution is uniform.

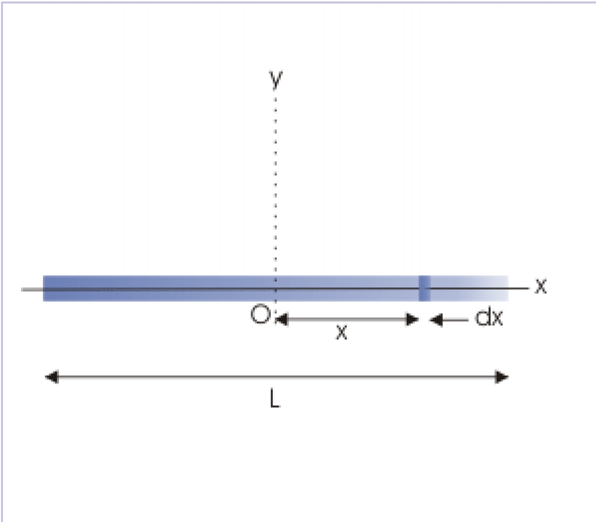
Evaluation of moment of inertia

In this section, we shall determine MI of known geometric bodies about the axis of its symmetry.

MI of a uniform rod about its perpendicular bisector

The figure here shows the small element with repect to the axis of rotation. Here, the steps for calculation are :

Moment of inertia



MI of a rod about perpendicular bisector

(i) Infinetesimally small element of the body :

Let us consider an small element "dx" along the length, which is situated at a linear distance "x" from the axis.

(ii) Elemental mass :

Linear density, λ, is the appropriate density type in this case.

$$\lambda = \frac{M}{L}$$

where "M" and "L" are the mass and length of the rod respectively. Elemental mass (đm) is, thus, given as :

$$đm = \lambda đx = \left(\frac{M}{L}\right) đx$$

(iii) Moment of inertia for elemental mass :

Moment of inertia of elemental mass is :

$$đI = r^2 đm = x^2 \left(\frac{M}{L}\right) đx$$

(iv) Moment of inertia of rigid body :

$$I = \int r^2 đm = \int x^2 \left(\frac{M}{L}\right) đx$$

While setting limits we should cover the total length of the rod. The appropriate limits of integral in this case are -L/2 and L/2. Hence,

$$I = \int_{\frac{-L}{2}}^{\frac{L}{2}} \left(\frac{M}{L}\right) x^2 đx$$

Taking the constants out of the integral sign, we have :

$$\Rightarrow I = \left(\frac{M}{L}\right) \int_{\frac{-L}{2}}^{\frac{L}{2}} x^2 đx$$

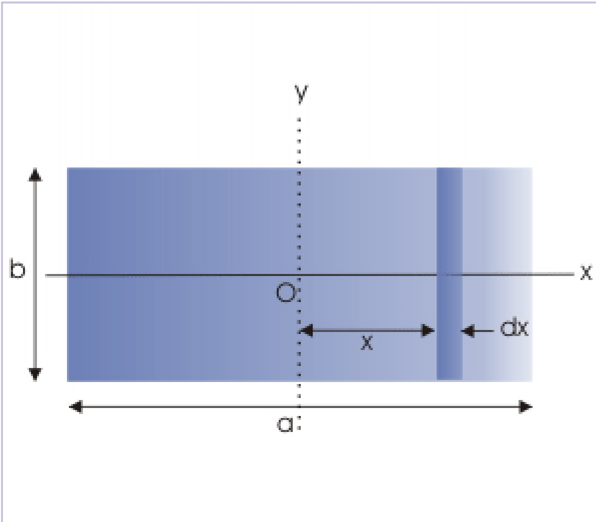
Equation:

$$\Rightarrow I = \left(\frac{M}{L}\right) \left[\frac{x^3}{3}\right]_{\frac{-L}{2}}^{\frac{L}{2}} = \frac{ML^2}{12}$$

MI of a rectangular plate about a line parallel to one of the sides and passing through the center

The figure here shows the small element with repect to the axis of rotation i.e. y-axis, which is parallel to the breadth of the rectangle. Note that axis of rotation is in the place of plate. Here, the steps for calculation are :

Moment of inertia



MI of rectangular plate about a line parallel to its length and passing through the center

(i) Infinetesimally small element of the body :

Let us consider an small element "dx" along the length, which is situated at a linear distance "x" from the axis.

(ii) Elemental mass :

Linear density, λ, is the appropriate density type in this case.

$$\lambda = \frac{M}{a}$$

where "M" and "a" are the mass and length of the rectangular plate respectively. Elemental mass (dm) is, thus, given as :

$$đm = \lambda đx = \left(\frac{M}{a}\right) đx$$

(iii) Moment of inertia for elemental mass :

Moment of inertia of elemental mass is :

$$đI = r^2 đm = x^2 \left(\frac{M}{a}\right) đx$$

(iv) Moment of inertia of rigid body :

Proceeding in the same manner as for the case of an uniform rod, the MI of the plate about the axis is given by :
Equation:

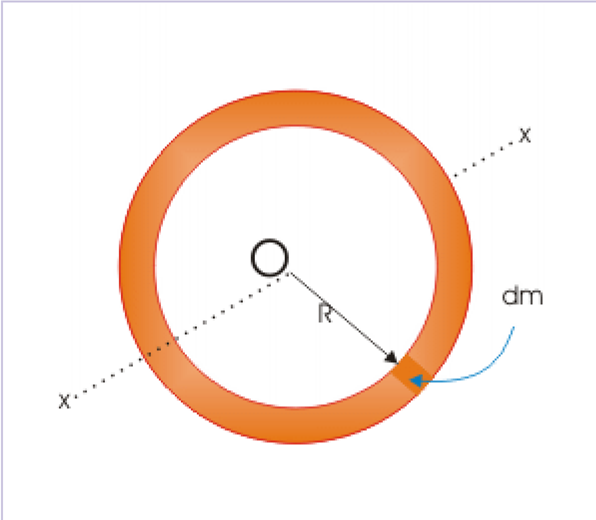
$$\Rightarrow I == \frac{Ma^2}{12}$$

Similarly, we can also calculate MI of the rectangular plate about a line parallel to its length and through the center,
Equation:

$$\Rightarrow I == \frac{Mb^2}{12}$$

MI of a circular ring about a perpendicular line passing through the center

The figure here shows the small element with respect to the axis of rotation. Here, we can avoid the steps for calculation as all elemental masses are at a fixed (constant) distance "R" from the axis. This enables us to take "R" directly out of the basic integral :
Moment of inertia



MI of a circular ring about a perpendicular line passing through the center

Equation:

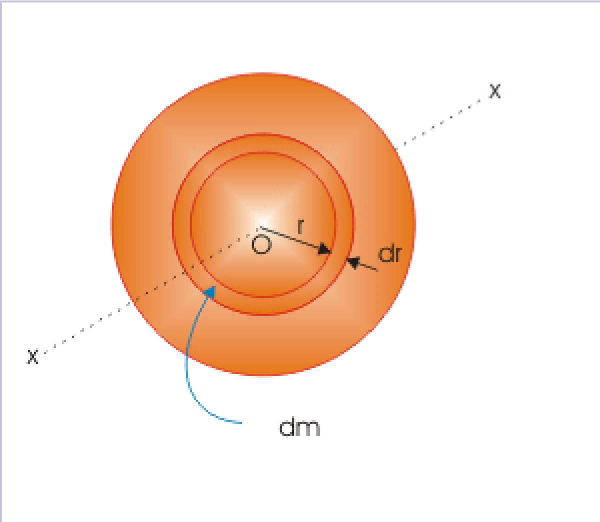
$$\text{đ}I = r^2\text{đ}m = R^2\text{đ}m$$
$$\Rightarrow I = \int \text{đ}I = R^2 \int \text{đ}m = MR^2$$

We must note here that it was not possible so in the case of a rod or a rectangular plate as elemental mass is at a variable distance from the axis of rotation. We must realize that simplification of evaluation in this case results from the fact that masses are at the same distance from the axis of rotation. This means that the expression of MI will be valid only when thickness of the ring is relatively very small in comparison with its radius.

MI of a thin circular plate about a perpendicular line passing through the center

The figure here shows the small ring element with repect to the axis of rotation. Here, the steps for calculation are :

Moment of inertia



MI of a thin circular plate about a perpendicular line passing through the center

(i) Infinetesimally small element of the body :

Let us consider an small circular ring element of width "dr", which is situated at a linear distance "r" from the axis.

(ii) Elemental mass :

Areal density, σ, is the appropriate density type in this case.

$$\sigma = \frac{M}{A}$$

where "M" and "A" are the mass and area of the plate respectively. Here, ring of infinitesimal thickness itself is considered as elemental mass (dm) :

$$đm = \sigma đA = \left(\frac{M}{\pi R^2}\right) 2\pi r đr$$
$$đm = \frac{2Mrđr}{R^2}$$

(iii) Moment of inertia for elemental mass

MI of the ring, which is treated as elemental mass, is given by :

$$đI = r^2 đm = r^2 \frac{2Mrđr}{R^2}$$

(iv) Moment of inertia of rigid body

$$I = \int r^2 đm = \int \frac{2r^2 Mrđr}{R^2}$$

Taking the constants out of the integral sign, we have :

$$I = \frac{2M}{R^2} \int r^3 đr$$

The appropriate limits of integral in this case are 0 and R. Hence,

$$I = \frac{2M}{R^2} \int_0^R r^3 đr$$

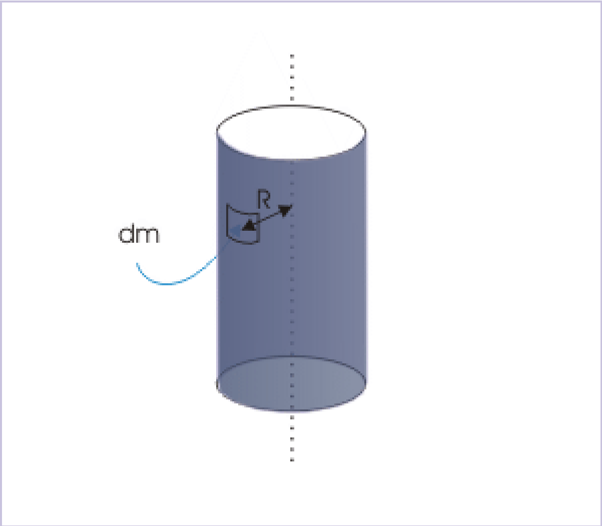
Equation:

$$\Rightarrow I = \frac{2M}{R^2} \left[\frac{x^4}{4} \right]_0^R = \frac{MR^2}{2}$$

Moment of inertia of a hollow cylinder about its axis

The figure here shows the small element with repect to the axis of rotation. Here, we can avoid the steps for calculation as all elemental masses composing the cylinder are at a fixed (constant) distance "R" from the axis. This enables us to take "R" out of the integral :

Moment of inertia



Moment of inertia of a hollow cylinder about its axis

Equation:

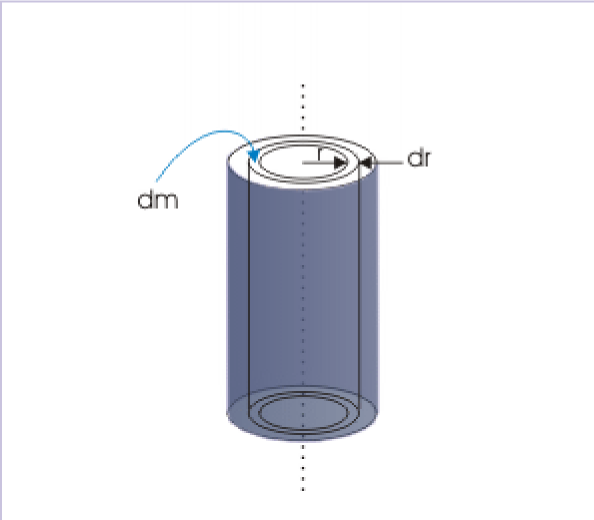
$$
\begin{aligned}
& dI = r^2 dm = R^2 dm \\
& \Rightarrow I = \int dI = R^2 \int dm = MR^2
\end{aligned}
$$

We note here that MI of hollow cyliner about its longitudinal axis is same as that of a ring. Another important aspect of MI, here, is that it is independent of the length of hollow cylinder.

Moment of inertia of a uniform solid cylinder about its longitudinal axis

The figure here shows the small element as hollow cylinder with repect to the axis of rotation. We must note here that we consider hollow cylinder itself as small element for which the MI expression is known. Here, the steps for calculation are :

Moment of inertia



Moment of inertia of a uniform solid cylinder about its longitudinal axis

(i) Infinetesimally small element of the body :

Let us consider the small mass of volume "dV" of the hollow cylinder of small thickness "dr" in radial direction, which is situated at a linear distance "r" from the axis.

(ii) Elemental mass :

Volume density, ρ, is the appropriate density type in this case.

$$\rho = \frac{M}{V}$$

where "M" and "V" are the mass and volume of the solid cylinder respectively. Here, cylinder of infinitesimal thickness itself is considered as elemental mass (dm) :

$$đm = \rho đV = \left(\frac{M}{V}\right) đV = \left(\frac{M}{\pi R^2 L}\right) 2\pi r L đr$$
$$đm = \frac{2M r đr}{R^2}$$

(iii) Moment of inertia for elemental mass

MI of the hollow cylinder, which is treated as elemental mass, is given by :

$$đI = r^2 đm = r^2 \frac{2Mrđr}{R^2}$$

(iv) Moment of inertia of rigid body

$$I = \int r^2 đm = \int \frac{2r^2 Mrđr}{R^2}$$

Taking out the constants from the integral sign, we have :

$$I = \left(\frac{2M}{R^2}\right) \int r^3 đr$$

The appropriate limita of integral in this case are 0 and R. Hence,

$$I = \left(\frac{2M}{R^2}\right) \int_0^R r^3 đr$$

Equation:

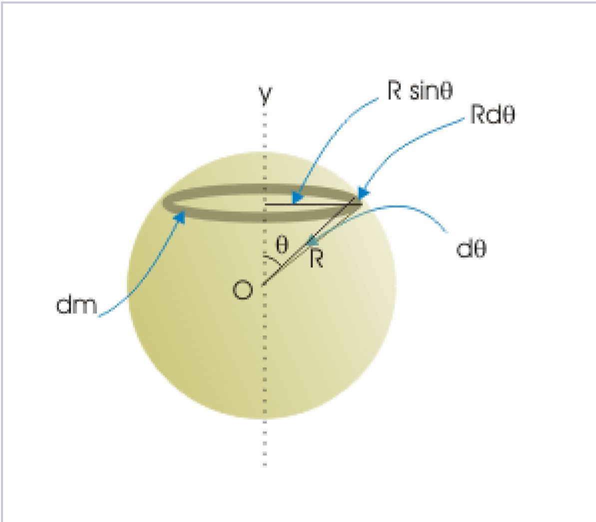
$$\Rightarrow I = \left(\frac{2M}{R^2}\right) \left[\frac{r^4}{4}\right]_0^R = \frac{MR^2}{2}$$

We note here that MI of solid cyliner about its longitudinal axis is same as that of a plate. Another important aspect of MI, here, is that it is independent of the length of solid cylinder.

Moment of inertia of a hollow sphere about a diameter

The figure here shows that hollow sphere can be considered to be composed of infinite numbers of rings of variable radius. Let us consider one such ring as the small element, which is situated at a linear distance "R" from the center of the sphere. The ring is positioned at an angle "θ" with respect to axis of rotation i.e. y-axis as shown in the figure.

Moment of inertia



Moment of inertia of a hollow sphere about a diameter

(i) Infinetesimally small element of the body :

Let us consider the small mass of area "dA" of the ring of small thickness "Rdθ", which is situated at a linear distance "R" from the center of the sphere.

(ii) Elemental mass :

Area density, σ, is the appropriate density type in this case.

$$\sigma = \frac{M}{A}$$

where "M" and "A" are the mass and surface area of the hollow sphere respectively. Here, ring of infinitesimal thickness itself is considered as elemental mass (dm) :

$$dm = \sigma dA = \left(\frac{M}{4\pi R^2}\right) 2\pi r \sin\theta R d\theta$$
$$dm = \left(\frac{M}{2}\right) \sin\theta d\theta$$

(iii) Moment of inertia for elemental mass

Here, radius of elemental ring about the axis is R sinθ. Moment of inertia of elemental mass is :

$$đI = R^2 \sin^2 \theta đm = R^2 \sin^2 \theta \left(\frac{M}{2}\right) \sin \theta đ\theta$$

(iv) Moment of inertia of rigid body

$$I = \int \left(\frac{M}{2}\right) R^2 \sin^2 \theta đ\theta$$

Taking out the constants from the integral sign, we have :

$$I = \left(\frac{MR^2}{2}\right) \int \sin^3 \theta đ\theta$$

The appropriate limits of integral in this case are 0 and π. Hence,

$$\Rightarrow I = \left(\frac{MR^2}{2}\right) \int_0^\pi \sin^3 \theta đ\theta$$

$$\Rightarrow I = \left(\frac{MR^2}{2}\right) \int_0^\pi \left(1 - \cos^2\theta\right) \sin \theta đ\theta$$

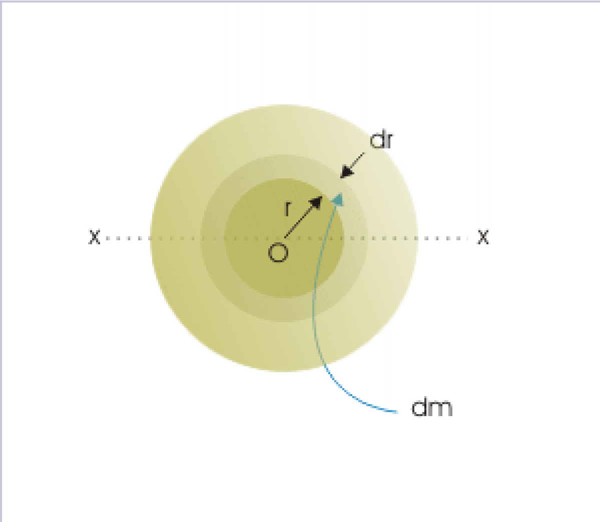
$$\Rightarrow I = \left(\frac{MR^2}{2}\right) \int_0^\pi -\left(1 - \cos^2\theta\right) đ\left(\cos \theta\right)$$

Equation:

$$\Rightarrow I = \frac{-MR^2}{2} \left[\cos \theta - \frac{\cos^3\theta}{3}\right]_0^\pi = \frac{2}{3} MR^2$$

Moment of inertia of a uniform solid sphere about a diameter

A solid sphere can be considered to be composed of concentric spherical shell (hollow spheres) of infinitesimally small thickness "dr". We consider one hollow sphere of thickness "dr" as the small element, which is situated at a linear distance "r" from the center of the sphere.
Moment of inertia



MI of a uniform solid sphere
about a diameter

(i) Infinetesimally small element of the body :

Let us consider the small mass of volume "dV" of the sphere of thickness "dr", which is situated at a linear distance "r" from the center of the sphere.

(ii) Elemental mass :

Volumetric density, ρ, is the appropriate density type in this case.

$$\rho = \frac{M}{V}$$

where "M" and "V" are the mass and volume of the uniform solid sphere respectively. Here, hollow sphere of infinitesimal thickness itself is considered as elemental mass (dm) :

$$đm = \rho đV = \left(\frac{M}{\frac{4}{3}\pi R^3}\right) 4\pi r^2 đr$$

(iii) Moment of inertia for elemental mass

Moment of inertia of elemental mass is obtained by using expresion of MI of hollow sphere :

$$đI = \frac{2}{3} r^2 đm$$

$$\Rightarrow đI = \frac{2}{3} r^2 \left(\frac{3M}{R^3}\right) r^2 đr = \left(\frac{2M}{R^3}\right) r^4 đr$$

(iv) Moment of inertia of rigid body

$$I = \int \left(\frac{2M}{R^3}\right) r^4 đr$$

Taking out the constants from the integral sign, we have :

$$I = \left(\frac{2M}{R^3}\right) \int r^4 đr$$

The appropriate limits of integral in this case are 0 and R. Hence,

$$\Rightarrow I = \left(\frac{2M}{R^3}\right) \int_0^R r^4 đr$$

Equation:

$$\Rightarrow I = \left(\frac{2M}{R^3}\right) \left[\frac{r^4}{4}\right]_0^R = \frac{2}{5} M R^2$$

Radius of gyration (K)

An inspection of the expressions of MIs of different bodies reveals that it is directly proportional to mass of the body. It means that a heavier body will require greater torque to initiate rotation or to change angular velocity of rotating body. For this reason, the wheel of the railway car is made heavy so that it is easier to maintain speed on the track.

Further, MI is directly proportional to the square of the radius of circular object (objects having radius). This indicates that geometric dimensions of the body have profound effect on MI and thereby on rotational inertia of the body to external torque. Engineers can take advantage of this fact as they

can design rotating part of a given mass to have different MIs by appropriately distributing mass either closer to the axis or away from it.

The MIs of ring, disk, hollow cylinder, solid cylinder, hollow sphere and solid sphere about their central axes are MR^2, $\frac{MR^2}{2}$, MR^2, $\frac{2MR^2}{3}$ and $\frac{2MR^2}{5}$ respectively. Among these, the MIs of a ring and hollow cylinder are MR^2 as all mass elements are distributed at equal distance from the central axis. For other geometric bodies that we have considered, we find thaheir MIs expressions involve some fraction of MR^2 as mass distribution is not equidistant from the central axis.

We can, however, assume each of other geometric bodies (not necessarily involving radius) equivalent to a ring (i.e a hoop) of same mass and a radius known as "radius of gyration". In that case, MI of a rigid body can be written in terms of an equivalent ring as :

Equation:

$$I = MK^2$$

Formally, we can define radius of gyration as the radius of an equivalent ring of same moment of inertia. In the case of solid sphere, the MI about one of the diameters is :

$$I = \frac{2MR^2}{5}$$

By comparison, the radius of gyration of solid sphere about its diameter is :

$$K = \sqrt{\left(\frac{2}{5}\right)} \text{ x } R$$

Evidently, radius of gyration of a rigid body is specific to a given axis of rotation. For example, MI of solid sphere about an axis parallel to its diameter and touching its surface (we shall determine MI about parallel axis in the next module) is :

$$I = \frac{7MR^2}{5}$$

Thus, radius of gyration of solid sphere about this parallel axis is :

$$K = \sqrt{\left(\frac{7}{5}\right)} \text{ x } R$$

The radius of gyration is a general concept for rotational motion of a rigid body like moment of inertia and is not limited to circularly shaped bodies, involving radius. For example, radius of gyration of a rectangular plate about a perpendicular axis passing through its COM and parallel to one of its breadth (length "a" and breadth "b") is :

$$I = \frac{Ma^2}{12}$$

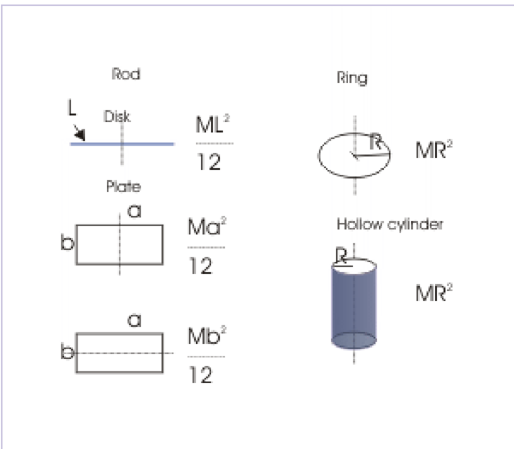
and the radius of gyration about the axis is :

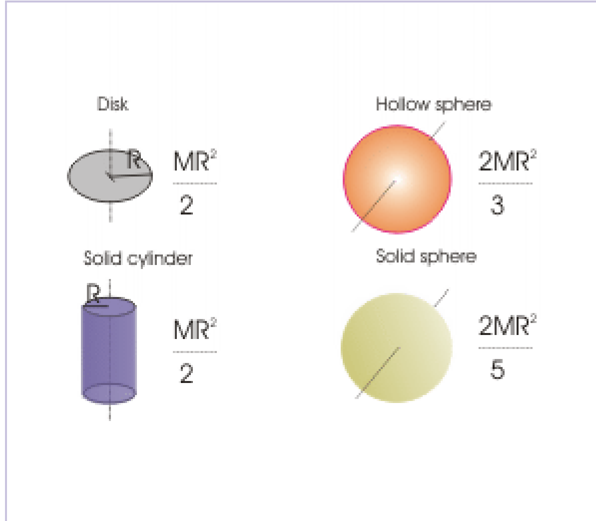
$$K = \frac{a}{\sqrt{12}}$$

Summary

1. The results of some of the known geometric bodies about their central axes, as derived in this module, are given in the figures below :
Moment of inertia of known geometric bodies

Moment of inertia of rod, plate, ring and hollow cylinders	Moment of inertia of circular disk, solid cylinder, hollow sphere and solid sphere





2. The radius of gyration (K) is defined as the radius of an equivalent ring of same moment of inertia. The radius of gyration is defined for a given axis of rotation and has the unit that of length. The moment of inertia of a rigid body about axis of rotation, in terms of radius of gyration, is expressed as :

$$I = MK^2$$

3. Some other MIs of interesting bodies are given here. These results can be derived as well in the same fashion with suitably applying the limits of integration.

(i)The MI of a thick walled cylinder (R_1 and R_2 are the inner and outer radii)

$$I = \frac{M\left(R_1^2+R_2^2\right)}{2}$$

(ii) The MI of a thick walled hollow sphere (R_1 and R_2 are the inner and outer radii)

$$I = \frac{2}{5} \text{ x } M \text{ x } \frac{\left(R_2^5-R_1^5\right)}{\left(R_2^3-R_1^3\right)}$$

Moments of inertia of rigid bodies (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the theoretical treatment of the topic. The idea is to provide a verbose explanation of the solution, detailing the application of theory. Solution presented here, therefore, is treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

- Moment of inertia is a scalar quantity without any directional property. This has important implication in solving problems, which involve removal of a part of the rigid body from the whole. We need to simply use the MI formula for a given shape with changed mass of the remaining body, ensuring that the pattern of mass distribution about the axis has not changed.
- Moment of inertia about an oblique axis involves MI integration with certain modification. Here, mass distribution and distance involve different variables. We are required to express integral in terms of one variable so that the same can be integrated by single integration process.
- In determining relative MIs, we should look how closely or how distantly mass is distributed. This enables us to compare MIs without actual calculation in some cases.
- So far, we have studied calculation of MI for uniform objects. However, we can also evaluate MI integral, if variation in mass follows certain pattern and the same can be expressed in terms of mathematical expression involving variable.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the calculation of moment of inertia of regularly shaped rigid bodies. For this reason,

questions are categorized in terms of the characterizing features pertaining to the questions as :

- Using MI formula
- Estimation of MI by inspection
- Part of a rigid body
- Non-uniform mass distribution

Using MI formula

Example 1

Problem : The moment of inertia of a straight wire about its perpendicular bisector and moment of inertia of a circular frame about its perpendicular central axis are I_1 and I_2 respectively. If the composition of wires are same and lengths of the wires in them are equal, then find the ratio ${}^{I_1}/_{I_2}$.

Solution : The MI of the straight wire about a perpendicular through the mid point is :

$$I_1 = \frac{ML^2}{12}$$

The MI of the circular frame about its perpendicular central axis is :

$$I_2 = MR^2$$

According to question, the lengths of the wires in them are equal. As the composition of the wires same, the mass of two entities are same. Also :

$$L = 2\pi R$$
$$\Rightarrow R = \frac{L}{2\pi}$$

Substituting in the expression of MI of circular frame,

$$I_2 = \frac{ML^2}{4\pi^2}$$

Now, the required ratio is :

$$\frac{I_1}{I_2} = \frac{ML^2 \text{x} 4\pi^2}{12 \text{x} ML^2}$$

$$\frac{I_1}{I_2} = \frac{\pi^2}{3}$$

Example 2

Problem : The moments of inertia of a solid sphere and a ring of same mass about their central axes are same. If R_s be the radius of solid sphere, then find the radius of the ring.

Solution : The MI of the solid sphere about its central axis is given as :

$$I_s = \frac{2M{R_s}^2}{5}$$

Let the radius of the ring is R_r . The MI of the ring about its central axis is given as :

$$I_r = M{R_r}^2$$

According to question, the moments of inertia of the solid sphere and the ring about their central axes are same :

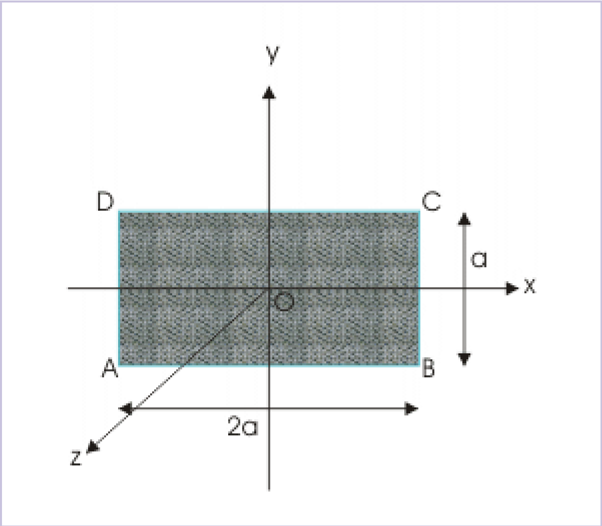
$$M{R_r}^2 = \frac{2M{R_s}^2}{5}$$

$$\Rightarrow R_r = \sqrt{\left(\frac{2}{5}\right)} \text{ x } R_s$$

Estimation of MI by inspection

Example 3

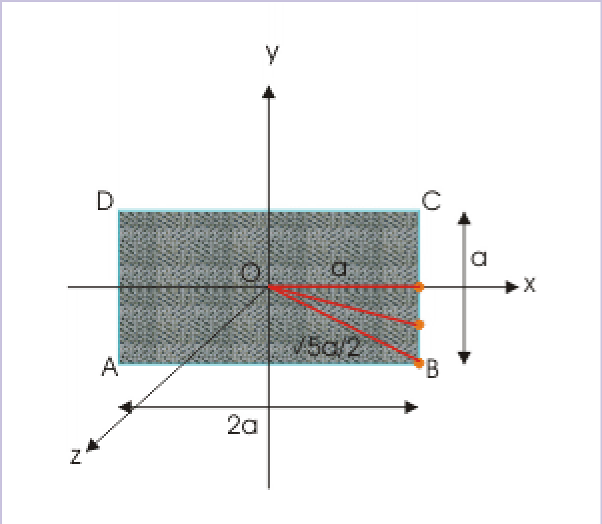
Problem : Find the axis of the coordinate system about which the moment of inertia of a uniform rectangle of dimensions "a" and "2a" is least.
Rectangular plate



MI of rectangular plate about an axis.

Solution : We can actually determine MI about the given axes by evaluating the integral of MI about each of the axes and then compare. However, we can as well estimate least MI as mass of the rectangle is uniformly distributed. MI is least for the axis about which the mass is distributed closest to it.

Rectangular plate



MI of rectangular plate about z-

axis.

We can see here that the most distant mass about x-axis is at a distance "a/2"; most distant mass about y-axis is at a distance "a"; most distant mass about z-axis at a distance between "a" and "√5a/2".

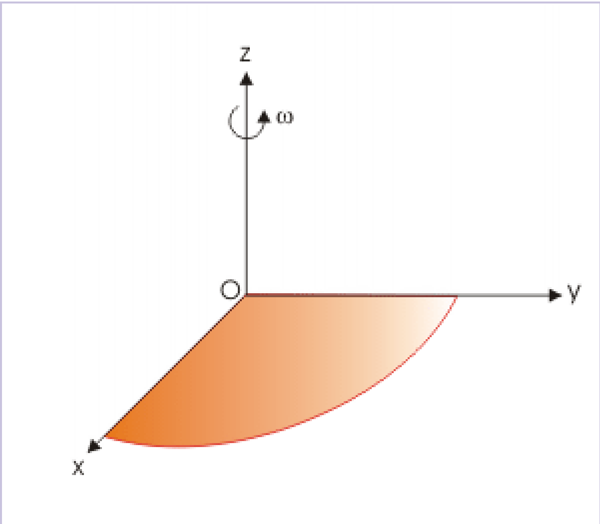
Here, mass distribution is closest about x-axis. Thus, moment of inertia about x-axis is least.

Part of a rigid body

Example 4

Problem : Find the the MI of one quarter of a circular plate of mass "M" and radius "R" about z-axis, as shown in the figure.

Quarter plate



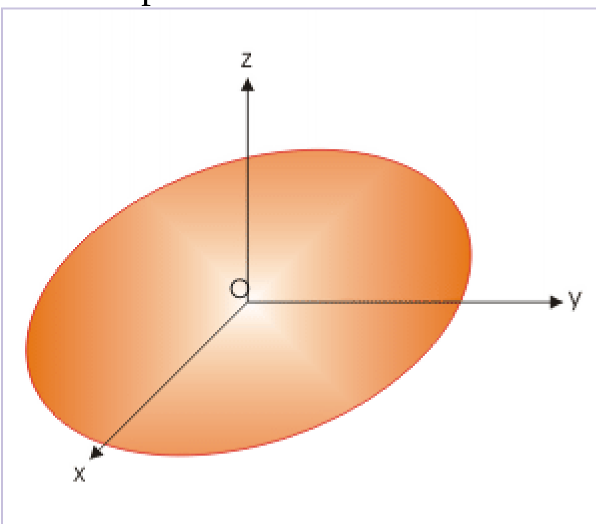
MI of quarter plate about z-axis.

Solution : We need to look closely the way MI about an axis is defined. MI of the rigid body about the an axis is given by :

$$I = \sum M_i {R_i}^2$$

This definition is essentially scalar in nature. The "R" in the expression is perpendicular distance of a particle from the axis of rotation. Does it matter whether the particle lie on left or right of the axis? Obviously no. It means that MI has no directional attribute. Thus, we can conclude that MI of the quarter plate is arithmetic quarter (1/4) of the MI of a complete disk, whose mass is 4 times greater than that of quarter plate.

Circular plate



MI of complete circular plate.

Hence, mass of the corresponding complete disk is 4M and radius is same R as that of quarter plate. The MI of the complete disk about perpendicular axis to its surface is :

$$I_O = \left(4M\right)\frac{R^2}{2}$$

Now, MI of the quarter plate is 1/4 of the complete disk,

$$I = \frac{1}{4} I_O = \frac{1}{4} \left(4M \right) \frac{R^2}{2} = \frac{MR^2}{2}$$

Note : We notice that the expression of MI has not changed for the quarter plate. We must, however, be aware that the symbol "M" represents mass of the quarter plate - not that of the complete circular plate.

QBA (Question based on above) : The MI of a uniform disk is 1.0 $\text{kg} - m^2$ about its perpendicular central axis. If a segment, subtending an angle of 120° at the center, is removed from it, then find the MI of the remaining disk about the axis.

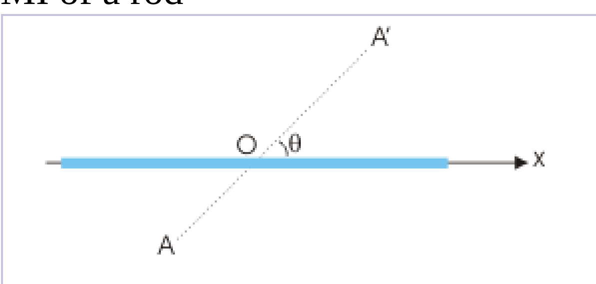
Hint : We see that 120°/360° = 1/3 part of the complete disk is removed. Therefore, the mass of the remaining part is 1-1/3 = 2/3. Answer is “2/3 $\text{kg} - m^2$ ”.

Oblique axis

Example 5

Problem : Determine the moment of inertia of a uniform rod of length "L" and mass "m" about an axis passing through its center and inclined at an angle "θ" as shown in the figure.

MI of a rod



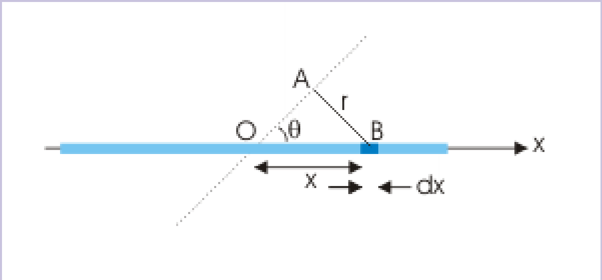
MI of a rod about an axis making an angle with the rod

Solution : We shall work out MI from the basic integral expression for elemental mass :

$$đI = r^2 đm$$

Let us consider an small element "dx" along the length, which is situated at a linear distance "x" from the axis. The elemental mass, however, is at a perpendicular distance AB from the axis of rotation – not “x’.

MI of a rod



MI of a rod about an axis making an angle with the rod

From ΔOAB,

$$r = x \sin\theta$$

The above expression allows us to convert all distances involved in the calculation of MI in terms of a single variable "x". Now, the elemental mass (dm) is, given as :

$$đm = \lambda đx = \left(\frac{M}{L}\right) đx$$

The MI of elemental mass about inclined axis AA' is :

$$đI = r^2 đm = x^2 \sin^2\theta \left(\frac{M}{L}\right) đx$$

The MI of rod about inclined axis AA', therefore, is :

$$I = \int đI = \int x^2 \sin^2 \theta \left(\frac{M}{L}\right) đx$$

We note here that the angle "θ" is constant for all elemental masses along the rod and, thus " $\sin^2 \theta$ " together with "M/L" can be taken out of the integral sign. The appropriate limits of integration covering the length of the rod are "-L/2" and "L/2" :

$$\Rightarrow I = \left(\frac{M}{L}\right) \sin^2 \theta \int_{\frac{-L}{2}}^{\frac{L}{2}} x^2 đx$$

$$\Rightarrow I = \left(\frac{M}{L}\right) \sin^2 \theta \left[\frac{x^3}{3}\right]_{\frac{-L}{2}}^{\frac{L}{2}}$$

$$\Rightarrow I = \left(\frac{M}{L}\right) \sin^2 \theta \left[\frac{L^3}{24} + \frac{L^3}{24}\right]$$

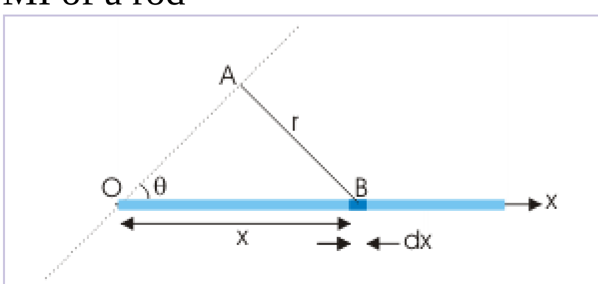
$$\Rightarrow I = \frac{ML^2}{12} \sin^2 \theta$$

Note1 : Here, $\frac{ML^2}{12}$ is MI of the rod about perpendicular line through COM, denoted by Ic. The moment of inertia about an axis making an angle with the rod, therefore, can be expressed in terms of "Ic" as :

$$\Rightarrow I = I_C \sin^2 \theta$$

Note2 : The MI of the rod at an inclined axis passing through one of its edge is obtained by integrating between limits "0" and "L" :

MI of a rod



MI of a rod about an axis making an angle with the rod and passing through one of the edges

$$I = \left(\frac{M}{L}\right)\sin^2\theta\left[\frac{x^3}{3}\right]_0^L$$

$$\Rightarrow I = \left(\frac{M}{L}\right)\sin^2\theta \text{ x } \frac{L^3}{3}$$

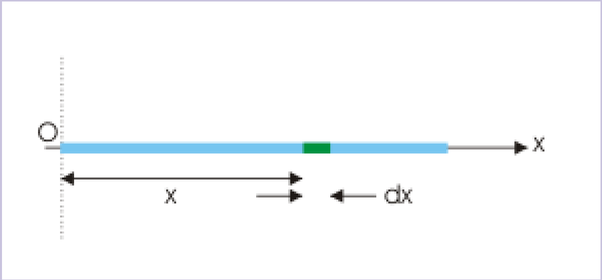
$$\Rightarrow I = \frac{ML^2}{3}\sin^2\theta$$

Non-uniform mass distribution

Example 6

Problem : The mass per unit length of a non-uniform rod of mass “M” and length “L” is given as “ax”, where “x” is measured from left end and “a” is a constant. What is its moment of inertia about an axis perpendicular to the rod and passing through its left end (point “O”) ?

MI of a rod



Mass per unit length of the rod varies from one end to another.

Solution : Let us now consider a small length “dx” at a distance “x” from the axis of rotation. The elemental mass is,

$$đm = \lambda đx = ax đx$$

In order to evaluate the constant “a”, we integrate the elemental mass for the whole length of the rod and equate the same to its mass as :

$$M = \int đm = \int axđx$$

Putting the appropriate limit,

$$\Rightarrow M = a\left[\frac{x^2}{3}\right]_0^L = \frac{aL^2}{2}$$

$$\Rightarrow a = \frac{2M}{L^2}$$

Thus, elemental mass is :

$$\Rightarrow đm = axđx = \left(\frac{2M}{L^2}\right)xđx$$

The moment of inertia of the elemental mass about the axis is :

$$đI = r^2 đm = x^2\left(\frac{2M}{L^2}\right)xđx$$

Integrating between the limits “0” and “L”, we have :

$$\Rightarrow I = \int đI = \int \left(\frac{2M}{L^2}\right)x^3 đx$$

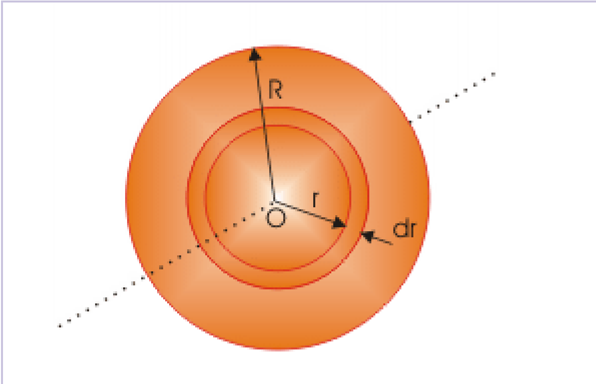
$$\Rightarrow I = \frac{2M}{L^2}\left[\frac{x^4}{4}\right]_0^L$$

$$\Rightarrow I = \frac{ML^2}{2}$$

Example 7

Problem : A circular disk of radius “R” and mass “M” has non-uniform surface mass density, given by $\sigma = ar^2$, where “r” is the distance from the center of the disk and “a” is a constant. What is its MI about the perpendicular central axis ?

Solution : Let us now consider a small concentric area of thickness “dr” at a distance “r” from the axis of rotation. The elemental mass is,
MI of a circular plate



Mass per unit areas of the circular plate varies in radial direction.

$$đm = σđA = ar^2 2\pi rđr$$

In order to evaluate the constant “a”, we integrate the elemental mass for the whole of the circular disk and equate the same to its mass as :

$$M = \int đm = 2\pi a \int r^3 đr$$

Putting the appropriate limit,

$$\Rightarrow M = 2\pi a \left[\frac{r^4}{4}\right]_0^L = \frac{\pi a R^4}{2}$$

$$\Rightarrow a = \frac{2M}{\pi R^4}$$

Thus, elemental mass is :

$$\Rightarrow đm = \left(\frac{2M}{\pi R^4}\right) x 2\pi r^3 đr$$

The moment of inertia of the ring of elemental mass about the axis is :

$$đI = r^2 đm = \left(\frac{2M}{\pi R^4}\right) x 2\pi r^5 đr$$

Integrating between the limits “0” and “R”, we have :

$$\Rightarrow I = \int đI = \frac{4M}{R^4} \int r^5 đr$$

$$\Rightarrow I = \frac{4M}{R^4} \left[\frac{r^6}{6} \right]_0^R$$

$$\Rightarrow I = \frac{4M}{R^4} \text{ x } \frac{R^6}{6} = \frac{2MR^2}{3}$$

Theorems on moment of inertia
Theorems on moment of inertia help to calculate it about additional relevant axes without any integral evaluation.

The use of integral expression of moment of inertia (MI) for a rigid body can be employed to calculate MI about different axes of rotation. However, such consideration is tedious because of difficulty in integral evaluation. There are two theorems, which avoid integral evaluation for calculating MI about certain other axes, when moment of inertia about an axis is known. The theorems are (i) theorem of parallel axes and (ii) theorems of perpendicular axes.

Theorem of parallel axes

This theorem enables us to calculate MI about any axis, parallel to the axis passing through center of mass (COM). The mathematical expression of this theorem is given as :
Equation:

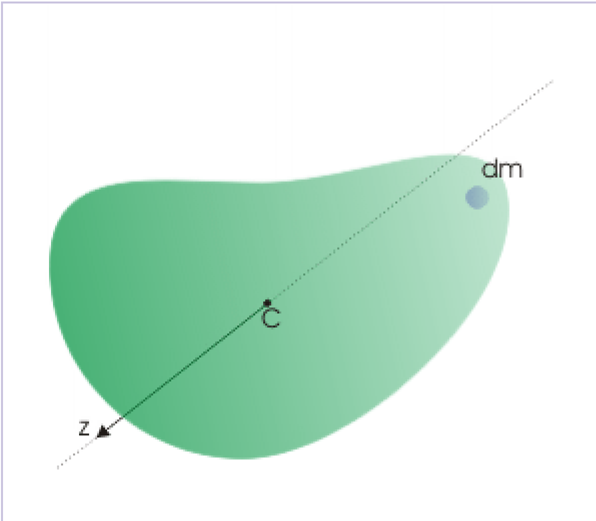
$$I = I_C + Md^2$$

where I_C is MI of the body about an axis passing through center of mass, "C"; I is MI of the body about an axis parallel to the axis passing through center of mass; "M" is the mass of the rigid body and "d" is the perpendicular distance between two parallel axes.

We shall here drive expression for the MI about parallel axis in a general case of arbitrarily shaped three dimensional rigid body. This derivation requires clear visualization in three dimensional coordinate space. We shall use three figures to bring out relative distances and angles in order to achieve the clarity as stated.

Let "M" be the mass of the rigid body and "dm" be the mass of a small element. Let "C" be the center of mass and "Cz" be an axis of rotation as shown in the figure. For convenience, we consider that "Cz" is along z-coordinate of the reference system.

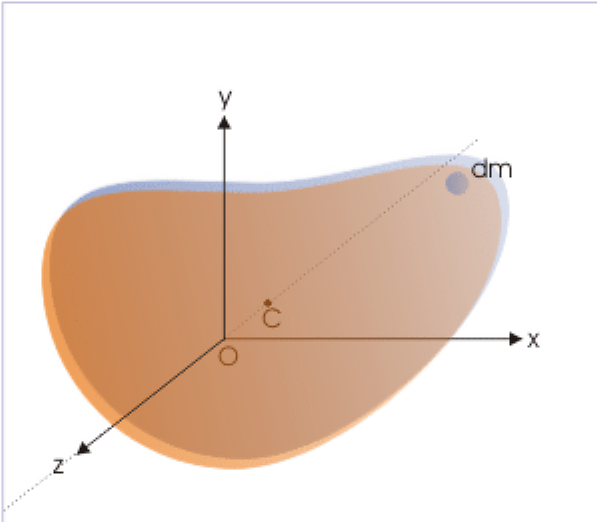
Theorem of parallel axes



MI about parallel axis of arbitrarily shaped three dimensional rigid body.

In general, the elemental mass "dm" may be situated anywhere in relation to the axis "cz". Here, we consider two cross sections of the rigid body in parallel planes, which are perpendicular to z - axis (axis of rotation). The two cross sections are drawn here such that one (on the rear) contains center of mass "C" and other (on the front) contains the elemental mass "dm". We designate the plane containing elemental mass "dm" as xy-plane of the coordinate system.

Theorem of parallel axes



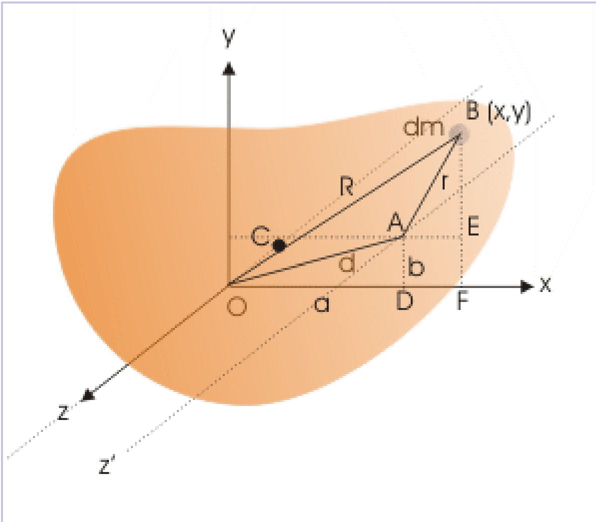
Two cross sections of rigid body.

Now we concentrate on the plane (the plane in the front) containing elemental mass "dm". The element of mass "dm" being in the xy-plane has z = 0 and its position is given by two coordinates "x" and "y".

The parallel axis through center of mass intersects the plane (xy) at point "O". Let Az' be the parallel axis about which, we have to calculate MI of the rigid body. This axis intersects the plane (xy) at point "A" as shown in the figure below.

In order to comprehend three dimensional aspect of perpendicular lines, we should realize that all lines drawn on the plane (xy) are perpendicular to two parallel axes (Oz or Az') as xy plane is perpendicular to them. The line BA in the xy plane, therefore, represents the perpendicular distance of elemental mass "dm" from the axis Az'. Similarly, the line BO represents the perpendicular distance of elemental mass "dm" from the axis Oz. Also, the line OA in xy-plane is perpendicular distance between two parallel axes, which is equal to "d".

Theorem of parallel axes



The cross section of rigid body containing elemental mass.

Now, MI of the rigid body about axis Az' in the form of integral is given by :

$$I = \int r^2 đm = \int \left(\mathrm{AE}^2 + \mathrm{BE}^2\right) đm = \int \left\{(x-a)^2 + (y-b)^2\right\} đm$$

$$\Rightarrow I = \int \left(x^2 + a^2 - 2xa + y^2 + b^2 - 2xb\right) đm$$

Rearranging, we have :

$$\Rightarrow I = \int \left(x^2 + y^2\right) đm + \int \left(a^2 + b^2\right) đm - 2a \int x đm - 2b \int y đm$$

However, the coordinates of the center of mass by definition are given as :

$$x_C = \frac{\int x đm}{M}$$

$$y_C = \frac{\int y đm}{M}$$

$$z_C = \frac{\int z đm}{M}$$

But, we see here that "x" and "y" coordinates of center of mass are zero as it lies on z - axis. It means that :

$$x_C = \frac{\int x đm}{M} = 0$$
$$\Rightarrow \int x đm = 0$$

Similarly,

$$\Rightarrow \int y đm = 0$$

Thus, the equation for the MI of the rigid body about the axis parallel to an axis passing through center of mass "C" is :

$$\Rightarrow I = \int \left(x^2 + y^2\right) đm + \int \left(a^2 + b^2\right) đm$$

From the figure,

$$x^2 + y^2 = R^2$$

and

$$a^2 + b^2 = d^2$$

Substituting in the equation of MI, we have :

$$\Rightarrow I = \int \left(x^2 + y^2\right) đm + \int \left(a^2 + b^2\right) đm = \int R^2 đm + \int d^2 đm$$

We, however, note that "R" is variable, but "d" is constant. Taking the constant out of the integral sign :

$$\Rightarrow I = \int R^2 đm + d^2 \int đm = \int R^2 đm + Md^2$$

The integral on right hand side is the expression of MI of the rigid body about the axis passing through center of mass. Hence,

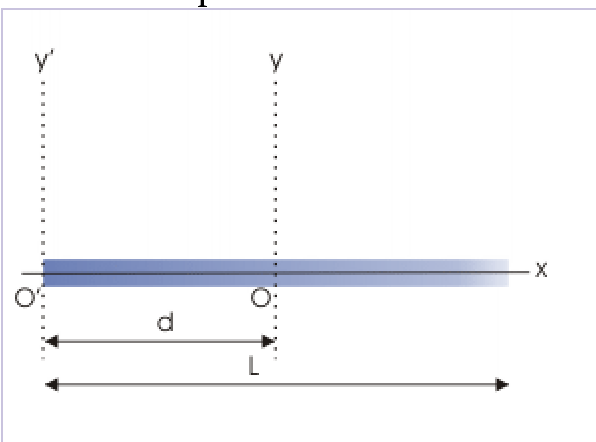
$$I = I_C + Md^2$$

Application of theorem of parallel axes

We shall illustrate application of the theorem of parallel axes to some of the regular rigid bodies, whose MI about an axis passing through center of mass is known.

(i) Rod : about an axis perpendicular to the rod and passing through one of its ends

Theorem of parallel axes



MI of rod about an axis perpendicular to the rod and passing through one of its ends.

MI about perpendicular bisector is known and is given by :

$$I_C = \frac{ML^2}{12}$$

According to theorem of parallel axes,

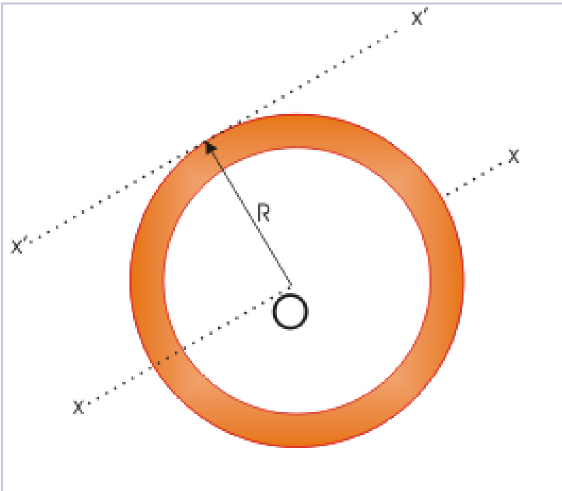
$$I = I_C + Md^2 = \frac{ML^2}{12} + M \text{ x } \left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$$

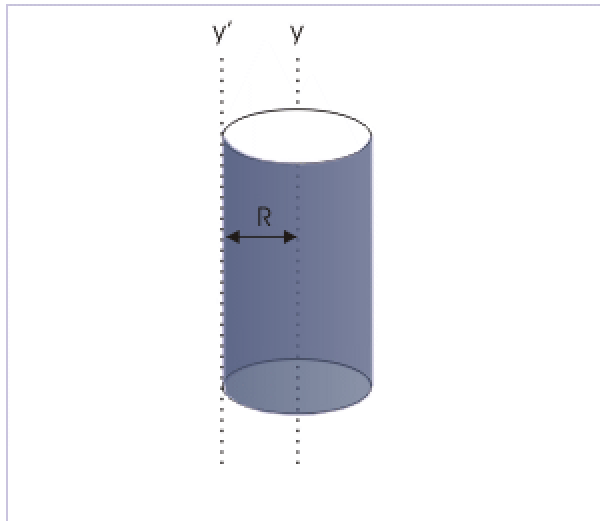
(ii) Circular ring : about a line tangential to the ring

Theorem of parallel axes

MI of circular ring about a line tangential to the ring.

MI of hollow cylinder about an axis tangential to the surface.





MI about an axis passing through the center and perpendicular to the ring is known and is given by :

$$I_C = MR^2$$

According to theorem of parallel axes,

$$I = I_C + Md^2 = MR^2 + MR^2 = 2MR^2$$

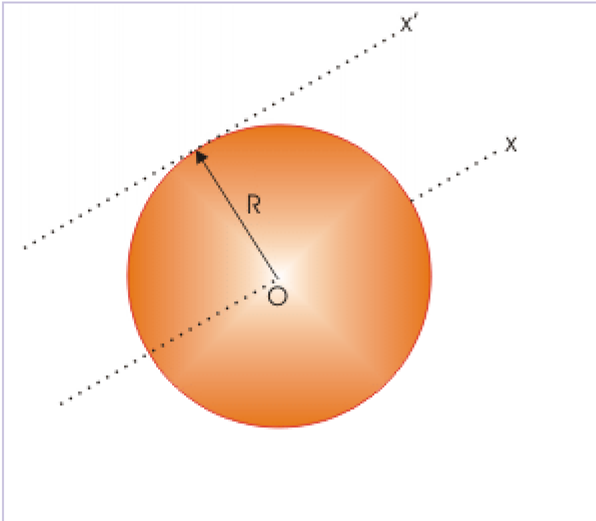
Since hollow cylinder has similar expression of MI, the expression for the MI about a line tangential to its curved surface is also same.

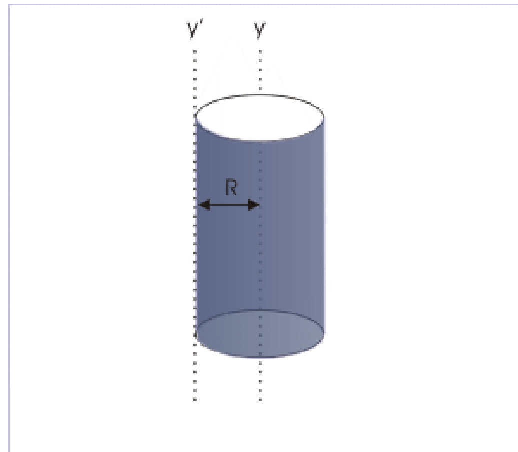
(i) Circular plate : about a line perpendicular and tangential to the circular plate

Theorem of parallel axes

MI of Circular plate about a line perpendicular and tangential to it.

MI of solid cylinder about a line tangential to it.





MI about an axis passing through the center and perpendicular to the circular plate is known and is given by :

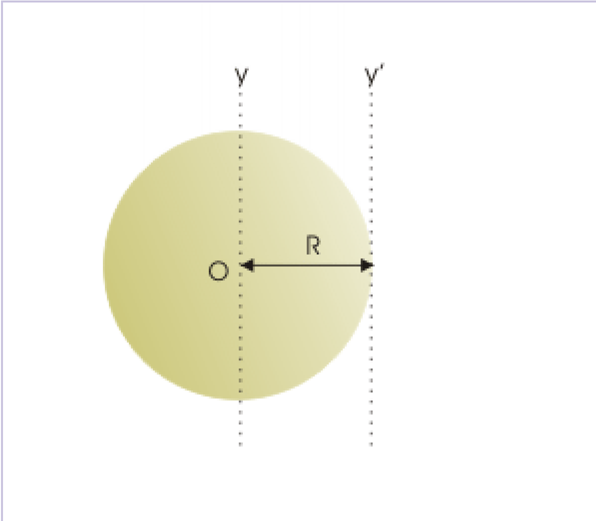
$$I_C = \frac{MR^2}{2}$$

According to theorem of parallel axes,

$$I = I_C + Md^2 = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}$$

Since solid cylinder has similar expression of MI, the expression for the MI about a line tangential to its curved surface is also same.

(iii) Hollow sphere : about a line tangential to the hollow sphere
Theorem of parallel axes



MI of hollow sphere about a line tangential to it.

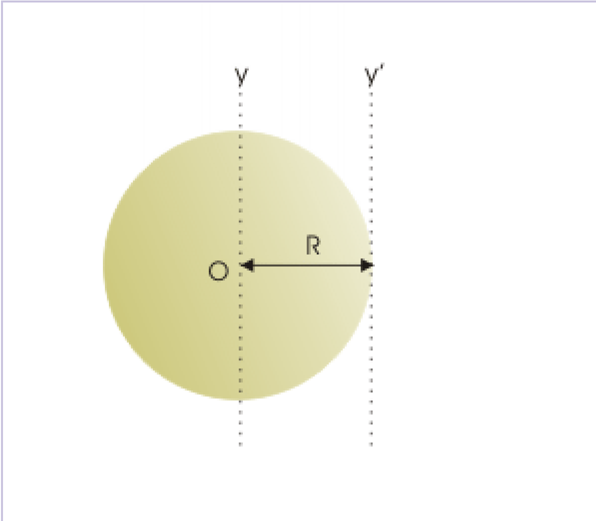
MI about one of its diameters is known and is given by :

$$I_C = \frac{2ML^2}{3}$$

According to theorem of parallel axes,

$$I = I_C + Md^2 = \frac{2MR^2}{3} + ML^2 = \frac{5ML^2}{3}$$

(iv) Solid sphere : about a line tangential to the solid sphere
Theorem of parallel axes



MI of solid sphere about a line tangential to it.

MI about one of its diameters is known and is given by :

$$I_C = \frac{2ML^2}{5}$$

According to theorem of parallel axes,

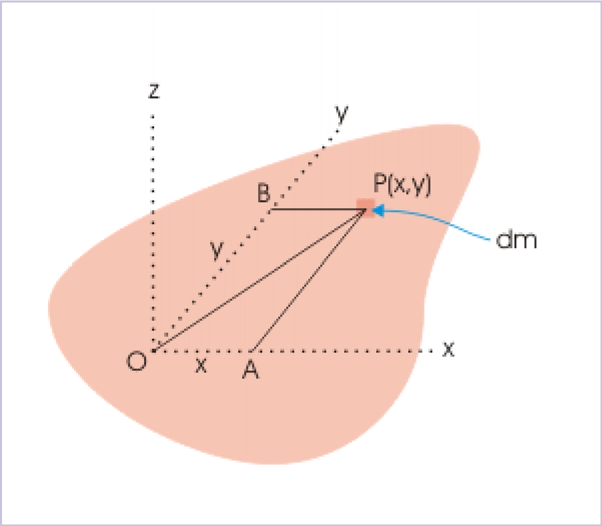
$$I = I_C + Md^2 = \frac{2MR^2}{5} + MR^2 = \frac{7ML^2}{5}$$

Theorem of perpendicular axes

Theorem of perpendicular axes is applicable to planar bodies only. According to this theorem, moment of inertia of a planar rigid body about a perpendicular axis (z) is equal to the sum of moments of inertia about two mutually perpendicular axes (x and y) in the plane of the body, which are themselves perpendicular to the z - axis.
Equation:

$$I_z = I_x + I_y$$

Theorem of perpendicular axes



MI of a plate of irregular shape about an axis perpenndicular to it.

Three axes involved in the theorem are mutually perpendicular to each other and form the rectangular coordinate system. It should be noted that there is no restriction that axes pass through center of mass of the rigid body as in the case of parallel axis theorem. In order to prove the theorem in general, we consider a planar body of irregular shape. We consider an elemental mass "dm" at P (x,y) as shown in the figure above.

The moment of inertia of the body about z - axis is given as :

$$I = \int r^2 đm$$

We drop two perpendiculars in the plane of body on "x" and "y" axes. Then,

$$r^2 = x^2 + y^2$$

Substituting in the equation of moment of inertia,

$$I_z = \int \left(x^2 + y^2 \right) đm = \int x^2 đm + y^2 đm$$

The expressions on the right hand side represent moments of inertia of the body about "x" and "y" axes. Hence,

$$I_z = I_x + I_y$$

Application of theorem of perpendicular axes

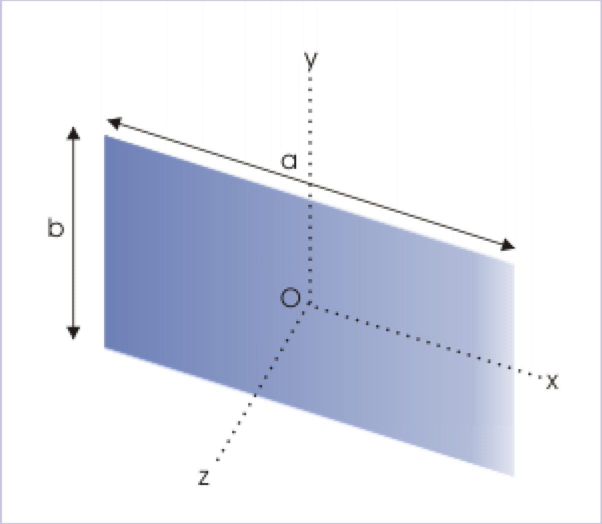
We shall determine MIs of objects about other axes, which are not symmetric to the body. As pointed out earlier, we shall limit the application of the theorem of perpendicular axes to planar objects like rectangular plate, ring and circular plate etc.

(i) Rectangular plate : about a line passing through center of mass and perpendicular to the plane

Moment of inertia about a line passing through center of mass and parallel to one of the side is given as :

$$I_x = \frac{Mb^2}{12}$$
$$I_y = \frac{Ma^2}{12}$$

Theorem of perpendicular axes



Moment of inertia of a rectangular plate about a line

passing through center of mass
and perpendicular to its surface.

Applying theorem of perpendicular axes, we have :

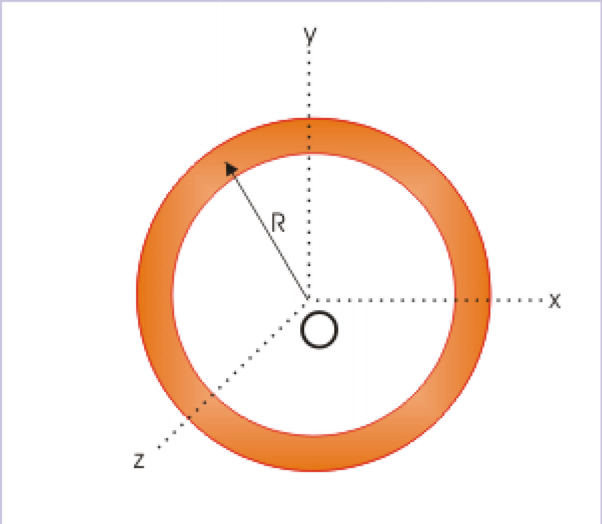
$$I_z = I_x + I_y = \frac{Mb^2}{12} + \frac{Ma^2}{12}$$
$$\Rightarrow I_z = \frac{M\left(a^2+b^2\right)}{12}$$

(ii) Circular ring : about one of the diameter

Moment of inertia about a perpendicular line passing through center of ring is given as :

$$I_z = MR^2$$

Theorem of perpendicular axes



Moment of inertia of a circular
ring about one of the diameters.

We consider two mutually perpendicular diameters as "x" and "y" axes of the cordinate system. Now, applying theorem of perpendicular axes, we have :

$$I_z = I_x + I_y$$

However, the circular ring has uniform mass distribution, which is symmetrically distributed about any diameter. Thus, MIs about the axes "x" and "y" are equal,

$$I_x + I_y = 2I_x$$

Thus,

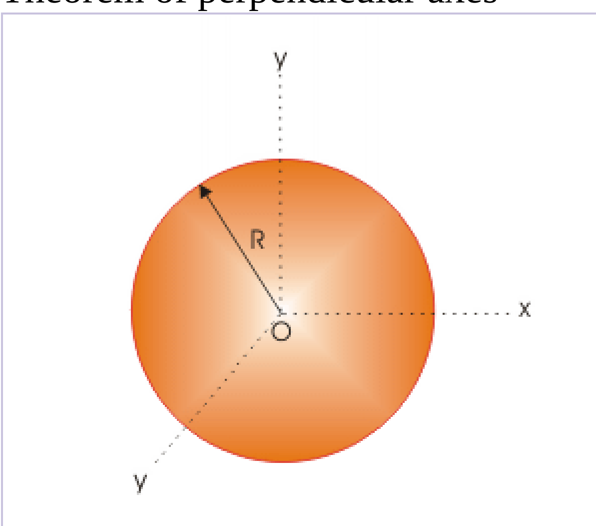
$$\Rightarrow 2I_x = I_z = MR^2$$
$$\Rightarrow I_x = \frac{MR^2}{2}$$

(iii) Circular plate : about one of the diameter

Moment of inertia about a perpendicular line passing through center of circular plate is given as :

$$I_z = MR^2$$

Theorem of perpendicular axes



Moment of inertia of a circular plate about one of the diameters.

We consider two mutually perpendicular diameters as shown in the figure. These diameters can be treated as two coordinates of the plane of the circular plate. Now, applying theorem of perpendicular axes, we have :

$$I_z = I_x + I_y$$

However, the circular plate has uniform mass distribution, which is symmetrically distributed about any diameter. Thus, MIs about the planar axes "x" and "y" are equal,

$$I_x + I_y = 2I_x$$

Thus,

$$\Rightarrow 2I_x = I_z = \frac{MR^2}{2}$$
$$\Rightarrow I_x = \frac{MR^2}{4}$$

Summary

1. Theorem of perpendicular axes

The mathematical expression of this theorem is given as :

$$I = I_C + md^2$$

where I_C is MI of the body about an axis passing through center of mass, "C"; I is MI of the body about an axis parallel to the axis passing through center of mass; "M" is the mass of the rigid body and "d" is the perpendicular distance between two parallel axes.

2. Theorem of parallel axes is valid for all shapes (regular and irregular).

3. Theorem of parallel axes expresses MI about a parallel axis in terms of the MI about a parallel axis passing through COM. Thus, we should ensure that one of the two parallel axes be through the COM of the rigid body.

4. Theorem of perpendicular axes

According to this theorem, moment of inertia of a planar rigid body about a perpendicular axis (z) is equal to the sum of moments of inertia about two mutually perpendicular axes (x and y) in the plane of the body, which are also perpendicular to the axis along z - axis.

$$I_z = I_x + I_y$$

4. Theorem of perpendicular axes is valid only for all planar shapes (regular and irregular).

Theorems on moment of inertia (Application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the theoretical treatment of the topic. The idea is to provide a verbose explanation of the solution, detailing the application of theory. Solution presented here, therefore, is treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the application of theorem on moment of inertia. For this reason, questions are categorized in terms of the characterizing features pertaining to the questions :

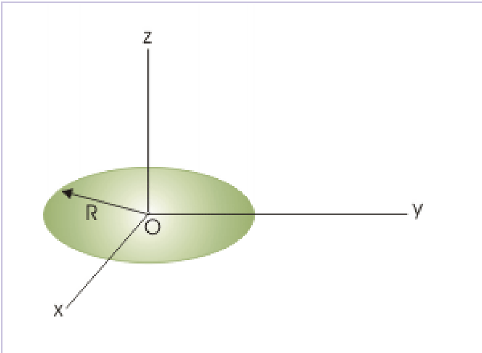
- Theorems on moment of inertia
- Nature of MI
- Part of a rigid body
- Geometric shapes formed from wire

Theorems on moment of inertia

Example 1

Problem : A uniform circular disk of radius "R" lies in "xy" - plane such that its center is at the origin of coordinate system as shown in the figure. Its MI about the axis defined as x = R and y = 0 is equal to its MI about axis defined as y = d and z = 0. Then, find “d”.

MI of uniform circular disk

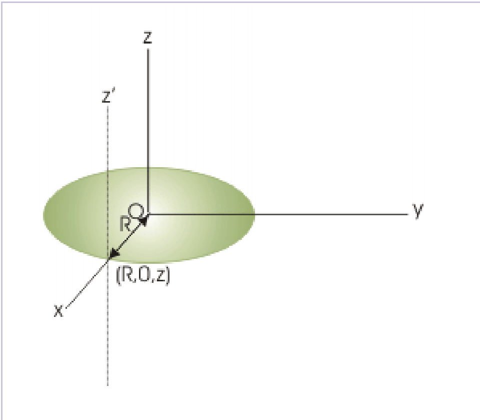


The circular disk lies in “xy” - plane.

Solution : First, we need to visualize the two axes as defined. Then, we find disk's MI about them. Finally, we equate the MIs as given in the question to find "d".

The coordinates x = R and y = 0 define the axis in "xz" - plane as "y" coordinate is zero. Further, the axis intersects x-axis at x = R = a constant. This axis (z'), therefore, is parallel to z-axis as shown in the figure.

MI of uniform circular disk



The location of first axis is on x-axis.

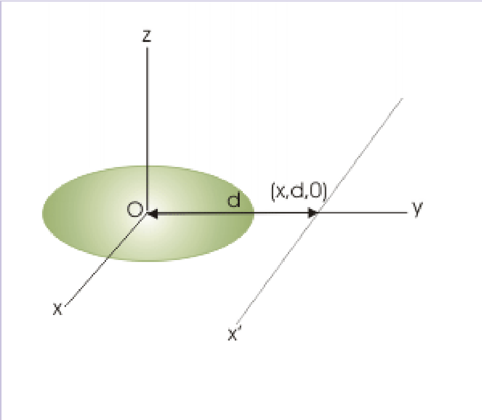
Note that z axis passes through center of mass. Thus by applying theorem of parallel axes, the MI of the disk about z' by the theorem of parallel axes is :

$$I_{\text{z'}} = I_z + MR^2$$

$$I_{\text{z'}} = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}$$

The coordinates y = d and z = 0 define the axis in "xy" - plane as "z" coordinate is zero. Further, this axis intersects y-axis at y = d = a constant. This axis (x'), therefore, is parallel to x-axis as shown in the figure.

MI of uniform circular disk



The location of second axis is on y-axis.

The MI of the disk about x' by the theorem of parallel axes :

$$I_{\mathrm{x'}} = I_x + MR^2$$

$$I_{\mathrm{x'}} = \frac{MR^2}{4} + Md^2$$

According to the question,

$$I_{\mathrm{x'}} = I_{\mathrm{z'}}$$

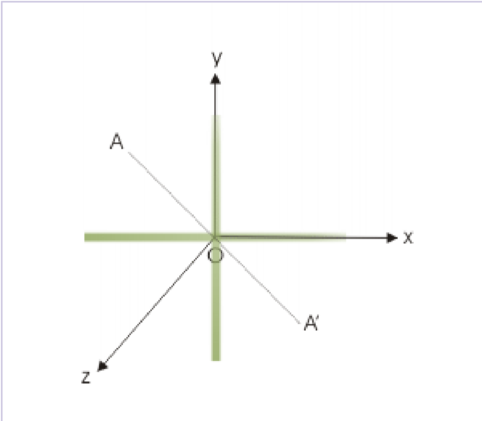
$$\frac{MR^2}{4} + Md^2 = \frac{3MR^2}{2}$$

$$Md^2 = \frac{3MR^2}{2} - \frac{MR^2}{4} = \frac{5MR^2}{4}$$

$$d = \frac{\sqrt{5}R}{2}$$

Example 2

Problem : Two identical uniform rods, each of mass "M" and length "L", are joined together to form a cross. Find the MI of the cross about the angle bisector "AA'" as shown in the figure.
MI of a cross



The MI of a cross of two rods at right angle.

Solution : MI of each of the rod about perpendicular axis, which passes through COM (O) is :

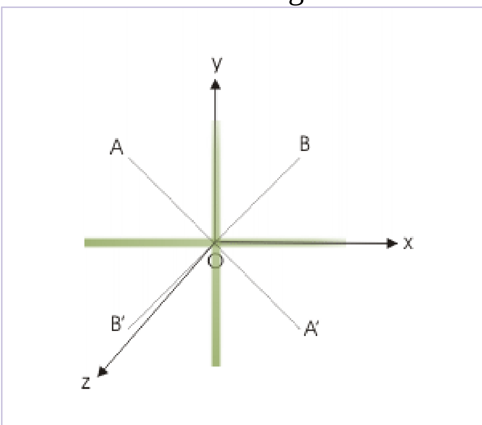
$$I_{\text{rod}} = \frac{ML^2}{12}$$

The MI of two rods joined together as a cross about perpendicular axis through the common point is :

$$\Rightarrow I_{\text{cross}} = 2I_{\text{rod}} = 2 \text{ x } \frac{ML^2}{12} = \frac{ML^2}{6}$$

The cross can be treated as a planar object. As such, we can apply theorem of perpendicular axes to find MI about the bisector of the right angle. We consider another axis BB' perpendicular to AA' in the plane of cross. Then, according to theorem of perpendicular axes,

MI of a cross about angle bisector



The MI of a cross of two rods at right angle.

$$I_{\text{cross}} = I_{\text{AA'}} + I_{\text{BB'}}$$

The MIs about the two bisectors are equal by the symmetry of the cross about them. Hence,

$$I_{\text{AA'}} = I_{\text{BB'}}$$

Thus,

$$\Rightarrow I_{\text{cross}} = I_{\text{AA'}} + I_{\text{BB'}} = 2I_{\text{AA'}}$$

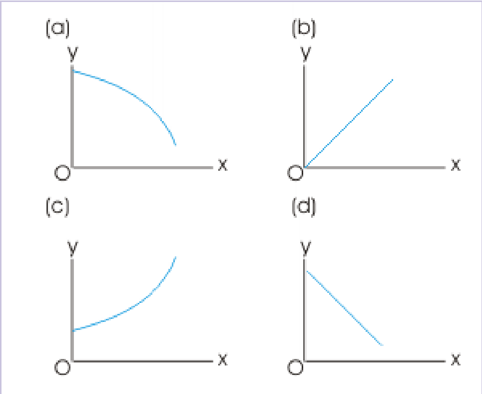
$$\Rightarrow I_{\text{AA'}} = \frac{I_{\text{cross}}}{2} = \frac{ML^2}{12}$$

Nature of MI

Example 3

Problem : The MI of a solid sphere is calculated about an axis parallel to one of its diameter, at a distance "x" from the origin. The values are plotted against distance. Which of the four plots, as shown below, correctly describes the variation of MI with respect to "x" for the solid sphere.

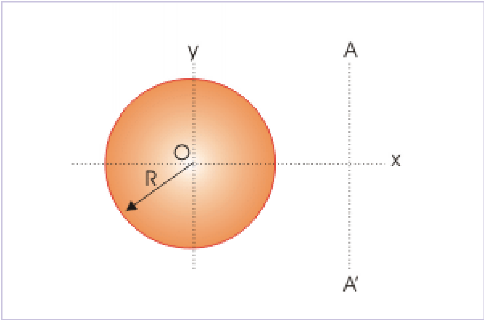
Plots of MI .vs. distance



The MI of a sphere about an axis parallel to central axis.

Solution : It is evident that farther the axis from the origin, greater is MI about that axis. We can, thus, conclude that the plot between MI and the distance should be increasing one with increasing distance ("x"). Thus, options (a) and (d) are incorrect. Now, the MI of solid sphere about an axis parallel to one of diameters, at a distance "x" from the origin is given as :

Plots of MI .vs. distance



The MI of a sphere about an axis parallel to central axis.

$$I = I_O + Mx^2$$

where I_O is MI about one of the diameters and is a constant for the given solid sphere. Thus, the equation above takes the form of the equation of parabola,

$$y = mx^2 + c$$

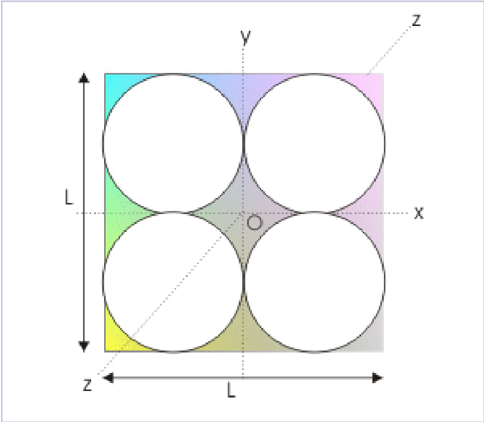
The correct plot, therefore, is an increasing parabola as shown in Figure (c).

Part of a rigid body

Example 4

Problem : Four holes of radius "L/4" are made in a thin square plate of mass "M" and side "L" in "xy" - plane, as shown in the figure. Find its MI about z – axis.

MI of remaining body



Four holes of radius "L/4" are made in a thin square plate of mass "M" and side "L."

mass "M" and side "L".

Solution : MI is a scalar quantity. It means that MI has no directional attribute. Thus, we can conclude that MI of the complete plate is equal to the sum of MIs of remaining plate and that of the four circular disks taken out from the plate about z-axis. Hence, MI of remaining plate is obtained as :

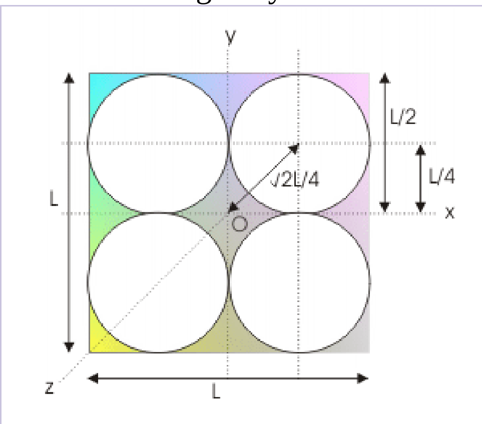
$$\text{I (MI of remaining plate)} = \text{Iz (MI of complete plate)} - 4 \text{ x } \text{Icz (MI of circular disks)}$$

Now, according to theorem of perpendicular axes, the MI of complete square about z-axis is two times its MI about x-axis :

$$\Rightarrow I_z = 2 \text{ x } \frac{ML^2}{12} = \frac{ML^2}{6}$$

Further, according to theorem of parallel axes, the MI of circular disk of mass "m" and radius "L/4" about the same z-axis is :

MI of remaining body



Four holes of radius "L/4" are made in a thin square plate of mass "M" and side "L".

$$\Rightarrow I_{\text{cz}} = m\left(\frac{L}{4}\right)^2 + m\left(\frac{\sqrt{2}L}{4}\right)^2$$

$$\Rightarrow I_{\text{cz}} = \frac{mL^2}{16} + \frac{2mL^2}{16} = \frac{3mL^2}{16}$$

The mass of the disk "m" is given by multiplying areal density with the area of the circular disk,

$$m = \frac{M}{L^2} \text{ x } \pi\left(\frac{L}{4}\right)^2 = \frac{\pi M}{16}$$

Substituting for "m", we have :

$$\Rightarrow I_{\text{cz}} = \frac{3mL^2}{16} = \frac{3\pi ML^2}{16\text{x}16}$$

Thus, the required MI is :

$$\Rightarrow I = \frac{ML^2}{6} - 4\text{ x } \frac{3\pi ML^2}{16\text{x}16} = \frac{16\text{x}16ML^2 - 12\text{x}6\pi ML^2}{6\text{x}16\text{x}16}$$

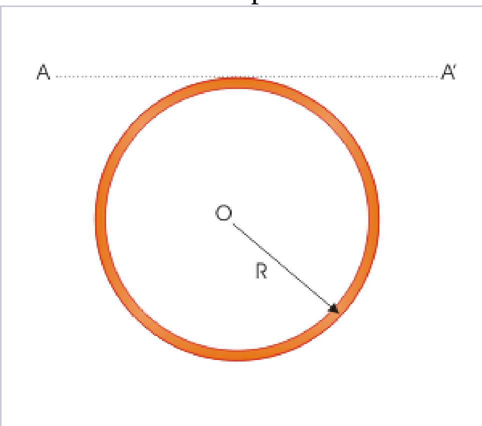
$$\Rightarrow I = \frac{32ML^2 - 9\pi ML^2}{192}$$

Geometric shapes formed from wire

Example 5

Problem : A thin wire of length "L" and uniform linear mass density "λ" is bent into a circular loop. Find the MI of the loop about an axis AA' as shown in the figure.

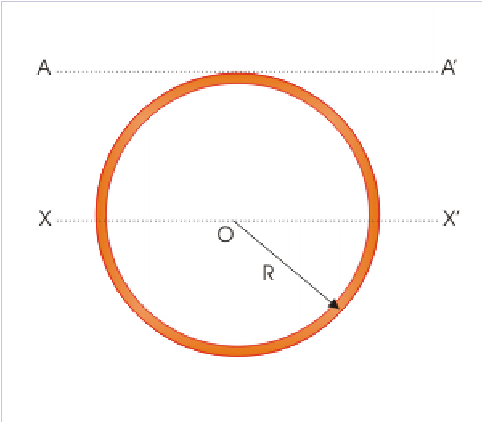
MI of a circular loop



MI of a circular loop about a tangent.

Solution : The MI of the circular loop about a tangential axis AA' is found out by the theorem of parallel axis :

MI of a circular loop



MI of a circular loop about a tangent.

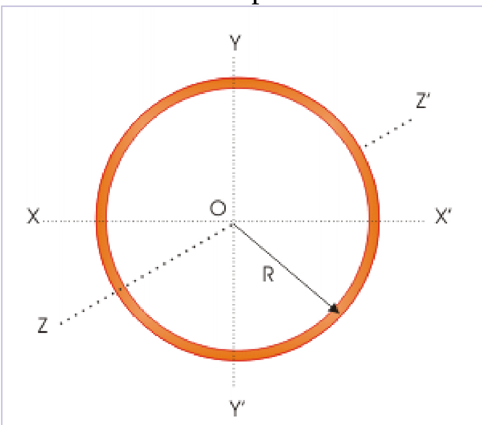
$$I_{\mathrm{AA'}} = I_{\mathrm{XX'}} + MR^2$$

where IXX' is the MI about an axis which is one of the diameters of the ring and is parallel to tangential axis AA'. Now, MI about the diameter can be calculated applying theorem of perpendicular axes :

$$\Rightarrow I_{\mathrm{XX'}} = \frac{I_O}{2}$$

where I_O is MI about an axis perpendicular to the surface of loop and passing through center of mass (ZZ' axis). It is given by the expression :

MI of a circular loop



MI of a circular loop about its perpendicular central axis.

$$\Rightarrow I_O = MR^2$$

Combining two equations, we have :

$$\Rightarrow I_{\text{XX'}} = \frac{MR^2}{2}$$

Putting this in the expression of $I_{\text{AA'}}$, we have :

$$\Rightarrow I_{\text{AA'}} = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}$$

Now, we are required to find mass and radius from the given data. Here,

$$M = \lambda L$$

Since the wire is bent into a circle, the perimeter of the circle is equal to the length of the wire. Hence,

$$2\pi R = L$$

$$\Rightarrow R = \frac{L}{2\pi}$$

Putting these values in the expression of MI about AA' axis is :

$$\Rightarrow I_{\text{AA'}} = \frac{3MR^2}{2} = \frac{3\lambda L \text{x} L^2}{2(2\pi)^2} = \frac{3\lambda L^3}{8\pi^2}$$

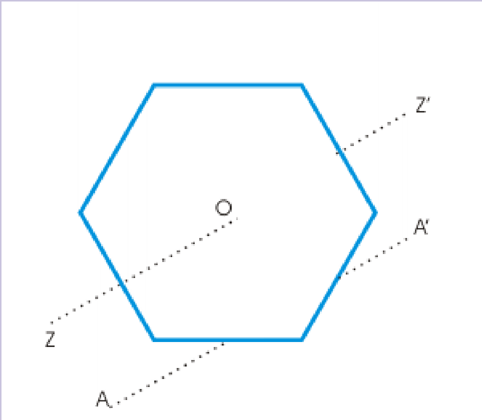
Example 6

Problem : A uniform wire of mass "M" and length "L" is bent into a regular hexagonal loop. Find its MI about an axis passing through center of mass and perpendicular to its surface.

Solution : We find MI of hexagonal loop in following three steps :

1. Treat each side as a wire of mass M/6 and length L/6 and find its MI about perpendicular axis through center of mass of the side.
2. Apply theorem of parallel axes to find MI of the side about perpendicular axis through center of mass of the loop.
3. Add MIs of individual sides to find the MI of the loop about perpendicular axis through center of mass.

The MI of the side about perpendicular axis through center of mass of the side is :
MI of a regular hexagonal loop

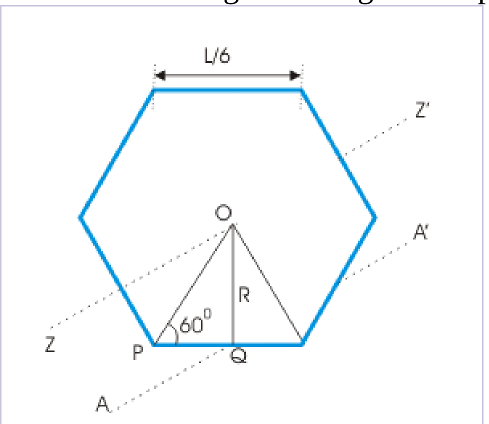


MI of a regular hexagonal loop
is sum of MIs of its sides.

$$\Rightarrow I_{\text{AA'}} = \frac{\frac{M}{6} \text{x} \left(\frac{L}{6}\right)^2}{12} = \frac{ML^2}{12\text{x}6^3}$$

By the geometry of a side with respect to center, the traingle OPQ is a right angle triangle. We have perpendicular distance "R" between axes AA' and ZZ' as :

MI of a side of regular hexagonal loop



MI of a regular hexagonal loop
is sum of MIs of its sides.

$$\Rightarrow \tan 60^\circ = \frac{R}{\frac{L}{12}}$$

$$\Rightarrow R = \frac{L}{12} \text{ x } \sqrt{3} = \frac{\sqrt{3}}{12} L$$

Applying theorem of parallel axes, the MI about perpendicular axis through center of mass of the loop is :

$$I_O = I_{\text{zz'}} = I_{\text{AA'}} + \frac{M}{6} \text{ x } R^2$$

$$\Rightarrow I_O = \frac{ML^2}{12\text{x}6^3} + \frac{M}{6} \text{ x } \left(\frac{\surd 3L}{12}\right)^2$$

$$\Rightarrow I_O = \frac{ML^2}{12\text{x}6^3} + \frac{3ML^2}{12^2\text{x}6}$$

$$\Rightarrow I_O = \frac{ML^2+9ML^2}{12\text{x}6^3} = \frac{10ML^2}{12\text{x}6^3}$$

Now, adding MIs of each side, the required MI about perpendicular axis through center of mass of the loop is :

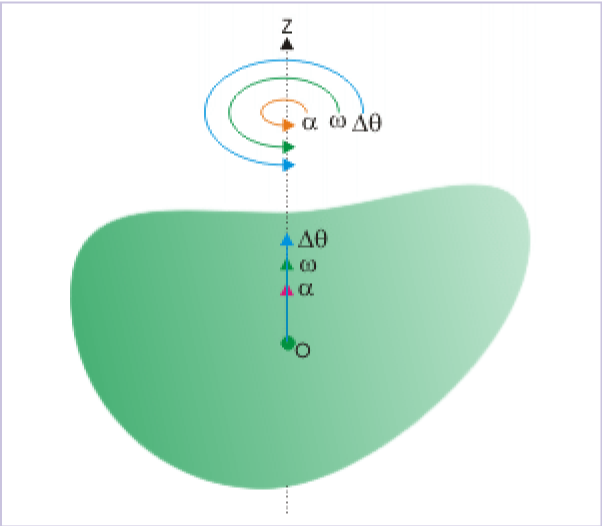
$$\Rightarrow I = 6I_O = \frac{6\text{x}10ML^2}{12\text{x}6^3} = \frac{5ML^2}{216}$$

Work - kinetic energy theorem for rotational motion
Rotation of a particle or rigid body about a fixed axis has unique motional attributes like angular position, displacement, velocity and acceleration.

There is one to one correspondence between quantities in pure translation and pure rotation. We can actually write rotational relations for work and kinetic energy by just inspecting corresponding relation of pure translational motion.

Beside similarity of relations, there are other similarity governing description of pure motions in translation and rotation. We must, first of all, realize that both pure motions are one-dimensional motions in description. It might sound a bit bizarre for rotational motion but if we look closely at the properties of vector attributes that describe pure rotational motion, then we would realize that these quantities are indeed one - dimensional.

Rotational quantities like angular displacement (θ), speed (ω) and accelerations (α) are one - dimensional quantities for rotation about a fixed axis. These vector attributes are either anticlockwise (positive) or clockwise (negative). If it is anticlockwise, then the corresponding rotational vector quantity is in the direction of axis; otherwise in the opposite direction.
Rotation

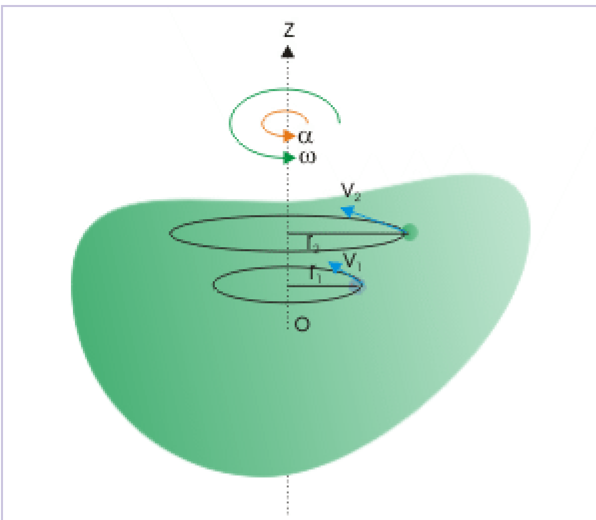


Rotational quantities like angular displacement (θ), speed (ω) and accelerations (α) are

one - dimensional quantities for rotation about fixed axis.

We observe yet another similarity between two pure motions. We have seen that the pure translation of a particle or a rigid body has unique (single value) motional attributes like position, displacement, velocity and acceleration. In the case of rotation, on the other hand, even if individual particles constituting the rigid body have different linear velocities or accelerations, but corresponding rotational attributes of the rotating body, like angular velocity and acceleration, have unique values for all particles.

Rotation



Individual particles constituting the rigid body have different linear velocities/ acceleration, but they have unique angular velocity/ acceleration.

There are, as a matter of fact, more such similarities. We shall discuss them in the appropriate context as we further develop dynamics of pure rotational motion.

Expressions of work and energy

The expressions of work in pure translation in x- direction (one dimensional linear motion) is given by :

$$đW = F_x đx$$
$$\Rightarrow W = \int F_x đx$$

The corresponding expressions of work in pure rotation (one dimensional rotational motion) is given by :

Equation:

$$đW = \tau đ\theta$$
$$\Rightarrow W = \int \tau đ\theta$$

Obviously, torque replaces component of force in the direction of displacement (Fx) and angular displacement (θ) replaces linear displacement (x). Similarly, the expression of kinetic energy as derived earlier in the course is :

Equation:

$$K = \frac{1}{2} I \omega^2$$

Here, moment of inertia (I) replaces linear inertia (m) and angular speed (ω) replaces linear speed (v). We, however, do not generally define gravitational potential energy for rotation as there is no overall change in the position (elevation) of the COM of the body in pure rotation. This is particularly the case when axis of rotation passes through COM.

This understanding of correspondence of expressions helps us to remember expressions of rotational motion, but it does not provide the insight into the angular quantities used to describe pure rotation. For example, we can not understand how torque accomplishes work on a rigid body, which does not have any linear displacement ! For this reason, we shall develop

expressions of work by considering rotation of a particle or particle like object in the next section.

We need to emphasize here another important underlying concept. The description of motion in pure translation for a single particle and a rigid body differ in one very important aspect. Recall that the motion of a rigid body, in translation, is described by assigning motional quantities to the "center of mass(COM)". If we look at the expression of center of mass in x-direction, then we realize that the concept of COM is actually designed to incorporate distribution of mass in the body.

Equation:

$$x_{\text{COM}} = \frac{\int x đm}{M}$$

We must note here that such distinction arising due to distribution of mass between a particle and rigid body does not exist for pure rotation. It is so because the effect of the distribution of mass is incorporated in the definition of moment of inertia itself :

Equation:

$$I = \int r^2 đm$$

For this reason, there is no corresponding "center of rotational inertia" or "center of moment of inertia" for rigid body in rotation. This is one aspect in which there is no correspondence between two motion types. It follows, then, that the expressions of various quantities of a particle or a rigid body in rotation about a fixed axis should be same. Same expressions would, therefore, determine work and kinetic energy of a particle or a rigid body. The difference in two cases will solely arise from the differences in the values of moments of inertia.

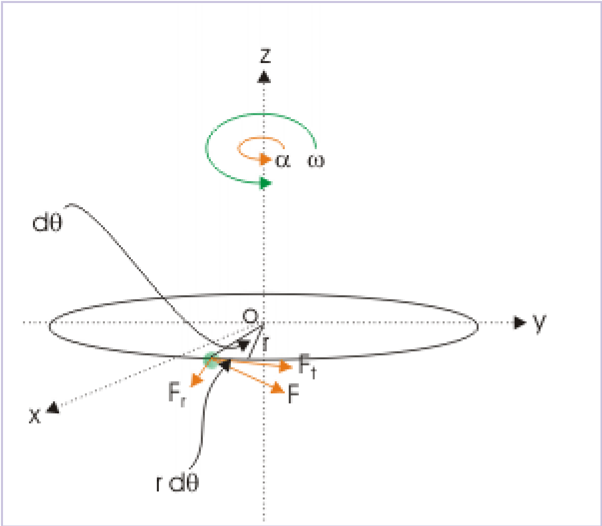
Work done in rotation

We consider a particle like object, attached to a "mass-less" rod, to simulate rotation of a particle. The particle like object rotates in a plane

perpendicular to the plane of screen or paper as shown in the figure. Here, we shall evaluate work by a force, "F", in xy-plane, which is perpendicular to the axis of rotation.

We know that it is only the tangential component of force that accelerates the particle and does the work. The component of force perpendicular to displacement does not perform work. For small linear displacement "ds", we have :

Rotation



A particle like object in rotation.

$$\text{đ}W = F_x \text{đ}x = F_t \text{đ}s$$

By geometry,

$$= r\text{đ}\theta$$

Combining two equations, we have :

$$\text{đ}W = F_t r \text{đ}\theta = \tau \text{đ}\theta$$

Work done by a torque in rotating the particle like body by an angular displacement, we have :

$$\Rightarrow W = \int \tau đ\theta$$

If work is done by a constant torque, then we can take torque out of the integral sign and :

$$\Rightarrow W = \tau \int đ\theta = \tau \Delta\theta$$

As discussed earlier, this same expressions is valid for work done on a rigid body in pure rotation.

Power in rotation

Power measures the rate at which external torque does work on a particle like object or a rigid body. It is equal to the first time differential of work done,

Equation:

$$P = \frac{đW}{đt} = \frac{\tau đ\theta}{đt} = \tau\omega$$

We must note the correspondence of this expression with its linear counterpart, where :

$$P = Fv$$

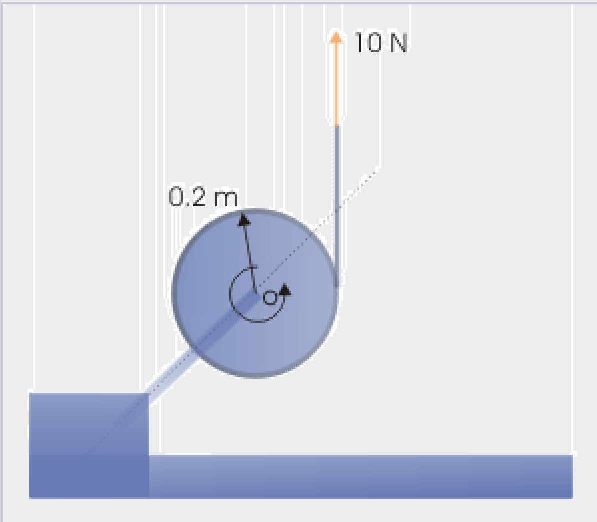
Clearly, torque (τ) replaces force (F) and angular speed (ω) replaces linear speed (v) in the expression of power.

Example:

Problem : A uniform disk of mass 5 kg and radius 0.2 m is mounted on a horizontal axle to rotate freely. One end of a rope, attached to the disk, is

wrapped around the rim of the disk. If a force of 10 N is applied vertically as shown, then find the work done by the force in rotating the disk for 2 s, starting from rest at t = 0 s.

Rotation



A uniform disk is mounted on a horizontal axle.

Solution : Here, torque due to vertical force is constant. Thus, in order to determine the work on the disk in rotation, we need to use the expression of work done by a constant torque as :

$$W = \tau\Delta\theta = \tau(\theta_2 - \theta_1)$$

It means that we need to calculate torque and angular displacement. Here, torque (positive as rotation is anti-clockwise) is given by :

$$\tau = F_t r = 10 \text{ x } 0.2 = 2 \text{ N-m}$$

This is a rotational motion with constant acceleration (due to constant torque). Thus, we can use the equation of motion for constant angular acceleration to find the displacement :

$$\theta_f = \theta_i t + \frac{1}{2}\alpha t^2$$

Here, disk is at rest in the beginning, θi = 0. Hence,

$$\Rightarrow \theta_f = \frac{1}{2}\alpha t^2$$

But, we do not know the angular acceleration. We can, however, use corresponding Newton's second law for rotation. Since, disk rotates freely, we can assume that there is no other torque involved :

$$\Rightarrow \alpha = \frac{\tau}{I}$$

On the other hand, the moment of inertia for the disk about an axis passing through the center of mass and perpendicular to its surface is given by :

$$I = \frac{MR^2}{2} = \frac{5 \text{ x } 0.2^2}{2} = 0.1 \text{ kg} - m^2$$

Putting this value of moment of inertia, we get the value of angular acceleration :

$$\Rightarrow \alpha = \frac{\tau}{I} = \frac{2}{0.1} = 20 \ \frac{\text{rad}}{s^2}$$

Now final angular position is :

$$\theta_f = \frac{1}{2} \text{ x } 20 \text{ x } 2^2 = 40 \text{ rad}$$

Hence, work done is :

$$W = \tau(\theta_2 - \theta_1) = \tau\theta_2 = 2 \text{ x } 40 = 80 \ J$$

Work - kinetic energy theorem for rotational motion

We have seen earlier that net work on a body is equal to change in its kinetic energy.

In translation, we have :

$$W = \Delta K = \frac{1}{2} m {v_f}^2 - \frac{1}{2} m {v_i}^2$$

A corresponding work - kinetic energy theorem for pure rotation, therefore, is :

Equation:

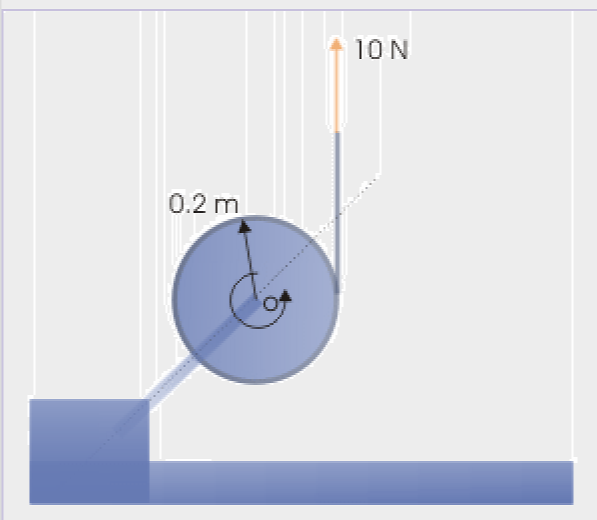
$$W = \Delta K = \frac{1}{2} I {\omega_f}^2 - \frac{1}{2} I {\omega_i}^2$$

where "W" represents the net work on the rigid body in rotation.

Example:

Problem : A uniform disk of mass 5 kg and radius 0.2 m is mounted on a horizontal axle to rotate freely. One end of a rope, attached to the disk, is wrapped around the rim of the disk. If a force of 10 N is applied vertically as shown, then find the work done by the force in rotating the disk for 2 s, starting from rest at t = 0 s. Use work - kinetic energy theorem to find the result.

Rotation



A uniform disk is mounted on a horizontal axle.

Solution : This is the same question as the previous one except that we are required to solve the question, using work-kinetic energy theorem :

$$W = \Delta K = \frac{1}{2} I {\omega_f}^2 - \frac{1}{2} I {\omega_i}^2$$

From the previous example, we have :

$$I = \frac{MR^2}{2} = \frac{5 \text{ x } 0.2^2}{2} = 0.1 \text{ kg} - m^2$$

and

$$\alpha = \frac{\tau}{I} = 20 \ \frac{\text{rad}}{s^2}$$

Now, as the disk starts from rest, $\omega_i = 0$. In order to find the final angular speed, we use equation of motion for constant angular acceleration :

$$\omega_f = \omega_i + \alpha t = 0 = 2 \text{ x } 20 = 40 \ \text{rad} / s$$

Thus,

$$W = \Delta K = \frac{1}{2} I {\omega_f}^2 - 0 = \frac{1}{2} I {\omega_f}^2$$
$$\Rightarrow W = \frac{1}{2} \text{ x } 0.1 \text{ x } 40^2 = 80 \ J$$

As expected, the result is same as calculated in earlier example.

Summary

1: Rotational quantities describing the motion are linear (one-dimensional) vectors.

2: Work done by a torque is given by :

$$W = \int \tau đ\theta$$

3: Work done by a constant torque is given by :

$$W = \tau \Delta \theta$$

4: We generally do not consider change in gravitational potential energy in the case of rotation about an axis that passes through its center of mass. Since there is no change in the position of center of mass, there is no corresponding change in its potential energy.

Rolling motion
Rolling motion is a composite motion, in which the body rotates about a moving axis as it translates from one position to another.

Rolling motion is a composite motion, in which a body undergoes both translational and rotational motion. The wheel of a car, for example, rotates about a moving axis and at the same time translates a net distance. Rolling motion being superposition of two types of motion, presents a different context of motion, which needs visualization from different perspectives. Importantly, different parts of the wheel are moving with different velocities, depending on their relative positions with respect to the center of mass or the point of contact with the surface.

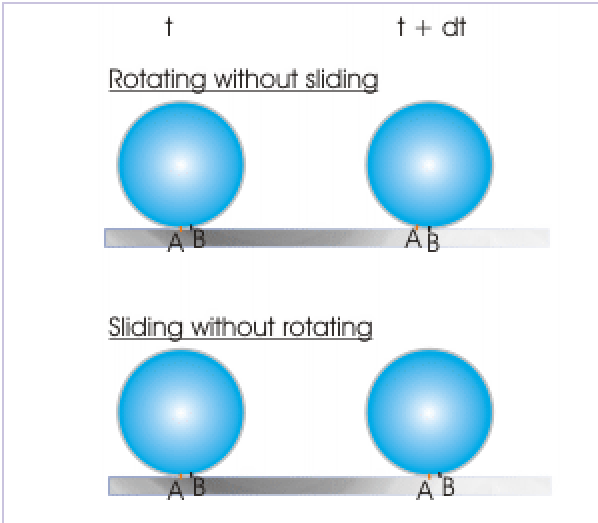
In this module, we shall examine rolling motion of a circular disc along a straight line. This study shall be valid for all other circular or spherical bodies capable of rolling such as sphere, cylinder, ring etc.

Characterizing features of rolling motion

In order to bring out characterizing aspects of rolling motion, we consider a disk, which is rolling without sliding (simply referred as rolling) smoothly on a horizontal surface such that its center of mass translates with a velocity "v_C" in x-direction.

Rolling without sliding

"Rolling without sliding" means that the point on the rim in contact with the surface changes continuously as the disk rotates while translating ahead. If point "A" is in contact at a given time "t", then another neighboring point "B" takes up the position immediately after, say, at a time instant, t+dt. In case the disk slides while translating, the point of contact remains same at the two time instants. The two cases are illustrated in the figure here.
Rolling without sliding



Comparison between "rolling without sliding" and "siliding without rolling".

The terms “Rolling without sliding”, "pure rolling" or simply "rolling" refer to same composite motion along a straight line.

Description of rolling in moving frame

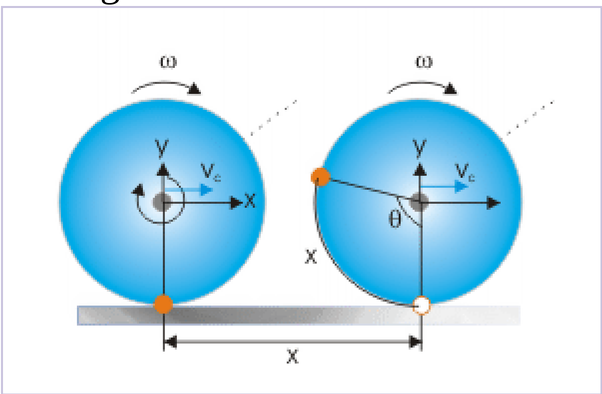
When rolling motion is seen from a frame of reference attached to the center of rolling disk i.e. a frame of reference which is moving with a velocity, "v_C", the particles constituting the disk undergo rotation with an angular velocity,ω, without translation. The axis of rotation is fixed in the moving frame of reference. As such, rolling is a pure rotation in the moving frame of reference attached to the center of disk.

In this frame of reference (attached to the rolling disk), a particle on the rim of the disk rotates through the same distance in a given time as distance covered by the moving frame of reference in x-direction. If the particle in contact with the surface describes an angle 2π in rolling motion, then both the particle and axis of rotation (also center of mass) travels a distance 2πr.

However, they describe different paths. The particle on the rim travels an arc as seen from the moving reference, whereas the center of mass or attached frame of reference travels along a straight line as seen from the ground.

In general, if the moving frame of reference travels a distance "x" in time "t", then the particle at the contact point also rotates through the same distance along the arc of the perimeter of rolling object. This is true for a particle on any position on the rim of the disk. The figure here captures the two situations corresponding to start and end of the observations. From the geometry shown in the figure,

Rolling motion



Rolling is a pure rotation in the moving frame of reference.

$$x = \theta R$$

where "θ" is the angle subtended by the arc and "R" is the radius of the disk. Now differentiating the above relation with respect to time, we have :

$$\frac{đx}{đt} = \left(\frac{đ\theta}{đt}\right) R$$
$$\Rightarrow v = \omega R$$

where "v" is the velocity of moving frame of reference (velocity of the observer) and "ω" is the angular velocity of the rotating disk. However, we

have assumed that frame of reference is moving with a velocity equal to that of center of mass, " v_C ". Hence,

$$\Rightarrow v_C = \omega R$$

We note from the figure that the linear velocity is along x-axis and is positive. On the other hand, angular velocity is clock-wise and is negative. In order to account for this, we introduce a negative sign in the relation as :
Equation:

$$\Rightarrow v_C = -\omega R$$

This equation relates the linear velocity of center of mass,"v_C", and angular velocity of the disk,"ω". We can drop negative sign if speeds (not velocities) are considered. Importantly, we must note here that this relation connects velocities which are measured in different frames of reference. The linear velocity of center of mass is measured in the ground reference, whereas angular velocity is measured in the moving reference.

We must also understand here that this relation does not relate linear velocities of the particles constituting the body to that of angular velocity. The linear velocities are, as a matter of fact, different for different particles. The velocities of the particles constituting the body depends on their relative position with respect to the center of mass or the point of contact. Thus, we should bear in mind that above relation specifically connects the linear velocity of center of mass (v_C) to the angular velocity of the rotating disk ("ω"), which is same for all particles, constituting the disk.

The relation “ $v_C = \omega R$ ” is the defining relation for pure rolling. The moment we refer a motion as pure rolling motion, then two velocities are related precisely as given by this equation. The caution is that it is not same as the similar relation available for pure rotational motion :

$$v = \omega r$$

This relation is a relation for any particle constituting the body. It is not so for pure rolling. The linear velocity in the equation (v_C) of pure rolling

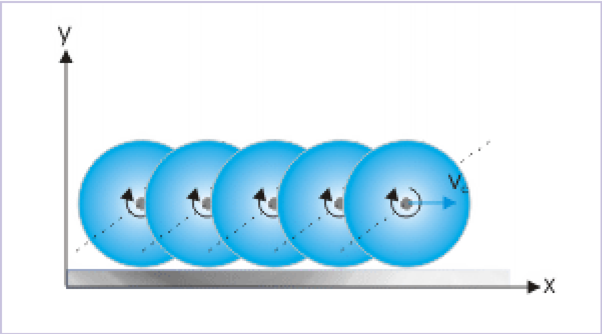
motion refers to the linear velocity of a specific point i.e. center of mass (not the velocity of any point,"v") and equation involves radius of the disk, "R", (not any "r" denoting the distance of any particle from the axis of rotation). These distinctions should always be kept in mind. In order to emphasize the distinction, we may write :

$$\Rightarrow \omega \ne \frac{v_C}{r} \text{ ; if } r \ne R$$

Description of rolling in ground frame

The description of rolling is different for different observers (also read frames of reference). We have observed that the motion of disk is pure rotation as seen in the moving reference attached to the rolling disk. The disk rotates about an axis, passing through the center of mass and perpendicular to its surface. When seen from the ground, however, the axis of rotation is not fixed as in pure rotation; rather it translates along x-direction with a velocity "v_C". It means that the rolling motion is not a pure rotation.

Rolling motion

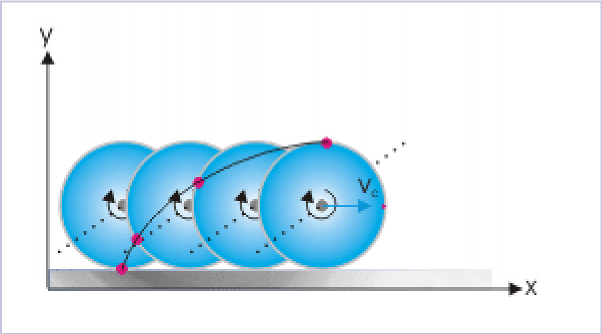


The axis of rotation is not fixed.

The different particles on the rim of the disk - what to talk of other particles constituting it - transverse (cover) different paths in the ground reference. The figure below captures the position of a particle at the contact point over a period, as the disk rolls on the horizontal surface. Evidently, the particle at

the rim describes a curved path, whereas the center of mass describes a straight line path. This means that rolling motion is not a pure translation either. Remember that all particles in pure translation have same motion along a straight line. In the nutshell, we can conclude that rolling of the disk is neither a pure rotation nor a pure translation as seen in the frame of reference attached to the ground.

Rolling motion



Path followed by a particle on the rim of the disk.

The path of the particle on the rim points to an interesting aspect of rolling motion. Since it has been assumed that the center of mass of the disk is moving with uniform velocity, we can infer from the nature of the path (as drawn in the figure above) that particle on the rim of the disk are actually moving with different velocities depending on its position with respect to the point of contact. The velocity of a particle near the point of contact is minimum as it covers smaller distance (length of the curve) and the velocity of the same particle near the top is greater as it covers the longer distance in a given time. This is also evident from the fact that curve is steeper near contact point and flatter near the top point. In simple words, we can conclude that the particles on the rim of the disk are having varying linear velocities - starting from zero at the contact to a maximum value at the top of the disk.

Analysis of rolling motion

Analysis of rolling motion can be undertaken with two different perspectives :

1. Rolling motion is considered as a combination of pure rotation (i.e rotation about a fixed axis) and pure translation (i.e. translation along a straight line).
2. Rolling motion is considered as a pure rotation only (i.e rotation about a fixed axis) about an axis passing through the point of contact and perpendicular to the plane containing point of contact and center of mass. This is essentially an equivalent frame work of analysis, which lets us calculate velocities of particles, constituting disk, in a simplified and identical manner. We shall study this alternative approach in the next module titled " Rolling as pure rotation ".

Rolling motion as a combined motion

Rolling is considered as the combination of pure rotation and pure translation.

1: Pure rotation

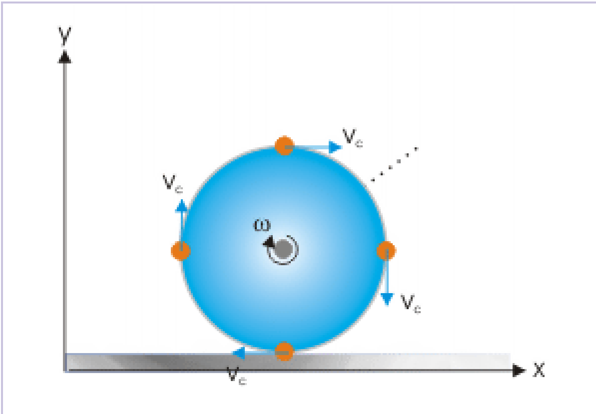
For pure rotation, we consider that the rotating disk rotates about a fixed axis with angular velocity, "ω" such that :

$$\Rightarrow \omega = \frac{v_C}{R}$$

Each particle of the rotating disk moves with same angular velocity. In the figure, we have shown the linear velocities of particles occupying four positions on the rim with appropriate vectors. The magnitude of velocity of these four particles on the rim, resulting from pure rotation, is given by :

$$\Rightarrow v = v_C = \omega R$$

Pure rotation

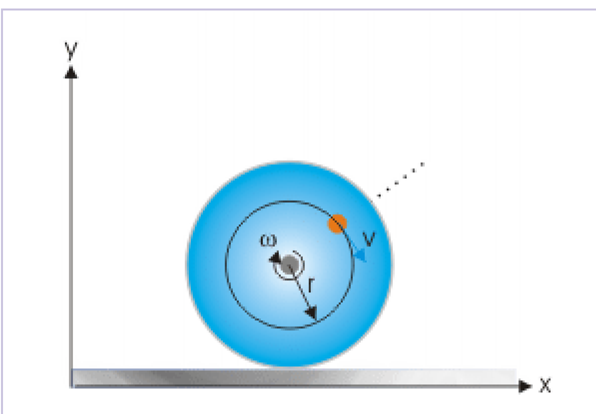


The disk is rotating about a fixed axis.

However, if we consider a particle inside the disk at radial distance "r", then its linear velocity resulting from pure rotation is given by :

$$\Rightarrow v = \omega r$$

Pure rotation



Linear velocity of a particle inside the disk due to pure rotation.

Substituting value of ω from earlier equation, we can obtain the velocity of a particle inside the rotating disk as :

Equation:

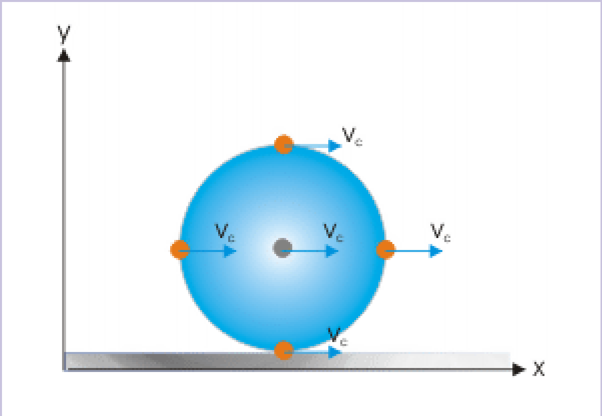
$$v = \left(\frac{v_C}{R}\right) r$$

where "r" is the linear distance of the position occupied by the particle from the axis of rotation. We must, however, clearly understand that angular velocities of all particles, constituting the rigid body, are same.

2: Pure translation

For pure translation, we consider that the rotating disk is not rotating at all. Each particle of the disk is translating with linear velocity that of center of mass, " v_C ". Unlike the case of pure rotation, each of the particle - whether situating on the rim or within the disk - is moving with same velocity. In the figure, we have shown the linear velocities of particles with appropriate vectors, occupying four positions on the rim.

Pure translation

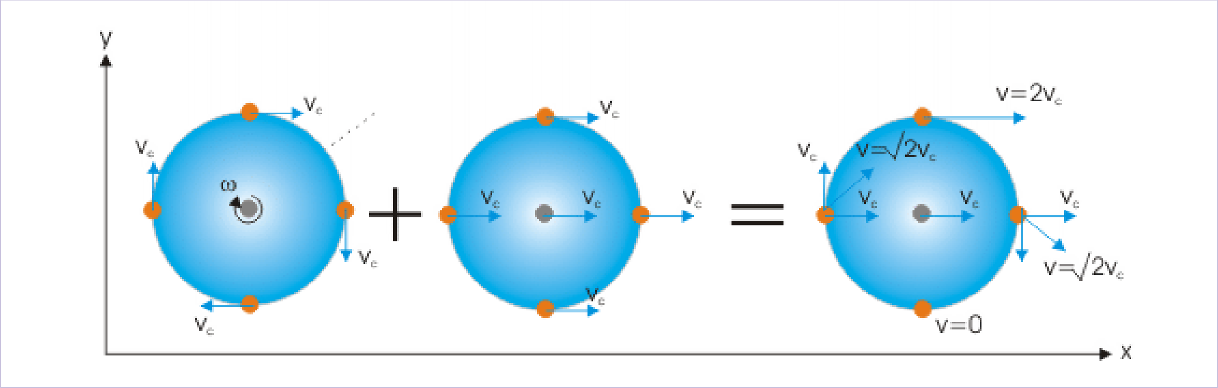


Linear velocities of all particles are equal to that of center of mass.

3: Resultant or combined motion of particles on the rim of the disk

We combine the two velocity vectors; one due to pure rotation and the other due to pure translation to find the velocities of particles.

Rolling motion



Rolling motion as a combination of pure rotation and pure translation.

We note here that particle at the point of contact has zero linear velocity. This result demands explanation as to how particle at contact point would move if its velocity is zero. It is considered that the particle at contact has zero instantaneous velocity resulting from equal and opposite linear velocities due to pure rotation and pure translation. Nevertheless, it has finite angular velocity, "ω ", that changes its position with the increment in time. On the very moment, the particle occupying contact position changes its position, it acquires finite linear velocity as the particle is no more at the contact point and velocities resulting from two constituent motions are not equal. Thus, the explanation based on the percept that rolling is combination of pure rotation and translation is consistent with the physical phenomenon of rolling.

The velocity of the particle on the rim increases as the position of the particle on the rim is away from the point of contact and it (velocity) reaches a maximum at the top of the disk, which is twice the velocity " $2v_C$ " that of center of mass" v_C ". It must be understood that linear velocity of a particle depends on its position on the rim. Farther the position from the point of contact, greater is the velocity.

In vector notation, the combined velocity of a particle on the rim or anywhere inside the rotating body (as seen in the ground reference) is given by :

Equation:

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_t$$
$$\mathbf{v} = \boldsymbol{\omega} \text{ x } \mathbf{r} + \mathbf{v}_C$$

In the expression, the symbol "r" refers position vector from the center of the disk, which is equal in magnitude to the radius "R" for particles on the rim of the disk.

Kinetic energy of rolling disk

We can determine kinetic energy of the rolling disk, considering it to be the combination of pure rotation and pure translation. Mathematically,

$$K = K_R + K_T$$

Equation:

$$K = \frac{1}{2} I_C \omega^2 + \frac{1}{2} M {v_C}^2$$

Example:

Problem : A uniform cylinder of mass 5 kg and radius 0.2 m rolls smoothly over a horizontal surface in a straight line with a velocity of 2 m/s. Find (i) the speed of the particle situated at the top of the cylinder and (ii) kinetic energy of the rolling cylinder.

Solution : The speed of the particle at the top of the cylinder is :

$$v = 2 v_C = 2 \text{ x } 2 = 4 \, m/s$$

The kinetic energy of the rolling solid cylinder is :

$$K = \frac{1}{2} I_C \omega^2 + \frac{1}{2} M {v_C}^2$$

In order to evaluate this equation, we need to find angular velocity and moment of inertia of the rotating cylinder. Here, the angular velocity of the cylinder is :

$$\omega = \frac{v_C}{R} = \frac{2}{0.2} = 10 \text{ rad} / s$$

Now, we need to find the moment of inertia of the cylinder about its axis :

$$I_C = \frac{MR^2}{2} = \frac{5 \text{ x } 0.2^2}{2} = 0.1 \text{ kg} - m^2$$

Putting values, we have :

$$K = \frac{1}{2} \text{ x } 0.1 \text{ x } 10^2 + \frac{1}{2} \text{ x } 5 \text{ x } 2^2 = 15 \; J$$

Summary

1. The terms “rolling”, “pure rolling” and “rolling without sliding” and “rolling without slipping” are used to convey same motion. If intended otherwise, the motion is described with qualification such as “rolling with sliding/ slipping”.

2. The particle at contact continuously changes with time in pure rolling. The distance covered by center of mass in pure translation is equal to the distance covered by a particle on the rim in pure rotation.

3. Pure rolling motion is characterized by the relation :

$$v_C = \omega R$$

This equation is termed as “equation of rolling motion”. This equation relates the linear velocity of center of mass,” v_C “, and angular velocity of the disk,”ω”, where "R" is the radius of the disk.

4. Rolling motion can be analyzed as a combination of (i) pure rotation and (ii) pure translation. Alternatively, rolling motion can be treated as pure rotation about a perpendicular axis through the point of contact (this equivalent description shall be discussed in the next module).

5. The resultant velocity of any particle of a rolling body is vector sum of velocity due to pure rotation ($\boldsymbol{\omega} \text{ x } \mathbf{r}$) and velocity due to translation ($\mathbf{v}_C$).

6. The reference to velocity, sometimes, results in confusion. For clarification, a general reference of velocity, in terms of rolling motion, is interpreted as :

(i) velocity : linear velocity of center of mass (vc);

(ii) velocity of a particle : resultant linear velocity of a particle, which is equal to the vector sum of velocities due to translation and rotation.

Rolling motion (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the theoretical treatment of the topic. The idea is to provide a verbose explanation of the solution, detailing the application of theory. Solution presented here, therefore, is treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the rolling motion. For this reason, questions are categorized in terms of the characterizing features pertaining to the questions :

- Velocity of center of mass
- Velocity of a point on the rolling body
- General equation for the coordinates of a particle
- General equation for the velocity of a particle

Velocity of center of mass

Example 1

Problem : What is the speed of center of mass, if a solid sphere of radius 30 cm rolls at 2 revolutions per second along a straight line?

Solution : The particle on the outer surface in contact with the surface moves the same distance in rotation as the center of mass moves in translation. In 1 second, the particle on the rim moves a distance :

$$x = 2 \text{ x } 2\pi R = 4\pi \text{ x } 0.3 = 3.77\, m$$

$$\Rightarrow v = \frac{x}{t} = \frac{3.77}{1} = 3.77\, m/s$$

Alternatively, we can find the speed of the center of mass, using equation of rolling as :

$$v_C = \omega R = 4\pi \text{ x } 0.3 = 3.77\, m/s$$

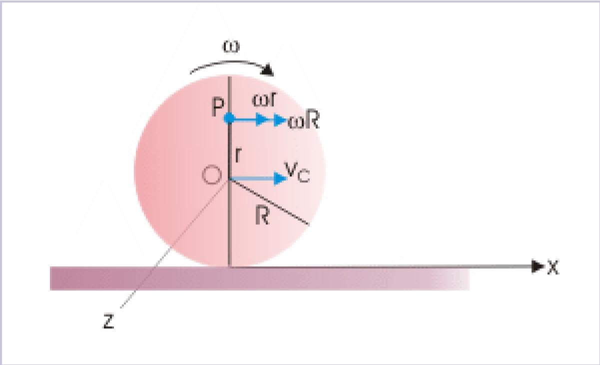
Velocity of a point

Example 2

Problem : A disk of radius "R" is rolling with an angular speed of "ω". If the center of mass at time t = 0 coincides with the origin of the coordinate system, then find the velocity of the point 0,r (r < R).

Solution : The position of the point is shown in the figure. It lies on the y-axis. The linear velocity of the center of mass is :

Pure rolling of a disk



Linear velocity of a point on the vertical diameter.

$$v_C = \omega R$$

The component of velocity due to pure translation, therefore, is :

$$\Rightarrow \mathbf{v}_T = \mathbf{v}_C = \omega R\mathbf{i}$$

On the other hand, linear velocity of the point due to pure rotation is tangential to the circle drawn with radius “r”,

$$\mathbf{v}_R = \omega r \mathbf{i}$$

Hence, velocity of the point is :

$$\mathbf{v} = \mathbf{v}_T + \mathbf{v}_R$$

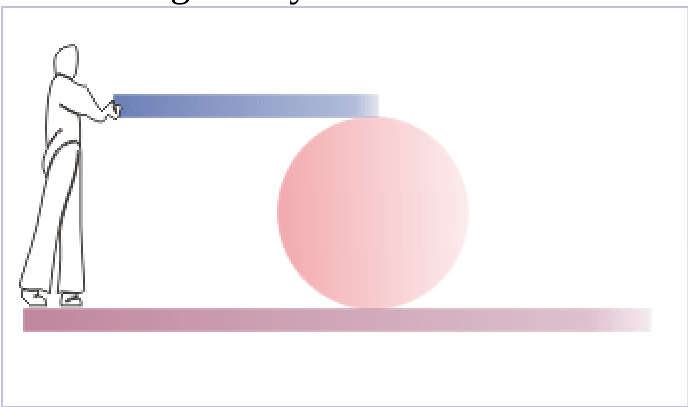
$$\Rightarrow \mathbf{v} = \omega (r + R) \mathbf{i}$$

This question highlights an important fact that velocity due to pure translations is constant for all particles in the rolling body. On the other hand, linear velocities of the particles due to rotation vary, depending upon their relative positions ("**r**") with respect to COM. The resultant velocities of particles, therefore, vary as a whole depending on their positions in the body.

Example 3

Problem : A person rolls a cylinder by pushing a board on top of the cylinder horizontally. The board does not slide and cylinder rolls without sliding. If the linear distance moved by the person is “L”, then find (i) the length of board that rolls over the top of the cylinder and (ii) linear distance covered by cylinder.

Pure rolling of a cylinder

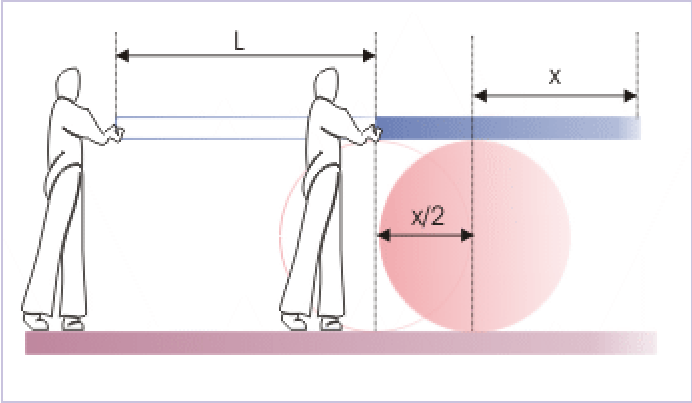


A person rolls a cylinder by pushing a horizontal board over it.

Solution : The person needs to move a distance, which is sum of the length of board pushed over and the linear distance moved by the central axis of the cylinder in horizontal direction. This is the requirement of physical situation.

The important thing to understand here is that speeds of the board and that of central axis of the cylinder are not same. The top of the cylinder moves at a speed “2v”, if the central axis of the cylinder moves at speed “v”. Now, the speed of the board is same as that of the top of the cylinder. Hence, if board moves by a distance “x”, the COM of the cylinder moves by “x/2”.

Pure rolling of a cylinder



A person rolls a cylinder by pushing a horizontal board over it.

The distance covered by the person is :

$$x_P = x + \frac{x}{2}$$

According to question,

$$\Rightarrow x_P = x + \frac{x}{2} = L$$

$$\Rightarrow x = \frac{2L}{3}$$

and distance moved by the cylinder is :

$$\Rightarrow \frac{x}{2} = \frac{L}{3}$$

General equation for the coordinates of a particle

Example 4

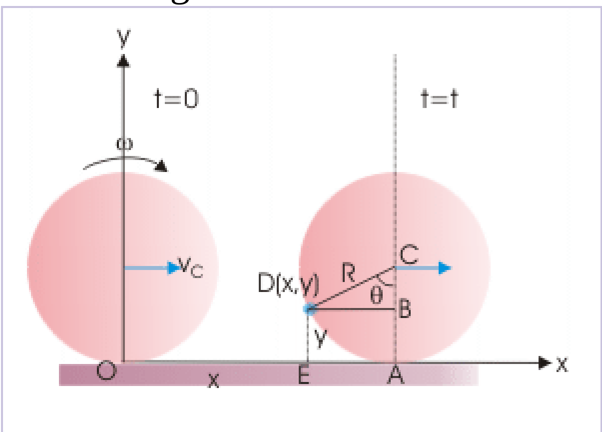
Problem : A disk of radius “R” rolls over a horizontal surface along a straight line with a constant velocity, “v”. If the point of contact with the surface at time t = 0 is the origin of the coordinate system, then determine the coordinates of the particle at the contact after time “t”.

Solution : In general, the center of mass of the disk transverses a linear distance, x, :

$$\mathrm{OA} = vt$$

Let the particle makes an angle “θ” with the vertical after time “t” as shown in the figure. If “x” and “y” be the coordinates of its position in the coordinate system, then

Pure rolling motion



General equation for the coordinates of a particle

$$x = \mathrm{OA} - \mathrm{BD} = vt - R\sin\theta$$
$$y = \mathrm{AC} - \mathrm{BC} = R - R\sin\theta$$

But, angular displacement “θ” is :

$$\theta = \omega t = \left(\frac{v_C}{R}\right)t = \frac{vt}{R}$$

Hence, coordinates are :

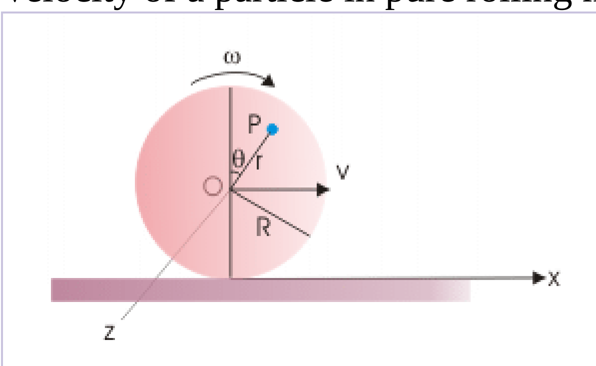
$$x = vt - R\sin\left(\frac{vt}{R}\right)$$
$$y = R - R\sin\left(\frac{vt}{R}\right)$$

General equation for the velocity of a particle

Example 5

Problem : A disk of radius “R” rolls on a horizontal surface with a velocity “v”. Find the velocity of a particle at a point “P” (as shown in the figure).

Velocity of a particle in pure rolling motion

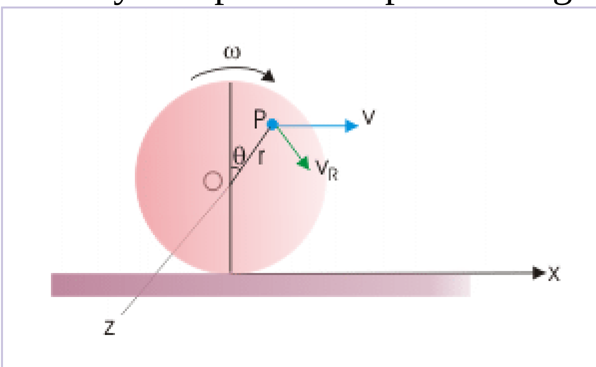


The center of the disk at t=0 is the origin of coordinate system.

Solution : The velocity of point “P” is the vector sum of velocities due to (i) pure translation and (i) pure rotation.

For pure translation, each particle constituting the body moves with the same speed as that of its COM, which is “v” as given in the question. Thus, velocity due to translation is :

Velocity of a particle in pure rolling motion



The velocity of the particle is vector sum of linear velocities due to translation and rotation.

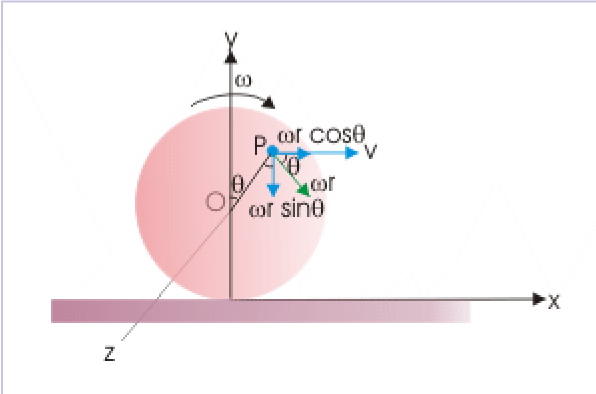
$$\mathbf{v}_C = v\mathbf{i}$$

For pure rotation, each particle constituting the body moves with the same angular velocity, “ω“. The tangential linear speed of point “P” due to rotation is :

$$v_T = \omega r$$

This velocity acts tangentially at the point “P” as shown in the figure. In component form, the linear velocity due to pure rotation is :

Velocity of a particle in pure rolling motion



The velocity of the particle is vector sum of linear velocities due to translation and rotation.

$$\mathbf{v}_R = \omega r \cos\theta\mathbf{i} - \omega r \sin\theta\mathbf{j}$$

Combining velocities due to translation and rotation, we have :

$$\mathbf{v}_P = \mathbf{v}_T + \mathbf{v}_R$$

$$\mathbf{v}_P = (v + \omega r \cos\theta)\mathbf{i} - \omega r \sin\theta)\mathbf{j}$$

For rolling motion, we know that angular velocity is related to velocity of COM as :

$$\omega = \frac{v_C}{R} = \frac{v}{R}$$

Substituting in the expression of velocity of the point “P”, we have :

$$\mathbf{v}_P = \left(v + \frac{vr\cos\theta}{R}\right)\mathbf{i} - \frac{vr\sin\theta}{R}\mathbf{j}$$

This equation is the general equation for the velocity of a particle constituting the rigid body in pure rolling. For a particle on the rim of the disk (r = R), the relation reduces to :

$$\mathbf{v}_P = (v + v\cos\theta)\mathbf{i} - v\sin\theta\mathbf{j}$$

It is interesting to evaluate this expression of the velocity for the particle on the rim for certain knows positions. This may be taken up as an exercise to calculate velocities at top most and at bottom most positions and to see whether results match with that determined earlier.

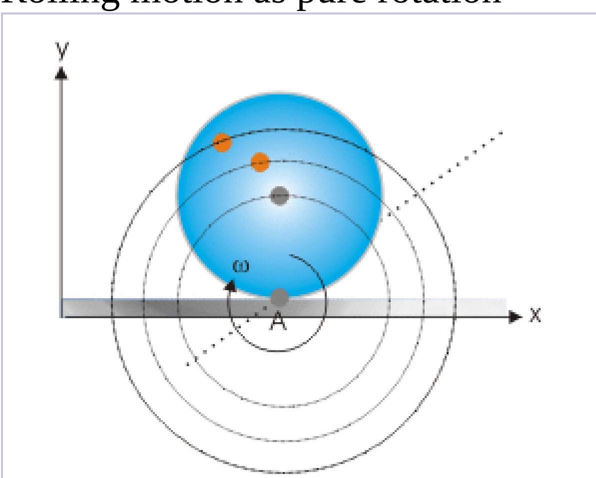
Rolling as pure rotation

Rolling motion is equivalent to pure rotation about the axis at the point of contact, which is parallel to central axis.

The pure rolling is a combination of pure rotation and translation. Obviously, it can not be termed as pure rotation as far as the actual motion is concerned. We can, however, exploit the fact that the disk in rolling is actually rotating at any given instant - not over a period but for an instant. This enables us with a very powerful alternative technique to analyze rolling motion.

The alternative consideration assumes rotation about an axis passing through the point of contact and perpendicular to the plane of rotating disk. Each particle can be considered to rotate about the axis instantaneously i.e. at a particular instant. Clearly, this is an equivalent analysis paradigm that gives the same result as when rolling motion is considered as combination of rotation and translation.

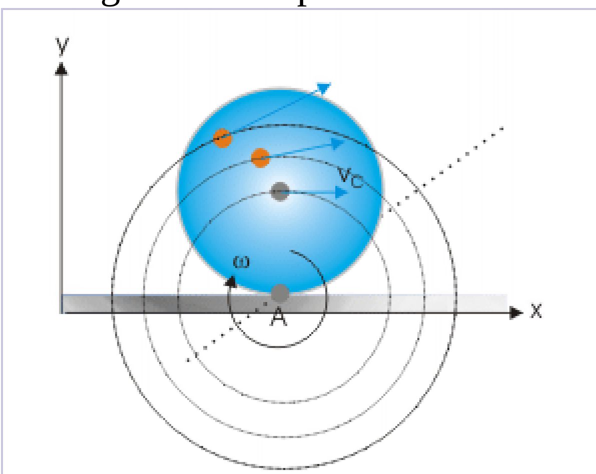
Rolling motion as pure rotation



Each particle of the body can be considered to rotate about the axis through the contact point with same angular velocity.

Under this alternative analysis framework, the angular velocities of all particles about this instantaneously stationary axis are considered same. Importantly, this unique angular velocity is equal to angular velocity of rolling (ω). Depending on the linear distance of the particles from the contact point, their linear velocity vary. This alternative framework is consistent with the fact that linear velocities of the particles away from the point of contact are indeed greater as worked out in previous module titled " Rolling motion ".

Rolling motion as pure rotation



The particles of the body away from the contact point moves with increasing speed.

Equivalence of pure rolling

The most important aspect of the equivalence of analysis frameworks, as described earlier, is that rolling motion, which is a combination of pure translation and rotation in physical sense, is described in terms of pure rotation only for the analysis of the motion. In this analysis, we do not need to consider translation at all, as if the rolling body were stationary.

In the nutshell, the equivalent description of rolling in terms of pure rolling has following important considerations :

1. The axis of rotation through contact point is parallel to central axis. This axis is instantaneously at rest and is also referred as "Instantaneous axis of rotation (IAOR)"

2. Each particle of the body is rotating about IAOR with an angular velocity given by :

Equation:

$$\Rightarrow \omega = \frac{v_C}{R}$$

where "v_C" is the linear velocity of center of mass and "R" is the radius of the disk.

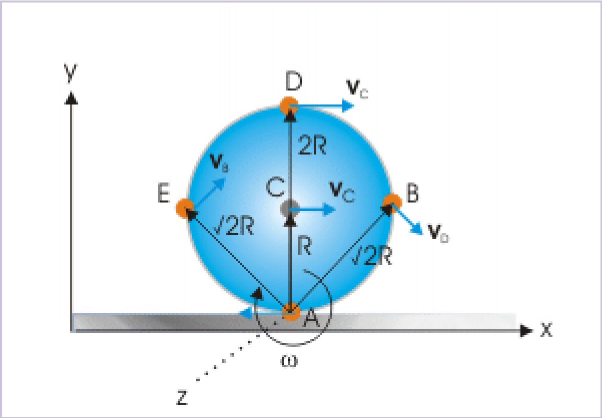
3. The linear velocity of any of the position within the rotating disk is obtained by using the relation,

Equation:

$$\mathbf{v} = \boldsymbol{\omega} \mathbf{\ x\ } \mathbf{r}$$

The vector notation of the relation above is important. It emphasizes that the linear velocity is directed such that it is perpendicular to both angular velocity vector (**ω**) and position vector (**r**). In the figure below, the vector **ω** is into the plane of figure i.e. angular velocity vector is perpendicular to xy-plane. The velocity vector being perpendicular to angular velocity vector, therefore, lies in xy-plane. Further, the velocity vector is perpendicular to the positions vector drawn from the point of contact. Note that this is the requirement of pure rotation. The linear velocity should be tangential to the circular path of the particle about the axis in pure rotation.

Rolling motion



The linear velocity is tangential to the circular path of the particle about the instantaneous axis.

We must note that all velocity vectors are drawn perpendicular to position vectors, which are drawn from the point of contact "A". The magnitude of the velocity is given by :

Equation:

$$v = \omega r \sin 90^0 = \omega r$$

where "r" is the linear distance of the position from the point of contact and "ω" is the angular velocity about the new axis through the point of contact or the axis through center of mass.

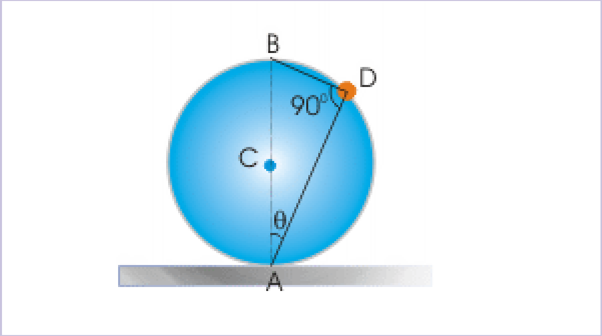
We can validate the assumption of rolling as pure rotation by calculating velocities of some of the positions (A,B,C,D and E) on the rim of the rotating disk.

$$v_A = \omega r = \omega \text{ x } 0 = 0$$
$$v_B = \omega r = \omega \text{ x } \sqrt{2}R = \sqrt{2}v_C$$
$$v_C = \omega r = \omega \text{ x } R = v_C$$
$$v_D = \omega r = \omega \text{ x } 2R = 2v_C$$
$$v_E = \omega r = \omega \text{ x } \sqrt{2}R = \sqrt{2}v_C$$

These results are same as obtained by considering pure rolling motion equivalent to the combination of pure rotation and pure translation. As such, the equivalence of pure rolling as pure rotation about an axis through the point of contact and perpendicular to the plane of rotating disk is indeed true. It is also evident from the calculation details that this approach of analyzing rolling motion is simpler than the earlier approach as far as calculation of linear velocities of different positions within the rotating disk is concerned.

For calculating velocities of the positions on the rim of the rotating disk, there is a convenient trigonometric relation that can be used with ease. We consider a position on the rim of the disk making an angle θ with the vertical at the point of contact. The base of the triangle AB is the diameter of the circle. As such the angle ADB is a right angle (angle subtended by diameter on the circumference is right angle).

Rolling motion



The ring rolls with constant velocity.

$$\cos\theta = \frac{\mathrm{AD}}{\mathrm{AB}} = \frac{\mathrm{AD}}{2\mathrm{R}}$$
$$\Rightarrow \mathrm{AD} = 2\mathrm{R}\cos\theta$$

The linear velocity of the particle at the rim of the rotating disk, therefore, is :

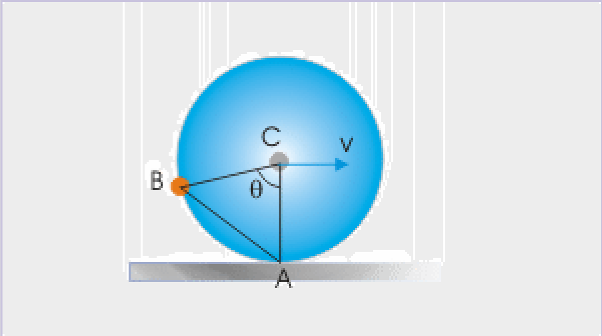
$$\Rightarrow v = \omega r = \omega 2\mathrm{R}\cos\theta = 2\omega R\cos\theta$$

Equation:

$$\Rightarrow v = 2v_C\cos\theta$$

Example:
Problem : A ring rolls on a horizontal plane with a constant velocity, "v". Find the speed of a particle on the ring, making an angle "θ" with the center as shown in the figure.

Rolling motion



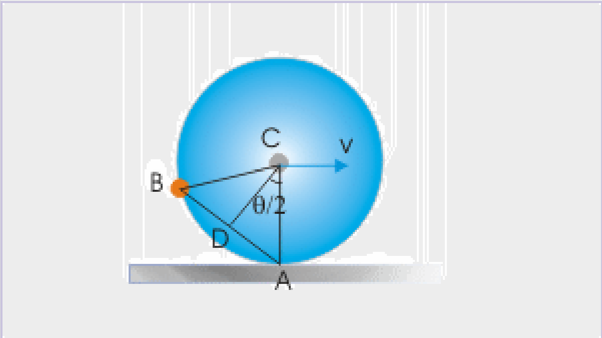
The ring rolls with constant velocity.

Solution : We can not apply the formula as derived earlier directly. Here, the given angle is at the center of the ring and not at the point of contact as needed to use the formula directly. Now, the linear velocity of the particle is given as :

$$v_B = \omega \text{ r}$$

In order to find "r" i.e. "AB", we drop a perpendicular from the center of the circle.

Rolling motion



The ring rolls with constant velocity.

In the right angle ACD, we have :

$$\text{AD} = R \sin \frac{\theta}{2}$$

$$\Rightarrow \text{AB} = \text{AD} = R \sin \frac{\theta}{2}$$

Putting this expression for "r" in the equation :

$$v_B = \omega \text{AB} = \omega R \sin \frac{\theta}{2} = \omega R \sin \frac{\theta}{2}$$

But, we know that "ωR" equals speed of the center of mass.

Equation:

$$v_B = v_C \sin \frac{\theta}{2}$$

As given in the question, the speed of the center of mass is "v". Hence,

$$v_B = v \sin \frac{\theta}{2}$$

Kinetic energy of rolling disk

We can determine kinetic energy of the rolling disk, considering it to be the pure rotation about an axis passing through point of contact and plane of disk. Mathematically,

$$K = K_R = \frac{1}{2} I_A \omega^2$$

where " I_A " is the moment of inertia of the disk about new axis. Now, using theorem of parallel axes, we have :

$$I_A = I_C + \text{M } R^2$$

where "I" is moment of inertia about an axis passing through center of mass and perpendicular to the plane of rotating disk. Putting in the equation of kinetic energy, we have :

Equation:

$$K = \frac{1}{2} I_A \omega^2$$

$$\Rightarrow K = \frac{1}{2} \; I_C + \text{M } R^2 \; \omega^2$$
$$\Rightarrow K = \frac{1}{2} I_C \omega^2 + \frac{1}{2} \text{ M } R^2 \omega^2$$
$$\Rightarrow K = \frac{1}{2} I_C \omega^2 + \frac{1}{2} \text{ M } {v_C}^2$$
$$\Rightarrow K = K_R + K_T$$

This result is same as that obtained by considering rolling as combined motion of pure rotation and pure translation. It again emphasizes the correctness of the original assumption that rolling can be treated as pure rotation about an axis through point of contact and perpendicular to the plane of disk.

Summary

1. Pure rolling is equivalent to pure rotation about an axis through point of contact and parallel to central axis.

2. The axis of rotation passing through point of contact and parallel to axis of rotation is instantaneously at rest and is known as “Instantaneous axis of rotation (IAOR)”.

3. The particles rotate about instantaneous axis of rotation with same angular velocity, which is given by :

$$\omega = \frac{v_C}{R}$$

4. The linear velocity of any of the position within the rotating disk is obtained by using the relation,

$$\mathbf{v} = \boldsymbol{\omega} \ \mathbf{x} \ \mathbf{r}$$

where “**r**” is the position vector drawn from instantaneous axis of rotation.

5. The magnitude of linear velocity i.e. speed of the particle is :

$$v = \omega r$$

where "r" is the linear distance of the particle from the point of contact. We must know that angular velocity vector and position vector are perpendicular to each other. As such, the "sinθ" term in the magnitude of a vector product always evaluates to "1".

6. The velocity is tangential to the circular path i.e. perpendicular to position vector.

7. The speed of a particle, which makes an angle "θ" with the vertical on the circumference (i.e circular path) is given by :

$$v = 2 v_C \cos\theta$$

8. The speed of a particle, which makes an angle "θ" with the vertical at the center of mass is given by :

$$v = \ \ v_C \sin \ \frac{\theta}{2}$$

9. The kinetic energy of the rolling is given by :

$$K = \frac{1}{2} I_A \omega^2 = \frac{1}{2} I_C \omega^2 + \frac{1}{2}\ \mathrm{M}\ {v_C}^2$$

Rolling as pure rotation (Exercise)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the theoretical treatment of the topic. The idea is to provide a verbose explanation of the solution, detailing the application of theory. Solution presented here, therefore, is treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, whose analysis is suited to the technique of treating rolling motion as pure rotation. For this reason, questions are categorized in terms of the characterizing features pertaining to the questions :

- Positions on the rolling body with a specified velocity
- Velocity of a particle situated at a specified position
- Distance covered by a particle in rolling
- Kinetic energy of rolling

Position on the rolling body with specified velocity

Example 1

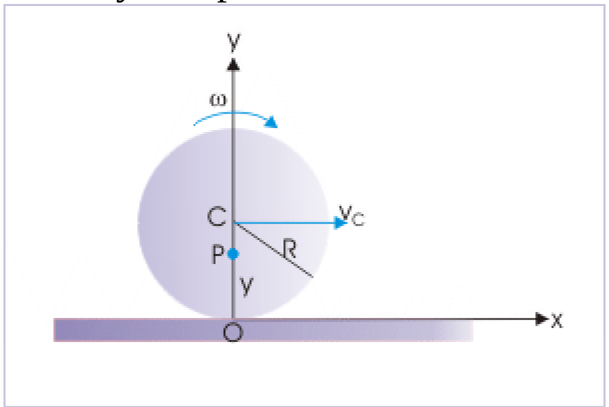
Problem : At an instant, the contact point of a rolling disk of radius "R" coincides with the origin of the coordinate system. If the disk rolls with constant angular velocity, "ω", along a straight line, then find the position of a particle on the vertical diameter, whose velocity is 1/ 2 of the velocity with which the disk rolls.

Solution : Here, the particle on the vertical diameter moves with a velocity, which is 1/ 2 of that of the velocity of the center of mass. Now, velocity of center of mass is :

$$v_C = \omega R$$

Let the particle be at a distance "y" from the point of contact on the vertical diameter. Then, velocity of the particle is :

Velocity of a particle



Velocity of a particle on the vertical diameter

$$v = \omega r = \omega y$$

According to question,

$$v = \frac{v_C}{2}$$

Putting values,

$$\Rightarrow \omega y = \frac{v_C}{2} = \frac{\omega R}{2}$$

$$\Rightarrow y = \frac{R}{2}$$

Note: This result is expected from the nature of relation " $v = \omega r$ ". It is a linear relation for vertical distance. The velocity varies linearly with the vertical distance.

Example 2

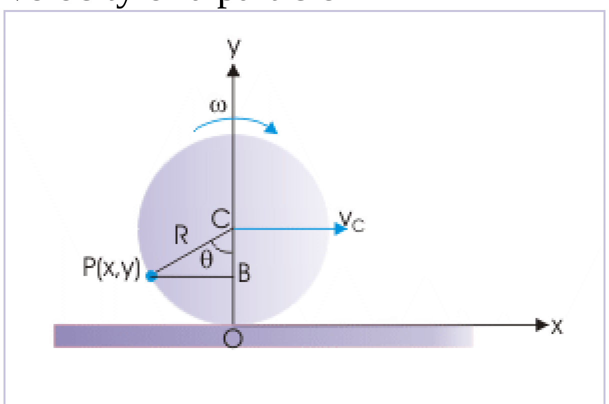
Problem : At an instant, the contact point of a rolling disk of radius "R" coincides with the origin of the coordinate system. If the disk rolls with constant angular velocity, "ω", along a straight line, then find the position of a particle on the rim of the disk, whose speed is same as the speed with which the disk rolls.

Solution : Here, the particle on the rim of the disk moves with the same velocity as that of the velocity of the center of mass. Now, velocity of center of mass is :

$$v_C = \omega R$$

Let the particle be at P(x,y) as shown in the figure. Then, velocity of the particle is :

Velocity of a particle



Velocity of a particle on the rim of the disk

$$v = 2v_C \sin\left(\frac{\theta}{2}\right) = 2\omega R \sin\left(\frac{\theta}{2}\right)$$

According to question,

$$v = v_C$$

Putting values,

$$2v_C \sin\left(\frac{\theta}{2}\right) = 2\omega R \sin\left(\frac{\theta}{2}\right) = \omega R$$

$$\Rightarrow \sin\left(\frac{\theta}{2}\right) = \frac{1}{2} = \sin 30^\circ$$
$$\Rightarrow \frac{\theta}{2} = 30^\circ$$
$$\Rightarrow \theta = 60^\circ$$

$$\Rightarrow x = -R \sin 60^\circ = -\frac{\sqrt{3}R}{2}$$

$$\Rightarrow y = R \cos 60^\circ = \frac{R}{2}$$

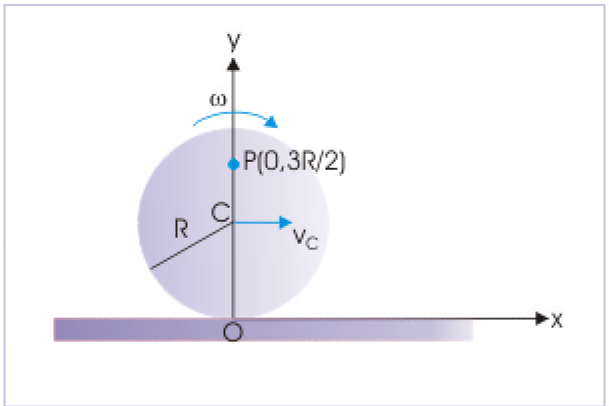
Since there are two such points on the rim on either side of the vertical line, the coordinates of the positions of the particles, having same speed as that of center of mass are :

$$-\frac{\sqrt{3}R}{2}, \ \frac{R}{2} \ \text{ and } \ \frac{\sqrt{3}R}{2}, \ \frac{R}{2}$$

Velocity of a particle situated at a specified position

Example 3

Problem : A circular disk of radius “R” rolls with constant angular velocity, “ω”, along a straight line. At an instant, the contact point of a rolling disk coincides with the origin of the coordinate system. Find the velocity of a particle, situated at a point (0,3R/2), as seen by (i) an observer on the ground (ii) an observer who is translating in the direction of rolling with a velocity “ωR/2” and (iii) an observer who is translating in the direction of rolling with a velocity equal to that of center of mass.
Velocity of a particle



Velocity of a particle on the vertical diameter

Solution : (i) The linear distance of the particle from the contact point is 3R/2. Since, particle is situated on the vertical diameter, the tangent to the circular path is horizontal. This means that particle is moving in horizontal direction (x – direction) with this speed. Hence, velocity of the particle as seen from the ground is :

$$v = \omega r$$
$$\Rightarrow v = \omega \text{ x } \frac{3R}{2} = \frac{3\omega R}{2}$$

(ii) The observer is moving with velocity "ωR/2" in the direction of rolling. Thus, relative velocity of the particle with respect to observer is :

$$v_{\text{PO}} = v_P - v_O$$
$$\Rightarrow v_{\text{PO}} = \frac{3\omega R}{2} - \frac{\omega R}{2} = \omega R$$

(iii) The observer is moving with velocity of center of mass "ωR" in the direction of rolling. Thus, relative velocity of the particle with respect to observer is :

$$v_{\text{PO}} = v_P - v_O$$
$$\Rightarrow v_{\text{PO}} = \frac{3\omega R}{2} - \omega R = \frac{\omega R}{2}$$

Note that the observer moving with the velocity of center of mass views rolling as pure rotation. The motion of the particle will be seen by the observer as the pure rotation of the particle about the perpendicular axis, which passes through center of mass. Now, particle is at a distance "3R/2 - R = R/2" from the central axis. Hence, linear velocity due to pure rotation only is :

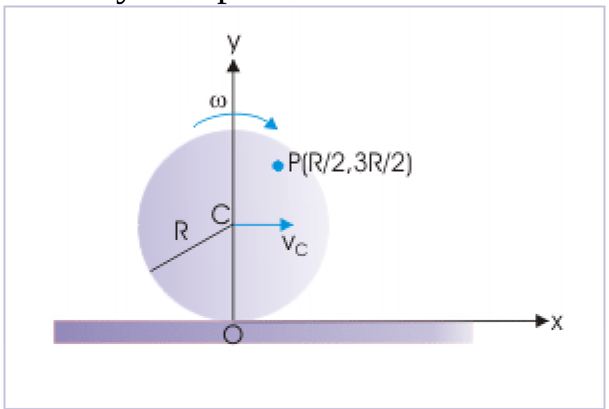
$$v_{\text{PO}} = \omega r = \frac{\omega R}{2}$$

Thus, the result is same as obtained when we consider rolling as pure rotation about instantaneous axis of rotation at contact point.

Example 4

Problem : A circular disk of radius “R” rolls with constant angular velocity, “ω”, along a straight line. At an instant, the contact point of a rolling disk coincides with the origin of the coordinate system. Find the velocity of a particle, situated at a point (R/2,3R/2), as seen by (i) an observer on the ground and (ii) an observer, who is translating in the direction of rolling with a velocity equal to that of center of mass.

Velocity of a particle



Velocity of a particle at a specified position

Solution : (i) In this case, velocity of the particle can be easily found out applying vector form of the relation,

$$\mathbf{v} = \boldsymbol{\omega} \text{ x } \mathbf{r}$$

Here, angular velocity is into the plane of disk and is, therefore, negative :

$$\boldsymbol{\omega} = -\omega \, \mathbf{k}$$

The position vector of the particle at the given instant is :

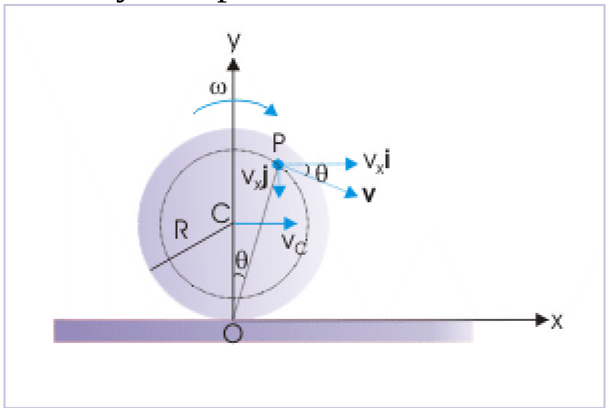
$$\mathbf{r} = \frac{R}{2}\mathbf{i} + \frac{3R}{2}\mathbf{j}$$

Hence, velocity of the particle is :

$$\mathbf{v} = -\omega\mathbf{k} \text{ x } \left(\frac{R}{2}\mathbf{i} + \frac{3R}{2}\mathbf{j}\right)$$
$$\mathbf{v} = -\frac{\omega R}{2}\mathbf{j} + \frac{3\omega R}{2}\mathbf{i}$$
$$\mathbf{v} = \frac{3\omega R}{2}\mathbf{i} - \frac{\omega R}{2}\mathbf{j}$$

It is significant to note that signs of the components involved in the result is consistent with the physical interpretation as shown in the figure.

Velocity of a particle



Velocity of a particle at a specified position

(ii) The observer moving with the velocity of center of mass in the x-direction. Its velocity is :

$$\mathbf{v}_O = \omega R\,\mathbf{i}$$

Thus, relative velocity of the particle with respect to observer is :

$$\mathbf{v}_{\mathrm{PO}} = \mathbf{v}_P - \mathbf{v}_O$$
$$\Rightarrow \mathbf{v}_{\mathrm{PO}} = \frac{3\omega R}{2}\mathbf{i} - \frac{\omega R}{2}\mathbf{j} - \omega R\mathbf{i}$$
$$\Rightarrow \mathbf{v}_{\mathrm{PO}} = \frac{\omega R}{2}\mathbf{i} - \frac{\omega R}{2}\mathbf{j}$$

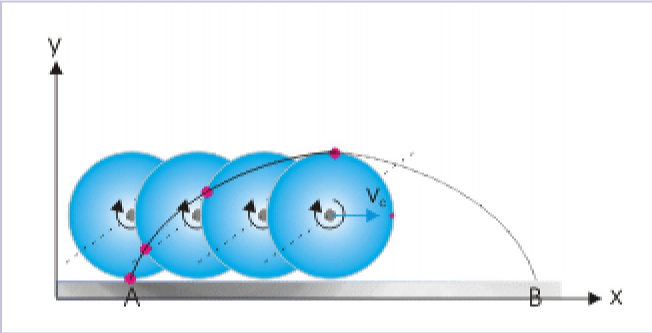
Distance covered by a particle in rolling

Example 5

Problem : Find the distance covered in one complete revolution by a particle on the rim of a circular disk, which is rolling with constant velocity along a straight line on a horizontal plane.

Solution : The question, here, seeks to find the distance covered. We know that each particle on the rim of the disk moves along a cycloid curve as shown in the figure.

Path of a particle in rolling motion



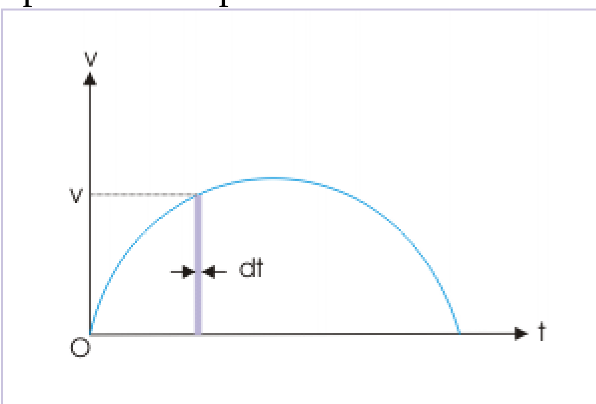
Path of a particle in rolling motion
is a cycloid.

The speed of the particle along the curve is not constant, but varies as :

$$v = 2v_C \sin\left(\frac{\theta}{2}\right) = 2\omega R \sin\left(\frac{\theta}{2}\right)$$

where "θ" is the angle that the particle makes with vertical at the center. During one rotation, the angle varies from 0° to 360°. For this range of values, sin θ/2 varies between 0 to 1 in the first half and then 1 to 0 in the second half of the motion. The plot of speed – time approximates as shown here.

Speed – time plot



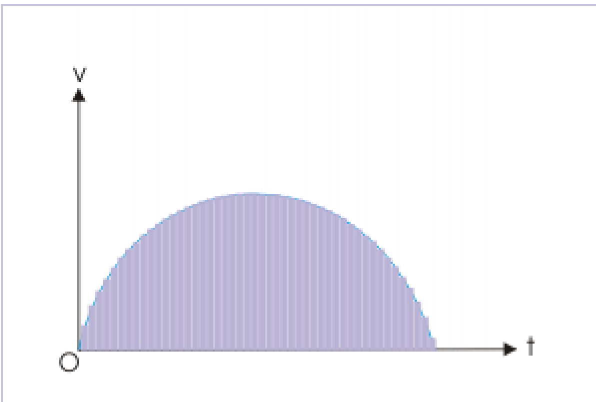
Speed of the particle first increases for first half, then decreases in the other half.

The differential distance covered is given as :

$$s = v dt$$

We know that the the area under speed – time curve gives the distance covered by the particle. Hence, distance is :

Speed – time plot



Area under plot gives the distance covered.

$$s = \int v đt = \int 2\omega R \sin\left(\frac{\theta}{2}\right) đt$$

In order to have only one variable in the integration, we use the relation,

$$\theta = \omega t$$

Substituting in the integration,

$$s = \int v đt = \int 2\omega R \sin\left(\frac{\omega t}{2}\right) đt$$

Putting the appropriate limits of time for one revolution and taking out the constants from the integration,

$$\Rightarrow s = 2\omega R \int_0^{\frac{2\pi}{\omega}} \sin\left(\frac{\omega t}{2}\right) đt$$

$$\Rightarrow s = 4R\left[-\cos\frac{\omega t}{2}\right]_0^{\frac{2\pi}{\omega}}$$

$$\Rightarrow s = -4R[\cos \pi - \cos 0^\circ]$$

$$\Rightarrow s = 8R$$

Kinetic energy of rolling

Example 6

Problem : A hollow sphere of mass "M" and radius "R" is rolling with a constant velocity "v". What is ratio of translational kinetic energy and total kinetic energy.

Solution : The kinetic energy of the rolling sphere is given by :

$$K = \frac{1}{2} I_A \omega^2$$

Here,

$$\omega = \frac{v_C}{R}$$

and

$$I_A = I_C + MR^2$$

For hollow sphere,

$$I_A = \frac{2MR^2}{3} + MR^2 = \frac{5MR^2}{3}$$

Putting these values in the equation of kinetic energy, we have :

$$K = \frac{1}{2} \text{ x } \frac{5MR^2}{3} \text{ x } \frac{{v_C}^2}{R^2}$$

$$\Rightarrow K = \frac{5M{v_C}^2}{6}$$

The translational kinetic energy of the rigid body is given by :

$$\Rightarrow K_T = \frac{M{v_C}^2}{2}$$

Hence, the required ratio is :

$$\Rightarrow \frac{K}{K_T} = \frac{5}{3}$$

Understanding rolling motion

Essentials of pure rolling motion are no different than that of pure translational and rotational motions, except that these two basic forms of motions occur simultaneously. A clear understanding of the two basic motion forms, therefore, is a perquisite for a clear understanding of pure rolling motion (referred simply as rolling also).

There are two distinct framework associated with the study of rolling motion :

- Uniform rolling
- Accelerated rolling

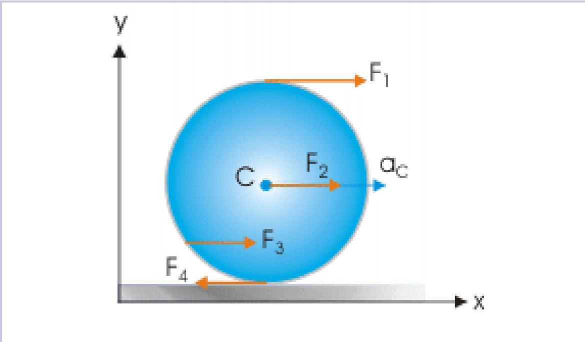
Independence of analysis

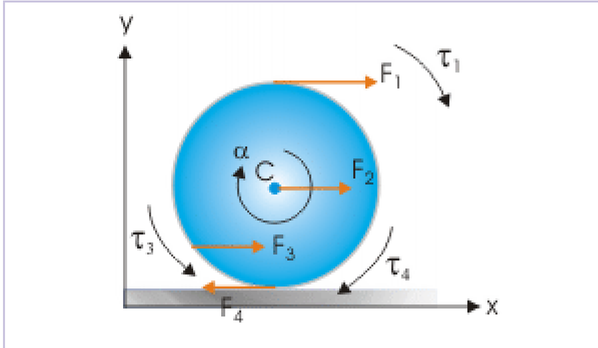
Rolling, being combination of translation and rotation, involves two “causes”, which might change its velocity. Two causes act to produce “effects” independently, but in tandem to satisfy the condition of rolling (we shall subsequently derive this condition in the module).

A net force causes acceleration of the center of mass of the rigid body. A rolling motion involves rigid body of finite size and, therefore, its translation should always be referred to the center of mass. Further, when we consider the effect of force, we treat translation as if the rigid body were not rotating at all.

Independence of analysis

Forces are analyzed as if rigid body were not rotating at all.	Torques are analyzed as if rigid body were not translating at all.





Similarly, a net torque causes rotational acceleration of the rigid body about its central axis passing through center of mass. When we consider the effect of torque, we treat rotation as if the axis of rotation were not translating at all.

In simple words, the analysis of rolling can be done independently for two motions types as if other motion did not exist. This independence of analysis of motion allows us to apply the familiar laws of motion for analyzing each motion types. We are required only to combine the results to describe rolling motion.

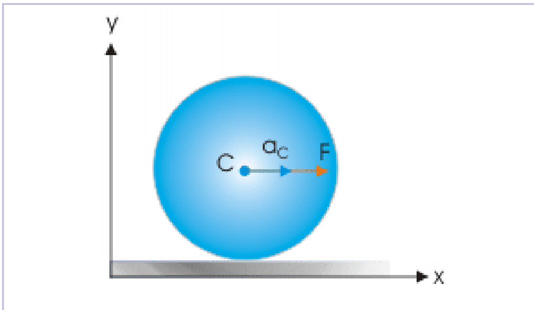
Force and torque

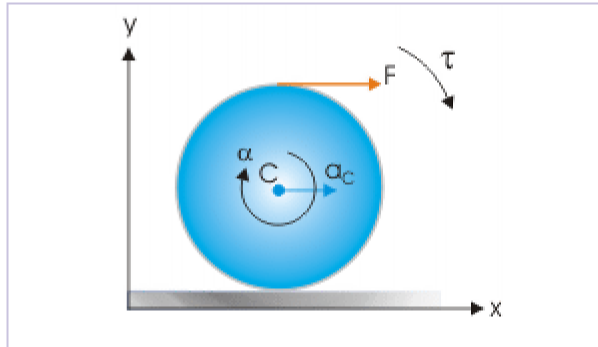
Treatment of force with respect to a rigid body capable of both translation and rotation is different than the case when only one type of motion is involved (i.e. not the combination). In pure translation along a straight line, the rigid body is constrained (or otherwise) not to rotate; similarly in pure rotation about a fixed axis, the rigid body is constrained not to translate.

A force, whose line of action passes through center of mass, is capable to produce only translational acceleration (a_C). A force, whose line of action does not pass through center of mass, works as "force" to produce translational acceleration (a_C) and simultaneously as "torque" to produce angular acceleration (α).
Force and torque

A force through the COM only produces linear	A force not through the COM produces both linear and angular

acceleration. accelerations.





Since there may be multiple effects (more than one) of a single force, it is always desirable to clearly understand the roles of the forces operating on the rolling body to accurately analyze its motion.

Rolling and Newton's first law

A pure rolling is equivalent to pure translation and pure rotation. It, therefore, follows that a uniform rolling (i.e. rolling with constant velocity) is equivalent to uniform translation (constant linear velocity) and uniform rotation (constant angular velocity).

According to Newton's first law for translation, if net external force is zero, then translation of the object i.e. linear velocity remains same. Similarly, according to Newton's first law for rotation, if net external torque is zero, then rotation of the object i.e. angular velocity remains same. It means, then, that a body in uniform rolling motion shall roll with the same velocity.

Note here that when we say that a body is rolling with a constant velocity, then we implicitly mean that it is translating at constant linear velocity and rotating at constant angular velocity. It is so because two motions are tied to each other with the following relation,

Equation:

$$v_C = \omega R$$

We had difficulty to visualize a real time situation to verify Newton’s first law in translation or rotation, as it was difficult to realize a “force – free” environment. However, we reconciled to the Newton’s first law as we experienced that a body actually moved a longer distance on a smooth surface and a body rotated longer without any external aid about an axle having negligible friction and resistance. In the case of rolling also, we need to extend visualization for the condition of rolling when neither there is net force nor there is net torque.

One such possible set up could be a smooth horizontal plane. If a rolling body is transitioned (i.e. released) on a smooth plane with pure rolling at certain velocity, then the body will keep rolling with same velocity. This statement, if we agree, can be construed to be the statement of Newton’s first law for pure rolling motion.

Uniform rolling

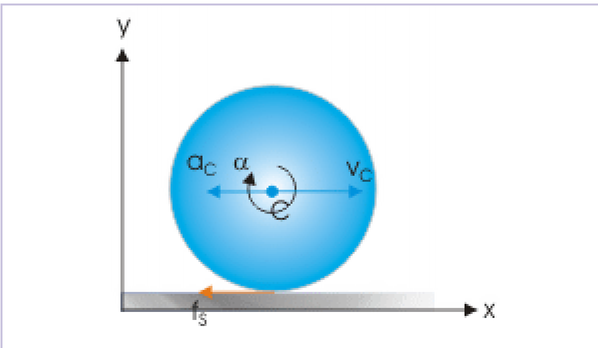
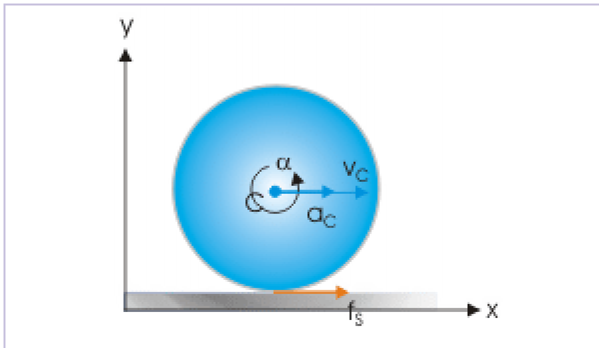
The similarity of uniform rolling in the absence of external force and torque to its constituent motion ends in the real time situation. There is a surprising aspect of rolling motion on a surface (which is not friction-less) : “Friction for uniform rolling (i.e. at constant velocity) on a surface is zero”. This is a special or characterizing feature of uniform rolling motion. This feature distinguishes this motion from either translation or rotation. For, we know that surface friction decelerates translation and rotation of a body. The rolling is exceptional in this regard.

We explain the situation in two ways. First, we shall revert to the definition of pure rolling. The motion of pure rolling is characterized by absence of sliding. Friction, on the other hand, comes in the picture only when there is sliding i.e. there is relative displacement of two surfaces. Here, there is only a point (not a surface) in contact with the surface, which is continuously being replaced by neighboring particles on the rim of the rolling body. Thus, there is no friction as there is no sliding of surfaces over one another.

In yet another way, we can think opposite and analyze the situation. Let a force of friction (contrary to the situation) operates on the body in a direction opposite to the motion as shown in the figure below on the left.

This friction decelerates translation. At the same time, friction constitutes a torque about center of mass. This torque accelerates rotation of the rigid body about the axis of rotation. This means that linear velocity decreases, whereas angular velocity increases. This contradicts the equation of rolling motion given by :

Uniform rolling

Imagine a friction force as shown.	Imagine a friction force as shown.
	

$$v_C = \omega R$$

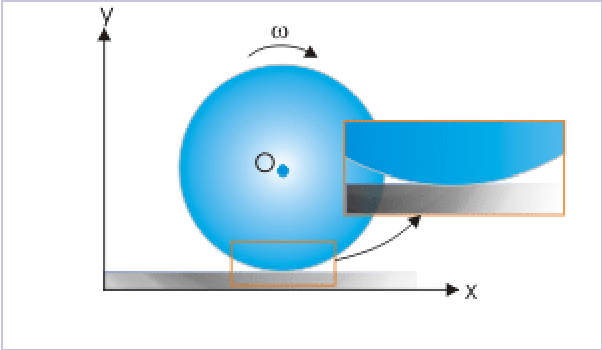
Looking at the motion of rolling (without sliding), it is easy to realize that it is not physically possible that the velocity of center of mass increases, but the angular velocity of the rolling body decreases. As a matter of fact, presence of friction shall contradict the physical reality of rolling itself!

According to the above relation, two velocities increase or decrease simultaneously. We can repeat this analysis for an opposite situation in which we assume that the friction operates in the direction of translation (not opposite) as shown in the figure above on the right. Even in this case, we shall find that the body is accelerated in translation, but decelerated in rotation – a contradiction of the condition of rolling. We, therefore, conclude that there is no friction when a body is rolling uniformly.

Absence of friction for rolling at constant velocity has a very significant implication as a disk in uniform rolling shall move indefinitely, if no net external force/ torque is acting. This is a slightly unrealistic deduction for

we know that all rolling disk is brought to rest ultimately unless external force is applied to maintain the speed. This needs explanation.

Pure rolling motion

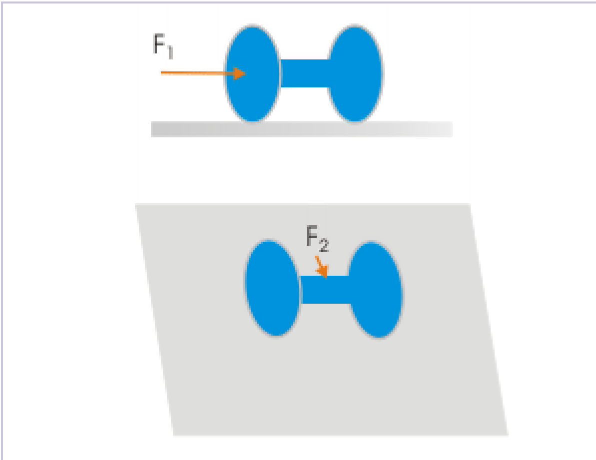


All rolling motion in our real world involves contact which spreads beyond a point.

As a matter of fact, it is not possible to realize an ideal pure rolling in the first place. All rolling motion in our real world involves contact which spreads beyond a point and there is some amount of deformation involved and, therefore, existence of normal force constituting a torque in the opposite direction to rotation of the object. As such, the rolling body decelerates.

We can have a direct feeling of the absence of friction in uniform rolling. We use a dumbbell that we often use for exercise. Just try to push across so that dumbbell slides without rolling at a constant speed. Then, push it to roll at a constant speed (approximately) without sliding. Experience the difference. We know that it is lot easier to roll than to slide the dumbbell. Had there been single point contact without deformation, the dumbbell would have continued rolling.

Friction in rolling motion



Experiencing force of sliding .vs. force of rolling.

Condition of accelerated rolling

We shall discuss the implications of the external force soon in terms of Newton's second law of motion. But, we first need to ascertain whether the body in pure rolling, when subjected to external force, shall retain the basic nature of the rolling motion or not? In simple words : can a rolling body shall continue rolling when external force or torque is applied?

Recall that rolling requires that linear and angular velocities are tied together by the equation of rolling motion :

$$v_C = \omega R$$

This means that if the motion retains the rolling character even after application of external force/ torque, then any change in velocities (i.e. linear and angular accelerations) should also be related (tied). We can use the above relation to obtain a conditional relation between linear and angular accelerations.

It must, however, be kept in mind that the equation of rolling was developed for the case of rolling. As such, any derivation based on this

relation will be valid only for pure rolling that does not involve sliding. Now, differentiating the equation with respect to time, we have :

$$\frac{dv_C}{dt} = \left(\frac{d\omega}{dt}\right)R$$

Equation:

$$\Rightarrow a_C = \alpha R$$

This is the conditional relation between linear and angular accelerations that should be maintained for the accelerated body to be in pure rolling. We call this relation as "equation of accelerated rolling" to distinguish the same with the "equation of rolling" derived earlier.

Like in the case of equation of rolling motion, this relation connects quantities, which are measured in two different references. Linear acceleration of center of mass is measured with respect to ground, whereas the angular acceleration is measured with respect to moving axis of rotation. This relation, therefore, also runs from sign syndrome. However, this should not cause concern as we shall use this relation mostly for magnitude purpose. If a particular condition (general derivation) requires to adjust the sign, then we will put a negative sign on the right hand of the equation to account for the direction.

The fact that axis of rotation is accelerated poses a serious problem with respect to application of Newton's second law for the analysis of rotation in the accelerated frame. Note here that rotation takes place in accelerated frame of reference – not the translation, which takes place in the inertial frame of ground. Thus, application of Newton's second law for translation is valid.

We know that Newton's second law for rotation is valid only in inertial frame of reference. However, an accelerated frame of reference can be rendered to an equivalent inertial frame of reference by applying a force (called pseudo force) at the center of mass. This pseudo force is equal to product of the mass of the rigid body and its linear acceleration. Whatever

be its magnitude, the important point is that this pseudo force acts through center of mass. Since force through center of mass does not constitute torque, the angular velocity of the rotating body is not affected.

We, therefore, conclude that application of Newton's second law of rotation even in accelerated frame of reference is valid for rolling.

Laws governing rolling

Corresponding to two motion types involved with pure rolling, there are two Newton’s second laws governing the motion. One (Newton’s law second law of translation) governs the linear motion of the center of mass, whereas the other (Newton’s law second law of rotation) governs the rotational motion of the rolling disk.

For pure translation of the center of mass, the second law of Newton's law is :

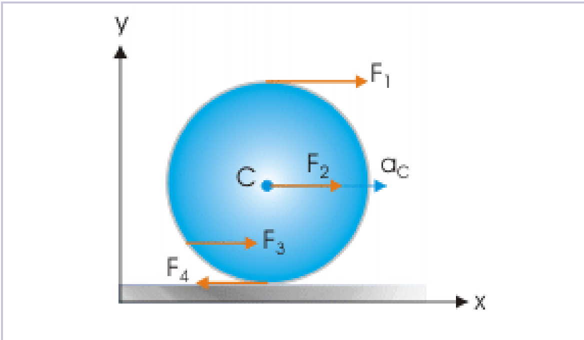
Equation:

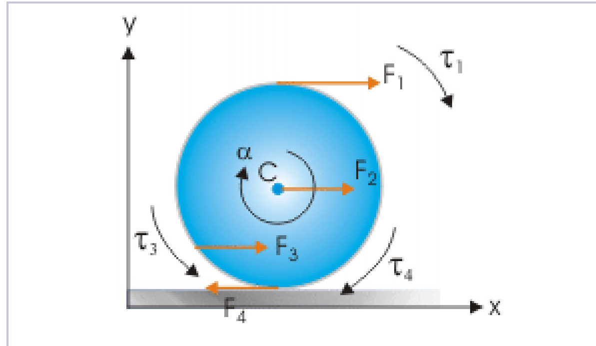
$$a = \frac{F}{M}$$

Here, "∑F" denotes the resultant force and " a_C " denotes the linear acceleration of the center of mass of the disk. For the sake of simplicity, we only consider rolling in one direction only and as such may avoid using vector notation.

Independence of analysis

Forces are analyzed as if rigid body was not rotating at all.	Torques are analyzed as if rigid body was not translating at all.





Similarly, for pure rotation of the disk about an axis passing through the center of mass and perpendicular to its surface, the second law of Newton's law for rotation is :

Equation:

$$\alpha = \frac{\tau}{I}$$

where "I" denotes the moment of inertia and "α" denotes angular acceleration about the axis of rotation through the center of mass.

Analysis of rolling, therefore, is carried out in terms of Newton’s second law of motion for translation and rotation. The application of the laws, however, is conditioned by a third relation between linear and angular accelerations,

$$a_C = \alpha R$$

What it means that we should apply Newton’s second laws for rotation and translation in conjunction with the equation of accelerated rolling.

Summary

1: Uniform rolling means pure rolling with constant velocity.

2: Rolling with constant velocity means constant linear velocity of COM and constant angular velocity of rigid body about the axis of rotation.

3: There is no friction involved in uniform rolling.

4: An accelerated rolling is described by "equation of accelerated rolling" as given by :

$$a_C = \alpha R$$

5: The equation of accelerated rolling underlines the fact that there can not be selective acceleration. In other words, if there is linear acceleration, there is corresponding angular acceleration as well.

6: Governing laws of rolling motion are Newton's second law of motion for translation and rotation. These two laws are subject to the condition imposed by equation of accelerated rolling as given above.

(i) Newton's second law for translation

$$a = \frac{F}{M}$$

(ii) Newton's second law for rotation

$$\alpha = \frac{\tau}{I}$$

Role of friction in rolling

The motion of a rolling body is altered due to external force/ torque. This can happen in following ways :

1. The line of action of external force passes through center of mass.
2. The line of action of external force does not pass through center of mass.

Depending on the line of action, a force causes linear or angular accelerations or both. The effect of external force or torque on rolling motion, however, is moderated by friction force between rolling body and surface.

In general, there is usually a bit of uncertainty with respect to the role and direction of friction in accelerated rolling. This module, therefore, aims to instill definite clarity with regard to the role and direction of friction. The friction in rolling is characteristically different and surprising in its manifestation with respect to the common negative perception (always negates motion) about it.

We have already learnt in the previous module that no friction is involved in uniform rolling. We shall find in this module that friction is actually the agent, which enforces the condition for accelerated rolling. In doing so, friction causes linear and angular accelerations or decelerations, depending on the requirement of rolling motion.

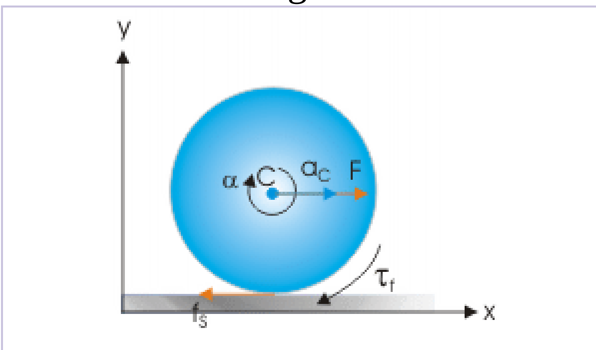
Role of friction

The line of action of external force passes through center of mass.

When the external force is passing through center of mass, it only produces linear acceleration as there is no moment arm and, thus, there is no torque on the body. Linear acceleration means linear velocity tends to increase. This, in turn, induces tendency of the rolling body to slide in the forward direction (i.e. in the direction of force/acceleration). Force of friction,

therefore, appears in the backward direction of external force to check the sliding tendency.

Accelerated rolling motion



The line of action of external force passes through center of mass.

We need to emphasize here that friction acts tangentially at the point of contact. Its direction is “backward” with respect to the component of external force parallel to this tangential direction. This clarification is required as the external force may not be parallel to the surface in contact.

So long the condition of pure rolling is met as given by the equation of rolling motion, there is no actual sliding – rather there is only a tendency to slide. As such the friction involved in accelerated rolling is static friction. We may recall our discussion of friction in translation. Friction is a self adjusting force. It adjusts with respect to external force. Importantly, the static friction is any intermediate value less than the maximum static friction ($\mu_s N$).

The friction acts to balance the changes in a manner so that the condition as imposed by the equation of accelerated rolling is met. First, it reduces the net external force ($F - f_S$) and hence the translational acceleration (a_C). Second, it constitutes a torque in clockwise direction inducing angular acceleration(α). In the nutshell, an increase in linear acceleration due to net external force acting through center of mass is moderated by friction by a two pronged actions and the rolling is maintained even when the body is

accelerated. Corresponding to linear acceleration, there is a corresponding angular acceleration such that :

Equation:

$$a_C = \alpha R$$

In plain words, it means that if there is an increase in linear velocity, then there shall be an increase in angular velocity as well. So is the correspondence for a decrease in either of two velocities.

We can understand the situation from yet another perspective. Since external force induces linear acceleration, there should be a mechanism to induce angular acceleration so that condition as imposed by the equation of accelerated rolling is met. In other words, the friction appears in magnitude and direction such that above relation is held for rolling.

Static friction in rolling differs to its counterpart in translation in one very important manner. In translation, friction adjusts to the external force parallel to the contact surface completely till the body is initiated. What it means that intermediate static friction is equal to the magnitude of the external force in opposite direction. Such is not the case in rolling i.e. ($f_S \neq F$).

The difference in the nature and magnitude of static friction can be easily understood. In pure translation like in sliding, the sole purpose of friction is to oppose relative motion between surfaces. In the case of rolling, on the other hand, friction coverts a part of one type of acceleration to another (from linear to angular as in this case). This statement may appear a bit awkward. The same can be put more elegantly; if we say that friction changes a part of translational kinetic energy into rotational kinetic energy. This sounds better as we are familiar with the conversion of energy - not conversion of acceleration.

Had it not been friction, then the force passing through COM would have only caused linear acceleration! But, in order to satisfy the physical requirement of rolling as defined by the equation of accelerated rolling - the effect of force is changed from one type to another.

Applying Newton's second law for translation, the linear acceleration of the center of mass is given by :

$$a_C = \frac{\sum F}{M} = \frac{F - f_S}{M}$$

Similarly applying Newton's second law for rotation, the angular acceleration of the center of mass is given by (note that external force causes clockwise rotation and hence negative torque) :

$$\alpha = \frac{\sum \tau}{I} = -\frac{R f_S}{I}$$

The two accelerations are such that they are linked by the equation of accelerated rolling (negative sign as linear and angular accelerations are in opposite directions) as :

$$a_C = -\alpha R$$

$$\Rightarrow \frac{F - f_S}{M} = \frac{R^2 f_S}{I}$$

Solving for “ f_S “, we have :
Equation:

$$\Rightarrow f_S = -\frac{IF}{(I + MR^2)}$$

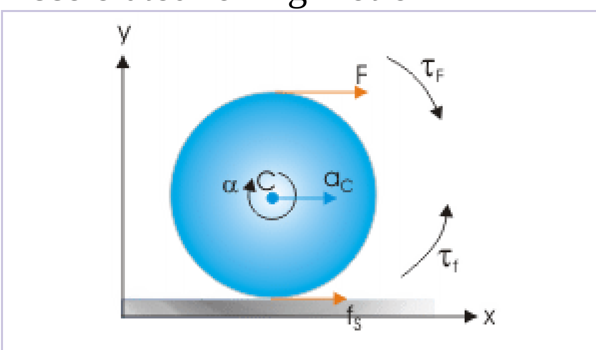
This relation can be used to determine friction in this case. Since all factors like moment of inertia, mass and radius of rotating body are positive scalars, the friction is negative and is in the opposite direction to the applied force.

The relationship as derived above, brings out an interesting feature of rolling friction. Its magnitude depends on the moment of inertia! This is actually expected. The requirement of friction in rolling is for causing angular acceleration, which, in turn, is dependent on moment of inertia. Thus, it is quite obvious that friction (a self adjusting force) should be affected by moment of inertia.

The line of action of external force does not pass through center of mass.

Now, we consider the second case. For the sake of contrast, we consider a tangential force acting tangentially at the top as shown in the figure below. The external force that does not pass through center of mass causes angular acceleration apart from causing linear acceleration. The force has dual role to play in this case. As far as its rotational form (torque) is concerned, resulting angular acceleration means that angular velocity tends to increase. This, in turn, induces tendency of the rolling body to slide in the backward direction (i.e. in the opposite direction of force). Force of friction, therefore, appears in the direction of external force to check the sliding tendency.

Accelerated rolling motion



The line of action of external force does not pass through center of mass.

The friction acts to balance the changes in a manner so that condition of rolling is met. First, it enhances the net external force ($F + f_S$) and hence the translational acceleration (a_C). Second, it constitutes a torque in anticlockwise direction inducing angular deceleration. In the nutshell, an increase in angular acceleration due to net torque is moderated by friction by a two pronged actions and the rolling is maintained even when the body is accelerated in rotation.

We can understand the situation from yet another perspective. Since external torque induces angular acceleration, there should be a mechanism

to induce linear acceleration so that condition of accelerated rolling is met.

Friction appears in magnitude and direction such that above relation is held for rolling. The linear acceleration of the center of mass is given by :

$$a_C = \frac{\sum F}{M} = \frac{F + f_S}{M}$$

The angular acceleration of the center of mass is given by (note that external force causes clockwise rotation and hence negative torque) :

$$\alpha = \frac{\sum \tau}{I} = \frac{R(f_S - F)}{I}$$

The two accelerations are such that they are related by the equation of accelerated rolling (negative sign as linear and angular accelerations are in opposite directions) as :

$$a_C = -\alpha R$$

$$\Rightarrow \frac{F + f_S}{M} = \frac{R(f_S - F)}{I}$$

Solving for " f_S ", we have :
Equation:

$$\Rightarrow f_S = \frac{(MR^2 - I)}{(MR^2 + I)} \text{ x } F$$

This relation (note that this is not a general case, but a specific case of force acting tangentially at the top of the rolling body) provides a reflection on the following aspects of friction force :

- Since $I > 0 \text{ and } I < MR^2$, friction force, " f_S ", is positive and is in the direction of external force. We should note here that for all rigid body MR^2 represents the maximum moment of inertia which corresponds to a ring or hollow cylinder. MIs of all other bodies like sphere, disk etc. are less than this maximum value.
- For ring and hollow cylinder, $I = MR^2$. Thus, friction is zero even for accelerated rolling in the case of these two rigid body. This is one

of the reasons that wheels are made to carry more mass on the circumference.

We conclude from the discussion as above that friction plays the role of maintaining the rolling motion in acceleration. As a matter of fact, had it not been the friction, it would have been possible to have accelerated rolling – it would not have been possible to accelerate bicycle, car, motor, rail etc! Can you imagine these vehicles moving with a constant velocity! There would not have been any car racing either without friction! Such is the role of friction in rolling.

The important aspect of the response of the body in rolling to external stimuli is that the "effect" takes place in both translation and rotation "together"– not selectively. For example, a force through center of mass is expected to produce translation alone. However, such is not the case. Friction ensures that external stimuli like force through center of mass works to affect both translation and rotation so that rolling continues.

We will strengthen our understanding of the role of friction from the perspective of energy in subsequent module. We shall find that friction by virtue of being capable to accelerate is actually capable of even doing positive work i.e. capable to impart kinetic energy in certain situation. Indeed, it is a totally different friction.

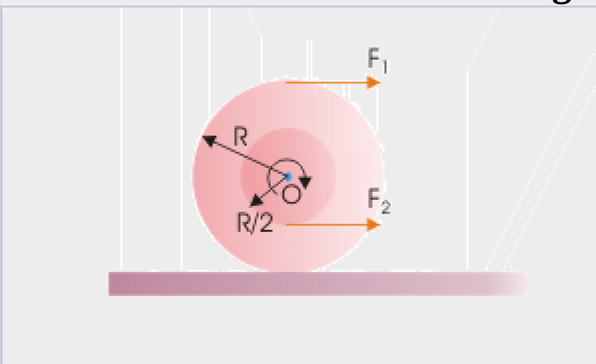
In case, the rolling is not maintained, the friction involved is kinetic friction as the body rotates with sliding. There can be many such situations in real life like applying a sudden brake to a moving car. We shall discuss these cases in a separate module.

We have discussed two extreme cases of the application of force to highlight its behavior on a rolling body. There can, however, be real situation in which external force may be a combination of forces or a force may be applied at an intermediate position between COM and the top or bottom of the rolling body. We need to evaluate effects of all such forces and arrive at the final conclusion about the role of friction and its direction. It is quite possible that some of the combinations yield zero friction even for accelerated rolling.

Example:

Problem : Two forces " F_1 " and " F_2 " are applied on a spool of mass "M", moment of inertia "I" and radius "R" as shown in the figure. If the spool is rolling on the surface, find the ratio of forces, F_1/F_2 , such that friction between spool and the surface is zero.

Two external forces on the rolling body



The rolling friction is zero.

Solution : A spool consists of two disk joined by a cylinder and is used to store flexible cables, ropes etc. Notably, this question involves more than one external force. However, one simplifying aspect here (as given in the question) is that friction is not required for the overall analysis.

For the overall analysis, friction of rolling is zero for the combined effect of two forces.

(i) For translation :

Applying Newton's second law for translation,

Equation:

$$a_C = \frac{F_1 + F_2}{M}$$

(ii) For rotation :

Applying Newton's second law for rotation,

Equation:

$$-F_1 R + F_2 \text{ x } \frac{R}{2} = I\alpha$$

Note that force “ F_1 “ constitutes a clockwise torque (negative in sign), whereas force “ F_2 “ constitutes an anticlockwise torque (positive in sign).
(iii) For rolling :
Equation:

$$a_C = -\alpha R$$
$$\alpha = -\frac{a_C}{R}$$

Putting this value of “α” in equation – 5,

$$-F_1 R + F_2 \text{ x } \frac{R}{2} = I \text{ x } \frac{a_C}{R}$$

Putting the value of linear acceleration “ a_C “ from equation – 4, we have :

$$-F_1 R + F_2 \text{ x } \frac{R}{2} = I \text{ x } \frac{F_1 + F_2}{MR}$$

Rearranging,

$$\Rightarrow F_1\left(R - \frac{I}{MR}\right) = F_2\left(\frac{R}{2} + \frac{I}{MR}\right)$$

Dividing both sides by “ F_2 “ ,

$$\Rightarrow \frac{F_1}{F_2} = \frac{\left(\frac{R}{2} + \frac{I}{MR}\right)}{\left(R - \frac{I}{MR}\right)}$$

$$\Rightarrow \frac{F_1}{F_2} = \frac{\left(MR^2 + 2I\right)}{2\left(MR^2 - I\right)}$$

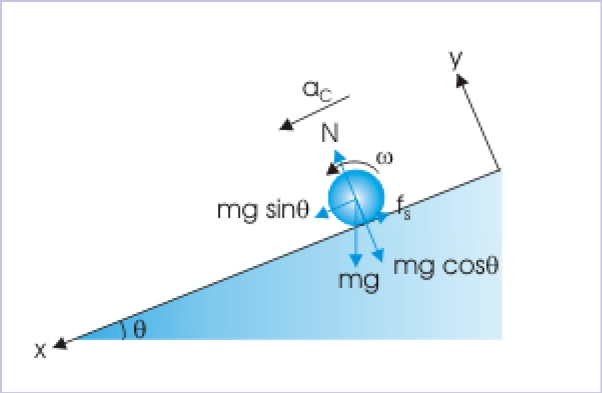
Direction of friction in real time motion

The discussion on the role of friction in rolling motion gives us definite clue about the direction of friction. Evidently it depends on the external force .vs. external torque situation.

We can accelerate rolling by applying force through its center of mass. As discussed earlier, friction in such case acts in the backward direction to counteract sliding in the forward direction. However, it is difficult to visualize a mechanical force that can be applied through center of mass of a rolling body. Incidentally, gravity provides a ready made arrangement in which force acts through center of mass.

A ball rolling along an incline is one such example, in which external force is gravity. Its component along the plane “mgsinθ” acts through COM of the rolling body.

Accelerated rolling motion and friction



Force of gravity acts through center of mass.

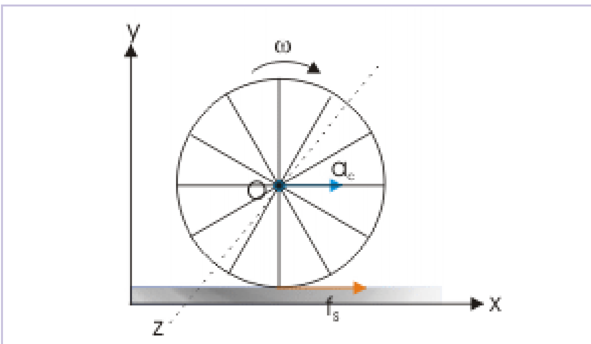
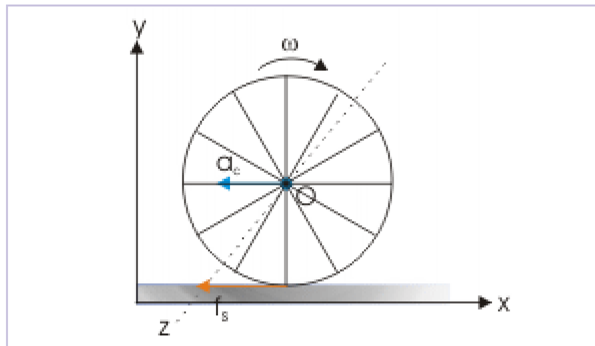
The friction, in this case, acts in the backward direction as shown in the figure above.

Another way to accelerate rolling is to impart a force such that its line of action does not pass through center of mass. In this case, force constitutes a torque that imparts angular acceleration in addition to translational acceleration.

Such is the case with all transporting vehicles having wheels. The internal drive rotates the shaft, which in turn tends to rotate the wheel. We can visualize the situation better for the case of bicycle. When a bicycle is peddled, torque is imparted to the wheel. The chain – socket arrangement

rotates the wheel. As discussed earlier, the friction, in this case, is in the direction of the acceleration of COM.

Accelerated rolling motion and friction

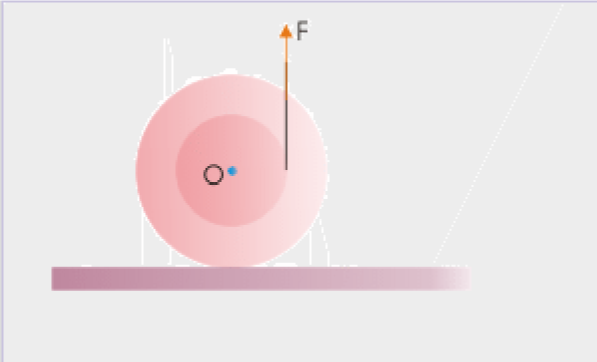
Torques causes acceleration in forward direction.	Torques causes acceleration in backward direction.
	

What if we apply the brake? The sole purpose of applying the brake is to apply a torque in the opposite direction to the direction of rotation. The brake pad jams on the rim. The tangential friction force constitutes the torque opposing the rotation. There is a corresponding linear deceleration of the wheel. A linear deceleration is equivalent to an acceleration in the backward direction. The friction between the wheel and surface, therefore, also acts backward along the backward direction of acceleration (i.e. deceleration) as shown in the right figure above.

Example:

Problem : A spool is pulled by a force, “F”, in vertical direction with the help of a rope wrapped on it as shown. If the spool does not loose contact with the surface, then what is the direction of friction.

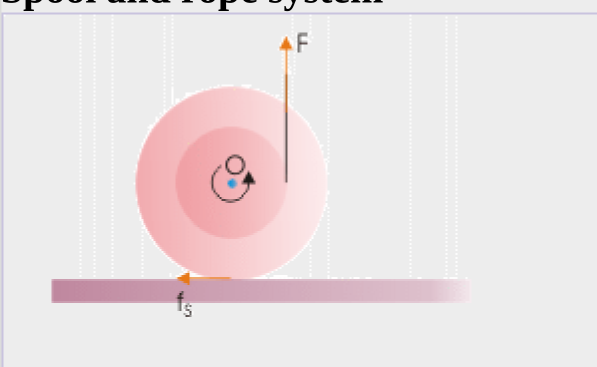
Spool and rope system



A force is applied on the spool
in the vertical direction.

Solution : The tension in the rope is equal to the force applied. This force does not pass through center of mass. The torque due to the force causes angular acceleration of the spool. Since there is no motion in the vertical direction (in the direction of force), there is no vertical linear acceleration involved.

Spool and rope system



A force is applied on the spool
in the vertical direction.

We must note here that the arrangement has created a special situation. In the normal case of a force not passing through center of mass has dual effects of rotation and translation. However, the mechanical arrangement here ensures that there is no vertical motion (weight of the spool does not allow vertical motion for the given situation).

Now, the torque is anticlockwise. As such, the spool tends to slide towards right. In response, static friction acts towards left as shown in the figure above. Though, it is not required in the question to ascertain the translational motion, but we should be aware that friction towards left will cause a translational acceleration in its direction.

Summary

1: The friction between surface and rolling body is self adjusting static friction for accelerated rolling.

2: The friction between surface and rolling body is less than that in the case of sliding.

3: Friction facilitates simultaneous occurrence of two types of acceleration as a response to external stimuli (force or torque or both).

4: Direction of friction :

(i) If the force passes through center of mass, then there is sliding tendency in the forward direction of applied force. In turn, friction acts in backward direction of the external force.

(ii) If the force does not pass through center of mass, then it constitutes torque. There is sliding tendency due to torque in the backward direction of applied force. The net result is that there is either no net friction (in special circumstance) or there is friction in the forward direction.

Role of friction in rolling (check your understanding)

The questions have been selected to enhance understanding of the topics covered in the module titled " Role of friction in rolling motion ". All questions are multiple choice questions with one or more correct answers. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

Understanding level (Role of friction in rolling motion)

Exercise:

Problem: In the front wheel driven car,

(a) friction at the rear wheel is in the direction of motion
(b) friction at the rear wheel is in the opposite direction of motion
(c) friction at the front wheel is in the direction of motion
(d) friction at the front wheel is in the opposite direction of motion

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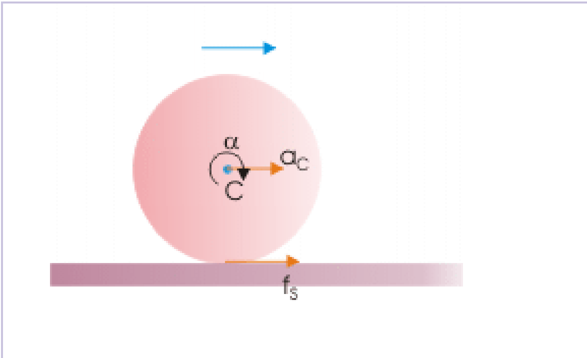
Solution:

The engine imparts torque to the front wheel to rotate. Friction converts angular acceleration into linear acceleration by providing a translational force to accelerate and a torque to counteract angular acceleration. The two functions as outlined here dictate that friction is in the direction of motion.

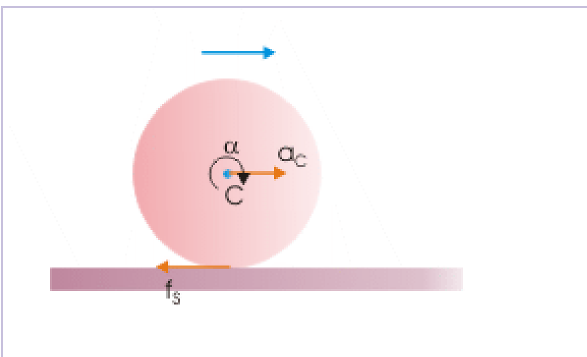
The wheel rolls in forward direction



The friction acts in the direction
of motion at the front wheel.

The translation of car, now, pulls the rear wheel to translate as well. Friction, here, converts linear acceleration into angular acceleration - by providing a torque to impart angular acceleration and a translation force to decelerate the wheel in translation. The two functions as outlined here dictate that friction is in the opposite direction of motion.

The wheel rolls in forward direction



The friction acts in the opposite
direction of motion at the rear
wheel.

Hence, options (b) and (c) are correct.

Exercise:

Problem: A sphere can roll on

(a) a smooth horizontal plane (b) a smooth incline plane
(c) a rough horizontal plane (d) a rough incline plane

Solution:

Rolling at constant velocity does not require external force and friction. Hence, a circular body can roll on a smooth horizontal plane with constant velocity. In the case of an incline, gravity works on the body through the center of mass to impart acceleration. This forms the external force to cause linear acceleration.

For a smooth incline, there is no friction. The external component of gravity along the direction of incline pulls the sphere in translation only as there is no friction. As such, smooth incline can not support accelerated rolling. On the other hand, rough horizontal plane and incline both support rolling by applying friction.

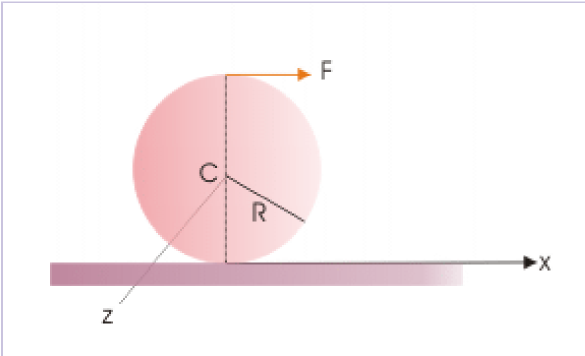
Hence, options (a), (c) and (d) are correct.

Application level (Role of friction in rolling)

Exercise:

Problem:

A disk of mass “M” and radius “R” is at rest on a horizontal surface. An external force “F” acts tangentially at the highest point as shown in the figure. If the disk rolls along a straight line as a result, then find the acceleration of the disk.
A disk in rolling motion



external force “F” acts
tangentially at the highest point.

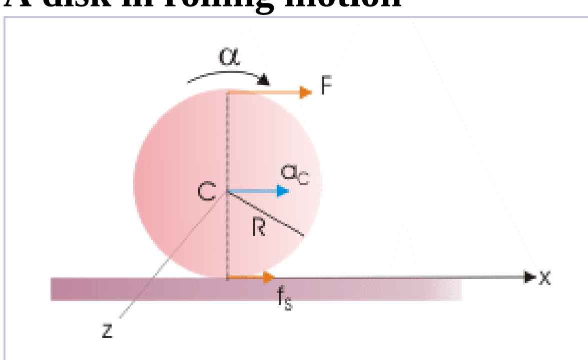
(a) $\frac{2F}{3M}$ (b) $\frac{F}{2M}$ (c) $\frac{4F}{3M}$ (d) $\frac{2F}{5M}$

Solution:

This question is same as discussed in the course. The only difference is that we need to solve the equation for the acceleration instead of friction.

First thing to note here is that external force acts tangentially and, therefore, constitutes a torque about axis of rotation. The torque imparts angular acceleration to the disk. This is opposed by the friction. Its direction is such that the torque due to friction opposes the torque due to external force. This means that friction also acts in the direction of external force.

A disk in rolling motion



external force "F" acts
tangentially at the highest point.

The net force on the disk, on the other hand, imparts linear acceleration. Considering translation of the disk, the second law yields :

Equation:

$$F + f_S = Ma_C$$

Similarly, considering rotation of the disk, the second law yields :

$$-FR + f_S R = I\alpha$$

For rolling motion, a = - αR (negative for clockwise rotation). Substituting for "α", we have :

Equation:

$$-FR + f_S R = -MR^2 \text{ x } \frac{a_C}{R}$$

$$-F + f_S = -\frac{Ma_C}{2}$$

Eliminating friction by solving two resulting (numbered) equations,

$$2F = Ma_C + \frac{Ma_C}{2} = \frac{3Ma_C}{2}$$

$$\Rightarrow F = \frac{3Ma_C}{4}$$

$$\Rightarrow a_C = \frac{4F}{3M}$$

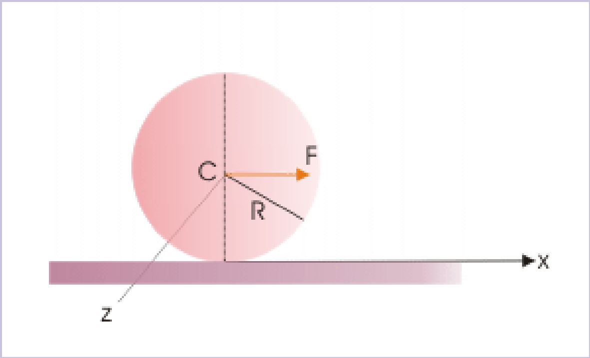
Hence, option (c) is correct.

Exercise:

Problem:

A horizontal force,"F", acts through the center of mass of a disk. If coefficient of friction between the surfaces be "μ" and mass of the disk be "M", then what should be the minimum value of "F" so that disk starts sliding.

A disk in rolling motion



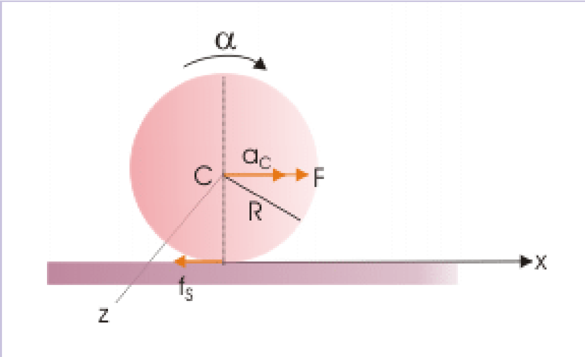
The external force "F" acts through center of mass of the disk.

$$(a)\ \mu Mg\ (b)\ 3\mu Mg\ (c)\ \frac{3\mu Mg}{2}\ (d)\ \frac{\mu Mg}{3}$$

Solution:

For sliding to start, the static friction between disk and the surface should be limiting stating friction.

A disk in rolling motion



The external force “F” acts through center of mass of the disk.

$$f_S = F_S$$

Now, limiting friction is given by :

$$F_S = \mu N = \mu M g$$

In order to find the static friction for rolling, we make use of two Newton’s laws of motion and equation of accelerated rolling. For translation, we have :

For translation, we have :
Equation:

$$F - f_S = M a_C$$

Similarly, for rotation :

$$\tau = f_S R = I\alpha = \frac{1}{2} \text{ x } MR^2 \frac{a_C}{R} = \frac{MRa_C}{2}$$

Equation:

$$f_S = \frac{Ma_C}{2}$$

Substituting in the equation of motion for translation, we have :

$$f_S = \frac{F}{3}$$

The disk starts sliding, when static friction becomes equal to the limiting friction,

$$f_S = F_S$$

$$\frac{F}{3} = \mu M g$$

$$\Rightarrow F = 3\mu M g$$

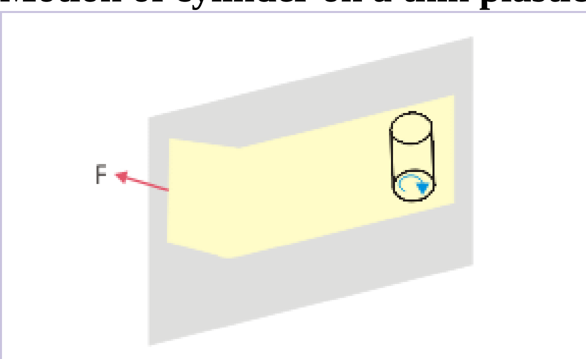
Hence, option (b) is correct.

Exercise:

Problem:

A hollow metal cylinder is placed on a thin rectangular plastic sheet, kept on a horizontal plane table. The plastic sheet is pulled to right rapidly from the bottom of the cylinder. As a consequence, the cylinder starts rotating and is carried along with the plastic sheet to the left as shown in the figure. Here,

Motion of cylinder on a thin plastic sheet



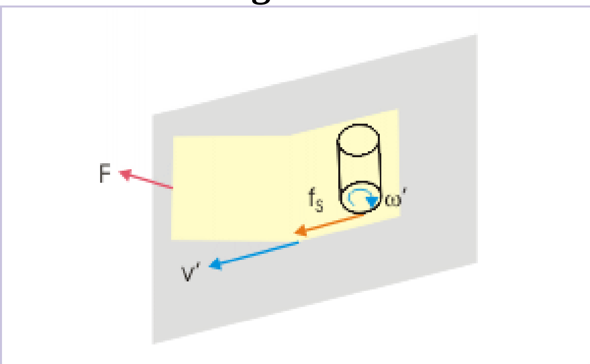
The plastic sheet is pulled out from beneath rapidly.

(a) rotation of cylinder is anticlockwise
(b) motion of cylinder is pure rolling
(c) the cylinder has tendency to slide to the right
(d) the friction acts from right to left

Solution:

As the plastic is pulled out from the bottom in the direction from left to right, the cylinder tends to slide in the opposite direction to the motion of plastic sheet (the cylinder tends to stay where it was). Friction, in turn, appears in the opposite direction i.e. from right to left to counter the sliding tendency.

A disk in rolling motion



The plastic sheet is pulled out from beneath rapidly.

A tangential friction from right to left constitutes a clockwise torque. This imparts a clockwise angular acceleration to the cylinder. This means that cylinder rotates in clockwise direction.

We should carefully consider the motion of the cylinder. The cylinder rotates clockwise. Since it is also carried with the plastic sheet from right to left, the cylinder has the translational velocity with respect to ground - that of the plastic sheet at that moment.

This observation, here, reveals that two motions are, as a matter of fact, opposite to the motions involved in rolling. What it means that a clockwise rotation should actually involve a left to right translation – not right to left translation as in this case.

Hence, options (c) and (d) are correct.

Answers

`1. (b) and (c) 2. (a), (c) and (d) 3. (c) 4. (b) 5. (c) and (d)`

Rolling with sliding

We encounter many real time examples, when circular body in motion rolls with sliding. When we drive the car and are required to decelerate quickly, we need to apply sudden brake. The car in response may skid before coming to a stop. The length of slide depends on the friction between the surfaces. On a wet road, we may find that car slides longer than anticipated (reduced friction). For this reason, we experience almost uncontrolled sliding on a surface covered with ice, when we apply brake hard.

On the other extreme of above example, we may encounter a situation when we may not be able to move a car on a wet land at all. As we press the gas (paddle), the wheel rotates in its place without translating. Only, the wheel keeps rotating. The car may be stuck getting deeper into a groove created there. In such situations, we are required to push the vehicle to translate, keeping some hard matter like bricks or wooden plate near stuck wheel.

Evidently, rolling (pure) is not taking place in these examples. Clearly, rolling with sliding is not pure rolling.

Implication of rolling with sliding

In this section, we outline differentiating aspects of rolling with sliding (impure rolling) via-a-vis pure rolling as :

1: In the case of applying sudden brake, we apply brake pad. The friction between brake pad and braking disk constitute a circumferential force. This force constitutes a torque that opposes angular motion of the rolling wheel. The net effect is that we disproportionately try to reduce angular velocity – not maintaining the relation between angular and linear velocity as given by equation of rolling.

When we accelerate or initiate a motion on a wet land without being able to translate, we apply torque and induce angular acceleration. But, friction between wheel and land is not sufficient to convert angular acceleration into linear acceleration as required for rolling.

2: For rolling with sliding, the distance covered in translation is not equal to the distance covered by a point on the rim of the body in rotation. This means that

Equation:

$$s \neq \theta R$$

In the case of applying “sudden brake”, the car moves a greater distance than 2πR before the wheel completes one revolution i.e. there are fewer revolutions than in the rolling motion.

$$s > \theta R$$

In the case of driving on a wet land, the car, in one revolution of wheel, moves a lesser distance than 2πR. There are more revolutions than in the rolling motion.

$$s < \theta R$$

3: The friction involved in the sliding is kinetic friction – not static friction.

4: In the case of rolling with sliding, the motion is still analyzed, using two forms of Newton’s second law (linear and angular). However, equations of rolling for velocity and acceleration are not valid :

Equation:

$$v_C \neq \omega R$$

Equation:

$$a_C \neq \alpha R$$

5 : When left to itself, motion of a body initially rolling with sliding ultimately changes to rolling due to friction.

Illustrations

We noted in earlier section that, when left to itself, a body initially rolling with sliding is rendered to roll without sliding due to friction. Here, we shall work out two examples to illustrate this assertion for two situations :

- The body initially has greater angular velocity.
- The body initially has greater linear velocity.

We shall first consider a case, in which the body is given initial spin (angular velocity) without any linear velocity. In this case, kinetic friction appears such that the angular velocity is decreased and linear acceleration is increased and over a period of time so that the equation of rolling is satisfied.

Second, we consider a case in which the rigid body initially has greater linear velocity than required for rolling without sliding i.e. for rolling motion. In this case, kinetic friction appears such that the linear acceleration is decreased and angular velocity is increased over a period of time so that the equation of rolling is satisfied. The friction works to balance combination of linear angular motion such that equation of rolling is satisfied.

The solution in each case requires us to apply Newton’s second law for translational and rotational motion separately. Then, we use equation(s) of rolling as limiting condition when motion of rolling with sliding of the body is rendered as pure rolling. In the subsections below, we consider two examples to illustrate the concept discussed here.

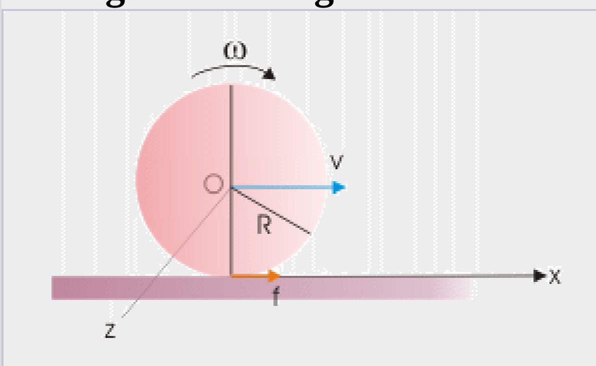
The rigid body initially having greater angular velocity.

Example:

Problem : A hollow sphere of mass “M” and radius “R” is spun and released on a rough horizontal surface with angular velocity, “ ω_0”. After some time, the sphere begins to move with pure rolling. What is linear and angular velocity when it starts moving with pure rolling ?

Solution : In the beginning, the sphere has only angular velocity about the axis of rotation. There is sliding tendency in the backward direction at the point of contact as it continues rotating. This is the natural tendency. This causes friction to come into picture. This friction, in the opposite direction of the sliding, does two things : (i) it works as a net force on the sphere and imparts translational acceleration and (ii) it works as torque to oppose angular velocity i.e. imparts angular deceleration to the rotating sphere. In simple words, the sphere acquires translation motion at the expense of angular motion. This process continues till the condition of pure rolling is met i.e.

Rolling with sliding



Linear velocity and angular velocity are related by equation of rolling.

$$v_C = \omega R$$

It is therefore clear that we must strive to find a general expression of translational and rotational velocities at a given time "t". Once we have these relations, we can use the condition as stated above and find out the required velocity, when sphere starts pure rolling. Now, we know that friction is causing acceleration. This force, incidentally, is a constant force for the given surfaces and allows us to use equation of motion for constant acceleration.

Let us first analyze translation motion. Here,

$$v = v_0 + at$$

Initially, the sphere has only rotational motion i.e. $v_0 = 0$. Hence,
Equation:

$$v = at$$

The translational acceleration is obtained from the Newton's law as force divided by mass :

$$a = \frac{f}{M}$$

Substituting in equation of motion (equation - 4),
Equation:

$$v = \frac{f}{M} \text{ X } t$$

Let us, now, analyze rotational motion. Here,

$$\omega = -\omega_0 + \alpha t$$

Initial angular velocity is clockwise and hence negative. The rotational acceleration is obtained from the Newton's law in angular form as torque divided by moment of inertia (angular acceleration is positive as it anti-clockwise) :

$$\alpha = \frac{\tau}{I} = \frac{fR}{I}$$

Torque and angular acceleration due to friction are anti-clockwise and hence are positive. Substituting in the equation of angular motion,
Equation:

$$\omega = -\omega_0 + \frac{fR}{I} \text{ x } t$$

Thus, we have two equations : one for translational motion and another for rotational motion. We need to eliminate time "t" from these equations. From equation of translational motion (equation – 5), we have :

$$t = \frac{vM}{f}$$

Substituting the value of time in the equation of rotational motion, we have :

$$\omega = -\omega_0 + \frac{vMR}{I}$$

Now, we use the condition of rolling, v = -ωR as "v" and ω have opposite signs. In this case, we multiply the equation for ω by “-R” to get the equation for translational velocity :

Equation:

$$R = -\omega R = \omega_0 R - \frac{vMR^2}{I}$$

Substituting the expression of MI of hollow sphere about a diameter,

$$\Rightarrow v = \omega_0 R - \frac{3vMR^2}{2MR^2}$$
$$\Rightarrow v = \omega_0 R - \frac{3v}{2}$$
$$\Rightarrow \frac{5v}{2} = \omega_0 R$$
$$\Rightarrow v = \frac{2\omega_0 R}{5}$$

Angular velocity is given by :

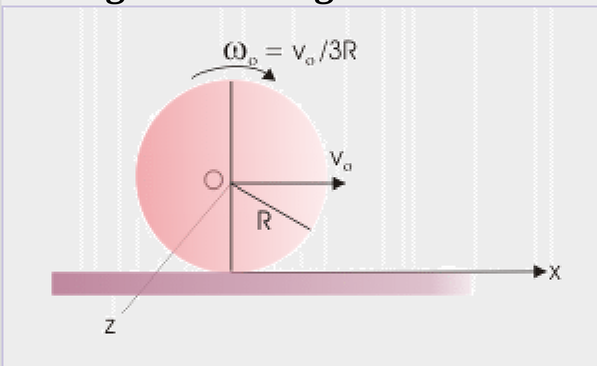
$$\Rightarrow \omega = -\frac{v}{R} = -\frac{\frac{2\omega_0 R}{5}}{R} = -\frac{2\omega_0}{5}$$

The rigid body initially having greater linear velocity.

Example:

Problem : A disk of mass “M” and radius “R” moves on a horizontal surface with translational speed “ v_0 ” and angular speed “ $\frac{v_0}{3R}$ ”. Find linear velocity, when the disk starts moving with pure rolling.

Rolling with sliding



Linear velocity is initially greater than angular velocity

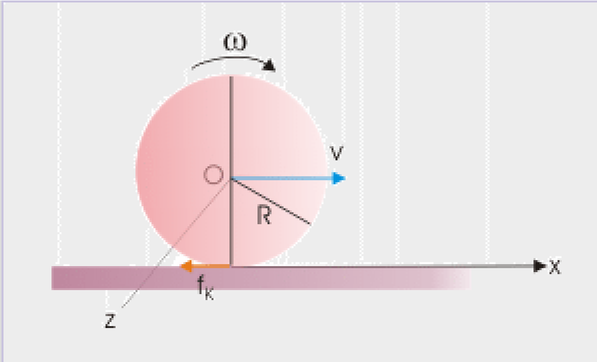
Solution : According to the question, the initial motion is not pure rolling. For pure rolling, translational speed is linked to angular speed by the equation,

$$v = \omega R$$

But here, translational velocity (v_0) is greater than the required angular velocity for rolling. The linear velocity corresponding to the given angular velocity is only " $\omega R = \frac{v_0}{3R}$ x $R = \frac{v_0}{3}$ ". This means that the disk translates more than "2πR" in one revolution. This, in turn, means that the disk also slides on the surface, besides rotating.

The sliding motion is opposed by friction acting in opposite direction to the motion. Now as the disk decelerates, let "a" be the deceleration. The deceleration, "a", is given by Newton's second law. Here, force responsible for deceleration is kinetic friction.

Rolling with sliding



Linear velocity and angular velocity are related by equation of rolling.

$$a = -\frac{F_K}{M}$$

Negative sign shows that it acts in the direction opposite to x-axis. The velocity of center of mass, “v”, at a time “t” is given by the equation of linear motion for constant acceleration (note that we have not used the subscript “C” for center of mass to keep notation simple) :

$$v = v_0 + at$$

Substituting the expression of acceleration,
Equation:

$$v = v_0 - \frac{F_K}{M} \text{ x } t$$

We can similarly find expression for angular velocity at a given time. First, we need to calculate angular acceleration resulting from the torque applied by the friction. Note here that friction causes angular acceleration – not deceleration.
Now, angular acceleration,”α”, is given as :

$$\alpha = \frac{\tau}{I} = -\frac{2F_K R}{MR^2} = -\frac{2F_K}{MR}$$

Angular acceleration is clockwise and hence negative. The angular velocity, “ω”, at a time “t” is given by :

$$\omega = -\omega_0 + \alpha t$$

Substituting expression of angular aceleration,

$$\omega = -\omega_0 - \frac{2F_K}{MR} \text{ x } t$$

Equation:

$$\omega = -\frac{v_0}{3R} - \frac{2F_K}{MR} \text{ x } t$$

According to equation of rolling (negative sign for relation of velocities as against unsigned relation for speeds),
Equation:

$$v = -\omega R = \frac{v_0}{3} + \frac{2F_K}{M} \text{ x } t$$

We have two expressions (equations 8 and 10) for translation velocity at time “t”. Now, "t" is obtained from equation - 8 as :

$$t = \frac{(v_0 - v)M}{2F_K}$$

Substituting expression of “t” in the equation - 10, we have :

$$v = \frac{v_0}{3} + 2\left(v_0 - v\right)$$
$$\Rightarrow 3v = \frac{7v_0}{3}$$
$$\Rightarrow v = \frac{7v_0}{9}$$

Acknowledgement

Author wishes to place special thanks to Wei Yu for their valuable suggestions on the subject discussed in this module.

Pulley in rolling

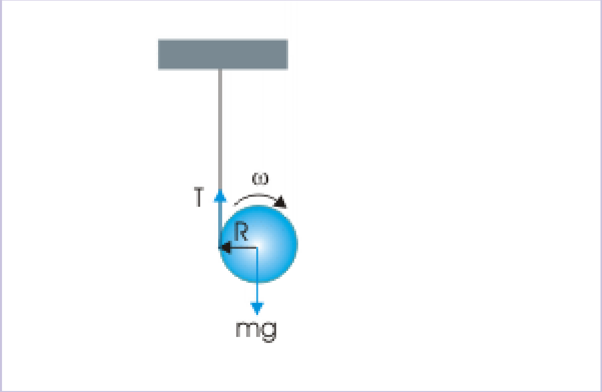
In our earlier treatment on "pulley" in this course, we had limited our consideration to "mass-less" pulley. Here, we shall consider pulley, which has finite mass and is characterized by rolling motion and presence of friction in certain cases.

We need to distinguish two situations (finite mass and mass-less cases) in order to analyze the motion correctly. Mass-less pulley is characterized by the fact that it does not affect the magnitude of tension in the string. It means that tensions in the string on either side of the pulley remains same. In general, a "mass-less" pulley changes the direction of force (tension) without any change in magnitude.

Analysis of the motion of Pulley in rolling

A pulley of finite mass, on the other hand, may rotate, fulfilling the condition of rolling. In this case, the length of rope/string released from the pulley is equal to the distance covered by a point on the rim. If the rolling is accelerated, there is friction between the pulley surface and the string/ rope, passing over it, enabling the pulley to accelerate/decelerate in rotation.

Pulley

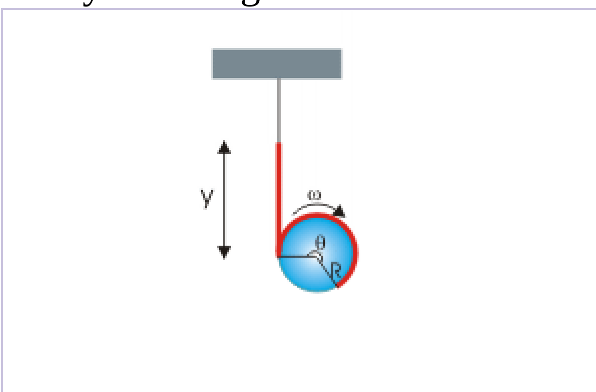


Pulley is rolling on the rope/ string in vertically downward direction.

Evidently, the acceleration of pulley of finite mass will be associated with a net force on the pulley. This, in turn, means that the tension in the strong are not same as in the case of "mass-less" pulley. The pulley, in the figure above, translates and rotates with acceleration, as the string wrapped over it unwinds.

The length of rope unwound is equal to the vertical distance traveled by the pulley/ disk as in the case of rolling and as shown in the figure. Hence,

Pulley in rolling



The length of rope unwound is equal to the vertical distance traveled by the pulley.

$$y = \theta R$$

The motion of pulley may not exactly look like rolling. But, we can see that string/rope plays the role of a surface in rolling. This analogy is not very obscure as string provides a tangential surface like horizontal surface for the pulley to roll. This is an analogous situation to the rolling of a disk. Differentiating the relation, as given above, with respect to time :

Equation:

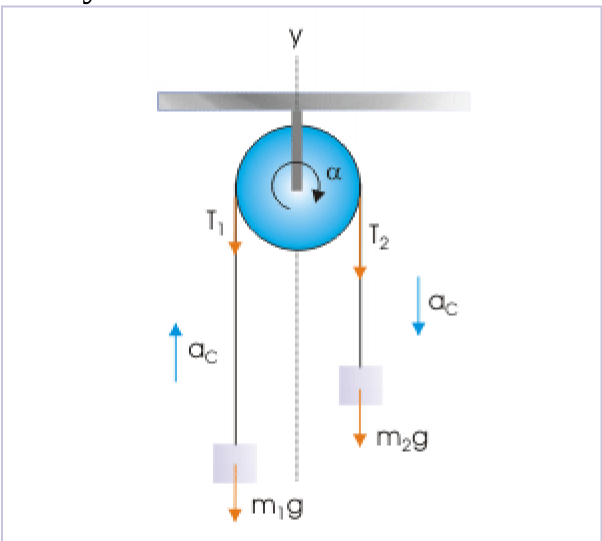
$$v_C = \omega R$$

Equation:

$$a_C = \alpha R$$

In certain situation, the pulley may be fixed to the ceiling as shown in the figure below and hence incapable of translation. We can not say here that pulley is actually not rolling. But, the rope translates as much as a point on the rim of the pulley and as such the rope translates at the same velocity and acceleration as that of the center of mass of the pulley, if it were free to translate. We can see that pulley is executing the rotational part of the rolling motion, whereas string, along with attached blocks, is executing the translational part of the rolling motion. Thus, motions of pulley and string together are equivalent to rolling motion.

Pulley



Fixed pulley and string together executes rolling.

We analyze motion of pulley in same manner as that of a rolling body with the help of two Newton's second laws – one for the linear motion and other for the angular motion. Such consideration of law of motion, however, is conditioned by the equation of rolling and equation of accelerated rolling.

Examples

In this section, we work out with few representative examples to illustrate the application of rolling motion in the case of pulley :

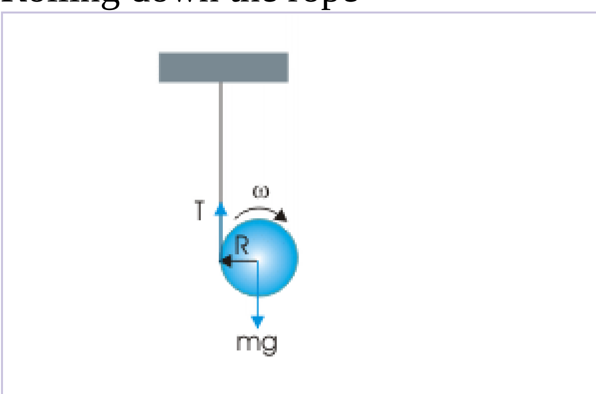
- Pulley or disk of finite mass is translating
- Pulley or disk of finite mass is stationary
- A mass-less pulley connects rolling motion of a disk to the translation of a block.

Pulley or disk of finite mass is translating

Example 1

Problem : A long rope of negligible mass is wrapped many times over a solid cylinder of mass "m" and radius "R". Other end of the rope is attached to a fixed ceiling and the cylinder is then let go at a given instant. Find the tension in the string and acceleration of the cylinder.

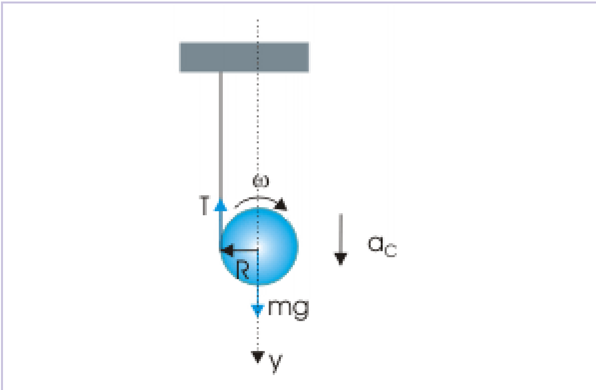
Rolling down the rope



The rolling of the cylinder is guided on the rope.

Solution : The rolling of the cylinder is guided on the rope. We select a coordinate system in which positive y-direction is along the downward motion as shown in the figure.

Rolling down the rope



Free body diagram.

As discussed earlier, we set out to write three equations and solve for the required quantity. Three equations for rolling motions are (i) Newton's second law for translation (ii) Newton's second law for rotation and (iii) Equation of accelerated rolling.

From the application of Newton's second law for translation in y - direction :

Equation:

$$\sum F_y = \mathrm{mg} - T = ma_C$$

From the application of Newton's second law for rotation :

$$\tau = TR = I\alpha$$

Equation:

$$T = \frac{I\alpha}{R}$$

Note that force due to gravity passes through center of mass and does not constitute a torque to cause angular acceleration. Also, we do not consider sign for angular quantities as we shall be using only the magnitudes here .

Now, from relation of linear and angular accelerations (equation of accelerated rolling), we have :

Equation:

$$a_C = \alpha R$$

Putting the value of "α" from above in the equation - 4, we have :

$$T = I\frac{a_C}{R^2}$$

The moment of inertia for solid cylinder about its axis is,

$$I = \frac{mR^2}{2}$$

Hence,

Equation:

$$\Rightarrow T = \frac{mR^2 a_C}{2R^2}$$
$$\Rightarrow T = \frac{ma_C}{2}$$

Substituting the expression of tension in the equation of force analysis in y-direction (equation - 3), we have :

$$\text{mg} - \frac{ma_C}{2} = ma_C$$
$$\Rightarrow g - \frac{a_C}{2} = a_C$$
$$\Rightarrow a_C\left(1 + \frac{1}{2}\right) = g$$
$$\Rightarrow a_C = \frac{2g}{3}$$

Putting this value in the expression of "T" (equation - 6), we have :

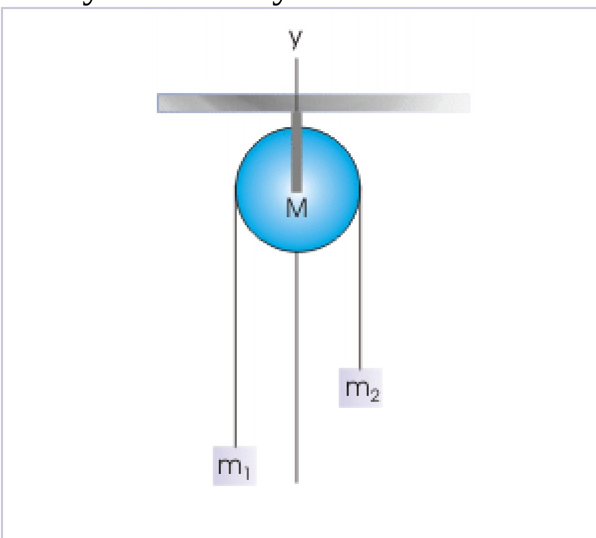
$$\Rightarrow T = \frac{ma_C}{2} = \frac{mg}{3}$$

Pulley or disk of finite mass is stationary

Example 2

Problem : In the “pulley – blocks” arrangement shown in the figure. The masses of the pulley and two blocks are “M”, “ m_1 ” and “ m_2 " respectively. If there is no slipping between pulley and rope, then find (i) acceleration of the blocks and (ii) tensions in the string.

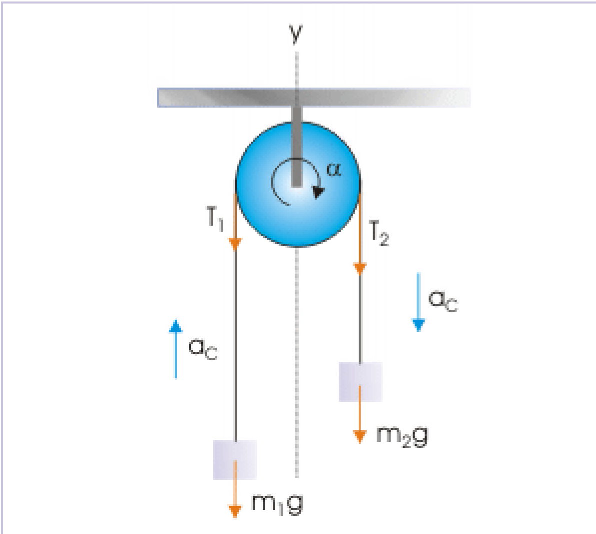
Pulley – blocks system



There is no slipping between pulley and rope.

Solution : The blocks have different masses (hence weights). The net force on the pulley due to difference in weight accelerates the pulley in rotation. Let the tensions in the string be “ T_1 “ and “ T_2 “. Since the rope is inextensible, the different points on the rope has same acceleration as that of the rolling i.e. a_C . Let $m_2 > m_1$.

Pulley – blocks system



There is no slipping between pulley and rope.

The magnitude of torque on the pulley (we neglect the consideration of direction) is :

$$T = (T_2 - T_1) \text{ x } R$$

The pulley is fixed to the ceiling. As such, it is constrained not to translate in response to the net vertical force on it. Instead, the net vertical force causes translational acceleration of the string and the masses attached to the string. The length of string released from the pulley is equal to the distance covered by a point on the rim of the pulley. It means that the linear acceleration of the string is related to angular acceleration of the pulley by the equation of accelerated rolling. Hence,

$$a_C = \alpha R$$

The directions of angular and linear accelerations are as shown in the figure. Now, considering pulley and blocks, we have three equations as :

(i) Rotation of pulley :

$$\left(T_2 - T_1\right) \text{ x } R = I\alpha = \frac{1}{2} \text{ x } MR^2\alpha$$

$$\Rightarrow \alpha = \frac{2(T_2 - T_1)}{MR}$$

and
Equation:

$$\Rightarrow a_C = \alpha R = \frac{2(T_2 - T_1)}{M}$$

(ii) Translation of m_1 :
Equation:

$$T_1 - m_1 g = m_1 a_C$$

(iii) Translation of m2 :
Equation:

$$m_2 g - T_2 = m_2 a_C$$

Putting values of " T_1 " and " T_2 " from equations 7 and 8, in the expression of acceleration (equation - 6), we have :
Equation:

$$\Rightarrow a_C = \frac{2(m_2 g - m_2 a_C - m_1 g - m_1 a_C)}{M}$$

Reaaranging,

$$\Rightarrow a_C(M + 2m_1 + 2m_2) = 2g(2m_2 - m_1)$$

$$\Rightarrow a_C = \frac{2g(2m_2 - m_1)}{(M + 2m_1 + 2m_2)}$$

Putting values of " a_C " from equation - 9, we have :

$$\Rightarrow T_1 = m_1 g + m_1 \text{ x } \frac{2g(2m_2 - m_1)}{(M + 2m_1 + 2m_2)}$$

Rearranging, we have :

$$\Rightarrow T_1 = \frac{m_1 g(M + 4m_2)}{(M + 2m_1 + 2m_2)}$$

Similarly,

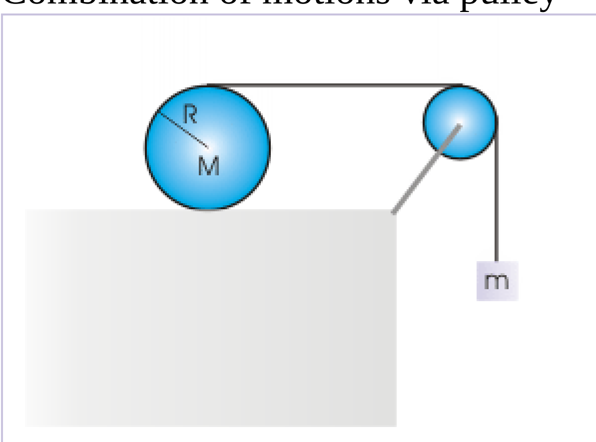
$$\Rightarrow T_2 = \frac{m_2 g(M + 4m_1)}{(M + 2m_1 + 2m_2)}$$

A mass-less pulley connects rolling motion of a disk to the translation of a block

Example 3

Problem : In the “pulley – blocks” arrangement, a string is wound over a circular cylinder of mass “M”. The string passes over a mass-less pulley as shown in the figure. The other end of the string is attached to a block of mass “m”. If the cylinder is rolling on the surface towards right, then find accelerations of the cylinder and block.

Combination of motions via pulley

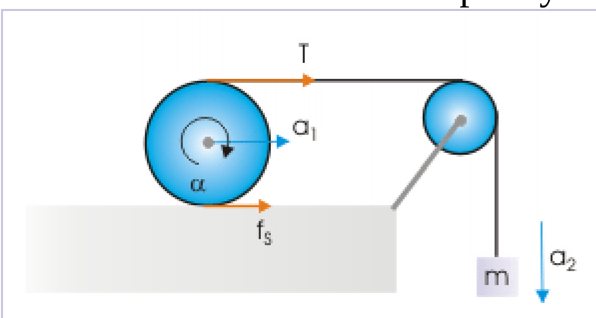


Rolling of cylinder and translation of block

Solution : In this question, string passes over a mass-less pulley. It means that the tension in the string through out is same. Since string is attached to the top of the cylinder, the tension force acts tangentially to the cylinder in rolling. In this case, the friction is acting in forward direction (as the cylinder tends to slide backward).

The linear acceleration of the top position and center of mass are different. As such, linear accelerations of the cylinder and blocks are different. Let the accelerations of cylinder and block be " a_1 "and " a_2 " respectively.

Combination of motions via pulley



Rolling of cylinder and translation of block

(i) Rolling of cylinder :

The magnitude acceleration of the rolling cylinder (i.e. its COM) is :

Equation:

$$a_C = a_1 = \alpha R$$

Now, we need to relate the acceleration of the cylinder to that of string (and hence that of block attached to it). For convenience, we consider only magnitudes here. Now, we know that the magnitude of the velocity of the top point of the cylinder is related to that of COM as :

$$v_T = 2v_C$$

Differentiating with respect to time, the magnitude of the acceleration of the string and the block, attached to it, is :
Equation:

$$a_2 = a_T = 2a_C = 2a_1 = 2\alpha R$$

(ii) Translation of cylinder :

Applying Newton’s second law for translation of the cylinder,
Equation:

$$T + f_S = Ma_1$$

(iii) Rotation of cylinder :

Applying Newton’s second law for rotation,

$$\left(T - f_S\right) \text{ x } R = I\alpha = \frac{1}{2} \text{ x } MR^2\alpha$$

$$\Rightarrow \alpha = \frac{2(T - f_S)}{MR}$$

$$\Rightarrow \alpha MR = 2T - f_S$$

For rolling, we use relation of accelerations as given by equation - 11 :

$$\Rightarrow 2T - 2f_S = Ma_1$$

Putting values of " f_S " from equation - 13,

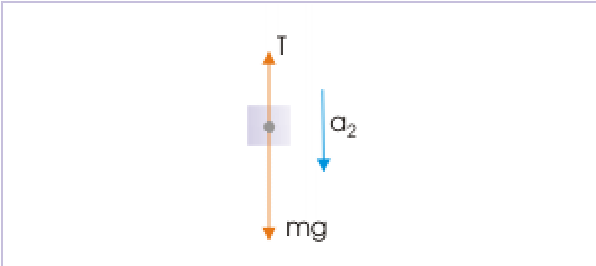
$$\Rightarrow 2T - 2\left(Ma_1 - T\right) = Ma_1$$

Equation:

$$\Rightarrow T = \frac{3Ma_1}{4}$$

(iv) Translation of block :

The free body diagram of the block is shown in the figure. Here,
Free body diagram of block



The block is accelerating down.

$$mg - T = ma_2$$

Substituting value of "T" from equation – 14,

$$ma_2 - mg = T = \frac{3Ma_1}{4}$$

Substituting this value of " a_1 " in terms of " a_2 " from equation – 11 and solving for " a_2 " ,

$$\Rightarrow a_2 = \frac{8mg}{(8m+3M)}$$

As the magnitude of " a_1 " is given by :

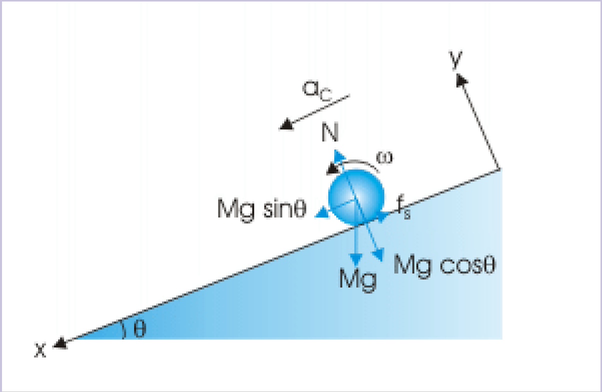
$$\Rightarrow a_1 = \frac{a_2}{2} = \frac{4mg}{(8m+3M)}$$

Rolling along an incline

An incline is an ideal arrangement to realize accelerated rolling motion. Force due to gravity acts through the center of mass of the rolling body. Component of gravity parallel to incline accelerates the body in translation as it goes down and decelerates the body as it goes up the incline.

We have already discussed the case of force, whose line of action pass through center of mass. For rolling of the body, the friction between rolling body and surface appears such that the condition as laid down by equation of accelerated rolling is satisfied. When a body rolls down, it has linear acceleration in downward direction. The friction, therefore, acts upward to counter sliding tendency as shown in the figure. This friction constitutes an anticlockwise torque providing the corresponding angular acceleration as required for maintaining the condition of rolling (if linear velocity is increasing, then angular velocity should also increase according to equation of accelerated rolling).

Rolling down an incline



The static friction acts up the incline.

Friction here plays a dual role :

- It decelerates translational motion.
- It accelerates rotational motion.

As the body rolls down, linear velocity increases with time such that its angular velocity also increases simultaneously in accordance with equation of rolling,

Equation:

$$v_C = \omega R$$

The linear acceleration of the COM of the rolling body is equal to the component of acceleration due to gravity in x direction,

Equation:

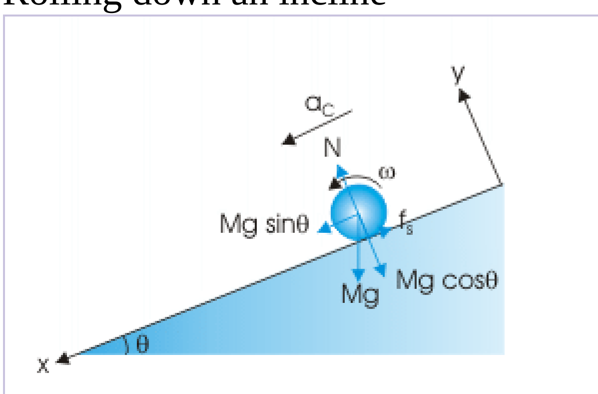
$$a_C = g \sin\theta = \alpha R$$

The fact that accelerations are constant has important implications. The motions (translational and rotational) of rolling body along an incline can be described by the constant acceleration kinematics i.e. by the equations of accelerated motion.

Analysis of rolling along an incline

The analysis of rolling involves applying Newton's second law for both translational and rotational motion and using equation of accelerated rolling. First of all, we select an appropriate pair of rectangular coordinates such that motion is along the positive direction of the x-coordinate. The various forces acting on the rolling disk are :

Rolling down an incline



Forces acting on the rolling disk.

1. Force of gravity, Mg, acting downward.
2. Normal force, N, perpendicular to the incline in y-direction.
3. Static friction, f_s , acting upward.

The force/ force components are acting in mutually perpendicular directions. As such, we can analyze motion in x-direction independently.

Thus, confining force analysis in x-direction, we apply Newton's law of motion for linear motion as :

Equation:

$$\sum F_x = \mathrm{Mg}\sin\theta - f_s = Ma_C$$

Similarly, we apply Newton's law for rotation :

$$\tau = I\alpha$$

We note here that force due to gravity and normal force pass through the center of mass. As such, they do not constitute torque on the rolling disk. It is only the friction that applies torque on the disk, which is given as :

Equation:

$$\tau = f_s R = I\alpha$$

At this stage, we can make use of third equation that connects linear and angular accelerations for rolling without sliding :

Equation:

$$a_C = \alpha R$$

Combining equations 4 and 5,

$$f_sR = I\alpha = I\frac{a_C}{R}$$
$$\Rightarrow f_s = \frac{Ia_C}{R^2}$$

Substituting the expression of friction as above in the equation – 3,

$$\Rightarrow \mathrm{Mg}\sin\theta - \frac{Ia_C}{R^2} = Ma_C$$
$$\Rightarrow a_C\left(M + \frac{I}{R^2}\right) = \mathrm{Mg}\sin\theta$$
$$\Rightarrow a_C = \frac{g\sin\theta}{\left(1+\frac{I}{MR^2}\right)}$$

Equation:

$$\Rightarrow a_C = \frac{g\sin\theta}{\left(1+\frac{I}{mR^2}\right)}$$

We can use this expression to find linear acceleration (hence angular acceleration also) for circular rolling bodies like ring, disk, cylinder and sphere etc. We only need to use appropriate expression of moment of inertia as the case may be.

Example:
Problem : A block and a circular body are released from the same height of two identical inclines. The block slides down the incline, whereas circular body rolls down the incline. If acceleration of circular body is $\frac{2}{3}$ rd that of the block, then identify the circular body.
Solution : The block slides down the incline without rotating. Its acceleration is :

$$a_B = g\sin\theta$$

The circular body rolls down the incline without sliding. Its acceleration is :

$$a_C = \frac{g \sin \theta}{\left(1 + \frac{I}{MR^2}\right)}$$

According to question,

$$a_C = \frac{2a_B}{3}$$

$$\frac{g \sin \theta}{\left(1 + \frac{I}{mR^2}\right)} = \frac{2g \sin \theta}{3}$$

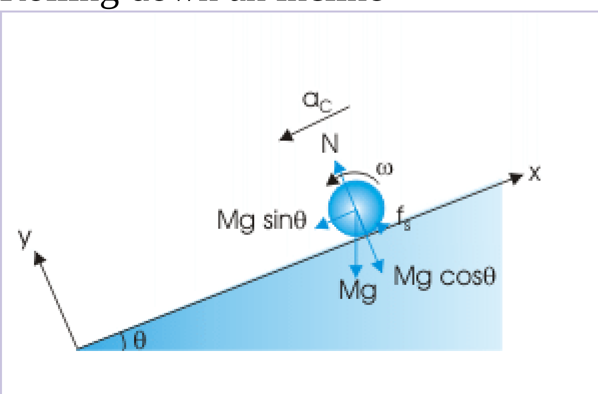
$$\left(1 + \frac{I}{MR^2}\right) = \frac{3}{2}$$

$$I = \frac{MR^2}{2}$$

This is MI of either a disk or a solid cylinder. Thus, the circular body in question is either of the two.

There is a caveat in the sign of the acceleration of rolling body down the incline. If we choose the orientation of x - coordinate is opposite direction to the one above, then the acceleration of the rolling body is given by the same expression, but with a negative sign preceding it :

Rolling down an incline



Changing reference direction.

$$a_C = -\frac{g \sin\theta}{\left(1+\frac{I}{MR^2}\right)}$$

The sign of the quantities should not be a source of concern. We should merely stick with the sign convention and choice of coordinate system. Negative sign essentially, though a source of confusion, does not ever change the physical meaning of the quantities in kinematics.

Example:

Problem : A ring, a disk, a solid cylinder and a solid sphere of same mass and radius roll down an incline simultaneously. Which of these will the reach the bottom first? Rank them in the order they reach the bottom.

Solution : The object with maximum acceleration will reach the bottom first. Now, acceleration of the rolling object is given by the following equation,

$$a_C = \frac{g \sin\theta}{\left(1+\frac{I}{MR^2}\right)}$$

From the equation, it is clear that the object with smaller moment of inertia (I) will have greater translational acceleration. Now, MI of different objects of same mass and radius are as given here :

$$I_{\text{ring}} = MR^2$$
$$I_{\text{disk/cyliner}} = \frac{MR^2}{2} = 0.5MR^2$$
$$I_{\text{solid}} = \frac{2MR^2}{5} = 0.4MR^2$$

Thus, solid sphere with minimum moment of inertia will reach the bottom first, followed by disk and cylinder. The ring will be the last to reach the bottom.

Summary

1: External force due to gravity on a rolling body acts through center of mass.

2: The component of gravity parallel to incline causes rolling body to slide down.

3: Friction acts opposite to the component of gravity parallel to incline.

4: The acceleration of the center of mass of the rolling body on an incline is given by :

$$a_C = \frac{g \sin \theta}{\left(1+\frac{I}{MR^2}\right)}$$

Rolling along an incline (check your understanding)

The questions have been selected to enhance understanding of the topics covered in the module titled " Rolling along an incline ". All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

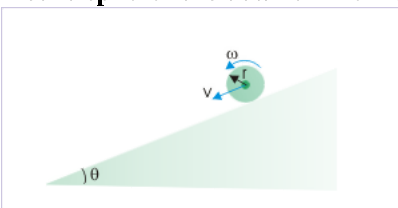
Understanding level (Rolling along an incline)

Exercise:

Problem:

A solid sphere of mass “m” and radius “r” rolls down an incline of angle “θ” without sliding. What is its acceleration down the incline?

A solid sphere rolls down an incline



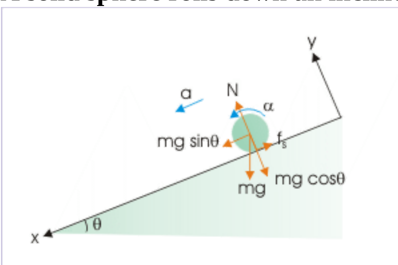
The sphere has mass “m” and radius “r”.

$$(a)\ g\sin\theta\ \ (b)\ \frac{2g\sin\theta}{3}\ \ (c)\ \frac{2g\sin\theta}{5}\ \ (d)\ \frac{5g\sin\theta}{7}$$

Solution:

For the analysis of rolling motion, we have three equations :

A solid sphere rolls down an incline



The sphere has mass “m” and radius “r”.

(i) Newton’s second law for translation

$$mg\sin\theta - f_S = ma$$

(ii) Newton's second law for rotation

$$\tau = f_S r = I\alpha$$
$$\Rightarrow f_S r = \frac{2mr^2\alpha}{5}$$
$$\Rightarrow f_S = \frac{2mr\alpha}{5}$$

(iii) Equation of accelerated rolling

$$a_C = a = \alpha r$$

Substituting value of "α" in the equation of rotation,

$$\Rightarrow f_S = \frac{2ma}{5}$$

Putting this value in the equation of translation,

$$\Rightarrow mg\sin\theta - \frac{2ma}{5} = ma$$

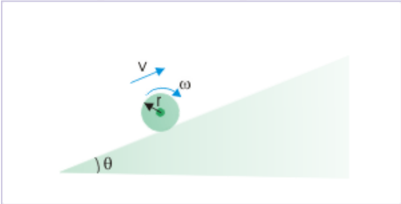
$$\Rightarrow a = \frac{5g\sin\theta}{7}$$

Hence, option (d) is correct.

Exercise:

Problem: A solid sphere of mass "m" and radius "r" is rolled up on an incline as shown in the figure. Then,

A solid sphere rolls up an incline



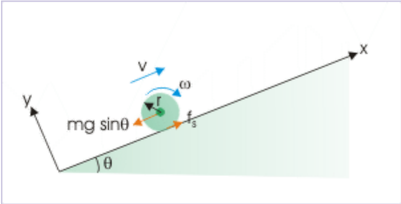
The sphere has mass "m" and radius "r".

$$\text{(a) Friction acts upwards (b) Friction acts downwards}$$
$$\text{(c) Friction accelerates translation (d) Friction accelerates rotation.}$$

Solution:

The solid sphere rotates clockwise to move up along the incline. The component of gravity along the incline acting downward decelerates translation. The friction, therefore, should appear in such a fashion that (i) there is linear acceleration to counter the deceleration due to gravity and (ii) there is angular deceleration so that equation of accelerated rolling is held.

A solid sphere rolls up an incline



The sphere has mass “m” and radius “r”.

This means that friction acts upward. It accelerates translation in the direction of motion.

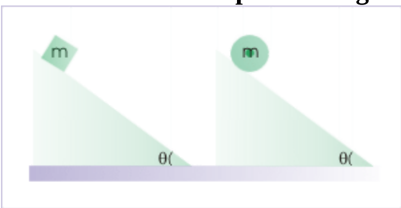
Hence, options (a) and (c) are correct.

Exercise:

Problem:

A block of mass “m” slides down a smooth incline and a sphere of the same mass rolls down another identical incline. If both start their motion simultaneously from the same height on the incline, then

A block and a solid sphere along identical inclines



The block and solid sphere move down the incline.

(a) block reaches the bottom first (b) sphere reaches the bottom first

(c) both reaches at the same time (d) either of them can reach first, depending on the friction betwee

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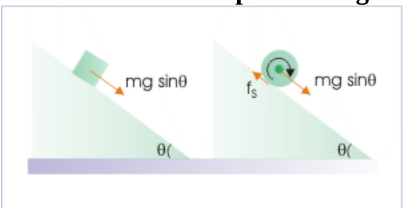
Solution:

The time taken to reach bottom depends on the translational velocity. In turn, velocity depends on the acceleration of the bodies during motion.

A block and a solid sphere along identical inclines



The block slides and solid sphere rolls down the incline.

In the first case, the block slides on the smooth plane. Hence, there is no friction between surfaces. The force along the incline is component of force due to gravity in the direction of motion. The translational acceleration of the block is :

$$a_B = g \sin \theta$$

In the second case, static friction less than maximum static friction operates at the contact point. The net force on the sphere, rolling down the incline, is :

$$F_{\text{net}} = mg \sin \theta - f_S$$

The corresponding translational acceleration of the sphere is :

$$a_S = g \sin \theta - \frac{f_S}{m}$$

Evidently,

$$a_B > a_S$$

Therefore, the block reaches the bottom first.

Hence, option (a) is correct.

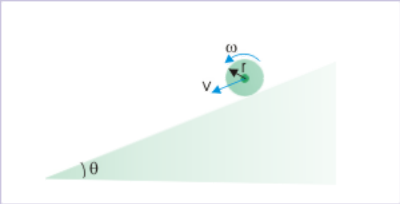
Application level (Rolling along an incline)

Exercise:

Problem:

A solid sphere of mass "m" and radius "r", rolls down an incline of angle "θ" without sliding. What should be the minimum coefficient of friction (μ) so that the solid sphere rolls down without sliding?

A solid sphere along an incline



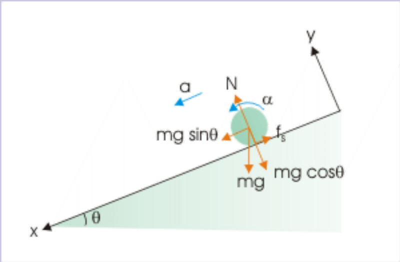
The solid sphere rolls down the incline.

$$\text{(a)}\ \mu > \frac{2g \tan \theta}{7}\ \text{(b)}\ \mu < \frac{2g \sin \theta}{5}\ \text{(c)}\ \mu < \frac{2g \cos \theta}{5}\ \text{(d)}\ \mu > \frac{2g \tan \theta}{3}$$

Solution:

Rolling requires a minimum threshold value of friction. If the friction is less than this threshold value, then the solid sphere will slide down instead of rolling down. Now, for the analysis of rolling motion, we have three equations :

A solid sphere along an incline



The solid sphere rolls down the incline.

(i) Newton's second law for translation

$$mg\sin\theta - f_S = ma$$

(ii) Newton's second law for rotation

$$\tau = f_S r = I\alpha$$
$$\Rightarrow f_S r = \frac{2mr^2\alpha}{5}$$
$$\Rightarrow f_S = \frac{2mr\alpha}{5}$$

(iii) Equation of accelerated rolling

$$a_C = a = \alpha r$$

Substituting value of "α" in the equation of rotation,

$$\Rightarrow f_S = \frac{2ma}{5}$$

Putting this value in the equation of translation,

$$\Rightarrow mg\sin\theta - \frac{2ma}{5} = ma$$

$$\Rightarrow a = \frac{5g\sin\theta}{7}$$

Thus,

$$\Rightarrow f_S = \frac{2ma}{5} = \frac{2mg\sin\theta}{7}$$

The limiting friction corresponding to a given coefficient of friction is :

$$F_S = \mu N = \mu mg\cos\theta$$

For rolling to take place, the coefficient of friction be such that limiting friction corresponding to it is greater than the static friction required to maintain rolling,

$$\mu mg\cos\theta > \frac{2mg\sin\theta}{7}$$
$$\Rightarrow \mu > \frac{2g\tan\theta}{7}$$

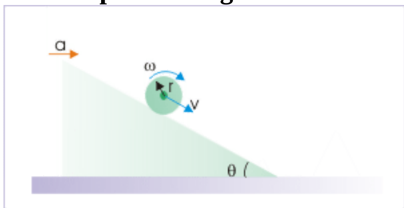
Hence, option (a) is correct.

Exercise:

Problem:

A solid sphere of mass “M” and radius “R” is set to roll down a smooth incline as shown in the figure. If the smooth incline plane is accelerated to the right with acceleration “a”, then what should be the horizontal acceleration of incline such that solid sphere rolls without sliding?

A solid sphere along an incline



The solid sphere rolls down the incline, which itself is accelerated toward right.

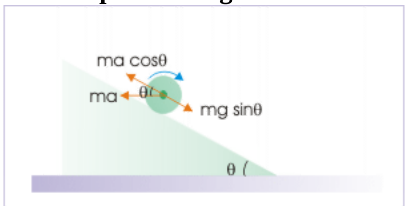
$$\text{(a)}\ g\sin\theta\ \ \text{(b)}\ g\cos\theta\ \ \text{(c)}\ g\tan\theta\ \ \text{(d)}\ g\cot\theta$$

Solution:

There is no friction between smooth incline and the solid sphere. The rolling sphere will roll on smooth plane at constant velocity. Acceleration will tend to slide the solid sphere on a smooth plane, where there is no friction. Hence, net force on the sphere along the incline should be zero so that the sphere keeps rolling with constant velocity.

Now, the incline itself is an accelerated frame of reference. For force analysis, we consider a pseudo force to covert the accelerated reference system to an equivalent inertia reference system. The pseudo force acts through the center of mass and its magnitude is :

A solid sphere along an incline



Pseudo force renders accelerated frame into an inertial frame.

$$F_{\text{Pseudo}} = ma$$

Analyzing force for translation along the incline, we have :

$$mg\sin\theta - ma\cos\theta = 0$$

$$\Rightarrow a = g\tan\theta$$

Hence, option (c) is correct.

Answers

`1. (d) 2. (a) and (c) 3. (a) 4. (a) 5. (c)`

Work and energy in rolling motion

Work and energy consideration in rolling has certain interesting aspects. It gives a new functional meaning to friction, which otherwise has always been associated with negative work. Also, work and energy consideration in rolling has two parts : one that pertains to translation and other that pertains to rotation.

In this module, we shall also find that work – energy consideration provides a very useful frame work to deal with situation in which rolling is not along a straight line, but a curved path.

Work done in rolling

Work is associated with force and displacement. When a body rolls on a horizontal plane at a constant velocity, there is no external force. Irrespective of the nature of surface, friction is zero. No work, therefore, is done on the body. There is no corresponding change in the kinetic energy of the rolling body.

For an accelerated rolling, an external force is applied on the body. Application of external force in rolling, however, is not very straight forward as in the case of translation. Static friction comes into picture – whenever there is sliding tendency. Depending on the situation, friction plays specific role to maintain rolling.

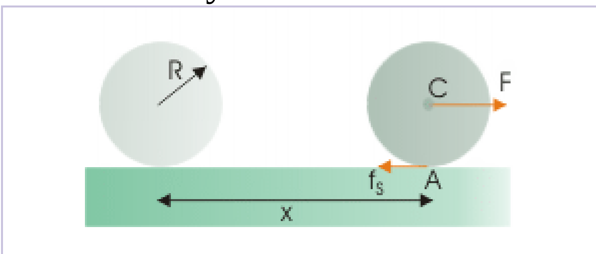
We shall begin with the simplest case of rolling along a horizontal line to the case of rolling along an incline to clearly understand work by different forces in rolling.

Accelerated rolling along a straight horizontal line

For illustration purpose, we consider an external force, “F”, that acts through the COM and parallel to the surface as shown in the figure below. Friction acts in the backward direction. Let the disk rolls a linear distance “x” in the x-direction.

(i) Work by external force "F"

Work done by external forces



Friction does negative translational work, but positive rotational work.

Equation:

$$W_F = Fx$$

(ii) Work by static friction

The static friction has dual role here. It negates translation i.e. does negative work. Also, it accelerates rotation. As such, friction does the positive rotational work. If "T" and "R" denotes translation and rotation respectively, then :

Equation:

$$W_{\mathrm{fT}} = Fr = -f_S x$$

and

Equation:

$$W_{\mathrm{fR}} = \tau\theta = f_S R\theta$$

where "θ" is total angle covered during the motion. For rolling motion,

Equation:

$$\theta = \frac{x}{R}$$

Putting the expression of angle in equation – 3,

Equation:

$$W_{\mathrm{fR}} = f_S R \text{ x } \frac{x}{R} = f_S x$$

The results for the work done by friction in translation and rotation are very significant. They are equal, but opposite in sign. The net work by friction, therefore, is zero.

$$\Rightarrow W_f = W_{\mathrm{fT}} + W_{\mathrm{fR}} = -f_S x + f_S x = 0$$

Thus, total work done by the external forces is :

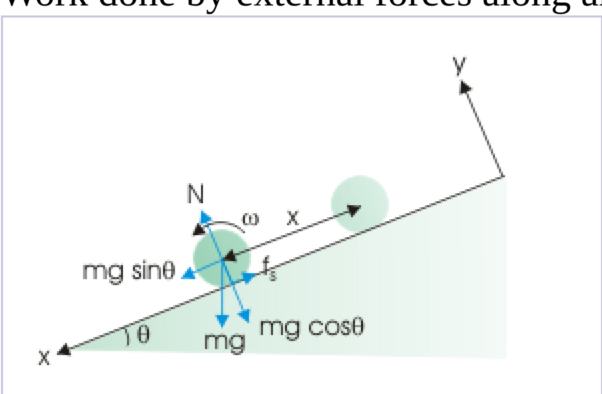
Equation:

$$\Rightarrow W = Fx$$

Accelerated rolling along an incline

In this case also, friction does negative work in translation and positive work in rotation.

Work done by external forces along an incline



Friction does negative

translational work, but positive rotational work.

(i) Work by gravity

The component of gravity perpendicular to motion is perpendicular to displacement. Hence, it does not do work. The work by the component of gravity parallel to incline is :

Equation:

$$W_g = Mgx\sin\theta$$

(ii) Work by static friction

The static friction has dual role here. If “T” and “R” denotes translation and rotation respectively, then :

Equation:

$$W_{\mathrm{fT}} = Fr = -f_S x$$

and

Equation:

$$W_{\mathrm{fR}} = \tau\theta = f_S R\theta$$

where “θ” is total angle covered during the motion. For rolling motion,

Equation:

$$\theta = \frac{x}{R}$$

Putting the expression of angle in equation – 8,

$$W_{\mathrm{fR}} = f_S R \text{ x } \frac{x}{R} = f_S \text{ x}$$

The work done by friction in translation and rotation are equal, but opposite in sign. The net work by friction is zero.

Thus, total work done by the external forces is :
Equation:

$$W = Mgx\sin\theta$$

We find that work by friction in rolling is zero. It is so because it does negative work in translation and equal positive work in rotation. There may be situations in which friction does positive work in translation and negative work in rotation. For example, if a body is initially given angular velocity greater than linear velocity required by equation of rolling, then the friction does the positive work in translation (accelerates translation) and negative work in rotation (decelerates rotation). Here also, net work by friction is zero in rolling.

Mechanical energy of rolling body

Mechanical energy comprises of potential and kinetic energy of the rolling body. These two forms of energy is described, in reference to rolling, as here :

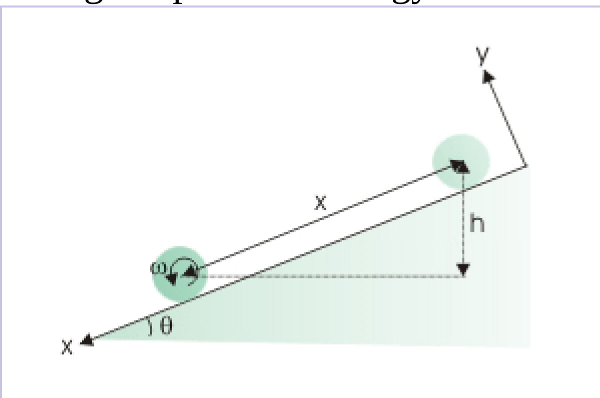
Potential energy

We must have observed the conspicuous absence of reference to the potential energy in rotation. The reason is simple. The rotating body does not change its center of mass over a period of time in pure rotation. There is no change in the body – Earth configuration due to rotation. This is a valid approximation for a relatively small body such that its mass distribution with respect to Earth remains same.

If a body is rolling along a straight line on a horizontal surface, then there is no change in potential energy as COM of the rolling body remains at same vertical height from the ground. In the case of rolling along an incline, the

COM of the rolling body changes elevation and as such there is change in the gravitational potential energy of the body – Earth system (Note that potential energy is always referred to a system – not to a single body like kinetic energy. When we assign potential energy to a body, the reference to Earth or other force field is implicit). It must be borne in mind that potential energy changes in rolling is on account of translation only.

Change in potential energy



The potential energy changes as vertical elevation of the body changes.

Equation:

$$\Delta U = Mgh$$

where “h” is the change in vertical elevation.

Kinetic energy

Unlike potential energy, kinetic energy arises from both translation and rotation. The kinetic energy of rolling body is a positive sum of translational and rotational kinetic energies. The total kinetic energy of a rolling body is given by :

$$K = K_T + K_R = \frac{1}{2} \text{ x } M{V_C}^2 + \frac{1}{2} \text{ x } I\omega^2$$

Looking at the expressions of translational and rotational kinetic energy, we find that we can convert the expression either in terms of linear or angular velocity, using equation of rolling,

$$v_C = \omega R$$

Total kinetic energy in terms of linear velocity,
Equation:

$$\Rightarrow K = \left(I + MR^2\right) \text{ x } \frac{{V_C}^2}{2R^2}$$

Total kinetic energy in terms of angular velocity,
Equation:

$$\Rightarrow K = \left(I + MR^2\right) \text{ x } \frac{\omega^2}{2}$$

Distribution of kinetic energy

The rolling motion is characterized by unique distribution of kinetic energy between translation and rotation. This distribution is dependent on the distribution of mass about the axis of rotation. If “T” and “R” subscripts denote translational and rotational motions respectively, then :

$$\frac{K_T}{K_R} = \frac{M{V_C}^2}{I\omega^2} = \frac{M{V_C}^2R^2}{I{V_C}^2}$$

Equation:

$$\frac{K_T}{K_R} = \frac{MR^2}{I}$$

For a ring,

$$I = MR^2$$

$$\Rightarrow \frac{K_T}{K_R} = \frac{MR^2}{MR^2} = 1$$

This means that energy is equally distributed between translation and rotation in the case of ring. Similarly, for a solid sphere,

$$\Rightarrow \frac{K_T}{K_R} = \frac{5MR^2}{2MR^2} = \frac{5}{2}$$

We can infer from the values of ratio in two cases that the ratio of kinetic energies in translation and rotation is equal to the inverse of the numerical coefficient of MI of the rolling body.

Work – kinetic energy theorem

Work by net external force is related to kinetic energy of the rolling body in accordance with work – energy theorem as :
Equation:

$$\Rightarrow W_{\text{ext}} = \Delta K$$

In rolling, the important feature of work by net external force on the left of the equation is that this net external force term does not include friction as work by friction is zero. This is the first difference between consideration of work – energy theorem in pure rolling as against in pure translation or pure rotation. Secondly, the change in kinetic energy in rolling refers to both translational and rotational kinetic energy.

Conservation of mechanical energy in rolling

The work by friction in rolling is zero. The presence of friction does not result in dissipation of energy like heat to the system and its surrounding. It means that total mechanical energy comprising of potential and kinetic energy of a rolling body system remains same or is conserved.

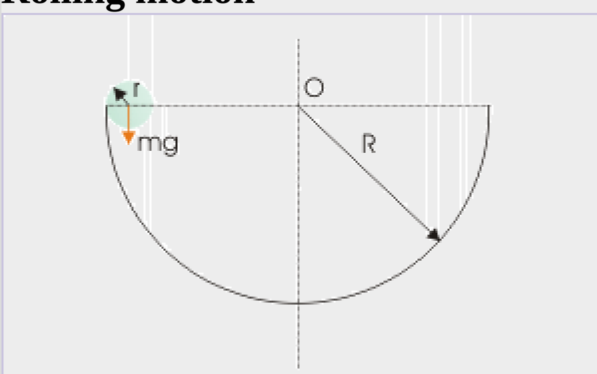
This is contrary to the motion of translation alone i.e. sliding - in which case, friction converts some of the mechanical energy into heat, sound etc, which can not be regained by the system as mechanical energy. Thus, mechanical energy of a body in translation only (sliding) is not conserved. In the case of pure rotation, mechanical energy of the rotating body is not conserved as the force of friction at the shaft (axis of rotation) results in dissipation of energy.

Conservation of mechanical energy is the characterizing feature of pure rolling. This is significant as mechanical energy is conserved even when friction is present.

Example:

Problem : A small spherical ball of mass “m” and radius “r” rolls down a hemispherical shell of radius “R” as shown in the figure. The ball is released from the top position of the shell. What is the angular velocity of the spherical ball at the bottom of the shell as measured about perpendicular axis through the center of hemispherical shell.

Rolling motion



A spherical ball rolls down the slope of a hemispherical shell.

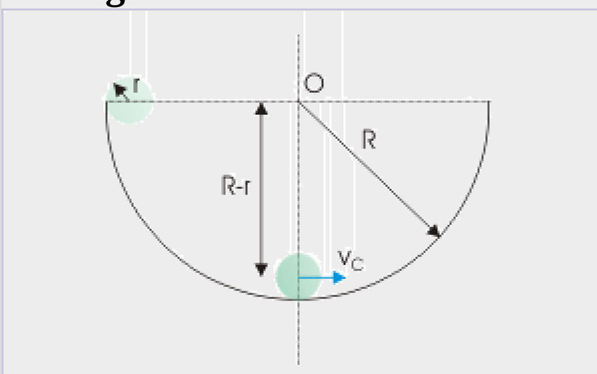
Solution : Let the ball reaches the bottom with certain velocity “ v_C ”. Then, the angular velocity of the ball about perpendicular axis through the center of hemispherical shell is :

Equation:

$$\omega_0 = \frac{v_C}{(R-r)}$$

Note that the linear distance between the center of ball and center of hemispherical shell is (R-r). In order to evaluate this equation, we need to find linear velocity of the spherical ball. Now, the ball is rolling along a curved path, while transcending a height of (R-r).

Rolling motion



A spherical ball rolls down the slope of a hemispherical shell.

According to conservation of mechanical energy,

$$K = \frac{1}{2} \text{ x } MV_C^2 + \frac{1}{2} \text{ x } I\omega^2 = Mg\ R - r$$

Using, $\omega = \frac{V_C}{R}$ and putting expression of MI of the sphere,

$$\Rightarrow \frac{1}{2} \text{ x } MV_C^2 + \frac{1}{2} \text{ x } \frac{2MR^2}{5} \text{ x } \frac{V_C^2}{R^2} = Mg\left(R - r\right)$$

$$\Rightarrow \frac{1}{2} \text{ x } MV_C^2 + \frac{1}{5} \text{ x } MV_C^2 = Mg\ R - r$$

$$\Rightarrow V_C = \sqrt{\left\{\frac{10g(R-r)}{7}\right\}}$$

The required angular velocity about perpendicular axis through the center of hemispherical shell is obtained by substituting for vC in equation – 17 :

$$\Rightarrow \omega_0 = \frac{v_C}{(R-r)} = \sqrt{\left\{\frac{10g}{7(R-r)}\right\}}$$

Summary

1: Work by friction is zero in rolling.

2: Work by friction in translation and rotation are equal in magnitude but opposite in direction.

3: Gravitational potential energy change in rolling motion is due to change in position of the COM due to translation – not rotation.

4: The rolling has both kinetic energies corresponding to translation and rotation. The distribution of kinetic energy between two motions depend on the distribution of moment of inertia as :

$$\frac{K_T}{K_T} = \frac{MR^2}{I}$$

5: Conservation of mechanical energy is the characterizing feature of pure rolling. This is significant as mechanical energy is conserved even when friction is present.

Work and energy in rolling (check your understnading)

The questions have been selected to enhance understanding of the topics covered in the module titled " Work and energy in rolling ". All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

Understanding level (Work and energy in rolling)

Exercise:

Problem:

A spherical ball rolls without sliding. Then, the fraction of its total mechanical energy associated with translation is :

$$\text{(a) } \frac{1}{2} \text{ (b) } \frac{2}{3} \text{ (c) } \frac{3}{4} \text{ (d) } \frac{5}{7}$$

Solution:

The kinetic energy for translation is distributed between translation and rotation. The ratio of two kinetic energy is given by :

$$\frac{K_R}{K_T} = \frac{I}{MR^2}$$

Adding “1” to either side of the equation and solving, we have :

$$\frac{K}{K_T} = \frac{I+MR^2}{MR^2}$$

Inverting the ratio,

$$\frac{K_T}{K} = \frac{MR^2}{I+MR^2}$$

For solid sphere,

$$I = \frac{2MR^2}{5}$$

Putting MI’s expression in the ratio,

$$\frac{K_T}{K} = \frac{MR^2}{\frac{2MR^2}{5} + MR^2} = \frac{5}{7}$$

Hence, option (d) is correct.

Note: Important to note here is that this distribution of kinetic energy between translation and rotation is independent of the path of rolling. It only depends on the MI of rolling body i.e. on mass and its distribution about the axis of rotation. As such, the distribution of kinetic energy between translation and rotation are different for different bodies. In the case of ring and hollow cylinder, the distribution is 50% - 50%.

Exercise:

Problem:

Two identical spheres start from top of two incline planes of same geometry. One slides without rolling and other rolls without sliding. If loss of energy in two cases are negligible, then which of the two spheres reachs the bottom first ?

(a) sphere sliding without rolling reaches the bottom first
(b) sphere rolling without sliding reaches the bottom first
(c) both spheres reaches the bottom at the same time
(d) sphere sliding without rolling stops in the middle

Solution:

Here, no loss of energy is involved, so we can employ law of conservation of energy for the two cases. B

Since both spheres move same vertical displacement, they have same gravitational potential energy. Further, there is no loss of energy. As such, spheres have same kinetic energy at the bottom.

In the case of sliding without rolling, the total kinetic energy is translational kinetic energy. On the other hand, kinetic energy is distributed between translation and rotation in the case of rolling without sliding. The sphere in rolling, therefore, has lesser translational kinetic energy.

Therefore, velocity of the sphere sliding without rolling reaches the ground with greater speed. Again, as geometry of the inclines are same, the sphere sliding without

rolling reaches the bottom first. With respect to option (d), we note that once sliding starts, the sphere does not stop on an incline.

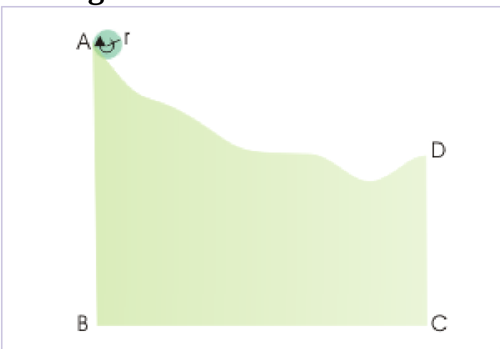
Hence, option (a) is correct.

Exercise:

Problem:

A hollow spherical ball of mass “m” and radius “r” at rest, rolls down the zig-zag track as shown in the figure. If AB = 10 m and CD = 4 m, then speed (m/s) of the sphere at the far right end of the path is (consider g = 10 m/s^2) :

Rolling motion



A hollow spherical ball from rest, rolls down the zig-zag track.

$$\text{(a)}\ 10\ \text{(b)}\ 5\sqrt{2}\ \text{(c)}\ 6\sqrt{2}\ \text{(d)}\ 3$$

Solution:

The hollow spherical ball is at rest in the beginning. Thus, hollow spherical ball has no kinetic energy, but only gravitational potential energy due to its place at a height from the ground.

There is no loss of energy due to friction and hence, mechanical energy of the ball is conserved. The hollow spherical ball acquires the kinetic energy, which is equal to the change in gravitational potential energy.

$$K = mg(\text{AB} - \text{CD}) = mg(10 - 4) = 6mg$$

In order to know the speed at the end of the track, we need to know the kinetic energy due to rolling. The kinetic energy of rolling, in terms, of linear velocity is given as :

$$K = \left(I + mr^2\right) \text{ x } \frac{v_C{}^2}{2r^2}$$

Solving for $v_C{}^2$ and putting value of kinetic energy as derived earlier, we have :

$$\Rightarrow v_C{}^2 = \frac{2Kr^2}{(I+mr^2)}$$

$$\Rightarrow v_C{}^2 = \frac{2 \text{ x } 6mgr^2}{\left(\frac{2mr^2}{3}+mr^2\right)}$$

$$\Rightarrow v_C = 6\sqrt{2}\, m/s$$

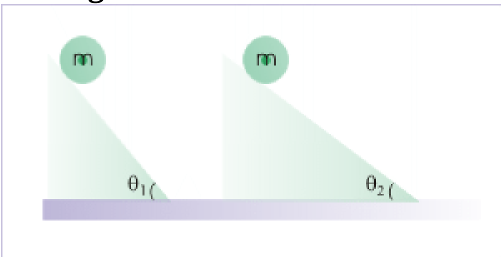
Hence, option (c) is correct.

Exercise:

Problem:

Two identical solid spheres roll down through the same height along two inclines of different angles. Then

Rolling motion



Two identical solid sphere rolls down through the same height along two inclines of different angles.

(a) the speed and time of descent are same in two cases

(b) for steeper incline, the speed is greater and time of descent is lesser

(c) the speed is same and time of descent for steeper incline is lesser

(d) the speed for steeper incline is greater and time of descent is same

---

Solution:

The mechanical energy of the rolling body is conserved in both cases. As such, the rolling of spheres along two inclines through same vertical displacement means that final kinetic energies of the solid spheres are same.

$$K = Mgh$$

We also know that distribution of kinetic energy between translation and rotation is fixed for a given geometry and mass of the body.

$$\frac{K_T}{K_R} = \frac{MR^2}{I}$$

This means that translational kinetic energy of the solid sphere will be same. In turn, it means that final speeds of the solid sphere in two cases will be same.

However, as the linear distance is less in the case of steeper incline, the sphere will take lesser time.

Hence, option (c) is correct.

Exercise:

Problem:

A ring of mass 0.3 kg and a disk of mass 0.4 have equal radii. They are given equal kinetic energy and released on a horizontal surface in such a manner that each of them starts rolling immediately. Then,

$$\begin{array}{l}\text{(a) ring has greater linear velocity (b) ring has lesser linear velocity} \\ \text{(c) ring has greater angular velocity (d) ring and disk have equal linear velocity}\end{array}$$

Solution:

The kinetic energy of a rolling body, in terms of linear velocity of center of mass, is given as :

$$K = \left(I + MR^2\right) \text{ x } \frac{{V_C}^2}{2R^2}$$

$$\Rightarrow {V_C}^2 = \frac{KR^2}{I+MR^2}$$

For ring, m = 0.3 kg and $I = MR^2$

$$\Rightarrow {V_C}^2 = \frac{KR^2}{MR^2+MR^2} = \frac{K}{M} = \frac{K}{0.3} = \frac{10K}{3}$$

For disk, m = 0.4 kg and $I = \frac{MR^2}{2}$.

$$\Rightarrow {V_C}^2 = \frac{KR^2}{\frac{MR^2}{2}+MR^2} = \frac{4K}{3m} = \frac{4K}{3\text{x}0.4} = \frac{10K}{3}$$

Thus, velocities of centers of mass for both ring and disk are same.

Hence, option (d) is correct

Exercise:

Problem:

A circular body of mass “M” and radius “R” rolls inside a hemispherical shell of radius “R”. It is released from the top edge of the hemisphere. Then, the angular kinetic energy at the bottom of the shell is maximum, if the circular body is :

$$\text{(a) a ring (b) a disk (c) a hollow sphere (d) a solid sphere}$$

Solution:

Since mechanical energy is conserved in rolling, the total kinetic energy for a circular body is a constant. Now, ratio of translational and rotational kinetic energy is :

$$\frac{K_T}{K_R} = \frac{MR^2}{I}$$

This ratio is independent of the position of the rolling body. Thus, rotational kinetic energy (and hence angular velocity) is maximum for the circular body having maximum moment of inertia. Here, ring for same mass and radius has the maximum moment of inertia. As such, angular velocity is maximum for the ring.

Hence, option (a) is correct.

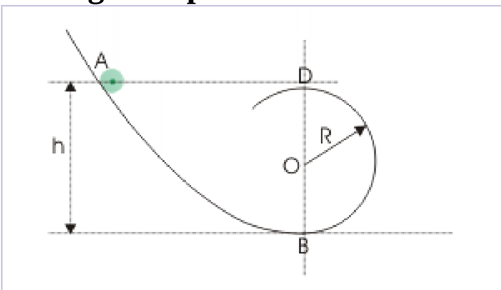
Application level (Work and energy in rolling)

Exercise:

Problem:

A small solid sphere of mass “m” and radius “r”, rolls down the incline and then move up the loop of radius “R” as shown in the figure. From what minimum height from the ground, the ball is released so that it does not leave the track at the highest point of the loop?

Rolling of a sphere



Sphere starts rolling from a height along the incline.

$$(a)\ \frac{2R}{3}\ \ (b)\ 3R\ \ (c)\ \frac{3R}{10}\ \ (d)\ \frac{7R}{10}$$

Solution:

We know that the minimum speed of the ball at the top (D) of loop, so that it does not leave the highest point, is :

$$v_D = \sqrt{}(gR)$$

Corresponding to this requirement, the velocity required at the bottom of the loop is :

$$v_B = \sqrt{}(5gR)$$

Now, the kinetic energy at the bottom of the loop, in terms of linear velocity, is :

$$K = \left(I + MR^2\right)\ \text{x}\ \frac{{V_C}^2}{2r^2}$$

$$\Rightarrow K = \frac{I+mr^2\ \ \text{x}\ 5gR}{2r^2}$$

Let “h” be the height, then (assuming r << R) :

$$\Rightarrow \frac{I+mr^2\ \ \text{x}\ 5gR}{2r^2} = mgh$$

$$\Rightarrow h = \frac{I+mr^2\ \ \text{x}\ 5gR}{2r^2mg}$$

For solid sphere,

$$\Rightarrow h = \frac{\left(\frac{2mr^2}{5}+mr^2\right)\text{x}5gR}{2r^2mg}$$

$$\Rightarrow h = \frac{\frac{7mr^2}{5}\text{x}5gR}{2r^2mg}$$

$$\Rightarrow h = \frac{7R}{10}$$

Hence, option (c) is correct.

Angular momentum
The angular momentum in rotation is a subset of angular momentum about a point in general motion.

Like linear momentum, angular momentum is the measure of the "quantity of motion". From Newton's second law, we know that first time derivative of linear momentum gives net external force on a particle. By analogy, we expect that this quantity (angular momentum) should have an expression such that its first time derivative yields torque on the particle.

Angular momentum about a point

Angular momentum is associated with a particle in motion. The motion need not be rotational motion, but any motion. Importantly, it is measured with respect to a fixed point.

Angular momentum of a particle about a point is defined as a vector, denoted as "ℓ".
Equation:

$$\boldsymbol{\ell} = \mathbf{r} \ \mathbf{x} \ \mathbf{p}$$

where "**r**" is the linear vector connecting the position of the particle with the "point" about which angular momentum is measured and "**p**" is the linear momentum vector. In case, the point coincides with the origin of coordinate system, the vector "**r**" becomes the position vector.

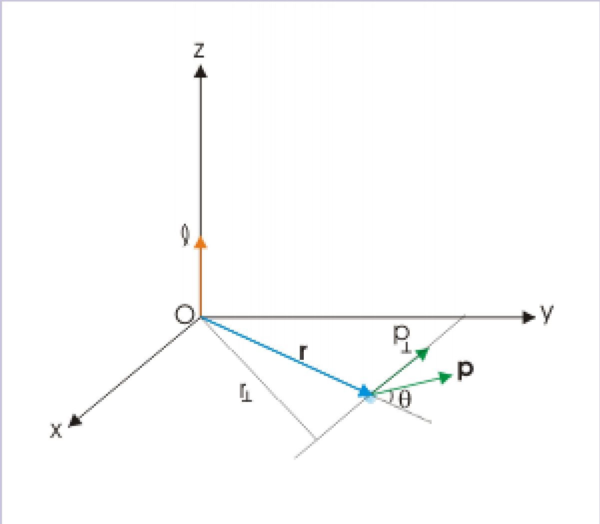
We should note here that small letter "ℓ" is used to denote angular momentum of a particle. The corresponding capital letter "L" is reserved for angular momentum of a system of particle or rigid body. This convention helps to distinguish the context and may be adhered to.

The SI unit of angular momentum is $\frac{\text{kg}-m^2}{s}$, which is equivalent to J-s.

Magnitude of angular momentum

Like in the case of torque, the magnitude of angular momentum can be obtained using any of the following relations :

Angular momentum of a particle



Angular momentum in terms of enclosed angle.

1: Angular momentum in terms of angle enclosed

Equation:

$$\ell = rp\sin\theta$$

2: Angular momentum in terms of force perpendicular to position vector

Equation:

$$\ell = rp_{\perp}$$

3: Angular momentum in terms of moment arm

Equation:

$$\ell = r_{\perp}p$$

If the particle is moving with a velocity "**v**", then the expression of angular momentum becomes :

Equation:

$$\boldsymbol{\ell} = \mathbf{r\ x\ p} = m(\mathbf{r\ x\ v})$$

Again, we can interpret this vector product as in the case of torque. Its magnitude can be obtained using any of the following relations :

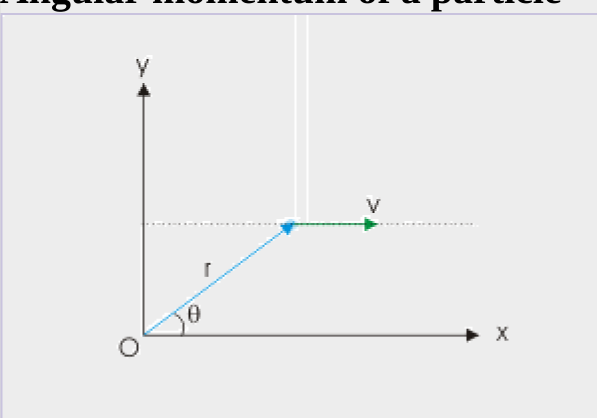
Equation:

$$\ell = mrv\sin\theta$$
$$\ell = mrv_{\perp}$$
$$\ell = mr_{\perp}v$$

Example:

Problem : A particle of mass, "m", moves with a constant velocity "v" along a straight line parallel to x-axis as shown in the figure. Find the angular momentum of the particle about the origin of the coordinate system. Also discuss the nature of angular momentum in this case.

Angular momentum of a particle



The particle is moving with a constant velocity.

Solution : The magnitude of the angular momentum is given by :

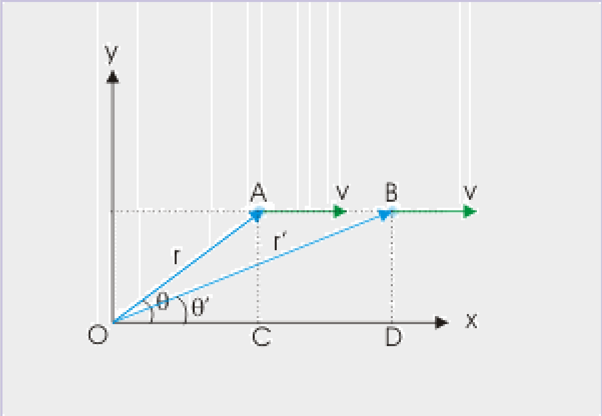
$$\ell = mrv \sin\theta$$

This expression can be rearranged as :

$$\ell = mv(r \sin\theta)$$

From the ΔOAC, it is clear that :

Angular momentum of a particle



The particle is moving with a constant velocity.

$$r \sin\theta = \mathrm{AC}$$

At another instant, we have :

$$r' \sin\theta' = \mathrm{BD}$$

But the perpendicular distance between two parallel lines are same (AC = BD). Thus,

$$\Rightarrow r \sin\theta = \text{a constant}$$

Also, the quantities "m" and "v" are constants. Therefore, angular momentum of the moving particle about origin "O" is a constant.

$$\ell = mv(r\sin\theta) = \text{a constant}$$

Since angular momentum is constant, its rate of change with time is zero. But, time rate of change of angular momentum is equal to torque (we shall develop this relation in next module). It means that torque on the particle is zero as time derivate of a constant is zero. Indeed it should be so as the particle is not accelerated. This result underlines the fact that the concept of angular momentum is consistent even for the description of linear motion as set out in the beginning of this module.

Direction of angular momentum

Angular momentum is perpendicular to the plane formed by the pair of position and linear momentum vectors or by the pair of position and velocity vector, depending upon the formula used. Besides, it is also perpendicular to each of operand vectors. However, the vector relation by itself does not tell which side of the plane formed by operands is the direction of torque.

In order to decide the orientation of the angular momentum, we employ right hand vector product rule. The procedure involved is same as that in the case of torque. See the module titled Torque about a point.

Angular momentum in component form

Angular momentum, being a vector, can be evaluated in component form with the help of unit vectors along the coordinate axes. The various expressions involved in the vector algebraic analysis are as given here :

1: In terms of position and linear momentum vectors

$$\boldsymbol{\ell} = \mathbf{r} \text{ x } \mathbf{p}$$

`ℓ = | i j k | | x y z | | px py pz|`

Equation:

$$\boldsymbol{\ell} = (yp_z - zp_y)\mathbf{i} + (zp_x - xp_z)\mathbf{j} + (xp_y - yp_x)\mathbf{k}$$

2: In terms of position and velocity vectors

$$\boldsymbol{\ell} = m(\mathbf{r}\ \mathbf{x}\ \mathbf{v})$$

`ℓ = m | i j k | | x y z | | vx vy vz|`

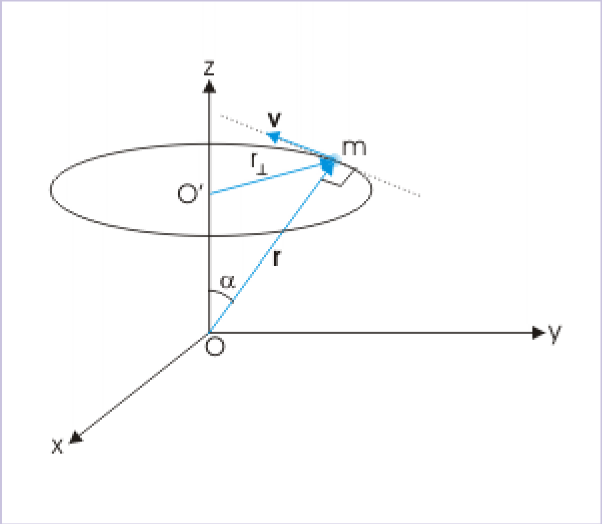
Equation:

$$\boldsymbol{\ell} = m(yv_z - zv_y)\mathbf{i} + (zv_x - xv_z)\mathbf{j} + (xv_y - yv_x)\mathbf{k}$$

Angular momentum for a particle in rotation

In rotation, a particle rotates about a fixed axis as shown in the figure. We consider haere a particle, which rotates about z-axis along a circular path in a plane parallel to "xy" plane. By the nature of the rotational motion, linear velocity,"**v**", and hence linear momentum, "**p**" are tangential to circular path and are perpendicular to the position vector, "**r**".

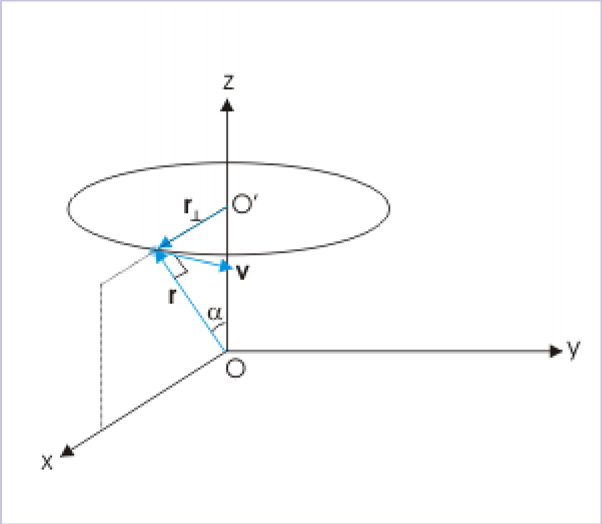
A particle rotating about an axis



The position vector is perpendicular to velocity vector.

The angle between position vector "**r**" and velocity vector "**v**" is always 90°. There may be some difficulty in visualizing the angle here. In order to visualize the same in a better perspective, we specifically consider a time instant when position vector and moment arm, $r_\perp$, are in "xz" plane. At this instant, the velocity vector, "**v**'", is tangential to the circle and is perpendicular to the "xz" plane. This figure clearly shows that position vector is indeed perpendicular to velocity vector.

Angular momentum



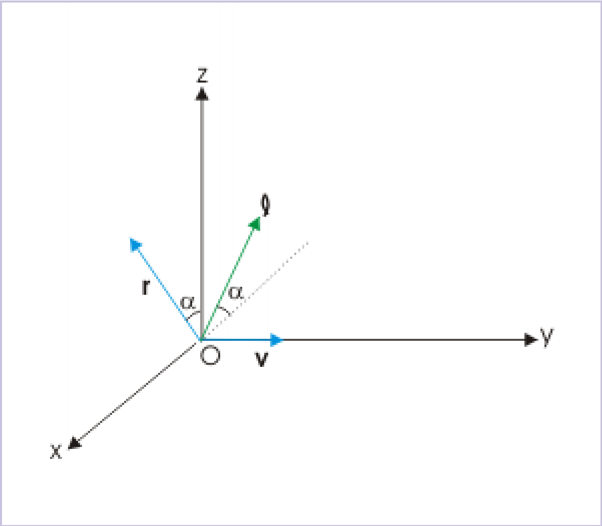
Angular momentum of a particle rotating about an axis.

The angular momentum of the particle, therefore, is :

$$\ell = mrv\sin\theta = mrv\sin 90^0 = mrv$$

The direction of angular momentum is perpendicular to the plane formed by position and velocity vectors. For the specific situation as shown in the figure above, the direction of angular momentum is obtained by first shifting the velocity vector to the origin and then applying right hand rule. Importantly, the angular momentum vector makes an angle with extended x-axis in opposite direction as shown here.

Angular momentum



Direction of angular momentum.

We observe that particle is restrained to move along a circular path perpendicular to axis of rotation i.e. z -axis. Thus, only the component of angular momentum in the direction of axis is relevant in the case of rotation. The component of angular momentum in z -direction is :

$$\ell_z = mrv \sin \alpha$$

From geometry, we see that :

$$r \sin \alpha = r_\perp$$

Hence,

$$\ell_z = mr_\perp v = mr_\perp \text{ x } \omega r_\perp = mr_\perp{}^2 \omega$$

But, we know that $mr_\perp{}^2 = I$. Hence,

$$\ell_z = I\omega$$

This has been the expected relation corresponding to p=mv for translational motion. The product of moment of inertia and angular velocity about a

common axis is equal to component of angular momentum about the rotation axis.

If we define angular momentum of the particle for rotation as the product of linear momentum and moment arm about the axis of rotation (not position vector with respect to point "O" as in the general case), then we can say that product of moment of inertia and angular velocity about a common axis is equal to the angular momentum about the rotation axis. Dropping the suffix referring the axis of rotation, we have :

Equation:

$$\ell = I\omega$$

We must ensure, however, that all quantities in the equation above refer to the same axis of rotation. Also, we should also keep in mind that the definition of angular momentum for rotation about an axis has been equal to the component of linear momentum about the axis of rotation and is different to the one about a point as in the general case. In the nutshell, we find that the angular momentum in rotation is a subset of angular momentum about a point in general motion.

It should be amply clear that the expression of angular momentum in terms of moment of inertia and angular velocity is valid only for rotational motion.

Summary

1: Angular momentum of a particle in general motion is given as :

$$\boldsymbol{\ell} = \mathbf{r} \textbf{ x } \mathbf{p} = m(\mathbf{r} \textbf{ x } \mathbf{v})$$

The interpretation of above relation differs for the reference with respect to which linear distance is measured. In the case of point reference, the vector “r” denotes position vector from the point, whereas it denotes radius vector from the center of circle in rotation.

2: The magnitude of angular momentum is evaluated, using any of the following six relations :

$$\ell = rp \sin\theta$$
$$\ell = rp_{\perp}$$
$$\ell = r_{\perp}p$$
$$\ell = mrv \sin\theta$$
$$\ell = mrv_{\perp}$$
$$\ell = mr_{\perp}v$$

3: The direction of the angular momentum, being a vector product, is evaluated in the same manner as that in the case of torque.

3: In the component form, the angular momentum is expressed as :

`ℓ = | i j k | | x y z | | px py pz|`

and

`ℓ = m | i j k | | x y z | | vx vy vz|`

4: For rotation of a particle, angular momentum has additional expression in terms of moment of inertia and angular velocity as :

$$\ell = I\omega = mr^2\omega$$

where “r “ is the radius of the circle from the center lying on the axis of rotation.

Angular mometum (check your understanding)
Objective questions, contained in this module with hidden solutions, help improve understanding of the topics covered under the module "Angular momentum".

The questions have been selected to enhance understanding of the topics covered in the module titled " Angular momentum ". All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

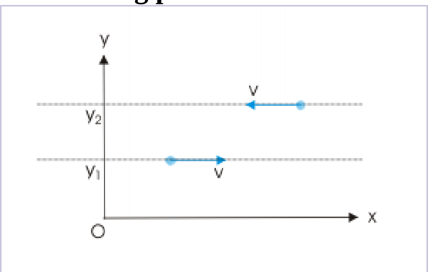
Understanding level (Angular momentum)

Exercise:

Problem:

Two particles of same mass, move with same speed along two parallel lines in opposite directions as shown in the figure. Then, magnitude of net angular moment of two particles about the origin of coordinate system :

Two moving particles



Two particles of same mass, move with same speed

(a) is directly proportional to the perpendicular distance between the parallel lines
(b) is inversely proportional to the perpendicular distance between the parallel lines
(c) is constant for a given pair of parallel paths
(d) is zero

Solution:

Here, moment arms of the particles are y_1 and y_1. Let the mass of each particle be “m” and speed of each particle is “v”. Then, the angular momentum of the individual particles about the origin of coordinate system are :

$$\ell_1 = m y_1 \mathbf{j} \ \mathbf{x} \ v_1 \mathbf{i} = -m y_1 v \mathbf{k}$$
$$\ell_2 = m y_2 \mathbf{j} \ \mathbf{x} \ (-v_2) \mathbf{i} = m y_2 v \mathbf{k}$$

Since angular momentum are along the same direction, we can find the net angular momentum by arithmetic sum as :

$$\ell = \ell_1 + \ell_2 = mv(y_2 - y_1)\mathbf{k}$$

We see that the magnitude of net angular momentum is directly proportional to the perpendicular distance between the parallel lines. Further, the perpendicular distance is fixed for a given pair of parallel paths. Hence, net angular momentum is constant for a given pair.

Hence, options (a) and (c) are correct.

> **Note:** Since we can always orient coordinates similar to the situation here, we can conclude that angular momentum about any point is constant for a given pair of parallel paths under similar conditions of mass and speed. Will the angular momentum be constant at a point between parallel paths? The answer is yes.

Exercise:

Problem:

A particle of mass "m" is moving with linear velocity **v** = (2**i** – **j** + **k**) m/s. At an instant, the particle is situated at a position (in meters) given by the coordinates 0,1,1. What is the angular momentum of the particle about the origin of coordinate system at that instant?

$$\text{(a)}\ \mathbf{i} + \mathbf{j} - 2\mathbf{k}\ \ \text{(b)}\ 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}\ \ \text{(c)}\ \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}\ \ \text{(d)}\ 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

Solution:

This question can be evaluated using cross product in component form as :

`ℓ = | i j k | | 0 1 1 | | 2 -1 1 |`

$$\Rightarrow \boldsymbol{\ell} = (1 \text{ x } 1 - 1 \text{ x } -1)\mathbf{i} + (1 \text{ x } 2 - 0)\mathbf{j} + (0 - 2 \text{ x } 1)\mathbf{k}$$
$$\Rightarrow \boldsymbol{\ell} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

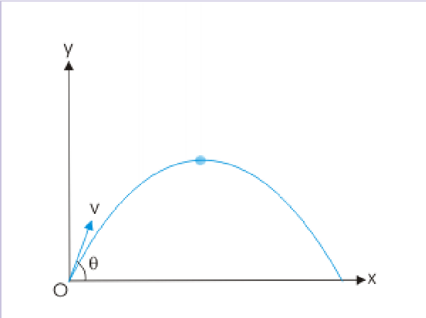
Hence, option (b) is correct.

Exercise:

Problem:

A projectile of mass "m" is projected with a velocity "v", making and angle "θ" with the horizontal. The angular momentum of the projectile about the point of projection, when it reaches the highest point is :

Projectile



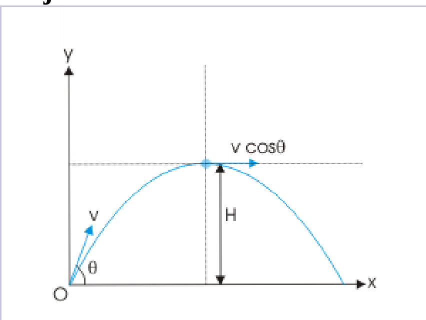
A projectile of mass “m” is projected with a velocity “v”, making and angle “θ” with the horizontal.

$$(a)\ \frac{mv^3 \sin^2\theta \cos\theta}{2g}\ (b)\ \frac{mv^3 \sin^2\theta}{2g}\ (c)\ \frac{mv^2 \sin^2\theta}{2g}\ (d)\ \frac{mv^3 \sin^2\theta \cos\theta}{g}$$

Solution:

The velocity of the projectile at the highest point is equal to the horizontal component of velocity as vertical component of velocity is zero. Further, there is no acceleration in the horizontal direction. Therefore, horizontal component of velocity is same as the horizontal component of the velocity at projection. It means that velocity of projection at the highest point is :

Projectile motion



The projectile is at maximum vertical displacement.

$$v_x = v\cos\theta$$

Here, the moment arm i.e. the perpendicular drawn from the point of projection on the extended line of velocity is equal to the height attained by the projectile. Thus,

$$r_\perp = H = \frac{v^2 \sin^2\theta}{2g}$$

The angular momentum as required is :

$$\Rightarrow \ell = mr_{\perp}v = \frac{mv^2 \sin^2\theta}{2g} \text{ x } v\cos\theta$$
$$\Rightarrow \ell = \frac{mv^3 \sin^2\theta \cos\theta}{g}$$

Hence, option (a) is correct.

Exercise:

Problem:

If “m”,”p” and “ℓ” denote the mass, linear momentum and angular momentum of the particle executing uniform circular motion along a circle of radius “r”, then kinetic energy of the particle is :

$$\text{(a) } \frac{p^2}{2m} \text{ (b) } \frac{p^2}{2mr} \text{ (c) } \frac{\ell^2}{2mr^2} \text{ (d) } \frac{\ell^2}{2mr}$$

Solution:

From the alternatives given, it is evident that we need to find kinetic energy in terms of linear and angular momentum. Now, the kinetic energy of the particle is expressed as :

$$K = \frac{1}{2} \text{ x } mv^2$$

The magnitude of linear momentum is given by :

$$p = mv$$

Squaring both sides, we have :

$$\Rightarrow p^2 = m^2v^2$$

Rearranging,

$$\Rightarrow mv^2 = \frac{p^2}{m}$$

Combining equations,

$$K = \frac{p^2}{2m}$$

Now, the magnitude of angular momentum is given by :

$$\ell = mvr$$

Squaring both sides, we have :

$$\Rightarrow \ell^2 = m^2v^2r^2$$

$$\Rightarrow mv^2 = \frac{\ell^2}{mr^2}$$

Combining equations,

$$K = \frac{\ell^2}{2mr^2}$$

Hence, options (a) and (c) are correct.

Exercise:

Problem: Dimensional formula of angular momentum is :

$$\text{(a) } ML^3T^{-1} \text{ (b) } ML^2T^{-2} \text{ (c) } ML^2T^{-1} \text{ (d) } MLT^{-1}$$

Solution:

In order to find dimensional formula, we use the expression of angular momentum :

$$\ell = mvr\sin\theta$$

The dimension of angular momentum is :

$$[K] = [m][v][r][\sin\theta] = [M][LT^{-1}][L] = [ML^2T^{-1}]$$

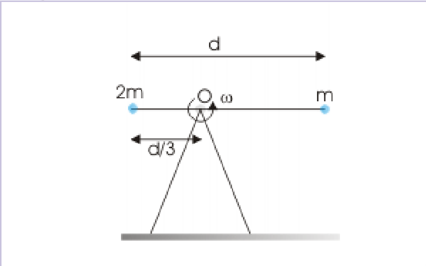
Hence, option (c) is correct.

Exercise:

Problem:

Two point masses “m” and “2m” are attached to a mass-less rod of length “d”, pivoted at point “O” as shown in the figure. The rod is free to move in vertical direction. If “ω” be the angular velocity, then the angular momentum of two particles about the axis of rotation is :

Angular momentum of particle



The rod is free to move in vertical plane.

$$\text{(a) } \frac{3m\omega d^2}{4} \text{ (b) } \frac{2m\omega d^2}{3} \text{ (c) } \frac{2m\omega d^2}{5} \text{ (d) } \frac{5m\omega d^2}{9}$$

Solution:

This is a case of rotation. We can solve this problem using either the general expression or specific expression involving moment of inertia. When we use general expression, we should measure moment arm from the axis of rotation.

Since rod is rotating in anticlockwise direction, angular momentum of each particle is positive.

The angular momentum of first particle of mass “m” is :

$$\ell_1 = mvr = m\omega \text{ x } \frac{2d}{3} \text{ x } \frac{2d}{3} = \frac{4m\omega d^2}{9}$$

The angular momentum of second particle of mass “2m” is :

$$\ell_2 = mvr = 2m\omega \text{ x } \frac{d}{3} \text{ x } \frac{d}{3} = \frac{2m\omega d^2}{9}$$

$$\ell = \ell_1 + \ell_2 = \frac{4m\omega d^2}{9} + \frac{2m\omega d^2}{9} = \frac{2m\omega d^2}{3}$$

Hence, option (b) is correct.

Application level (Angular momentum)

Exercise:

Problem:

A particle of mass 100 gram is moving with a constant speed at √2 m/s along a straight line y = x + 2 in "xy" – plane. Then, the magnitude of the angular momentum (in $\frac{\text{kg}-m^2}{s}$) about the origin of the coordinate system, if “x” and “y” are in meters, is :

$$\text{(a) } 0.1 \text{ (b) } 0.2 \text{ (c) } 0.3 \text{ (d) } 0.4$$

Solution:

The angular momentum of a particle about a point is defined as :

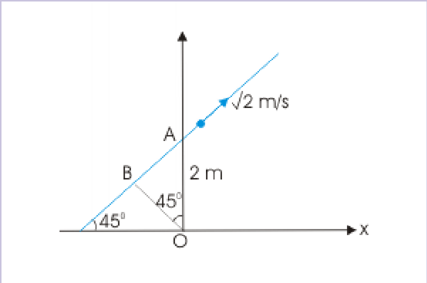
$$\boldsymbol{\ell} = m(\mathbf{r} \text{ x } \mathbf{v})$$

The magnitude of angular momentum can be evaluated using any of the three methods discussed in the module on the subject. In this case, evaluation of angular momentum involving moment arm is most appropriate. Thus,

$$\ell = m r_\perp v$$

We, now, analyze the situation by drawing the figure. The path of the motion is shown. The moment arm i.e. the perpendicular distance of velocity vector from the origin is :

Angular momentum of a particle



The path of the moving particle.

$$r_\perp = \text{OB} = 2 \sin 45^\circ = \sqrt{2}$$

Hence,

$$\Rightarrow \ell = 0.1 \text{ x } \sqrt{2} \text{ x } \sqrt{2} = 0.2 \, \frac{\text{kg}-m^2}{s}$$

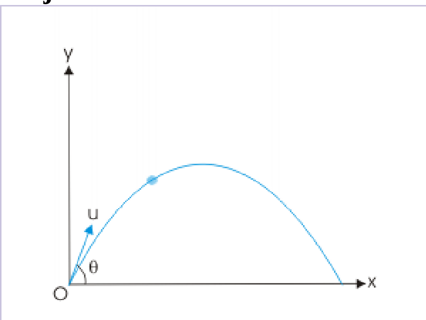
Hence, option (b) is correct.

Exercise:

Problem:

A projectile of mass “m” is projected with a velocity “u”, making and angle “θ” with the horizontal. The angular momentum of the projectile about the point of projection at a given instant “t” during its flight is :

Projectile motion



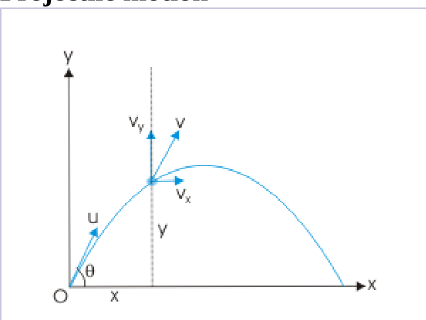
Angular momentum of a projectile at a given time.

$$(\text{a}) \;\; -mugt^2 \cos\theta\mathbf{i} \;\; (\text{b}) \;\; -2mugt^2 \cos\theta\mathbf{j} \;\; (\text{c}) \;\; -mugt^2 \cos\theta\mathbf{k} \;\; (\text{d}) \;\; -\frac{1}{2} \text{ x } mugt^2 \cos\theta\mathbf{k}$$

Solution:

We need to find a general equation for angular momentum for a given time “t”. We observe here that projectile motion is a two dimensional motion for which we can have information in component form. It is, therefore, required to obtain component information for position and the velocity.

Projectile motion



Angular momentum of a projectile at a given time.

$$x = u\cos\theta \text{ x } t$$
$$y = u\sin\theta \text{ x } t - \tfrac{1}{2}\, gt^2$$
$$v_x = u\cos\theta$$
$$v_y = u\sin\theta - gt$$

Now, the angular momentum is :

`ℓ = | i j k | | utcosθ utsinθ – 1/2g t*2 0 | | u cosθ usinθ – gt 0 |`

$$\Rightarrow \ell = m\left\{u\cos\theta \text{ x } t\left(u\sin\theta - gt\right) - u\cos\theta\left(u\sin\theta - \tfrac{1}{2} \text{ x } gt^2\right)\right\}\mathbf{k}$$

$$\Rightarrow \ell = m\left(u\cos\theta \text{ x } t \text{ x } u\sin\theta - u\cos\theta \text{ x } t \text{ x } gt - u\cos\theta \text{ x } u\sin\theta \text{ x } t + \tfrac{1}{2} \text{ x } gt^2 \text{ x } u\cos\theta\right)\mathbf{k}$$

$$\Rightarrow \ell = m\left(u^2t\cos\theta \text{ x } \sin\theta - ugt^2\cos\theta - u^2t\cos\theta \text{ x } \sin\theta + \tfrac{1}{2} \text{ x } ugt^2\cos\theta\right)\mathbf{k}$$

$$\Rightarrow \ell = -\tfrac{1}{2} \text{ x } mugt^2\cos\theta\mathbf{k}$$

Hence, option (d) is correct.

Answers

`1. (a) and (c) 2. (b) 3. (a) 4. (a) and (c) 5. (c) 6. (b) 7. (b) 8. (d)`

Second law of motion in angular form

The heading of this module (law of motion in angular form - not law of motion in rotational form) points to the subtle difference about angular quantities as applicable to general motion vis-a-vis rotation. The title also indicates that law of motion in angular form is just yet another form of Newton's law - not specific to any particular type of motion. What it means that we can use angular concepts and angular law (including the one being developed here) to even pure translational motion along a straight line.

In earlier module titled Rotation of rigid body , we wrote Newton's second law for rotation as :
Equation:

$$\tau = I\alpha$$

This relation is evidently valid for rotation, which involves circular motion of particles constituting the rigid body or that of a particle in a plane perpendicular to a fixed axis of rotation. This is so because, moment of inertia is defined with rotational motion only. It is evident that we can not use this form of Newton's second law for any other motion.

In this module, we shall develop the form of Newton's law, which is based on the concept of angular momentum of a particle about a fixed point in the coordinate reference. The law so derived, ofcourse, will then be shown to yield the version of Newton's second law for rotation, which is a special case.

Newton's second law for a particle in general motion

It was stated in the previous module that angular momentum is defined such that its first time derivative gives torque on the particle. This condition was specified keeping in mind about Newton's second law for translation. Following the logic, let us consider angular momentum of a moving particle moving with respect to a point and find whether first derivative actually yields torque as expected or not?

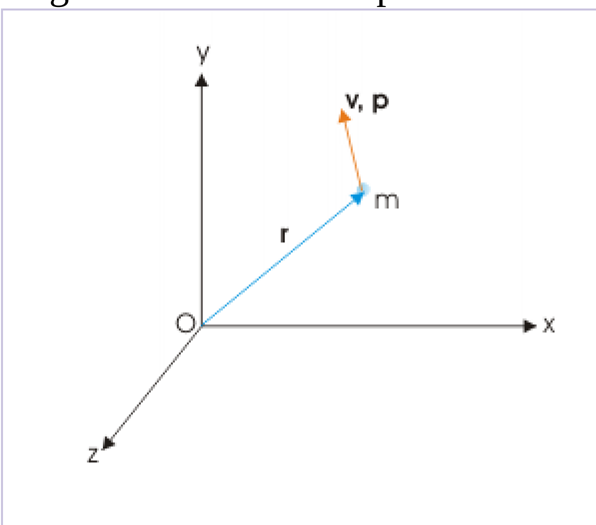
Angular momentum of the particle about the origin of the coordinate system is :

Equation:

$$\boldsymbol{\ell} = \mathbf{r} \text{ } \mathbf{x} \text{ } \mathbf{p} = m(\mathbf{r} \text{ } \mathbf{x} \text{ } \mathbf{v})$$

The "**r**" and "**v**" vectors represent position and velocity vectors respectively as shown in the figure. Taking differentiation of the terms with respect to time, we have :

Angular momentum of particle in motion



The particle is moving with a velocity in 3-D reference system.

$$\frac{d\boldsymbol{\ell}}{dt} = \frac{d}{dt}\left\{m\left(\mathbf{r} \text{ } \mathbf{x} \text{ } \mathbf{v}\right)\right\}$$

For constant mass,

$$\Rightarrow \frac{d\boldsymbol{\ell}}{dt} = m\frac{d}{dt}\left(\mathbf{r} \text{ } \mathbf{x} \text{ } \mathbf{v}\right)$$

$$\Rightarrow \frac{d\ell}{dt} = m\left(\mathbf{r}\ \mathbf{x}\ \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt}\ \mathbf{x}\ \mathbf{v}\right)$$

By definition, the first time derivative of velocity is the acceleration and first time derivative of position vector is the velocity of the particle. Putting the appropriate terms for these quantities,

$$\Rightarrow \frac{d\ell}{dt} = m\left(\mathbf{r}\ \mathbf{xa} + \mathbf{v}\ \mathbf{xv}\right)$$

But, the vector product of a vector with itself is equal to zero as sinθ = sin0° = 0. Hence,

$$\Rightarrow \frac{d\ell}{dt} = m\left(\mathbf{r}\ \mathbf{xa}\right)$$

Rearranging, we have :

$$\Rightarrow \frac{d\ell}{dt} = \mathbf{r}\ \mathbf{x}\ m\ \mathbf{a} = \mathbf{r}\ \mathbf{xF}$$

Indeed the first derivative of angular momentum equals the toque on the particle as expected. Since force on the particle is external force, we can qualify the above relation that first time derivative of angular momentum equals external torque applied on the particle. Next, we should think about a situation when more than one force acts on the particle. According to Newton's second law in translation, we have :

$$\Rightarrow \mathbf{F}_{\text{net}} = m\ \mathbf{a}$$

Substituting the same in the equation above, we have :

$$\Rightarrow \frac{d\ell}{dt} = \mathbf{r}\ \mathbf{x}\ m\ \mathbf{a} = \mathbf{r}\ \mathbf{xF}_{\text{net}} = \mathbf{r}\ \mathbf{x} \sum \mathbf{F}_i$$

We note here that all forces are acting at the same point (occupied by the particle). As such, position vectors for all forces are same. Thus, we can enclose the vector product inside the summation sign,

$$\Rightarrow \frac{d\ell}{dt} = \sum \mathbf{r}\ \mathbf{xF}_i = \sum \mathbf{t}_i = \mathbf{t}_{\text{net}}$$

Equation:

$$\Rightarrow \mathbf{t}_{\text{net}} = \frac{d\boldsymbol{\ell}}{dt}$$

In words, Newton's second law of motion for a particle in angular form is read as :

Newton's second law of motion in angular form

> The net torque on a particle is equal to the time rate of change of its angular momentum.

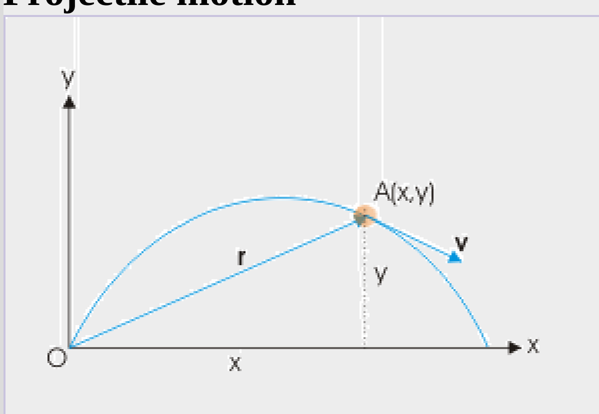
Example:

Problem : Consider projectile motion of a particle of mass "m" and show that time derivative of angular moment about point of projection is equal to torque on the particle about that point.

Solution : For this question, we express various vector quantities in terms of unit vectors, because two directions involved in projectile motion i.e. horizontal and vertical are mutually perpendicular. This makes it easy to represent vectors in the unit vector form.

For the problem in hand, we consider horizontal and vertical directions as "x" and "y" reference directions respectively. Now, the torque, τ, on the particle about the point of projection "O" is :

Projectile motion



Angular moment about point of

projection.

$$\boldsymbol{\tau} = \mathbf{r} \text{ } \mathbf{x} \text{ } \mathbf{F}$$

The position vector of the position of the projectile at a given time "t" is :

$$\mathbf{r} = x \text{ } \mathbf{i} + y\mathbf{j}$$

We note here that the external force on the projectile is force due to gravity. Hence,

$$\mathbf{F} = -\text{mg } \mathbf{j}$$

Putting in the equation of the torque, we have :

$$\mathbf{t} = (x \text{ } \mathbf{i} + y\mathbf{j}) \text{ } \mathbf{x} \text{ } -\text{mg } \mathbf{j}$$

$$\mathbf{t} = -x\text{mg } \mathbf{k}$$

Negative sign shows that torque is clockwise i.e. into the plane of the projection "xy" plane. Next, we evaluate torque about "O" in terms of angular momentum. Here, angular momentum is given as :

$$\boldsymbol{\ell} = m(\mathbf{r} \text{ } \mathbf{x} \text{ } \mathbf{v}) = m\{(x \text{ } \mathbf{i} + y\mathbf{j})) \text{ } \mathbf{x} \text{ } \mathbf{v}$$

Differentiating with respect to time "t", we have :

$$\mathbf{t} = \frac{d\boldsymbol{\ell}}{dt} = m\left\{\left(\frac{dx}{dt} \text{ } \mathbf{i} + \frac{dx}{dt}\mathbf{j}\right)\right) \text{ } \mathbf{x} \text{ } \frac{d\mathbf{v}}{dt}$$

But $\frac{d\mathbf{v}}{dt}$ is equal to acceleration. In the case of projectile, acceleration due to gravity is the acceleration of the projectile,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -g \text{ } \mathbf{j}$$

Putting this value in the equation,

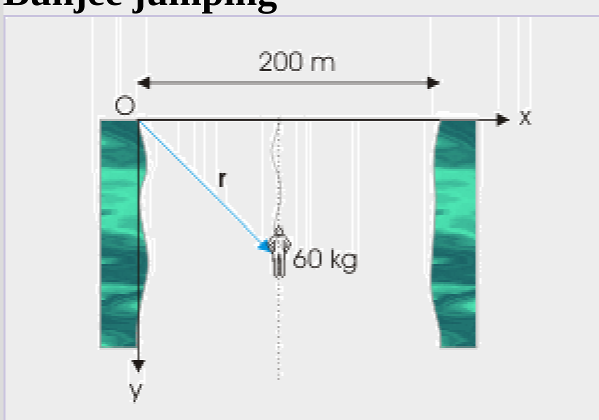
$$\mathbf{t} = m\left\{\left(\frac{dx}{dt}\ \mathbf{i} + \frac{dx}{dt}\mathbf{j}\right)\right)\ \mathbf{x} - g\ \mathbf{j}$$

$$\mathbf{t} = -x\text{mg}\ \mathbf{k}$$

The quantities torque and angular momentum are both measured or calculated about a common point "O". If it is not so, then the relationship is not valid. This aspect should always be kept in mind while applying the law.

We have emphasized that the angular form of Newton's second law is defined about a point - not an axis and as such not limited to rotational motion. Hence, we can employ this law even in situation where motion is strictly along a straight line. We shall, as a matter of fact, work out an example to illustrate this.

Example:
Problem : A bunjee jumper of mass 60 kg jumps from the middle of a bridge 200 m in length. Find the torque as time derivative of angular momentum about a point "O" as shown in the figure, before the rope attached to the jumper becomes taught.
Bunjee jumping



The bubjee jumper jumps from

the middle of bridge.

Solution : Initially, the jumper is at a horizontal distance of 100 m from the end of the bridge. We must observe here that the jumper is not tied to the bridge. The rope is loose during the fall. He or she falls through a vertical straight path. It is a linear motion for which we are required to calculate torque. It is also evident that we can treat the jumper as point mass as all body parts are moving with same motion.
Now, angular momentum of the jumper with respect to the point, O, is :

$$\ell = mrv\sin\theta$$

Here, r sinθ = L/2, where "L" is the length of the brifge.

$$\ell = mv\frac{L}{2}$$

However, we do not know the speed of the jumper. We note here that the force on the jumper is force due to gravity. He/she falls with a constant acceleration. Thus, applying equation of motion, speed of the jumper after time "t" is :

$$v = u + at = 0 + gt = gt$$

Substituting in the equation above :

$$\Rightarrow \ell = mgt\frac{L}{2}$$

Differentiating with respect to time,

$$\Rightarrow \tau = \frac{d\ell}{dt} = mg\frac{L}{2} = 60 \text{ x } 10 \text{ x } \frac{200}{2} = 60000\,N - m$$

In a similar situation in the earlier module titled Angular momentum , we had found that a particle moving in a straight line parallel to one of the axes of coordinate system has constant angular momentum and zero torque. The difference has arisen as velocity was constant in that case, whereas velocity is not constant in this case. We need force (hence torque)

to cause acceleration or deceleration. The result, therefore, is consistent with the basic percept of Newton's law.
Further, we can also calculate torque using relation $\boldsymbol{\tau} = \mathbf{r} \ \mathbf{x} \ \mathbf{F}$, where F = mg. This relation should also give same result.

Newton's second law for a particle in rotation

Newton's law for a particle in rotation is established by taking first derivative of the angular momentum of a particle about an axis of rotation. As derived in previous module titled Angular momentum . it is given by :
Equation:

$$\ell = I\omega$$

Differentiating with respect to time,

$$t_{net} = \frac{d\ell}{dt} = \frac{d(I\omega)}{dt}$$

Since particle moves around a circular path of a constant radius, $I = mr_{\perp}{}^{2}$ is a constant and the same can be taken out of differentiation,
Equation:

$$t_{net} = \frac{d(I\omega)}{dt} = I\frac{d(I\omega)}{dt} = I\alpha$$

This derivation underlines that the above relation is specific to rotation about an axis - not for general motion. On the other hand, the definition of torque as the time rate of angular momentum about a point is a general form of Newton's second law.

Summary

Since discussion for angular momentum and corresponding Newton's second law are considered for different situations, it would be of great help

to summarize the main results and special points for each of the cases so far covered. For clarity, we summarize as follows :

General motion of a particle

1: The reference for measurement of various angular quantities is a fixed point in the coordinate system. All angular quantities are measured about the same point.

2: Angular momentum of a particle in motion is given as :

$$\boldsymbol{\ell} = \mathbf{r} \textbf{ x } \mathbf{p} = m(\mathbf{r} \textbf{ x } \mathbf{v})$$

3: Newton's second law of motion in angular form for a particle is expressed mathematically as :

$$\mathbf{t}_{\text{net}} = \frac{d\boldsymbol{\ell}}{dt}$$

In particular we must note that the form of Newton's second expressed in terms of moment of inertia and angular acceleration is not valid for general motion. It is so because, moment of inertia is defined about an axis - not about a point. For this reason, we shall not use the following form for general motion,

$$\text{t}_{\text{net}} = I\alpha$$

Rotational motion of a particle

1: For rotational motion, the reference for measurement of various angular quantities is a fixed axis of rotation. All angular quantities are measured about the same axis of rotation.

2: Angular momentum of a particle in motion is given as :

$$\ell = r_{\perp}p = mr_{\perp}v = mr_{\perp} \text{ x } \omega r_{\perp} = m{r_{\perp}}^2\omega = I\omega$$

where "I" and "ω" denote moment of inertia and angular velocity respectively. Here, " $r_{\perp}$ " is moment arm about the axis of rotation.

Since there are only two directions invoved (one in the direction of axis and other opposite to it), it is possible to represent the directional features of angular quantities in rotation with a preceding "plus" or "minus" sign for anti-clockwise and clockwise rotational motions respectively. This is also the reason why do we write equations of rotation in scalar form.

3: Newton's second law of motion is expressed mathematically as :

$$\mathrm{t}_{\mathrm{net}} = \frac{d\ell}{dt} = I\alpha$$

Law of motion in angular form for a system of particles

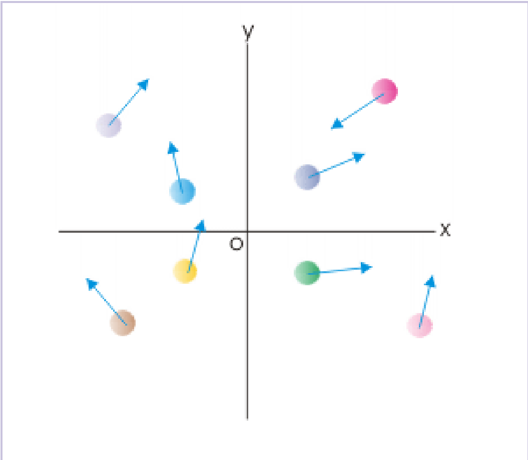
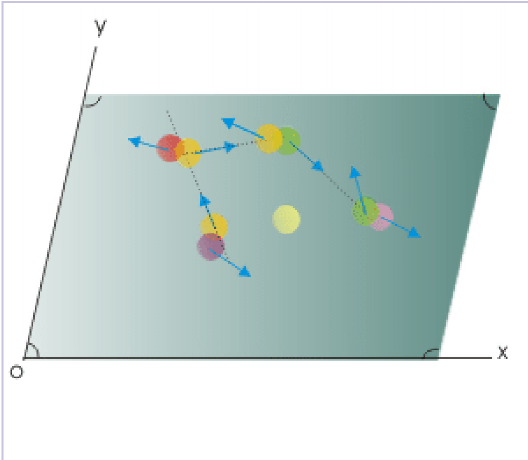
This module is an extension of previous module in the sense that we move from consideration of a "particle" to that of a "collection of particles". If we have understood the derivation of angular momentum and second law of motion for a particle, then it would be near replica of reasoning for a system of particles here in this module. However, there are certain characteristic features about system of particles as different from a single particle. These characteristic features will be pointed out (emphasized) in the appropriate context.

Newton's second law for a system of particles in general motion

The concept of angular momentum can easily be extended to include particles, constituting a system. Each of the particles can be associated with certain velocity. In angular parlance, we can assign angular momentum to each of them, provided they are moving. They may be subjected to internal as well as external forces. This is very important difference between a "particle" and a "system of particles". When we consider a single particle, the force can only be external. A particle occupying a point of zero dimension can not be associated with internal force.

For visualization, we can again consider the example of particles or billiard balls that we used in the context of linear momentum. The situation, here, is same with one exception that we shall refer to a point (origin "O" as shown) for calculating angular momentum as against calculation of linear momentum that does not require any such reference point.

Angualr momentum of asystem of particles

Particles moving in different directions	Billiard balls moving in different directions
	

The angular momentum of a particle in three dimensional space is defined by the vector relation :

$$\boldsymbol{\ell} = m(\mathbf{r} \ \mathbf{x} \ \mathbf{v})$$

where "**r**" and "**v**" denotes the position and velocity vectors respectively. We can combine angular momentum of one particle provided angular momentums are measured about a common point. Though, we can calculate angular momentum about different points, but then no physical meaning can be assigned to such calculation. This requirement is actually the reason that angular momentum, in general, is defined about a point - not about an axis. It would have not been possible to associate motion of particles with a common axis.

Since angular momentum is a vector quantity, the angular momentum of all the particles is equal to the vector sum of all the individual angular momentums. It is imperative that the summation would require application of vector addition rules to get the sum of the angular momentums.

Equation:

$$\mathbf{L} = \sum \boldsymbol{\ell}_i = \sum m_i(\mathbf{r}_i \ \mathbf{x} \ \mathbf{v}_i)$$

The angular momentum of the system of particles is denoted by capital "**L**". The particles may change their velocities subsequent to collisions among themselves (due to internal forces) or because of external forces. Consequently, angular moment of the system may change with time. The first time derivative of the angular momentum of a particle is equal to the torque on it :

$$\frac{d\mathbf{L}}{dt} = \sum \boldsymbol{\tau}_i$$

The most critical aspect of this equation is that the sum of torques involves both internal as well as external torques. When we talk of torques on the system of particles, the classification of internal and external depends on the boundary of closed system. This, in turn, depends which of the particles are included and which of the particles are excluded from the system. The forces (constituting torques) applied by particles included in the system constitute internal torques. The remaining ones are external torques. A particle can not have internal torque anyway. We must also understand that when we say a torque is applied on a particle (without any qualification), we implicitly mean that torque is external to the particle.

According to Newton's third law, the internal forces (torques) appear always in the pair of equal and opposite forces (torques). As such, they cancel each other. The expression of angular momentum, then, reduces to :

$$\frac{d\mathbf{L}}{dt} = \sum \boldsymbol{\tau}_i = \boldsymbol{\tau}_{\text{net}}$$

where " $\boldsymbol{\tau}_{\text{net}}$ " represents net external torue on the system of particles. Finally, the second law of motion in angular form for a system of particles is expressed as :
Equation:

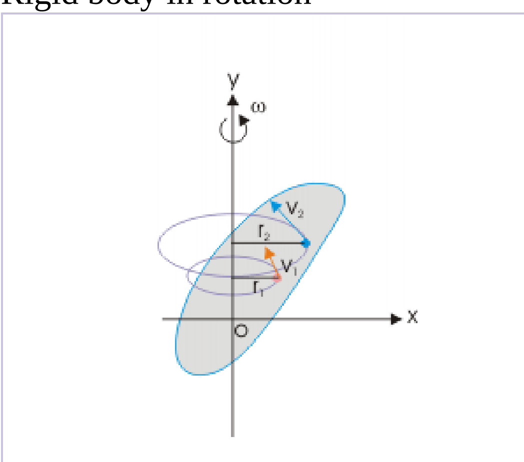
$$\boldsymbol{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt}$$

This relationship is same as that of a single particle. Only difference is that net external torque, now, is equal to the first time derivative of the vector sum of angular momentum (L) of the system of particles as against a single particle. Evidently, this relation holds for measurement of angular momentum and torque about the same reference point.

Newton's second law for a rigid body in rotational motion

If the particles in a system are closely packed and its mass distribution about the axis of rotation is fixed, then the aggregation of particles is known as rigid body. The case of rigid body differs to a system of particles in following aspects :

Rigid body in rotation



Every particle composing the rigid body executes circular motion.

1. The rigid body moves about an axis of rotation instead of a point.

2. The angular quantities can be written in scalar forms with appropriate signs, as they have only two directions.

The angular momentum (ℓ) is the product of moment arm ($r_\perp$) and linear momentum. The angular momentum of the rigid body (as a system of particles) is given by a scalar sum as given here :

$$L = \sum \ell_i = \sum m_i p_i = \sum m_i r_{\mathrm{i}\perp} v_i = \sum m_i {r_{\mathrm{i}\perp}}^2 \omega$$

Scalar summation is possible in the case of rotation as angular momentum is always along the axis of rotation - either in the positive or negative direction. We must be careful to remember that $r_{\mathrm{i}\perp}$ is moment arm of about the axis of rotation - not about the point as in the general case. Now, each of the particle rotates with same angular velocity. As such, it can be taken out of the summation sign.

$$L = \omega \sum m_i {r_{\mathrm{i}\perp}}^2 = I\omega$$

We note that moment of inertia of the rigid body does not change with time. The only quantity that may change with is angular velocity. Hence, the first time derivative of the total angular momentum is :

$$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha$$

However, we have derived the following relation earlier for rotation of a rigid about an axis :

$$\boldsymbol{\tau}_{\text{net}} = I\alpha$$

Combining, two equations we have two expressions for torque :
Equation:

$$\boldsymbol{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt} = I\alpha$$

Example:
Problem : A wheel of mass 10 kg and radius 0.2 m is rotating at an angular speed of 100 rpm with the help of a motor drive. At a particular moment, the motor is turned off. Neglecting friction at the axle, find the constant tangential force required to stop the wheel in 10 revolutions.
Solution : We need to find the tangential force, which is related to torque as :

$$\tau = rF$$

If we know torque, then we can find the force required to stop the wheel. Now, torque is given as (using Newton's second law in angular form),

$$\tau = I\alpha$$

Here, MI of the rotating wheel is :

$$I = \frac{1}{2}MR^2 = \frac{1}{2}10 \text{ x } 0.2^2 = 0.2\,\text{kg-}m^2$$

By inspecting equation of torque, we realize that we need to find angular acceleration to calculate torque on the wheel. Since the external force is constant, we can conclude that the resulting torque is also constant. This means that the wheel is decelerated at constant rate. Thus, applying equation of motion,

$$\omega_2^2 = \omega_1^2 - 2\alpha\theta$$

Here,

$$\omega_1 = \frac{100\text{x}2\pi}{60} = \frac{10\pi}{3} \text{ rad}/s, \omega_2 = 0, \theta = 10\,\text{rev} = 2\pi \text{ x } 10 = 20\pi \text{ rad}$$

$$\Rightarrow \alpha = -\frac{\omega_2^2}{2\theta} = -\frac{\left(\frac{10\pi}{3}\right)^2}{2\,\text{x}\,20\pi} = -\frac{5\pi}{18} \text{ rad}/s^2$$

Putting this value in the equation, we have :

$$\Rightarrow \tau = I\alpha = 0.2 \text{ x } -\frac{5\pi}{18} = -\frac{\pi}{18}\, N - m$$

Thus,

$$F = \frac{\tau}{r} = -\frac{\pi}{18\text{x}0.2} = -0.87\,N$$

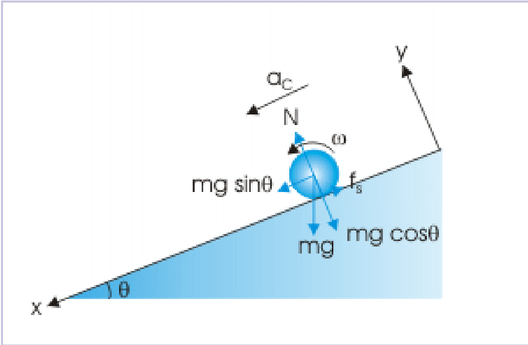
Negative sign indicates that force is opposite to the direction of velocity.

Does Newton’s law of motion in angular form hold in the accelerated frame ?

Answer to this question is very relevant in the case of combined (translation and rotation) motion. Consider the case of rolling. The axis of rotation i.e. the reference of motion itself may be accelerating down an incline. We may also consider the example of a spring board diver. In this case, the diver accelerates down the spring board under

gravity while performing eye catching twists and turns. Here, also the reference frame of rotation attached to the diver is accelerating.

Rolling along an incline



The disk rotates about an axis passing through center of mass.

Now, let us have the look the way we define a torque :

$$\mathbf{t} = \mathbf{r} \ \mathbf{x} \ \mathbf{F}$$

where,

$$\mathbf{F} = m \, \mathbf{a}$$

We ofcourse know that the relation of force and acceleration (which is Newton's second law of motion) is valid in inertial frame and is not valid for non-inertial frame. We may, therefore, infer that the Newton's second law in angular forms as derived here for system of particles and rigid body may also be not valid for non-inertial frames.

The question is, then, how do we apply these equation forms in cases of rolling and spring board driver ? The answer is that accelerated rolling or combined motion provides an unique situation for dealing motion in accelerated frame of reference.

If we recall, then we realize that we can render a non-inertial (accelerating reference) into an inertial frame by applying a psuedo force at the center of mass of the system of particle or the rigid body as the case may be. This has important bearing on the outcome of net torque. If the reference point is the center of mass of the system of particle, then the torque due to pseudo force about center of mass is zero. It means that applying Newton's law to a system of particle about the center of mass will be valid even if center of mass i.e. reference point itself is accelerating. Similar is the situation

for rotation of a rigid body about an accelerating axis of rotation, which is passing through the center of mass. The torque due to pseudo force about the axis passing through center of mass is zero. Therefore, application of Newton's law for rotation of a rigid body about an accelerating axis passing through center of mass is valid.

In retrospect, we find that application of Newton's law to determine torque on a rolling body with accelerated axis was premediated as we had not considered the non-inertial frame of reference. From the discussion here, it is clear that we had been not been wrong.

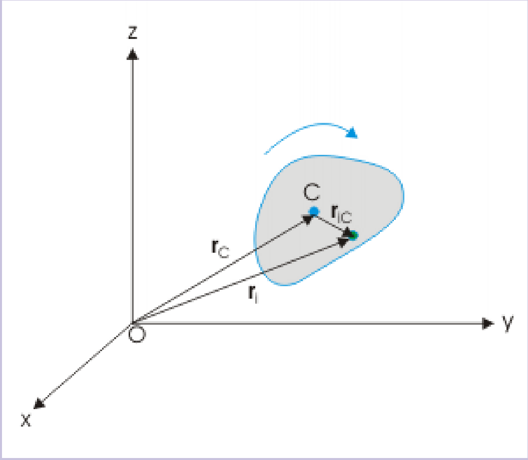
Angular momentum of a body in combined motion

Here, we consider combined motion of a rigid body comprising of both translational and rotational motions. This is a general consideration, in which the body may or may not be in pure rolling. We shall derive the required expression, proceeding from the basic expression of angular momentum with respect to a point.

Let a body of some shape is in combined motion at an instant as shown in the figure below. For our consideration, it is not required that the body of regular geometric shape as we intend to develop the expression in very general term. Let the point, about which angular momentum is considered, be the origin of coordinate system.

In this situation, we further consider that the center of mass is defined by the position vector ($\mathbf{r}_C$) and its velocity is represented by " $\mathbf{v}_C$ " . We, now, consider a particle of mass " m_i ", which is specified given the position vector " $\mathbf{r}_i$ ". According to the defining equation of angular momentum of a particle,

A body in combine motion



The body may translate as well as rotate.

$$\boldsymbol{\ell} = m(\mathbf{r} \ \mathbf{x} \ \mathbf{v})$$

Angular momentum of the body about the origin is :

$$\mathbf{L} = \sum \boldsymbol{\ell}_i = \sum m_i(\mathbf{r}_i \ \mathbf{x} \ \mathbf{v}_i)$$

By the triangle formed by the vectors,

$$\mathbf{r}_i = \mathbf{r}_C + \mathbf{r}_{\text{iC}}$$

Substituting in the expression of angular momentum of the particle, we have :

$$\mathbf{L} = \sum m_i(\mathbf{r}_i \ \mathbf{x} \ \mathbf{v}_i) = \sum m_i(\mathbf{r}_C + \mathbf{r}_{\text{iC}}) \ \mathbf{x} \ (\mathbf{v}_C + \mathbf{v}_{\text{iC}})$$

Expanding the expression, we have :

$$\mathbf{L} = \sum m_i \mathbf{r}_C \ \mathbf{x} \ \mathbf{v}_C + \sum m_i \mathbf{r}_C \ \mathbf{x} \ \mathbf{v}_{\text{iC}} + \sum m_i \mathbf{r}_{\text{iC}} \ \mathbf{x} \ \mathbf{v}_C + \sum m_i \mathbf{r}_{\text{iC}} \ \mathbf{x} \ \mathbf{v}_{\text{iC}}$$

We should be careful to maintain the sequence of vectors with respect to vector cross sign as a change will mean change in direction. Further, we should understand that we determine angular momentum for a given instant. At a given instant, the position vector of the center of mass, however, is unique and as such, we can take the same out from the summation sign. Rearranging,

$$\mathbf{L} = (\sum m_i)\mathbf{r}_C \ \mathbf{x} \ \mathbf{v}_C + \mathbf{r}_C \ \mathbf{x} \ (\sum m_i \mathbf{v}_{\text{iC}}) + (\sum m_i \mathbf{r}_{\text{iC}}) \ \mathbf{x} \ \mathbf{v}_C + \sum m_i \mathbf{r}_{\text{iC}} \ \mathbf{x} \ \mathbf{v}_{\text{iC}}$$

Looking at the individual terms in the expression, we realize that some of the terms are part of the definition of center of mass. The center of mass of the body with respect to the origin of a coordinate system is defined as :

$$\mathbf{R}_O = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$$

Now, the center of mass of the body with respect to “itself” is zero :

$$\mathbf{R}_C = \frac{\sum m_i \mathbf{r}_{\text{iC}}}{\sum m_i} = 0$$

$$\Rightarrow \sum m_i \mathbf{r}_{\text{iC}} = 0$$

Going by the similar logic or by taking time derivative of the above equation, we can conclude that :

$$\Rightarrow \sum m_i \mathbf{v}_{\text{iC}} = 0$$

Using these zero values and simplifying, we have :
Equation:

$$\Rightarrow \mathbf{L} = M\mathbf{r}_C \;\; \mathbf{x} \;\; \mathbf{v}_C + \sum m_i \mathbf{r}_{\mathrm{iC}} \;\; \mathbf{x} \;\; \mathbf{v}_{\mathrm{iC}}$$

We note here that first term is the angular momentum of the body with respect to origin, as if the mass of the body were concentrated at center of mass. This measurement of angular momentum is a measurement in ground reference i.e. in the coordinate system. The second term, however, is equal to angular momentum of the body ($\mathbf{L}_C$) as seen from the reference system attached to “center of mass” and about “center of mass” – a point.

$$\Rightarrow \mathbf{L} = M\mathbf{r}_C \;\; \mathbf{x} \;\; \mathbf{v}_C + \mathbf{L}_C$$

We should emphasize that angular momentum about center of mass is not the same as about an axis of rotation. However, the angular momentum about center of mass is same as angular momentum about axis of rotation passing through center of mass in cases where point (origin) and center of mass are in the plane of motion. In such case,
Equation:

$$\Rightarrow \mathbf{L} = M\mathbf{r}_C \;\; \mathbf{x} \;\; \mathbf{v}_C + \mathbf{I}\omega$$

Summary

1: Newton’s second law of motion in angular form for a system of particles about a point is given as :

$$\boldsymbol{\tau}_{\mathrm{net}} = \frac{d\mathbf{L}}{dt}$$

In words, the net external torque on the system of particles is equal to the time rate of change of angular momentum.

2: Newton’s second law of motion for rotation of a rigid body is given as :

$$\boldsymbol{\tau}_{\mathrm{net}} = \frac{d\mathbf{L}}{dt} = I\alpha$$

3: The differences in comparison to Newton’s second law of motion in angular form for a system of particles about a point are :

- Angular momentum is measured with respect to an axis of rotation.
- The moment arm is measured in the plane of rotation from the axis of rotation.

- Angular momentum about an axis is equal to the component of angular momentum about a point in the direction of axis of rotation.
- The relation that connects torque to moment of inertia and angular acceleration is valid only for the case of rotation.
- We can apply Newton's second law of motion for rotation of a rigid body - even in the cases where the axis of rotation is accelerating.

4: Angular momentum of Combined motion

The expression of angular momentum for combined motion is given by :

$$\mathbf{L} = M\mathbf{r}_C \;\; \mathbf{x} \;\; \mathbf{v}_C + \mathbf{L}_C$$

The first term is the angular momentum of the body with respect to origin, as if the mass of the body were concentrated at center of mass. The second term (LC) is equal to angular momentum of the body as seen from the reference system attached to "center of mass" and about "center of mass" – a point.

The angular momentum about center of mass is same as angular momentum about axis of rotation passing through center of mass in cases where point (about which angular momentum is measured) and center of mass are in the plane of motion like. In such case,

$$\Rightarrow \mathbf{L} = M\mathbf{r}_C \;\; \mathbf{x} \;\; \mathbf{v}_C + \mathbf{I}\omega$$

Law of motion in angular form for a system of particles (check your understanding)

The questions have been selected to enhance understanding of the topics covered in the module titled " Law for system in angular form for a system of particles ". All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

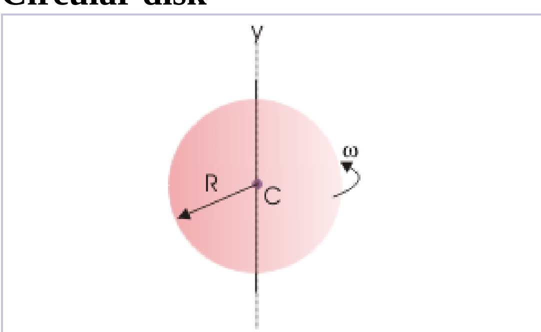
Understanding level (Law for motion in angular form for a system of particles)

Exercise:

Problem:

A uniform circular disc of mass “M” and radius “R” rotates about one of its diameters at an angular speed “ω”. What is the angular momentum of the disc about the axis of rotation?

Circular disk



The circular disk rotates about one of its diameters.

$$(a)\ MR^2\omega \quad (b)\ \frac{MR^2\omega}{2} \quad (c)\ \frac{MR^2\omega}{3} \quad (d)\ \frac{MR^2\omega}{4}$$

Solution:

The angular momentum of a rigid body is given by the equation,

$$L = I\omega$$

In order to find MI about diameter, we first calculate moment of inertia, " I_C ", of the disk about a perpendicular axis passing through the center as :

$$I_C = \frac{MR^2}{2}$$

Applying theorem of perpendicular axes, the moment of inertia about the axis of rotation (i.e. about one of the diameters) is :

$$I = \frac{I_C}{2} = \frac{MR^2}{4}$$

Therefore, the angular momentum of the disk about one of its diameter is :

$$L = I\omega = \frac{MR^2\omega}{4}$$

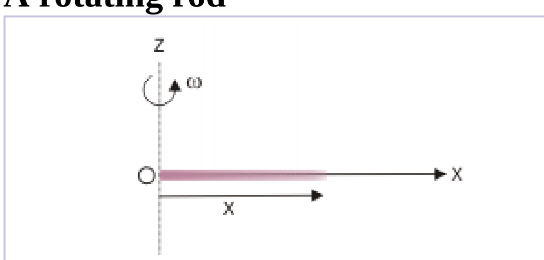
Hence, option (d) is correct.

Exercise:

Problem:

A thin uniform rod of mass "M" and length "x" rotates in a horizontal plane about a vertical axis. If the free end of the rod has a constant tangential velocity "v", then angular momentum of the rod about the axis of rotation is :

A rotating rod



The rod rotates about a perpendicular axis at one end.

(a) $\frac{Mv^2x}{3}$ (b) $\frac{Mvx}{3}$ (c) $\frac{Mvx^2}{3}$ (d) $\frac{Mv^2x^2}{6}$

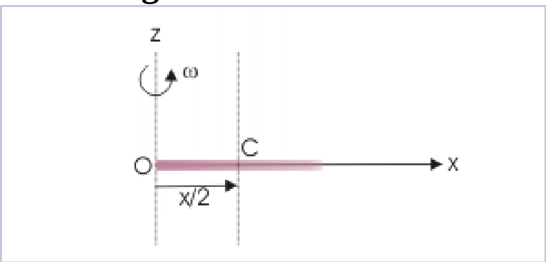
Solution:

The angular momentum of a rigid body is given by the equation,

$$L = l\omega$$

Here, the rod rotates about a perpendicular axis at its end. Thus, we need to find MI of the rod, "I", about this axis. Using theorem of parallel axes, the MI about the axis of rotation is :"

A rotating rod



The rod rotates about a perpendicular axis at one end.

$$I = I_{\mathrm{C}} + M \text{ x } \left(\frac{x}{2}\right)^2$$
$$\Rightarrow I = \frac{Mx^2}{12} + \frac{Mx^2}{4} = \frac{Mx^2}{3}$$

Putting in the expression of angular momentum :

$$\Rightarrow L = \frac{Mx^2\omega}{3}$$

The angular velocity is expressed in terms of tangential velocity as :

$$\omega = \frac{v}{x}$$

Substituting this value in the expression of angular momentum, we have

$$\Rightarrow L = \frac{Mvx}{3}$$

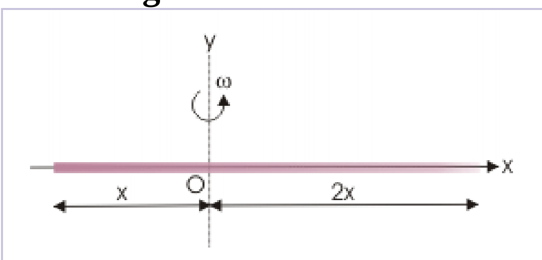
Hence, option (b) is correct.

Exercise:

Problem:

A thin uniform rod of mass "M" and length "3x" rotates in a horizontal plane with an angular velocity “ω” about a perpendicular vertical axis as shown in the figure. Then, the angular momentum of the rod about the axis of rotation is :

A rotating rod



The rod rotates about a perpendicular axis.

$$\text{(a)}\ Mx^2\omega\ \ \text{(b)}\ \frac{Mx^2\omega}{2}\ \ \text{(c)}\ \frac{2Mx^2\omega}{3}\ \ \text{(d)}\ \frac{Mx^2\omega}{4}$$

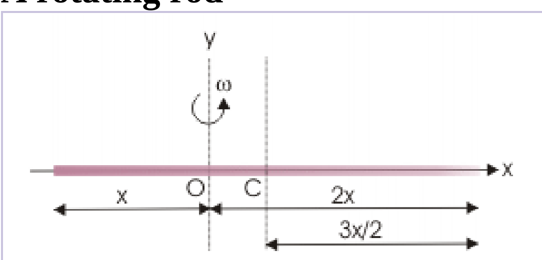
Solution:

We know that angular momentum of a rigid body is given by :

$$L = l\omega$$

We need to calculate the MI of the rod about the axis of rotation. Now, the MI of the rod about central axis is given by :

A rotating rod



The rod rotates about a perpendicular axis.

$$\Rightarrow I_C = \frac{M(3x)^2}{12} = \frac{3Mx^2}{4}$$

Applying theorem of parallel axis, the MI about the axis of rotation passing through point "O" is :

$$\Rightarrow I_O = I_C + Md^2$$

Here,

$$d = \frac{3x}{2} - x = \frac{x}{2}$$

$$\Rightarrow I_O = \frac{3Mx^2}{4} + \frac{Mx^2}{4} = Mx^2$$

Therefore, angular momentum about the axis of rotation is :

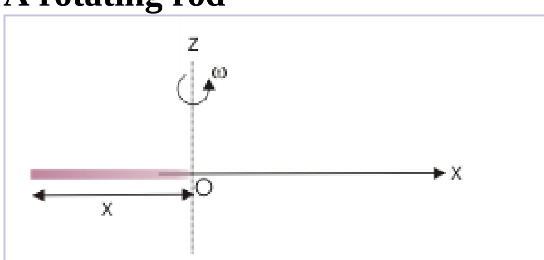
$$\Rightarrow L = I\omega = Mx^2\omega$$

Hence, option (a) is correct.

Note 1 : Alternatively, we can also calculate the angular momentum of the rod in terms of angular momentum of the parts on either side of the axis of rotation. Since each part rotates about the same axis of rotation, the net angular momentum of the rod about the axis of rotation is algebraic sum of the angular momentum of the parts.

The MI of the left rod, about a perpendicular axis at one of the ends, is :

A rotating rod



The MI of the left rod about a perpendicular axis at one end.

$$I_{\text{OL}} = \frac{\left(\frac{M}{3}\right)x^2}{3} = \frac{Mx^2}{9}$$

$$\Rightarrow L_{\text{OL}} = I\omega = \frac{Mx^2\omega}{9}$$

Similarly, the MI of the right rod is :

$$I_{\text{OR}} = \frac{\left(\frac{2M}{3}\right)(2x)^2}{3} = \frac{8Mx^2}{9}$$

$$\Rightarrow L_{\text{OR}} = I\omega = \frac{8Mx^2\omega}{9}$$

$$\Rightarrow L = L_{\text{OL}} + L_{\text{OR}}$$
$$\Rightarrow L = \frac{Mx^2\omega}{9} + \frac{8Mx^2\omega}{9} = Mx^2\omega$$

Note 2 : It must be understood here that we can not treat the rod as a particle at its center of mass and then find the angular momentum. For example, in this case, the center of mass is at distance of "3x/2" from either end of the rod and at a distance of "x/2" from the point "O" through which axis of rotation passes. The MI of this particle equivalent rod about the axis of rotation is :

$$\Rightarrow I_O = \frac{Mx^2}{4}$$

and

$$\Rightarrow L = I\omega = \frac{Mx^2\omega}{4}$$

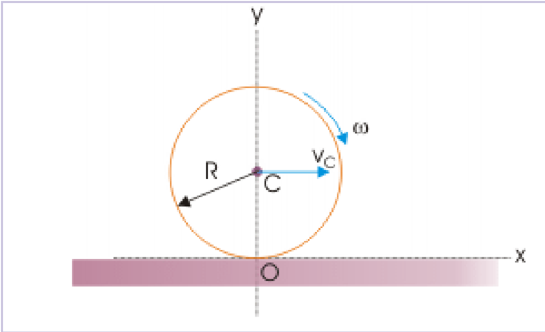
Obviously, this approach is conceptually wrong. The concept of center of mass by definition is valid for translation - not rotation.

Exercise:

Problem:

A circular ring of mass “M” and radius “R” rolls on a horizontal surface without sliding at a velocity “v”. What is the magnitude of angular momentum of the ring about point of contact?

Rolling of a circular ring



The circular ring of mass “M” and radius “R” rolls on a horizontal plane.

$$(a)\ MRv\ (b)\ 2MRv\ (c)\ \frac{2MRv}{3}\ (d)\ \frac{3MRv}{2}$$

Solution:

Here, the point of contact and center of mass are in the plane of motion. Hence,

$$\mathbf{L} = M\mathbf{r}_C \text{ x } \mathbf{v}_C) + I\boldsymbol{\omega}$$

Now, the magnitude of angular momentum of the body, considering it as a particle at center of mass, with respect to point of contact is :

$$\mathbf{L} = Mr_C v_C = MRv$$

This angular momentum is clockwise and into the page.

The magnitude of angular momentum about center of mass is same as that about its central axis i.e. perpendicular axis passing through center of mass :

$$I\omega = MR^2\omega$$

For rolling motion, $\omega = \frac{v}{R}$

$$I\omega = MR^2 \text{ x } \frac{v}{R} = MRv$$

This angular momentum is also clockwise and into the page. Therefore, the magnitude of net angular momentum is :

$$L = 2MRv$$

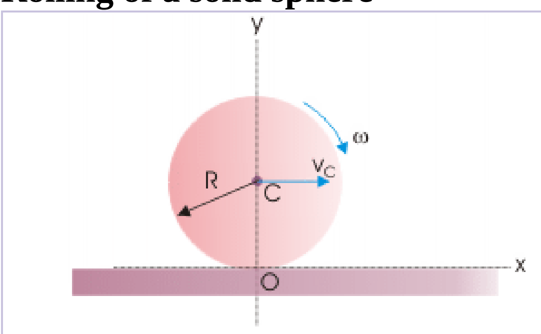
Hence, option (b) is correct.

Exercise:

Problem:

Q A solid sphere of mass “M” and radius “R” rolls on a horizontal surface without sliding at a velocity “v”. What is the magnitude of angular momentum of the ring about point of contact?

Rolling of a solid sphere



The solid sphere of mass “M” and radius “R” rolls on a horizontal plane.

$$\text{(a) } 2MRv \text{ (b) } \frac{3MRv}{5} \text{ (c) } \frac{4MRv}{5} \text{ (d) } \frac{7MRv}{5}$$

Solution:

This question is similar to the previous one - with one exception that sphere is a three dimensional body as against the disk, which is two dimensional. However, the situation is same as far as the point about angular momentum is calculated and the center of mass. The point of contact and center of mass are in the same middle section of the sphere, along which motion takes place. Thus, the point of contact and center of mass of the solid sphere are in the plane of motion. Hence,

$$\mathbf{L} = M(\mathbf{r}_C \ \mathbf{x} \ \mathbf{v}_C) + I\boldsymbol{\omega}$$

Now, the magnitude of angular momentum of the body as particle at center of mass with respect to point of contact is :

$$M(r_C v_C) = MRv$$

This angular momentum is clockwise and into the page.

The magnitude of angular momentum about center of mass is same as that about its central axis i.e. perpendicular axis passing through center of mass :

$$I\omega = \frac{2MR^2\omega}{5}$$

For rolling motion, $\omega = \frac{v}{R}$

$$I\omega = \frac{2MR^2v}{5R} = \frac{2MRv}{5}$$

This angular momentum is also clockwise and into the page. The magnitude of net angular momentum, therefore, is :

$$L = MRv + \frac{2MR^2v}{5} = \frac{7MR^2v}{5}$$

Hence, option (d) is correct.

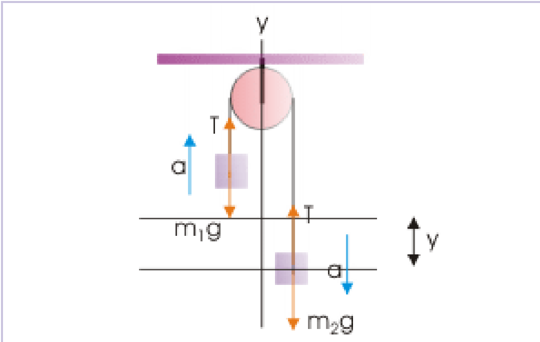
Application level (Law of motion in angular form for a system of particle)

Exercise:

Problem:

Two blocks of masses “ m_1 ” and “ m_2 ” ($m_2 > m_1$), connected by a mass-less pulley of radius “r”, are released from rest. What is the angular momentum of the system about the axis of rotation of the pulley, if the block travels a vertical distance of “y”.

Pulley – blocks system



Blocks are connected with a string passing over a pulley.

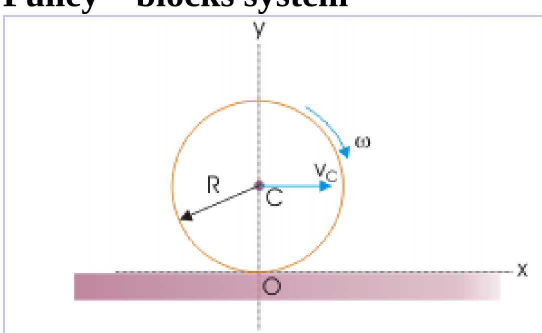
(a) $\sqrt{\frac{2\left({m_2}^2-{m_1}^2\right)gr^2y}{(m_1+m_2)}}$ (b) $\sqrt{\frac{2(m_1+m_2)^2gry}{(m_2-m_1)}}$ (c) $\sqrt{\frac{2m_1 \text{ x } m_2gr^2y}{(m_1+m_2)}}$ (d) $\sqrt{\frac{2m_1\text{x}m_2gr^3y}{(m_1+m_2)}}$

Solution:

The pulley and string are both considered "mass-less". Therefore, these elements do not contribute to the angular momentum of the system. Further, we also observe that we are required to find angular momentum about an axis - not about a point. We need to know the linear velocity of the blocks in order to calculate their angular momentum about the axis of rotation of the pulley. This, in turn, requires us to find the acceleration of the blocks in vertical direction.

Here, pulley is "mass-less". Thus, tension in the string on either side of the pulley is same. From force analysis of each block :

Pulley – blocks system



Force diagram.

$$m_2 g - T = m_2 a$$
$$T - m_1 g = m_1 a$$

$$\Rightarrow a = \frac{(m_2 - m_1)}{(m_1 + m_2)} g$$

For vertical motion of block " m_2 ", we have :

$$v^2 = u^2 + 2ay = 0 + 2ay = 2ay$$

$$\Rightarrow v^2 = 2ay = \frac{2(m_2 - m_1)gy}{(m_1 + m_2)}$$

Now, angular momentums of the blocks are both clockwise. Hence, we can get magnitude of net angular momentum as arithmetic sum :

$$L = m_1 vr + m_2 vr = (m_1 + m_2)vr$$

Substituting the value of the speed,

$$\Rightarrow L = \left(m_1 + m_2\right) \text{ x } r \text{ x } \sqrt{\frac{2(m_2 - m_1)gy}{(m_1 + m_2)}}$$

$$\Rightarrow L = \sqrt{\frac{2\left(m_2{}^2 - m_1{}^2\right)gr^2 y}{(m_1 + m_2)}}$$

Hence, option (a) is correct.

Answers

`1. (d) 2. (b) 3. (a) 4. (b) 5. (d) 6. (a)`

Conservation of angular momentum
Conservation of angular momentum of an isolated system is a general and fundamental law.

Conservation of angular momentum is a powerful general law that encapsulates various aspects of motion as a result of internal interactions. The conservation laws, including this one, actually addresses situations of motions, which otherwise can not be dealt easily by direction application of Newton's law of motion. We encounter rotation, in which the rotating body is changing mass distribution and hence moment of inertia. For example, a spring board diver or a skater displays splendid rotational acrobatics by manipulating mass distribution about the axis of rotation. Could we deal such situation easily with Newton's law (in the angular form)? Analysis of motions like these is best suited to the law of conservation of angular momentum.

Conservation of angular momentum is generally believed to be the counterpart of conservation of linear momentum as studied in the case of translation. This perception is essentially flawed. As a matter of fact, this is a generalized law of conservation applicable to all types of motions. We must realize that conservation law of linear momentum is a subset of more general conservation law of angular momentum. Angular quantities are all inclusive of linear and rotational quantities. As such, conservation of angular momentum is also all inclusive. However, this law is regarded to suit situations, which involve rotation. This is the reason that we tend to identify this conservation law with rotational motion.

The domain of application for different conservation laws are different. The conservation law pertaining to linear momentum is broader than force analysis, but limited in scope to translational motions. Conservation of energy, on the other hand, is all inclusive kind of analysis framework as we can write energy conservation for both pure and impure motion types, involving translation and rotation. Theoretically, however, energy conservation is limited at nuclear or sub-atomic level or at high speed translation as mass and energy becomes indistinguishable. These exceptional limits of energy conservation can, however, be overcome simply by substituting energy conservation by an equivalent "mass-energy"

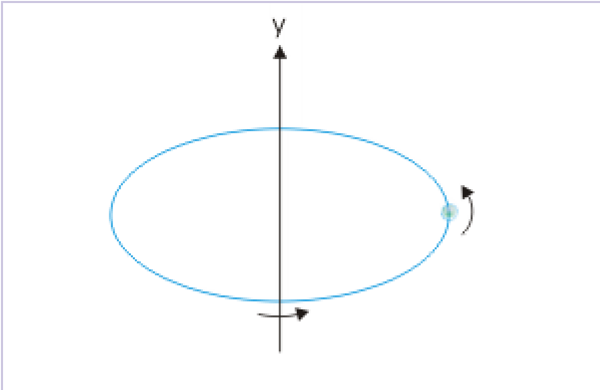
conservation law. In this context, conservation of angular momentum is as general as "mass-energy" conservation law to the extent that it is valid at extreme bounds of sub-atomic, nuclear and high speed motions.

Exercise:

Problem:

What is conserved in the uniform circular motion of a particle?

Uniform circular motion



A particle executing uniform circular motion.

(a) speed of the particle
(b) velocity of the particle
(c) linear momentum of the particle
(d) angular momentum of the particle

Solution:

The uniform circular motion is characterized by constant speed. Hence, speed is conserved.

The particle continuously changes direction. Hence, velocity is not conserved.

The particle would move along a straight line if there is no external force. However, the particle changes direction. It means that there is an external force on the particle. This force is called centripetal force. As there is an external force on the particle, linear momentum (m**v**) is not conserved. We can also simply conclude the same saying that since velocity is not conserved, the linear momentum of the particle (m**v**) is not conserved.

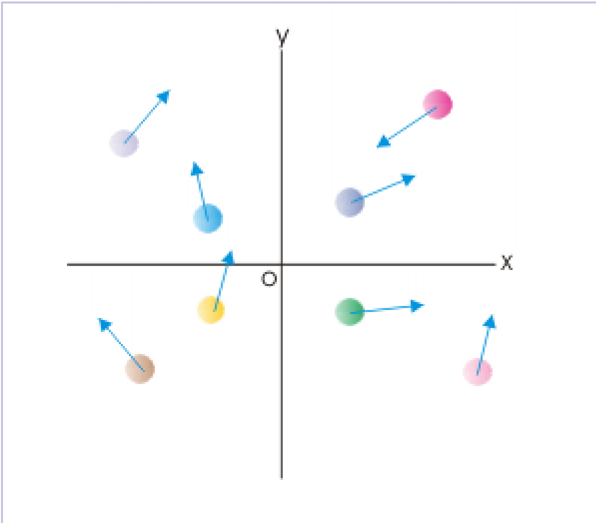
The particle has constant angular velocity (ω) and constant moment of inertia (I) about the axis of rotation. Hence, angular momentum (Iω) is conserved. We can look at the situation in yet another way. The only external force involved here is centripetal force, which is radial and passes through the axis of rotation of the particle. Thus, there is no external torque (**τ**) on the particle. As such, angular momentum of the particle is conserved.

Hence, options (a) and (d) are correct.

Conservation of angular momentum

An aggregate of objects may have combination of motions. Some of the subjects may be translating, others rotating and remaining may be undergoing a mix of motions. Conservation of angular momentum encompasses to analyze such complexities in motion. The generality of the conservation law is actually the reason why angular momentum has been defined about a point against an axis. This provides flexibility to combine all motion types. Had the angular quantities been defined only about an axis, then it would not have been possible to associate different types of motions with angular momentum. For example, it would have been difficult to apply law of conservation of angular momentum for randomly moving particles as shown in the figure below.

System of particles



Particles move randomly or may rotate about an axis.

Conservation of angular momentum, like conservation of linear momentum and energy, is fundamental to the nature and laws governing it. It is more fundamental than classical laws as it holds where Newton's law breaks down. It holds at sub-atomic level and also in the realm of motion, when it nears the speed of light.

We have studied that the time rate of change of angular momentum of a system of particles is equal to net external torque on the system. This is what is known as Newton's second law in angular form for a system of particles. It, then, follows that the angular momentum of the system will be conserved, if net external torque on the system is zero. Though, there is no external torque on the system, the particles inside the system may still be subjected to forces (torques) and, therefore, may undergo multiple change in velocity (angular velocity). Evidently, we can analyze resulting motions of the system with the help of the conservation of angular momentum.

There are many ways or forms in which this law can be stated. Mathematically,

$$\mathbf{t}_{\text{net}} = 0$$

$$\Rightarrow \mathbf{t}_{\text{net}} = \frac{d\mathbf{L}}{dt} = 0$$

$$\Rightarrow d\mathbf{L} = 0$$

From these result, we can state conservation of angular momentum in following equivalent ways :

Conservation of angular momentum
: If net external torque on a system is zero, then the angular momentum of the system can not change.

Equation:

$$\Rightarrow \mathbf{\Delta L} = 0$$

Conservation of angular momentum
: If net external torque on a system is zero, then the angular momentum of the system remains same.

Equation:

$$\Rightarrow \mathbf{L}_i = \mathbf{L}_f$$

Application of conservation of angular momentum is required to be made under the circumstance of zero torque. This condition, however, does not imply that net force on the system is zero. The two conditions are different. There may be net external force on the system, but the torque on it may be zero. This is the case, when net external force acts through center of mass of the system of particles or the rigid body or their combination.

Conservation of angular momentum in component form

Application of the law of conservation of angular momentum is not as straight forward as it may appear. Theoretically, though, it is possible to conceive or define a system such that there is no external torque, but in real time situation it is not advisable to adjust the system to reduce net external

torque to zero. If we also consider the fact that we have to consider angular momentums of all objects within the system about certain points and/or axes, then we realize that it is not actually possible to apply this law in most of the real time situation - unless when we have some complex algorithm with a powerful computer at our disposal.

However, there is the fact that motions along mutually perpendicular axes are independent of each other. This is an experimental fact, which has been described in detail in the module titled Projectile motion. This independence is a characteristic feature of motion and comes to our rescue in analyzing motion of an isolated system in the context of conservation of angular momentum.

The angular momentum is a vector quantity having direction as well. As such, we can express conservation of angular momentum along three mutually perpendicular axes of a rectangular coordinate system.
Equation:

$$
\begin{aligned}
L_{\mathrm{xi}} &= L_{\mathrm{xf}}, \;\; \text{when}\, t_x = 0 \\
L_{\mathrm{yi}} &= L_{\mathrm{yf}}, \;\; \text{when}\, t_y = 0 \\
L_{\mathrm{zi}} &= L_{\mathrm{zf}}, \;\; \text{when}\, t_z = 0
\end{aligned}
$$

Looking at the above formulations, we realize that application of conservation law in three mutually perpendicular directions is a powerful paradigm. Even if external torque is not zero on a system, it is likely and possible that net component of external torques in a particular direction is zero. The component form of the conservation law allows us to apply conservation in that particular direction, irrespective of consideration in other mutually perpendicular directions. This is a great improvisation as far as application of the conservation of angular momentum is concerned. We, therefore, can state the component form of the law :

Conservation of angular momentum in component form

> If the net component of external torques on a system along a certain direction is zero, then the component of angular momentum of the system in that direction can not change.

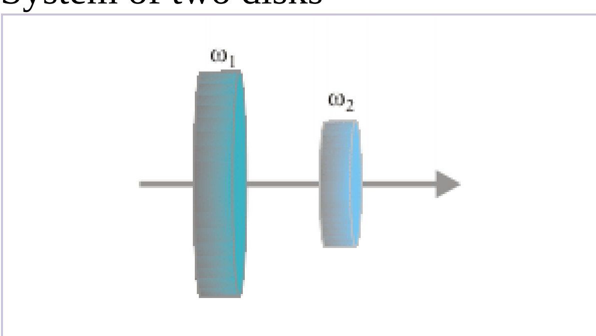
Conservation of angular momentum for isolated body system about a common axis

Isolated body system is a special case of general conservation law of angular momentum. It may occur to us that the rotational description can very well be analyzed in terms of Newton's second law. Why do we need to consider such eventuality for conservation of angular momentum? As pointed out early in this module, there are situations of rigid bodies, which are capable to change their mass distribution and it would be very difficult to analyze motion in terms Newton's second law of motion. Conservation law, on the other hand, can elegantly provide the solution.

Humans are one such body. We can change our body configuration by manipulating arms and legs. This is what dancers, skaters and spring board divers do. They change their body configuration, while in motion. This changes their moment of inertia about the axis of rotation. However as there is no external torque involved, there is corresponding change in their angular velocity to conserve angular momentum of the body system.

There is yet another situation, when conservation of angular momentum for body system can be helpful in analyzing motion. We can consider multiple parts of the body system which may selectively undergo rotation about a common axis of rotation. For example, motions of two discs along a common spindle can be analyzed by considering conservation of angular momentum of the isolated body system.

System of two disks



The disks rotate about a common axis.

The statement of conservation of angular momentum for isolated body system can take advantage of the relation valid for the rigid body. Here,

$$L_i = L_f$$

Equation:

$$I_i \omega_i = I_f \omega_f$$

We must, however, always keep in our mind that this form of conservation law is valid only for the rigid body for which angular momentum is measured about an axis.

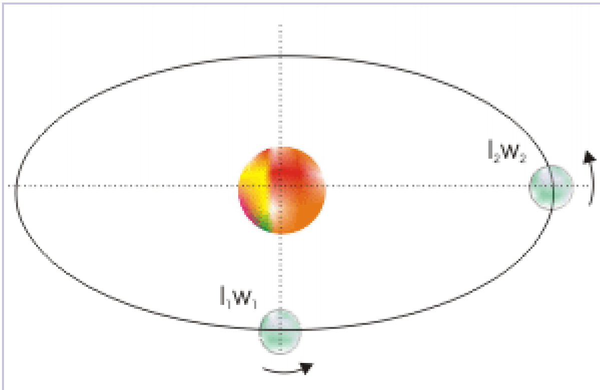
Examples

We have all along emphasized the general nature of angular momentum. But when we think about examples of real world, which can be analyzed with the help of this conservation law - we realize that most of them are actually the rotational cases. This, however, does no reduce the importance of generality. The physical examples for general motion require complex analysis tool beyond the scope of this course and hence are not considered.

Here, some examples of rotational motion are given to illustrate conservation of angular momentum.

1: The revolution of planets around Sun

The planets like Earth move around Sun along an elliptical orbit. The gravitational pull provides the necessary centripetal force for the curved elliptical path of motion. This gravitational force, however, passes through center of mass of the Earth and the Sun. As such, it does not constitute a torque. Thus, no external torque is applied to the Earth - Sun system. We can, therefore, apply conservation of angular momentum to the system.
Earth revolving around Sun



The Earth moves around Sun in an elliptical path.

When the Earth comes closer to the Sun, the moment of inertia of the Earth about an axis through the center of mass of the Sun decreases. In order to conserve its angular momentum, it begins to orbit the Sun faster. Similarly, the Earth rotates slower when it is away from the Sun. All through Earth's revolution, following condition is met :

$$I_i \omega_i = I_f \omega_f$$

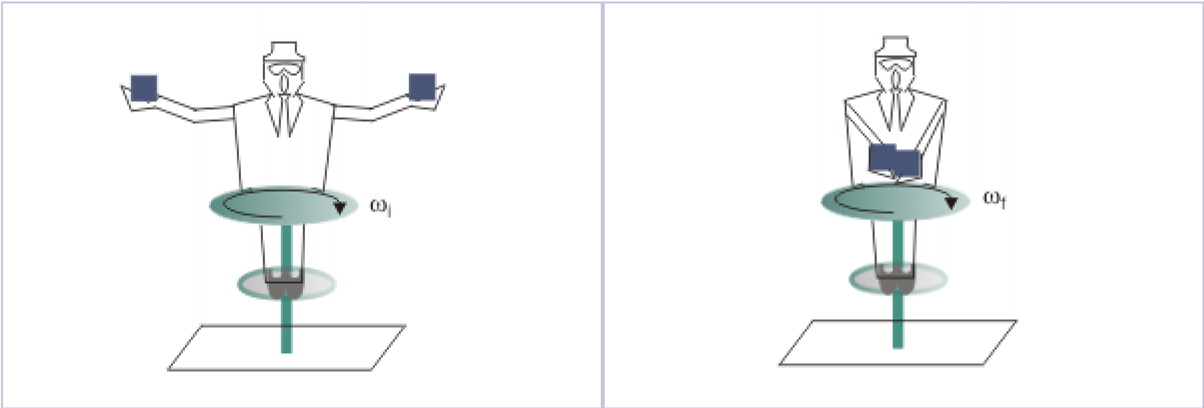
We should note here that we are considering rotation of the Earth about the Sun - not the rotation of Earth about its own axis of rotation. If we recall Kepler's law, we can see the convergence of results. This law states that the line joining Sun and Earth sweeps equal area in equal time and, thereby predicts greater tangential velocity (in turn, greater angular velocity), when closer to the Sun.

2: A person sitting on a turn table

A person sitting on a turn table can manipulate angular speed by changing moment of inertia about the axis of rotation without any external aid. In order to accentuate the effect, we consider that person is holding some weights in his outstretched hands. Let us consider that the system of the turntable and the person holding weights in the outstretched hands, is

rotating about vertical axis at certain angular velocity in the beginning. The moment of inertia of the body and weights about the axis of rotation is :
A person sitting on a turn table

The person with extended arms The person with folded arms



$$I = \sum m_i r_i^2$$

When the person folds his hands slowly, the moment of inertia about the axis decreases as the distribution of mass is closer to the axis of rotation. In order to conserve angular momentum, the turn table and person starts rotating at greater angular velocity.

$$\omega_f > \omega_i$$

3: Ice skater

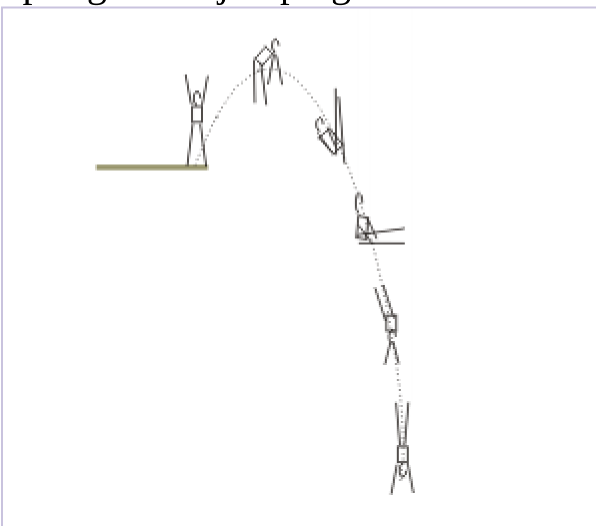
An ice skater rotates with outstretched hands on one leg. When he/she folds the hands and two legs closer to the axis of rotation, the moment of inertia of the body decreases. As a consequence, ice skater begins to spin at much greater angular velocity.

4: Spring board jumper

The spring board jumper follows a parabolic path of a projectile. In this case, the motion is about an accelerated axis of rotation, which moves along the parabolic path. In the beginning, the jumper keeps hands and legs

stretched. During the flight, he/she curls the body to decrease moment of inertia. This results in increased angular velocity i.e. more turns before the jumper hits the water.

Spring board jumping



The jumper follows a parabolic path under gravity.

When the spring board jumper nears the water surface, he/she stretches hands and legs so that he/she she has straight line posture ensuring minimum splash while hitting water surface.

Measurement of angular momentum

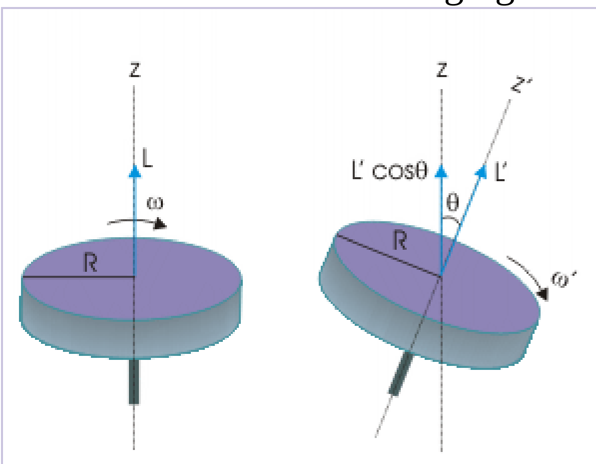
For applying conservation of angular momentum, we measure angular momentums in the context of some events like change in the distribution of mass, angular velocity etc. that arises from internal forces (torques).

The fundamental aspect of measurement of angular momentum is that its measurement should be about the same reference before and after the event. This is the basic requirement for applying law of conservation of angular momentum. We know that measurements of angular momentums about different points are different. Hence, we should stick to same set of points

for measuring angular momentum so that the single value property of this physical quantity could be maintained.

If the axis of rotation (for the case of rotation) changes direction, then we should consider conservation in terms of the components of angular momentum about mutually perpendicular axes of rotation. Since angular momentum is a vector quantity, we can always find its component in the reference direction.

Rotation about an axis changing orientation



We measure the component of angular momentum about the same .

The measurement of angular momentum of a system, however, could involve complexity for the following two reasons :

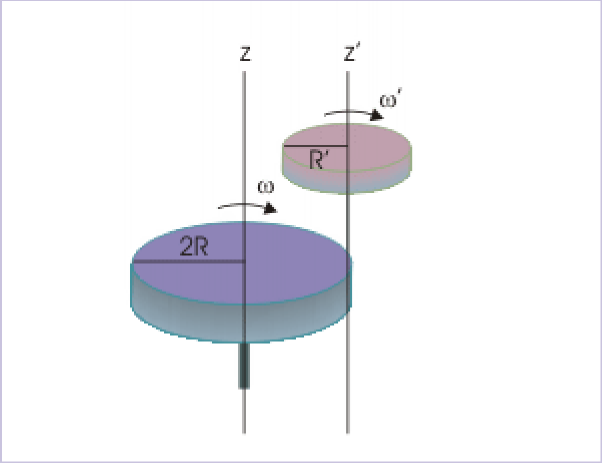
1. The system may have constituents involving both rotation and translation.
2. There may be more than one axes of rotations.

As far as rotation is concerned, we deal it about an axis of rotation. There is no ambiguity involved here. What about translation? The measurement of angular momentum for non-rotational motion is about a point in the plane of motion of the particle or the particle like body. Usually, we would prefer

(not required) that the point is on the axis of rotation, if the particle interacts with a rotating part of the system.

If a system consists of bodies rotating about different axes, then we should stick to the same axes for subsequent (after case) time for calculating angular momentum. As per the requirement of problem, we may, then, combine angular momentums, using vector addition to find the net or resultant angular momentum.

System with more than one axes of rotation



The combined system rotates about “z” axis, whereas smaller disk rotates about parallel “z’” axis.

All these aspects of measurement of angular momentum are illustrated with detailed explanation in the next module titled " Conservation of angular momentum (application) ".

Summary

1. Law of conservation of angular momentum

Definition : If there is no external torque on a system, then the angular momentum of the system can not change.

In general,

$$L_i = L_f$$

For rotation,

$$I_i\omega_i = I_f\omega_f$$

The rotational form of conservation law is suited for rotation of rigid body that changes its mass distribution due to internal forces or where constituent parts of the system change their angular velocities.

2. Law of conservation of angular momentum in component form

Definition : If the net component of external torques on a system along a certain direction is zero, then the component of angular momentum of the system in that direction can not change.

If “x”,”y” and “z” represent three mutually perpendicular axes, then :

$$\begin{aligned} L_{\text{xi}} &= L_{\text{xf}}, \ \text{when}\, t_x = 0 \\ L_{\text{yi}} &= L_{\text{yf}}, \ \text{when}\, t_y = 0 \\ L_{\text{zi}} &= L_{\text{zf}}, \ \text{when}\, t_z = 0 \end{aligned}$$

By corollary, component form of conservation law means that consideration of angular momentum in a given direction is not affected by torques in directions perpendicular to that direction.

Conservation of angular momentum (application)
Solving problems is an essential part of the understanding process.

Questions and their answers are presented here in the module text format as if it were an extension of the theoretical treatment of the topic. The idea is to provide a verbose explanation of the solution, detailing the application of theory. Solution presented here, therefore, is treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Hints on solving problems

- The first thing in attempting questions, based on conservation of angular momentum, is to know the torques operating on the system. If there is no torque on the system, then we can apply law of conservation of angular momentum in any convenient direction as we choose in accordance with the inputs given in the problem.
- In most of the situations, we find that there is external torque in certain direction. In general, however, we can apply law of conservation of momentum in a particular direction, if net torque on the system has no component in that direction. In the case of rotation, we can apply law of conservation, if the torque is perpendicular to the axis of rotation.
- Application of law of conservation of angular momentum in component form can be used in scalar form as there are only two directions, which can be represented by appropriate sign convention.
- We need to specify direction (assign sign) of angular quantities like angular velocity, momentum and torque with the help of right hand rule vector product rule.
- When the system involves both rotation and translation, then we should assign angular momentum with respect to an axis for rotation and a point for non-rotational motion of particles.
- The force at the axis of rotation is external force on the system. However, this force does not have moment arm and, therefore, does not constitute a torque on the system.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the conservation of angular momentum. For this reason, questions are categorized in terms of the characterizing features pertaining to the questions :

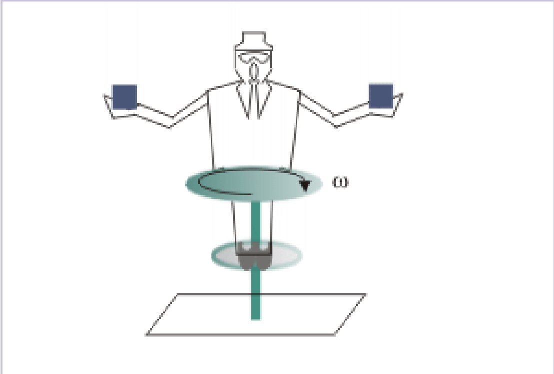
- Change in the distribution of mass about the axis of rotation
- Conservation of angular momentum about two parallel axes
- System consisting of both rotational and translational motion

Change in the distribution of mass about the axis of rotation

Example 1

Problem : A person sitting on a turn-table is free to rotate about a vertical axis. Initially, he rotates with angular velocity, "ω", holding two weights in his outstretched hands. The person, then, folds his hand horizontally to move weights towards axis of rotation. In the process, MI of the system of "turn-table, person and weights" is reduced from "I" to "I/2". Find (i) the final angular velocity and (ii) work done by the hands in moving the weights (neglect friction and air resistance).

Rotation of the system



A person sitting on a turn-table rotates about a vertical axis.

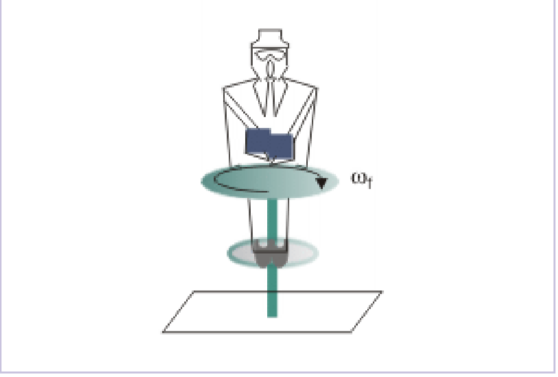
Solution : The system in question is composed of rigid bodies. However, internally the system can change its MI by virtue of changing positions of hands and weights held in them. The hand maneuvering, however, is internal to the system and does not affect angular momentum of the system.

Now, let us determine if there is external force and thereby external torque on the system. Force, resulting from rotation at the axis, does not constitute torque. On the other hand, we can consider the whole system as one, the weight of which acts

through the center of mass. Thus, we can say that the weight of the system do not constitute torque.

We may argue that the weights of outstretched hands and weights kept in them constitute torque about the center of mass of the system. Even if we look at the system parts in this fashion, two torques acts in opposite directions due to each of the weights. They act in horizontal plane (apply right hand rule of vector product) and balance each other. We, therefore, can safely conclude that there is no external torque on the system and apply law of conservation of momentum in component form in the vertical direction.

Rotation of the system



A person folds his hands drawing weights towards the axis.

If subscripts "i" and "f" correspond to initial and final values of the system, then :

$$L_i = L_f$$
$$I_i\omega_i = I_f\omega_f$$

$$\Rightarrow \omega_f = \frac{I_i\omega_i}{I_f}$$

From the question, $I_i = I$ and $I_f = \frac{I}{2}$,

Putting these values, we have :

$$\Rightarrow \omega_f = 2\omega_i = -2\omega$$

Negative sign indicates that angular velocity is clockwise.

In order to answer the second part of the question, we need to understand the process of folding hands. We see that folding of hands has resulted in the increase of angular kinetic energy of the system as angular speed has doubled. The change in angular kinetic energy of the system is given by :

$$\Delta K = \frac{1}{2} I_f {\omega_f}^2 - \frac{1}{2} I_i {\omega_i}^2$$

Putting values,

$$\Rightarrow \Delta K = \frac{1}{2} \text{ x } \frac{I}{2} \text{ x } (2\omega)^2 - \frac{1}{2} I\omega^2 = \frac{1}{2} I\omega^2$$

Note that expression of kinetic energy involves magnitude of angular velocity. Hence, sign is not considered.

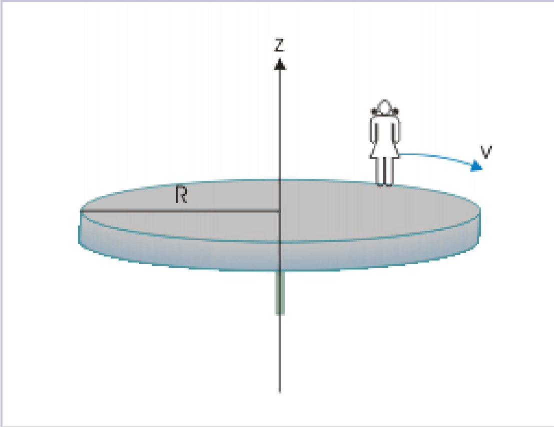
Now, the basic question is "where from this kinetic energy has come?" We see here that the person is applying force on the weights on his hands. This results in doing work on a part of the system i.e. weights. Work, as we know, transfers energy from one system to another. Work, here, draws muscular energy of the hands (internal energy of the person) and transfers the same to the whole system as angular kinetic energy. As no friction is involved in the process – there is no dissipation of energy either. Further, there is no vertical elevation of the weights involved as weights are moved horizontally. As such, there is no change in gravitational potential energy of the weights. Therefore, we can conclude that the kinetic energy change due to moving weights is equal to the work done on them by muscular force in accordance with Work – angular kinetic energy theorem (in order to conserve energy of the system) :

$$\Rightarrow W = \Delta K = \frac{1}{2} I\omega^2$$

Example 2

Problem : A uniform circular platform of mass 100 kg and radius 3 m is mounted on a frictionless vertical axle and is initially stationary. A girl, weighing 60 kg, stands on the rim of the platform. She begins to walk along the rim at a speed of 1 m/s relative to the ground in clockwise direction. Is the angular momentum of the system of girl and disk-axle conserved? What is the resulting angular velocity of the platform.

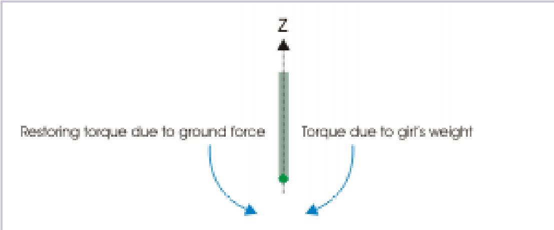
Relative motion of a girl along the rim of a disk



The girl begins to walk along the rim at a speed of 1 m/s relative to the ground in clockwise direction.

Solution : Here, mass is not distributed symmetrically about the axle. The weight of girl constitutes a torque perpendicular to the axis of rotation about the point at which axle is fixed on the ground. However, torque on axle by the ground prevents rotation due to the torque resulting from girl's weight in vertical plane. Thus, there is no external torque on the system and, therefore, angular momentum of the system of "girl and disk-axle" is conserved.

Torque due to the weight of girl



The orque due to the weight of girl is balanced by restoring force of ground on the axle.

Applying law of conservation of angular momentum in vertical direction (axis of rotation), we have :

$$L_i = L_f$$

Both platform and the girl are initially stationary with respect to ground. As such, initial angular momentum of the system is zero. From conservation law, it follows that the final angular momentum of the system is also zero.

But final angular momentum has two constituents : (i) final angular momentum of the platform and (ii) final angular momentum of the girl. Sum of the two angular momentums should, therefore, be zero. Let “P” and “G” subscripts denote platform and girl respectively, then :

$$L_f = L_{\text{Pf}} + L_{\text{Gf}} = 0$$

Now, MI of the circular platform about the axis of rotation is :

$$I_P = \frac{MR^2}{2} = \frac{100\text{x}\mathit{3}^2}{2} = 450\,\text{kg} - m^2$$

The mass of the particles constituting the girl are at equal perpendicular distances from the axis of rotation. Its MI about the axis of rotation is :

$$I_G = mR^2 = 60 \text{ x } \mathit{3}^2 = 540\,\text{kg} - m^2$$

From the question, it is given that final speed of the girl around the rim of the platform is 1 m/s in clockwise direction. It means that girl has linear tangential velocity of 1 m/s with respect to ground. Her angular velocity, therefore, is :

$$\omega_{\text{Gf}} = -\frac{v}{R} = -\frac{\mathit{1}}{3}\,\text{rad}\,/s$$

Negative sign indicates that the girl is rotating clockwise. Now, putting these values, we have :

$$L_{\text{Pf}} + L_{\text{Gf}} = I_P\omega_{\text{Pf}} + I_G\omega_{\text{Gf}} = 0$$

$$\omega_{\text{Pf}} = -\frac{-\mathit{1}\,\text{x}\,540}{\mathit{3}\,\text{x}\,450} = 0.4\ \text{rad}\,/s$$

The positive value of final angular velocity of platform means that the platform rotates in anti-clockwise direction.

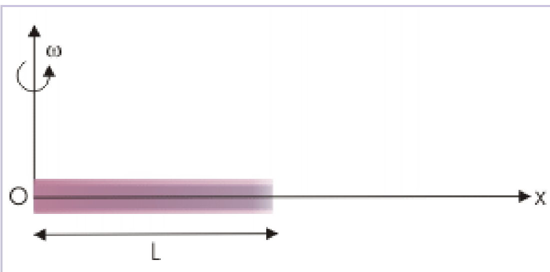
Note : Alternatively, we can find angular momentum of the girl, using the defining equation for angular momentum of a point (considering girl to be a particle like mass) as :

$$L_{\mathrm{Gf}} = -mvr_{\perp} = -60 \text{ x } 1 \text{ x } 3 = -180\,\mathrm{kg} - \frac{m}{s^2}$$

Example 3

Problem : A thin rod is placed coaxially within a thin hollow tube, which lies on a smooth horizontal table. The rod, having same mass “M” and length “L” as that of tube, is free to move within the tube. The system of "tube and rod" is given an initial angular velocity, "ω", about a vertical axis at one of its end. Considering negligible friction between surfaces, find the angular velocity of the rod, when it just slips out of the tube.

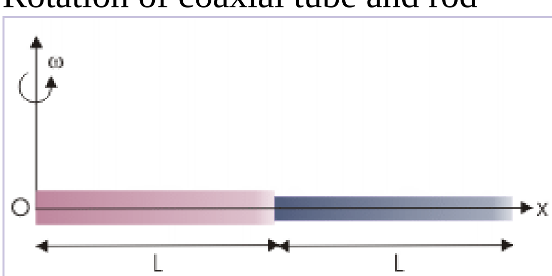
Rotation of coaxial tube and rod



The rod is free to move within the tube.

Solution : The first question that we need to answer is “why does rod slip out of the tube?”. To understand the process, let us concentrate on the motion of the rod only. Each particle of the rod, say at the far end, tends to move straight tangentially (natural tendency). However, there is no radial force available that could bend its path toward the axis of rotation. Hence, the particle tends to keep its path in tangential direction. As such, the particles have the tendency to keep themselves away from the axis of rotation. Eventually, the rod slips away from the axis of rotation.

Rotation of coaxial tube and rod



The rod slips away from the axis of rotation.

We may question why the same does not happen with a particle of rigid body in rotation. We know that rigid body does not allow relative displacement of particles. The particles are in place due to intermolecular forces operating on them. The tendency of the particle to move away from the axis of rotation is counteracted by the net inter-molecular force on the particle that acts towards the center of rotation. In this case, the inter-molecular force meets the requirement of centripetal force for circular motion of the particle about the axis of rotation.

Now, we must check about external torque on the system. The system lies on the smooth horizontal plane. It means that there is no net vertical force (weights of the bodies are balanced by normal forces). The force at the axis does not constitute torque as its moment arm is zero. We, therefore, conclude that there is no external torque on the system and that the problem can be analyzed in terms of conservation of angular momentum.

In the beginning, when the system is given initial angular velocity, the system has certain angular momentum, “ L_i ”. During the motion, however, rod begins to slip away from the axis. As such, mass distribution of the system changes. In other words, MI of the system changes. But, the angular momentum of the system (" L_f ") remains unchanged as there is no external torque on the system.

Now, applying conservation of angular momentum in vertical direction :

$$L_i = L_f$$
$$I_i \omega_i = I_f \omega_f$$

Here, $\omega_i = \omega$. According to theorem of parallel axes, the initial MI of the system, “ I_i ”, is :

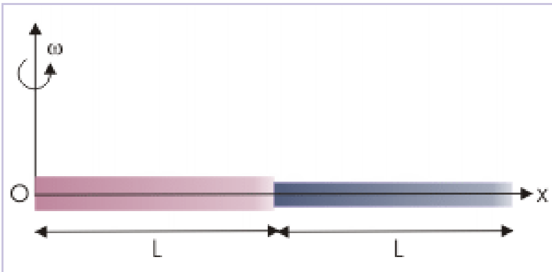
$I_i =$ MI of the system about perpendicular at mid-point
$+$ mass of the system x square of perpendicular distance between the axes

$$I_i = \left\{ \left(\frac{2ML^2}{12} + 2M\left(\frac{L}{2}\right)^2 \right\}$$

$$\Rightarrow I_i = \frac{ML^2}{6} + \frac{ML^2}{2} = \frac{2ML^2}{3}$$

Note that we have used "2M" to account for both tube and rod. Now, we again employ theorem of parallel axes in order to obtain MI of the system for the final position, when the rod just slips out of the tube. Final MI of the system, " I_f ", is :

Rotation of coaxial tube and rod



The rod slips away from the axis of rotation.

$$I_f = \left\{\left(\frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 + \frac{ML^2}{12} + M\left(\frac{3L}{2}\right)^2\right\}$$

$$\Rightarrow I_f = \frac{ML^2}{6} + \frac{ML^2}{4} + \frac{9ML^2}{4}$$

$$\Rightarrow I_f = \frac{2ML^2 + 3ML^2 + 27ML^2}{12} = \frac{8ML^2}{3}$$

Putting values in the equation of conservation of angular momentum, we have :

$$\Rightarrow \frac{2ML^2}{3} \text{ x } \omega = \frac{8ML^2}{3} \text{ x } \omega_f$$

$$\Rightarrow \omega_f = \frac{\omega}{4}$$

QBA (Question based on above) : A thin rod is placed coaxially within a thin hollow tube, which lies on a smooth horizontal table. The rod and tube are of same mass "M" and length "L". The rod is free to move within the tube. The system of tube and rod is given an initial angular velocity, "ω", about a vertical axis passing through the center of the system. Considering negligible friction between surfaces, find the angular velocity of the rod, when it just slips out of the tube.

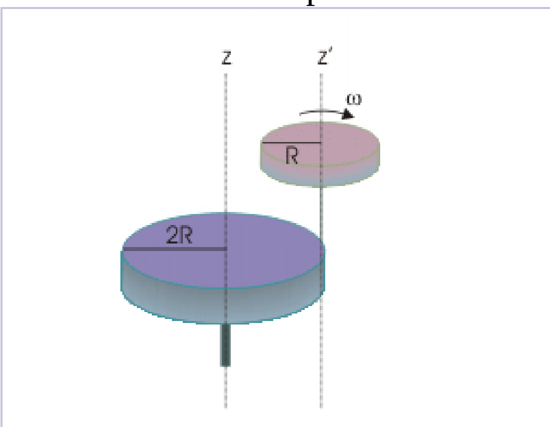
Hint : The question differs in one respect to earlier question. The axis of rotation here is vertical axis through the center of the system instead of the axis at the end of the system. This changes the calculation of MIs before and after. Answer is "ω/7".

Conservation of angular momentum about two parallel axes

Example 4

Problem : Two uniform disks of masses “4M” and “M” and radius “2R” and “R” respectively are connected with a mass-less rod as shown in the figure. Initially, the smaller disk is rotating clockwise with angular velocity “ω” with the help of an electric motor mounted on it, whereas the larger disk is stationary. At an instant, the smaller disk reverses the direction of rotation, keeping its angular speed same. Find the angular velocity of the larger disk.

Rotations about two parallel axes



The question involves two axes of rotation : one for larger plus smaller disks (z-axis) and other for smaller disk (z’-axis).

Solution : This question involves two axes of rotation : one for "larger plus smaller disks" (z-axis) and other for smaller disk (z’-axis). The treatment of angular momentum, in this case, differs from other cases in one important aspect. The constituents of the system are defined with respect to axis of rotation – not with respect to objects.

The body about the z - axis is a composite body comprising of larger and smaller disks. The smaller disk is also part of this composite body. The smaller disk (part of the composite body) rotates about its own axis (z’) parallel to z-axis. In the nutshell, the system comprises of two rotating bodies :

1. Composite body comprising of larger and smaller disk rotating about z-axis

2. Smaller disk rotating about z’-axis

We should note that the MI of the composite body is constant and is independent of the motion of smaller disk, because mass distribution about z-axis does not change by the rotation of smaller disk. It is so, because disks are uniform and circular in shape.

Now, let us check external torques on the system. The weight of smaller disk constitutes a torque, but it is perpendicular to the axis of rotation. Further, the torque due to smaller disk is balanced by an equal an opposite torque applied on the axle by the ground. This is confirmed, as there is no rotation of smaller disk in vertical plane. As such, we can conclude that there is no torque on the system in vertical direction and, therefore, we can apply law of conservation of momentum about z and z’ axes. Since two axes point in the same direction, the net angular momentums "before" and "after" are simply the arithmetic sum of angular momentums about the two axes. Let “C” and “S” subscripts denote the composite body and smaller disk respectively, then :

$$L_i = L_f$$

$$L_{\text{Ci}} + L_{\text{Si}} = L_{\text{Cf}} + L_{\text{Sf}}$$

The moment of inertia of the smaller disk about z’-axis is :

$$I_S = \frac{MR^2}{2}$$

Now, the moment of inertia, “ I_C ”, of the composite body is given by the theorem of parallel axes as :

$$I_C = \frac{4M\text{x}(2R)^2}{2} + \frac{MR^2}{2} + M(2R)^2 = 12.5MR^2$$

Let “ ω_f ” be the final angular velocity of the composite body. Substituting in the equation of conservation of angular momentum,

$$\Rightarrow I_C \text{ x } 0 - I_S \text{ x } \omega = I_C \text{ x } \omega_f + I_S \text{ x } \omega$$

Negative sign indicates that smaller disk rotates in clockwise direction.

$$\Rightarrow I_C \text{ x } \omega_f = 2I_S\omega$$

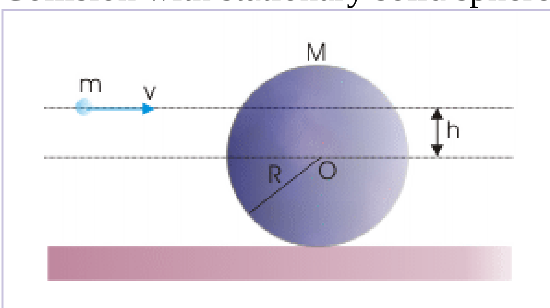
$$\Rightarrow \omega_f = \frac{2I_S \text{ x } \omega}{I_C} = \frac{2\left(\frac{MR^2}{2}\right)\omega}{12.5MR^2} = \frac{\omega}{12.5}$$

System consisting of rotational and translational motion

Example 5

Problem : A solid sphere of mass 1 kg and radius 10 cm lies on a horizontal surface. A particle of mass 0.01 kg moving with velocity 10 m/s parallel to horizontal surface hits the sphere at a vertical height 6 cm above the center of the sphere and sticks to it. Find (i) the angular velocity of the sphere just after the collision and (ii) the translation velocity of the sphere, if its motion is pure rolling after collision.

Collision with stationary solid sphere



A particle, moving parallel to horizontal surface, strikes the sphere at a vertical elevation "h" with respect to center.

Solution : First, we need to check for external torque to ensure that there is no external torque on the system. Here, the disk and particle are acted by gravity. The force of gravity on the disk passes through its COM and does not constitute torque. The force of gravity on the particle, however, constitutes torque about the axis of rotation, but question neglects this torque as particle in moving horizontally. As such, we can proceed to apply conservation of angular momentum in the direction of axis of rotation, using scalar component form. Let "P", "S" and "C" denote particle, sphere and combined mass respectively, then :

$$L_i = L_f$$
$$\Rightarrow L_{\text{Pi}} + L_{\text{Si}} = L_{\text{Cf}}$$

Let us consider that "m" and "M" be the masses of particle and sphere respectively and "R" be the radius of the sphere. Let us also consider that the vertical elevation

of the point above center is "h". Now, the initial angular momentum of the particle is :

$$L_{\text{Pi}} = mvr_{\perp} = -mvh$$

Negative sign shows that the angular momentum of the particle is perpendicular and into the plane of the disk (clockwise). The initial angular momentum of the sphere (L_{Si}) is zero.

Subsequent to collision, the moment of inertia of the combined mass (disk and particle) after collision is :

$$I_C = \frac{2MR^2}{5} + mR^2$$

We should note that though particle sticks at a vertical height “h” above COM, but its distance from COM is “R”. For this reason, its MI about COM involves “R” – not “h”.

Let the angular velocity of the composite system of sphere and the particle be “ω”. Now, putting values in the equation of conservation of angular momentum, we have :

$$-mvh + 0 = I_C\omega = \left(\frac{2MR^2}{5} + mR^2\right)\omega$$

$$\Rightarrow \omega = -\frac{mvh}{\left(\frac{2MR^2}{5} + mR^2\right)}$$

Putting numerical values,

$$\Rightarrow \omega = -\frac{0.01 \text{ x } 10 \text{ x } 0.06}{\left(\frac{2 \text{ x } 1 \text{ x } 0.1^2}{5} + 0.01 \text{ x } 0.1^2\right)} = -1.463 \text{ rad}/s$$

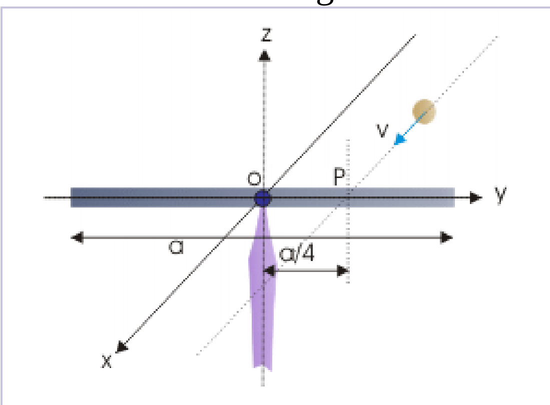
The negative sign of angular velocity shows that sphere along with stuck particle rotates in clockwise direction. According to the second part of the question, the motion of the combined mass is pure rolling. Hence, translational velocity after collision is :

$$v = \omega R = 1.463 \text{ x } 0.1 = 0.1463\, m/s$$

Example 6

Problem : A rod of mass "M" and length, "a", is pivoted at middle point "O" such that it can rotate in horizontal plane. A mud ball of mass "M/12", moving in horizontal plane strikes the rod at the point "P" with a speed "v" as shown in the figure. If the mud ball sticks to the rod after collision, find the magnitude of the angular velocity of the rod immediately after the ball strikes the rod.

Collision with rotating rod



A mud ball, moving in horizontal plane strikes the rod at the point "P" with a speed "v".

Solution : We first need to determine whether there is any external torque on the system of rod and mud ball. The forces during collision constitute internal torques. On the other hand, the force at the pivot passes through the axis of rotation. It has no moment arm about it and as such, it does not constitute a torque. The gravitation pull on the mud ball is in vertical direction. This force constitute a torque in "-x"-direction (apply right hand rule to assess the direction). We can, therefore, conclude that there is no component of external torque in z -direction on the system before, during and after the collision.

We observe here that the rod is capable to rotate in the horizontal plane. Also, the motion of the mud ball is in the horizontal plane. The directions of angular momentums for both rod and the mud ball are along vertical direction (z-direction). We can, therefore, apply conservation of angular momentums along the direction of axis of rotation i.e. perpendicular to the planes of motion. This enables us to use the component form of conservation law in z-direction as scalar expression with only two directions. Let “R” and “B” subscripts denote rod and the mud ball respectively, then :

$$L_i = L_f$$
$$\Rightarrow L_{\text{Ri}} + L_{\text{Bi}} = L_{\text{Rf}} + L_{\text{Bf}}$$

Here, the initial angular momentum of the rod is zero as it is stationary. Initial angular momentum of the mud ball just before it strikes the rod is given by,

$$L_{\text{Bi}} = -mr_{\perp}v = -\frac{Mva}{12\text{x}4}$$

Note in the above equation that mass of the mud ball is M/12 and its moment arm is "a/4". Negative sign to show that initial angular momentum of the mud ball is clockwise.

Since the mud ball sticks to the rod, two entities become one and move with a common angular velocity. Let us now consider their common angular velocity is "ω". The final angular momentum of the rod is given by :

$$L_{\text{Rf}} = I_R\omega = \frac{Ma^2\omega}{12}$$

The angular momentum of the mud ball as part of the rotating rod is given by :

$$L_{\text{Bf}} = \frac{M}{12}\text{ x }\left(\frac{a}{4}\right)^2$$

Putting these values in equation of conservation of angular momentum, we have :

$$0 - \frac{Mva}{12\text{x}4} = \left(\frac{Ma^2}{12} + \frac{M}{12}\text{ x }\frac{a^2}{16}\right\}\omega = \frac{17Ma^2\omega}{12\text{x}16}$$

$$\omega = -\frac{12\text{x}16Mva}{48\text{x}17Ma^2} = -\frac{4v}{17a}$$

Thus, magnitude of angular velocity of the mud ball just after the collision is :

$$\omega = \frac{4v}{17a}$$

Conservation of angular momentum (check your understanding)
Objective questions, contained in this module with hidden solutions, help improve understanding of the topics covered under the module "Conservation of angular momentum".

The questions have been selected to enhance understanding of the topics covered in the module titled " Conservation of angular momentum ". All questions are multiple choice questions with one or more correct answers. There are two sets of the questions. The “understanding level” questions seek to unravel the fundamental concepts involved, whereas “application level” are relatively difficult, which may interlink concepts from other topics.

Each of the questions is provided with solution. However, it is recommended that solutions may be seen only when your answers do not match with the ones given at the end of this module.

Understanding level (Conservation of angular momentum)

Exercise:

Problem: If the polar ice caps completely melt due to warming, then :

(a) the earth will rotate faster
(b) the earth will rotate slower
(c) there will be no change in the angular speed
(d) duration of a day on the Earth will increase

Solution:

The gravitational pull of the Sun passes through COM of the Earth, providing centripetal force required for the rotation of the Earth about it. Thus, there is no external torque on the Earth. It means that the angular momentum of the Earth remains constant.

The melting of ice cap will result in the rise of sea level. From the point of view of MI of the Earth, it means redistribution of mass. The water equivalent of ice moves away from the axis of rotation, which passes through the poles. This results in an increase in the MI of the Earth.

Now, applying law of conservation of angular momentum :

$$L_i = L_f$$
$$I_i \omega_i = I_f \omega_f$$

$$\Rightarrow \omega_f = \frac{I_i \omega_i}{I_f}$$

Hence, increase in the MI of the Earth, due to melting of ice, will decrease angular velocity. On the other hand, the duration of a day on the Earth is equal to its time period of rotation, which is given as :

$$\Rightarrow T = \frac{2\pi}{\omega_f}$$

As the Earth rotate slowly (lesser angular speed, ω_f), the duration of a day on the Earth will increase.

Hence, options (b) and (d) are correct.

Exercise:

Problem:

A circular disk of mass "M" and radius "R" is rotating with angular velocity "ω" about its vertical axis. When two small objects each of mass "m" are gently placed on the rim of the disk, the angular velocity of the ring becomes :

$$\text{(a) } \frac{M\omega}{(M+4m)} \quad \text{(b) } \frac{M\omega}{(M+2m)} \quad \text{(c) } \frac{M\omega}{(\frac{M}{2}+4m)} \quad \text{(d) } \frac{\mathrm{Mm}\,\omega}{(M+4m)}$$

Solution:

Since no external torque is operating on the disk – objects system, the angular momentum of the system is conserved.

Let " ω_f " be the common angular velocity of the composite system after objects are placed on the disk. Let subscripts "D", "O" and "C" represent disk, objects and composite system respectively , then according to conservation of angular momentum,

$$I_{\text{Di}}\omega_{\text{Di}} + I_{\text{Oi}}\omega_{\text{Oi}} = I_{\text{Ci}}\omega_f$$

Here,

$$I_{\text{Di}} = \frac{MR^2}{2} \quad ; \quad \omega_{\text{Di}} = \omega \quad ; \quad I_{\text{Oi}} = 0 \quad ; \quad \omega_{\text{Oi}} = 0$$

Also, the MI of the composite system is :

$$I_{\text{Cf}} = \left(\frac{MR^2}{2} + 2mR^2\right) = \left(\frac{M}{2} + 2m\right)R^2$$

Putting in the equation of conservation law,

$$\frac{MR^2\omega}{2} = \left(\frac{M}{2} + 2m\right)R^2\omega_f$$

$$\omega_f = \frac{M\omega}{2\left(\frac{M}{2}+2m\right)} = \frac{M\omega}{(M+4m)}$$

Hence, options (a) is correct.

Exercise:

Problem:

What would be the duration of day, if earth shrinks to half its radius with two – third of its original mass ? Consider motion of the Earth about the Sun along a circular path.

$$\text{(a) 2 hrs (b) 4 hrs (c) 8 hrs (d) 16 hrs}$$

Solution:

The Earth rotates around the Sun as gravitational pull provides the necessary centripetal force. This force passes through the center of mass of the Earth. As such, gravitational pull does not constitute torque on the Earth. Therefore, we can consider that angular momentum of the Earth remains unchanged. Now, let us use "i" and "f" subscripts to denote initial and final values, then according to law of conservation of angular momentum :

$$L_i = L_f$$
$$I_i\omega_i = I_f\omega_f$$

$$\Rightarrow \omega_f = \frac{I_i\omega_i}{I_f}$$

We know that time period of the Earth is :

$$T_i = 24 \text{ hrs}$$

Hence, its angular velocity is :

$$\Rightarrow \omega_i = \frac{2\pi}{T_i} = \frac{2\pi}{24} = \frac{\pi}{12} \text{ rad}/s$$

Since data on mass and radius are missing, it is not possible to calculate MIs in two cases separately. However, we can find the ratio of MIs :

$$\frac{I_i}{I_f} = \frac{\frac{2MR^2}{5}}{\frac{2}{5}\text{x}\frac{2M}{3}\text{x}\left(\frac{R}{2}\right)^2} = \frac{1\text{x}3\text{x}4}{2} = 6$$

Putting in the equation,

$$\omega_f = 6\omega_i = 6 \text{ x } \frac{\pi}{12} = \frac{\pi}{2} \text{ rad}/s$$

The new time period of rotation is :

$$T_f = \frac{2\pi}{\omega_f} = \frac{2\pi \text{ x } 2}{\pi} = 4 \text{ hrs}$$

Hence, options (b) is correct.

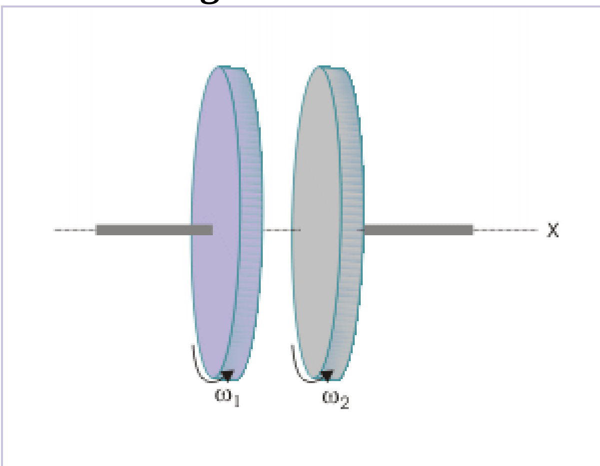
Note: This question underlines the fact that centripetal force does not constitute a torque for rotating body. Further this question describes a hypothetical situation in which mass of the body itself has changed – not its distribution, resulting in the change of MI and hence angular velocity to conserve angular momentum of the system.

Exercise:

Problem:

Two disks of moments of inertia I_1 and I_2 and having angular velocities ω_1 and ω_2 respectively are brought in contact with each other face to face such that their axis of rotation coincides. The situation is as shown in the figure just before the contact. What is the angular velocity of the combined system of two rotating disks, if they acquire a common angular velocity?

Two rotating disks



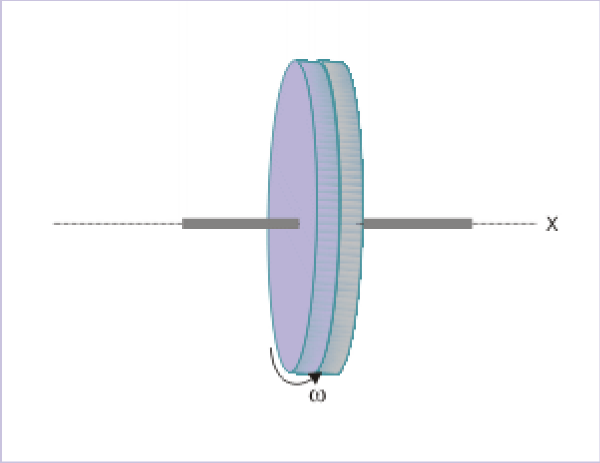
Two disks are rotating about a common axis.

(a) $\frac{I_1+I_2}{\omega_1+\omega_2}$ (b) $\frac{I_1\omega_1+I_2\omega_2}{\omega_1+\omega_2}$ (c) $\frac{I_1\omega_1+I_2\omega_2}{I_1+I_2}$ (d) $\frac{I_1 I_2\omega_1\omega_2}{I_1+I_2}$

Solution:

The bodies acquire equal angular velocity due to friction operating between the surfaces in contact. However, friction here is internal to the system of two rotating disks. Thus, there is no external torque on the system and we can employ law of conservation of angular momentum :

Two rotating disks



Two disks are brought in contact along common axis of rotation.

$$L_i = L_f$$

Here,

$$L_i = I_1\omega_1 + I_2\omega_2$$

When disks come in contact and rotate about a common axis with equal angular velocity, they acquire common angular velocity, say ”ω”. Since, two disks are rotating about a common axis of rotation, the MI of the combination is arithmetic sum of individual MIs. The moment of inertia of the composite system, “ I_C ” is given as :

$$I_{\mathrm{C}} = I_1 + I_2$$

Thus, final angular momentum of the system is :

$$L_{\mathrm{f}} = (I_1 + I_2)\omega$$

Putting values in the equation of conservation of angular momentum, we have :

$$\Rightarrow I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$$

$$\Rightarrow \omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

Hence, option (c) is correct.

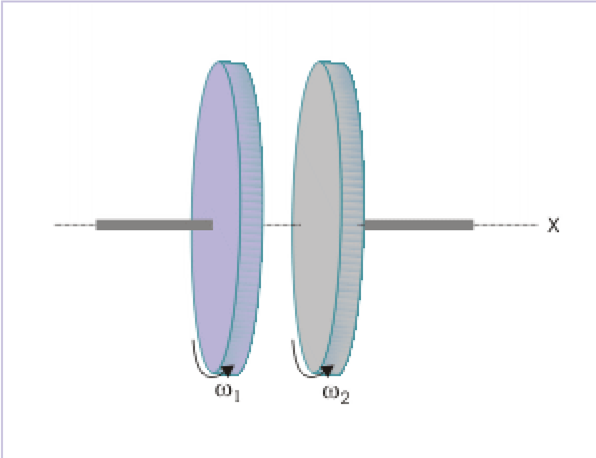
Application level (Conservation of angular momentum)

Exercise:

Problem:

Two disks of moments of inertia I_1 and I_2 and having angular velocities ω_1 and ω_2 respectively are brought in contact with each other face to face such that their axis of rotation coincides. The situation is as shown in the figure just before the contact. After sometime, the combined system of two rotating disks acquire a common angular velocity. The energy dissipated due to the friction is :

Two rotating disks



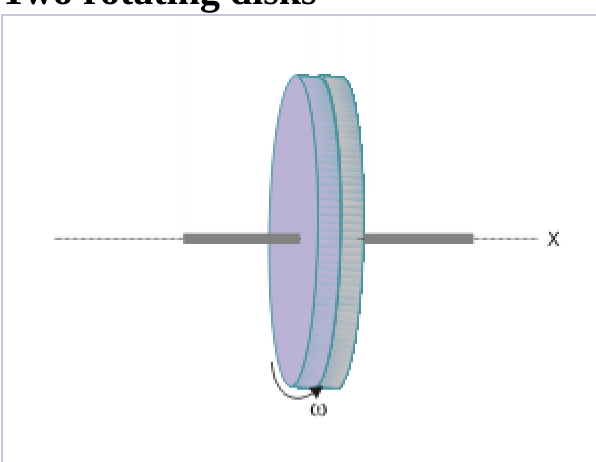
Two disks are rotating about a common axis.

(a) $\frac{\omega_1\omega_2(I_1-I_2)^2}{2(I_1+I_2)}$ (b) $\frac{I_1I_2(\omega_1-\omega_2)^2}{2(I_1+I_2)}$ (c) $\frac{I_1I_2(\omega_1-\omega_2)^2}{2(\omega_1+\omega_2)}$ (d) $\frac{2I_1I_2(\omega_1-\omega_2)^2}{(I_1+I_2)}$

Solution:

Applying law of conservation of angular momentum, the expression of common angular velocity is given as (as derived in the earlier question) :

Two rotating disks



Two disks are brought in contact along common axis of rotation.

$$\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

The energy dissipated due to the friction is equal to the change in the angular kinetic energy. Also as there is dissipation of energy, final angular kinetic energy is less than initial angular kinetic energy of the system. The change in angular kinetic energy, therefore, is :

$$\Delta K = \frac{1}{2} \text{ x } I_i\omega_i^2 - \frac{1}{2} \text{ x } I_f\omega_f^2$$

Here, final common angular speed and moment of inertia of the combined system are :

$$\omega_f = \omega \quad ; \quad I_f = I_1 + I_2$$

Putting values, we have :

$$\Rightarrow \Delta K = \frac{1}{2} \text{ x } I_1\omega_1^2 - \frac{1}{2} \text{ x } I_2\omega_2^2 = \frac{1}{2} \text{ x } \left(I_1 + I_2\right)\omega^2$$

Substituting value of common angular velocity and solving, we have :

$$\Rightarrow \Delta K = \frac{I_1 I_2 (\omega_1 - \omega_2)^2}{2(I_1 + I_2)}$$

Hence, option (b) is correct.

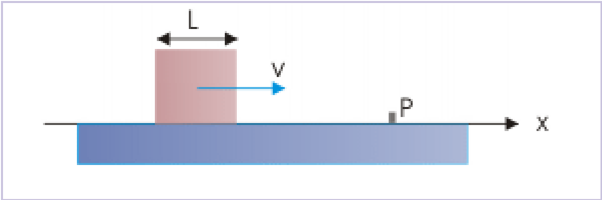
Note: We observe here that $(\omega_1 - \omega_2)^2$ is a square of difference of angular velocities. This means that this term is always positive. In turn, it means coupling two rotating disks due to friction will involve dissipation of angular kinetic energy of the system. In terms of angular kinetic energy, $K_i > K_f$.

Exercise:

Problem:

A cubical block of mass "M" and sides "L" hits a small obstruction at "p", while moving with a velocity "v" as shown in the figure. If the the block topples, then the angular speed of the block about "P", just after it hits the obstruction, is :

A block hitting an obstruction



The cubical block moves with a velocity "v" on a horizontal surface.

$$(a)\ \frac{3v}{4L}\quad (b)\ \frac{4v}{3L}\quad (c)\ \frac{3L}{4v}\quad (d)\ \frac{4L}{3v}$$

Solution:

The line of action of the force on the block due to obstruction passes through the axis of rotation about which block suddenly rotates i.e. about the obstruction for a brief period. These forces (operating at the time of collision) do not constitute torque. We can, therefore, conclude that there is no torque on the block. According to law of conservation of angular momentum,

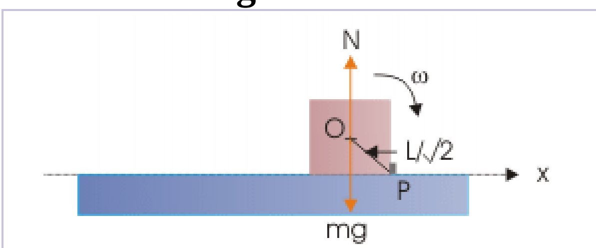
$$L_i = L_f$$

The angular momentum of the block, treating it as a particle like body in translation, about the point of obstruction,"P", is :

$$L_i = -Mvr_\perp = -\frac{MvL}{2}$$

Negative sign indicates that the angular momentum of the block about "P" is clockwise. The angular momentum of the block after collision, now treating it as a rigid body rotating about the axis perpendicular to the plane of motion and passing through the point of obstruction, is :

A block hitting an obstruction



The cubical block overturns at P.

$$L_f = I\omega$$

where “ω” is the angular velocity of the block about the axis of rotation.

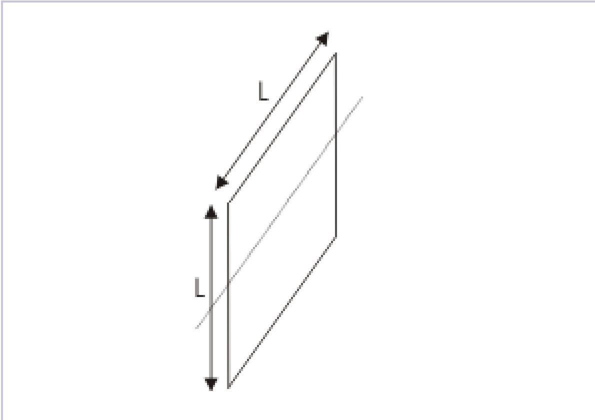
Thus,

$$\Rightarrow -\frac{MvL}{2} = I\omega$$

For moment of inertia of the block, it can be treated like a square plate. Its MI about the axis passing through COM is,

$$\Rightarrow I_{\text{COM}} = \frac{(L^2+L^2)}{12} = \frac{ML^2}{6}$$

Axis of rotation



Axis of rotation lies in the
central plane as seen from any
of the six sides.

Recall that the formula used for MI here is that of plate about an axis that passes through COM but which is in the plane of plate. It is a valid assumption as cube can be considered to be a cylinder with square face. Since MI of the plate and its corresponding cylinder are same, MI of the cube is same as that of corresponding square plate.

Here, the axis can be considered to lie in the central plane consisting of its COM. A central plane with perpendicular axis is shown for an enlarged cube. Since cube is symmetric about any such axis as seen from any of the six faces, we can conclude that MI as calculated above is about an axis, which is perpendicular to the plane of motion and passing through its COM. Now, using theorem of parallel axes, we have MI about the point of obstruction :

$$\Rightarrow I = I_{\mathrm{COM}} + Mr^2$$

From the geometry,

$$\mathrm{OP} = r = \frac{L}{\sqrt{2}}$$

$$\Rightarrow r^2 = \frac{L^2}{2}$$

Putting in the expression,

$$\Rightarrow I = \frac{ML^2}{6} + \frac{ML^2}{2} = \frac{2ML^2}{3}$$

Now, putting in the equation of conservation of angular momentum, we have :

$$\Rightarrow -\frac{MvL}{2} = \frac{2ML^2}{3} \text{ x } \omega$$

$$\Rightarrow \omega = -\frac{3v}{4L}$$

The negative sign indicates that the cubical block overturns in clockwise rotation about point "P". The angular speed of the block about obstruction, therefore, is equal to the magnitude of angular velocity.

Hence, option (a) is correct.

Answers

`1. (b) and (d) 2. (a) 3. (b) 4. (c) 5. (b) 6. (a)`

Equilibrium

We have already used this term in reference to balanced force system. We used the concept of equilibrium with an implicit understanding that the body has no rotational tendency. In this module, we shall expand the meaning by explicitly considering both translational and rotational aspects of equilibrium.

A body is said to be in equilibrium when net external force and net external torque about any point, acting on the body, are individually equal to zero. Mathematically,

$$\Sigma \mathbf{F} = 0$$

$$\Sigma \boldsymbol{\tau} = 0$$

These two vector equations together are the requirement of body to be in equilibrium. We must clearly understand that equilibrium conditions presented here only ensure absence of acceleration (translational or rotational) – not rest. Absence of acceleration means that velocities are constant – not essentially zero.

Now, we study translational motion of rigid body with respect to its center of mass, the linear and angular velocities under equilibrium are constants :

$$\mathbf{v}_C = \text{constant}$$

$$\boldsymbol{\omega} = \text{constant}$$

We need to analyze equilibrium of a body simultaneously for both translational and rotational equilibrium in terms of conditions as laid down here.

Equilibrium types

We are surrounded by great engineering architectures and mechanical devices, which are at rest in the frame of reference of Earth. A large part of engineering creations are static objects. On the other hand, we also seek equilibrium of moving objects like that of floating ship, airplane cruising at

high speed and such other moving mechanical devices. In both cases – static or dynamic, external forces and torques are zero.

An equilibrium in motion is said be “dynamic equilibrium”. Similarly, an equilibrium at rest is said be “static equilibrium”. From this, it is clear that static equilibrium requires additional conditions to be fulfilled.

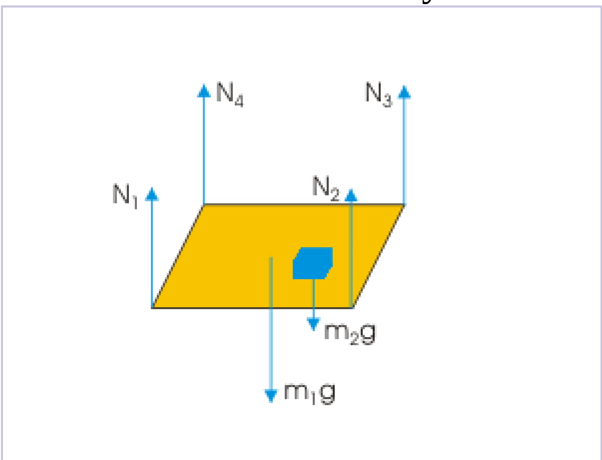
$$\Rightarrow \mathbf{v}_C = 0$$

$$\Rightarrow \boldsymbol{\omega} = 0$$

Equilibrium equations

In general, a body is subjected to sufficiently good numbers of forces. Consider for example, a book placed on a table. This simple arrangement actually is subjected to four normal forces operating at the four corners of the table top in the vertically upward direction and two weights, that of the book and the table, acting vertically downward.

External forces on the body



There are six external forces acting the table top.

If we want to solve for the four unknown normal forces acting on the corners of the table top, we would need to have a minimum of four

equations. Clearly, two vector relations available for equilibrium are insufficient to deal with the situation.

We actually need to write two vector equations in component form along each of thee mutually perpendicular directions of a rectangular coordinate system. This gives us a set of six equations, enabling us to solve for the unknowns. We shall, however, see that this improvisation, though, helps us a great deal in analyzing equilibrium, but is not good enough for this particular case of the book and table arrangement. We shall explain this aspect in a separate section at the end of this module. Nevertheless, the component force and torque equations are :

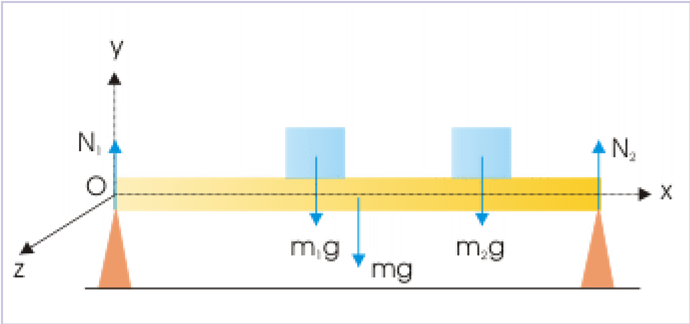
$$\sum F_x = 0; \quad \Sigma F_y = 0; \quad \Sigma F_z = 0$$

$$\sum \tau_x = 0; \quad \Sigma \tau_y = 0; \quad \Sigma \tau_z = 0$$

The first set of equations are called “force balance equations”, whereas second set of equations are called “torque balance equations”.

We must understand that mere existence of component equations does not guarantee that we would have all six equations to work with. The availability of relations depend on the orientation of force system operating on the body and dimensions of force – whether it is in one, two or three dimensions. To understand this aspect, let us consider the case of static equilibrium of the beam as shown in the figure. The forces acting on the beam are coplanar in xy- plane.

Torque on the body



Forces have tendencies to rotate beam about z-axis only.

Coplanar forces can rotate rigid body only around an axis perpendicular to the plane of forces. This is z-direction in this case. We can not write equations for torque around "x" or "y" axes. Thus, we have only torque balance equation in z-direction :

$$\Sigma\tau_z = 0$$

Further, all available balance equations may not be fruitful. Since forces lie in xy-plane in the example here, we would expect two force balance equations :

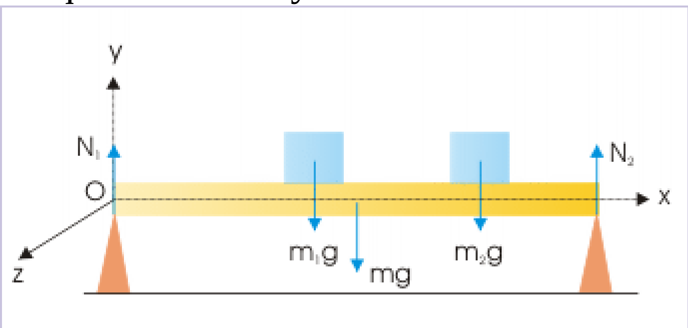
$$\Sigma F_x = 0$$

$$\Sigma Fy = N_1 + N_2 - mg - m_1 g - m_2 g$$

Clearly, forces have no components in x-direction. As such the force balance equation in x-direction is not meaningful.

We should apply available meaningful balance equations with certain well thought out plan. The most important is to select the point/ axis about which we calculate torque. In general, we choose the coordinate system in such a manner that moment arms of the most numbers of unknown forces become zero. This technique has the advantage that the resulting equation has least numbers of unknown and is likely to yield solution. In the figure shown here, the torque about a point "O" lying on the line of action of force " N_1 " results in zero moment of arm for this force.

Torque on the body



Force have no components in x-

direction.

If “a”, “b”, “c” and “d” be the linear distances of force " m_1g ", mg, " m_2g " and " N_2 " respectively from z-axis, then considering anticlockwise torque positive and anti-clockwise torque negative, we have :

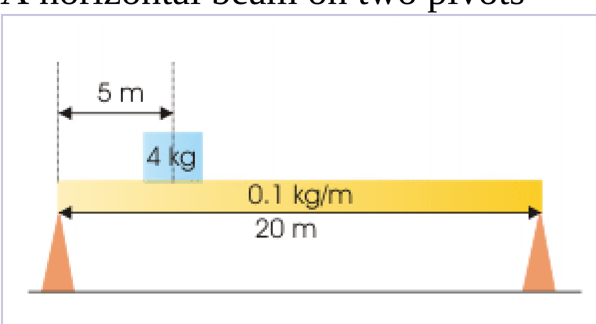
$$\Sigma\tau_z = 0XN_1 - aXm_1g - bXmg - cXm_2g + dXN_2$$

We can see that this equation eliminates “ N_1 ” and as such reduces numbers of unknowns in the equation. We must, however, understand that the selection of appropriate axis for calculation of torque is critical to evaluate the unknown. For example, if we consider to calculate torque through a perpendicular axis passing through center of mass of the beam, then we loose a known quantity (mg) and include additional unknown quanity, " N_2 " in the equation.

Example

Problem 1 : A uniform beam of length 20 m and linear mass density 0.1 kg/m rests horizontally on two pivots at its end. A block of mass of 4 kg rests on the beam with its center at a distance 5 m from the left end. Find the forces at the two ends of the beam.

A horizontal beam on two pivots

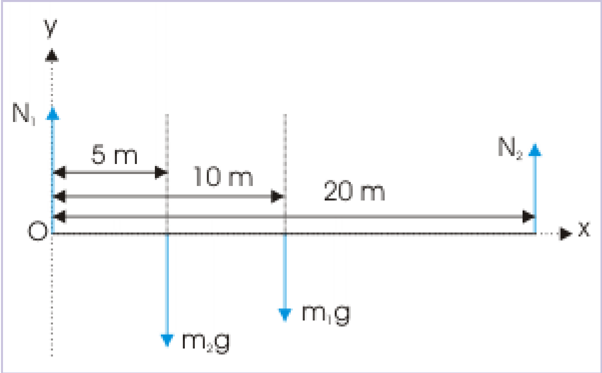


A block is placed on the beam.

Solution : The beam and block system is in static equilibrium. In order to write force balance equation, we first need to draw the free body diagram of the beam.

The normal forces at the ends act in vertically upward direction. Weights of the beam and block, on the other hand, acts downward through the corresponding centers of mass. Clearly, all forces are acting in vertical direction. As such, we can have one force balance equation in vertical direction. But, there are two unknowns. We, therefore, need to write torque balance equation about an axis perpendicular to the plane of figure (z-axis). The important aspect of choosing an axis is here that we should aim to choose the axis such that moment arm of one of the unknown forces is zero. This eliminates one of the forces from the torque balance equation.

A horizontal beam on two pivots



Forces on the beam are shown.

Force balance equation in vertical direction is :

$$\Sigma F_y = N_1 + N_2 - m_1 g - m_2 g = 0$$

Where " m_1 " and " m_2 " are the masses of beam and block respectively. According to question,

$$m_1 = \lambda L = 0.1X20 = 2 \quad kg$$

$$m_2 = 4 \quad kg$$

Putting values,

$$\Rightarrow N_1 + N_2 - 2X10 - 4X10 = 0$$

Considering anticlockwise torque positive and clockwise torque negative, the torque balance equation is :

$$\Sigma\tau_z = 0XN_1 + LXN_2 - L_1Xm_1 - L_2Xm_2 = 0$$

where “L”, “ L_1 ” and “ L_2 ” are moment arms of normal force “ N_2 ”, weight of beam, “ m_1g ” and weight of block, “ m_2g ” respectively. Putting values,

$$\Rightarrow 20XN_2 - 10X2X10 - 5X4X10 = 0$$

$$\Rightarrow N_2 = 20 \quad N$$

Since one of the unknown normal forces is, now, known, we can solve force balance equation by substituting this value,

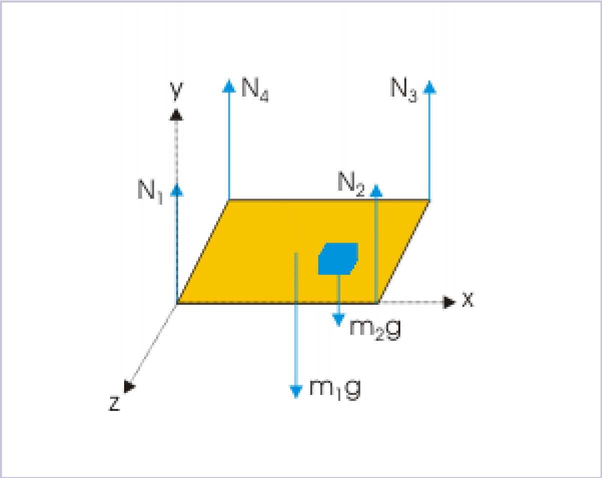
$$\Rightarrow N_1 + 20 - 2X10 - 4X10 = 0$$

$$\Rightarrow N_1 = 40 \quad N$$

Indeterminate condition of equilibrium

We had made the reference that equilibrium of a book on the table can not be analyzed for four unknown normal forces at the corners of the table. We shall have a closer look at the problem now. The figure below shows the forces on a table top on which a book lies in stable equilibrium.

A table top and a book



Forces on the table top.

The forces are non-planar, but directed in one direction i.e y – direction. We can, therefore, have only one meaningful force balance equation,

$$\Sigma F_y = N_1 + N_2 + N_3 + N_4 - m_1 g - m_2 g = 0$$

The forces in y-direction can have tendency to rotate table top either around x-axis or z-axis, which are perpendicular directions to moment arm and forces involved here. As such we can have two torque balance equations in these directions. If table top be a square of side “a” and the coordinates of the book be x,z in “xz” - plane, then :

$$\Sigma \tau_z = 0 X N_1 + a X N_2 + a X N_3 + 0 X N_4 - \frac{a}{2} X m_1 g - x X m_2 g$$

$$\Sigma \tau_x = 0 X N_1 + 0 X N_2 + a X N_3 + a X N_4 - \frac{a}{2} X m_1 g - z X m_2 g$$

Thus, we see that it is not possible to solve these three equations of static equilibrium to find four unknown normal forces. Either we need to measure one of the four normal forces physically with some device like a balance attached to the leg of the table or we measure certain properties of the material of the rigid body involved (elastic deformation, strain or stress etc.) that relate the properties to the normal force.

Momentum and equilibrium

The equilibrium of a body can also be stated in terms of linear and angular momentum. The conditions of equilibrium in terms of external force and torque are :

$$\Sigma\mathbf{F} = 0$$

$$\Sigma\boldsymbol{\tau} = 0$$

Since force is time rate of change of linear momentum, we can restate the force condition of equilibrium. Here,

$$\Rightarrow \Sigma F = \frac{d\mathbf{P}_C}{dt} = 0$$

$$\Rightarrow d\mathbf{P}_C = 0$$

$$\Rightarrow \mathbf{P}_C = \text{constant}$$

Similarly torque is time rate of change of angular momentum, we have :

$$\Rightarrow \Sigma\tau = \frac{d\mathbf{L}}{dt} = 0$$

$$\Rightarrow d\mathbf{L} = 0$$

$$\Rightarrow \mathbf{L} = \text{constant}$$

In static equilibrium, not only the net external force and torque are zero, but velocities are also zero in the inertial frame of reference of measurement. Since linear momentum is product of mass and velocity; and angular momentum is product of moment of inertia and angular velocity, the conditions of static equilibrium, in an inertial frame of reference, are given by :

$$\Rightarrow \mathbf{P}_C = m\mathbf{v}_C = 0$$

$$\Rightarrow \mathbf{L} = I\boldsymbol{\omega}_C = 0$$

Nature of equilibrium

We encounter different kinds of equilibrium. Take the case of a tennis ball and a paper weight, which are placed on a table. A slight force on tennis ball makes it to roll and finally drop off the table, whereas paper weight hardly moves from its position on the application of same force. Two bodies, therefore, are equilibrium of different nature. In this module, we shall not go for details of every aspects of equilibrium; but will limit ourselves to broader classification, which is based on energy concept.

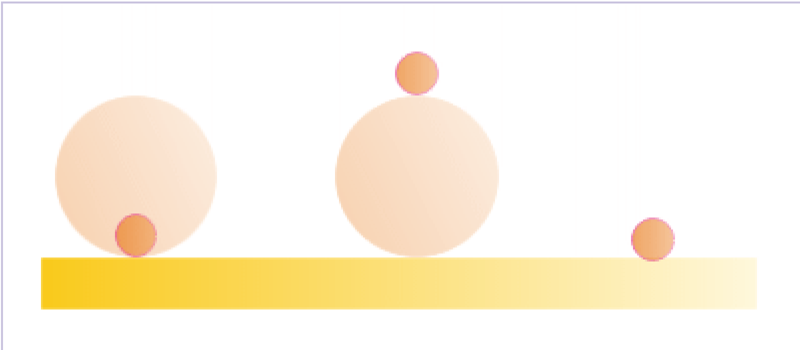
We shall further limit the analysis of equilibrium to the context of equilibrium in conservative force field like gravity. A body in conservative force field possesses mechanical energy in the form of kinetic and potential energy. Potential energy, as we know, is exclusively defined for conservative force field and is a function of position. In this module, we shall attempt to correlate potential energy with the nature of equilibrium in following categories :

- Stable
- Unstable
- Neutral

Disturbance and equilibrium

In order to appreciate the role of potential energy about equilibrium, we first need to visualize equilibrium in the light of external disturbance. We should keep this in mind that disturbance that we talk about is a relatively small force.

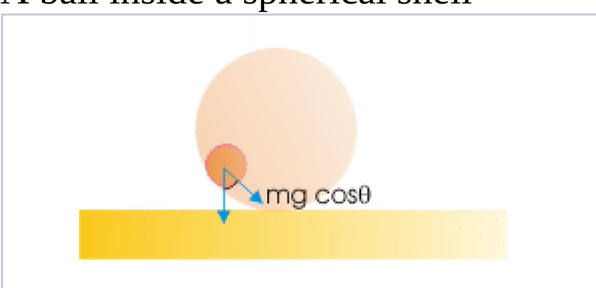
A typical set of example to illustrate the nature of equilibrium consist of three settings of a small ball (i) inside a spherical shell (ii) over the top of a sphere/ spherical shell and (iii) over a horizontal surface. These three settings are shown in the figure.
Nature of equilibrium



The equilibrium of a small spherical ball in three different positions.

What do we expect when the ball inside the shell is slightly disturbed to its left. A component of gravity acts to decelerate the motion; brings the ball to a stop; and then accelerates the ball back to its original position and beyond. Restoration by gravity continues till the ball is static at the original position, depending upon the friction. The equilibrium of the ball inside the shell is “stable” equilibrium as it is unable to move out of its setting. The identifying nature of this equilibrium is that a restoring force comes into picture to restore the position of the object.

A ball inside a spherical shell



Force on the small spherical ball.

Let us now consider the second case in which the ball is placed over the shell. We can easily visualize that it is difficult to achieve this equilibrium

in the first place. Secondly, when the ball is disturbed with a smallest touch, it starts falling down. The gravity here plays a different role altogether. It aids in destabilizing the equilibrium by pulling the ball down. The equilibrium of the ball over the top of the sphere is called “unstable” equilibrium. The identifying nature of this equilibrium is that once equilibrium ends, there is no returning back to original position as there is no restoring mechanism available.

In the last case, when ball is disturbed, it moves on the horizontal surface. If the surface is smooth it maintains the small velocity so imparted. The distinguishing aspect is that component of gravity in horizontal direction (direction of motion) is zero. As such gravity neither plays the role of restoring force nor that of an aid to the disturbance. The equilibrium of the ball on the horizontal surface is called “neutral” equilibrium. The identifying nature of this equilibrium is that once equilibrium ends, there is neither the tendency of returning back nor the tendency of moving away from the original position.

Potential energy and equilibrium

We are interested here to establish the characterizing features of equilibrium. As such, we shall keep our discussion limited to one dimension and attempt to find the required correlation.

For the conservative force system, force is related to potential energy as :

$$F = -\frac{dU}{dx}$$

For translational equilibrium,

$$F = -\frac{dU}{dx} = 0$$

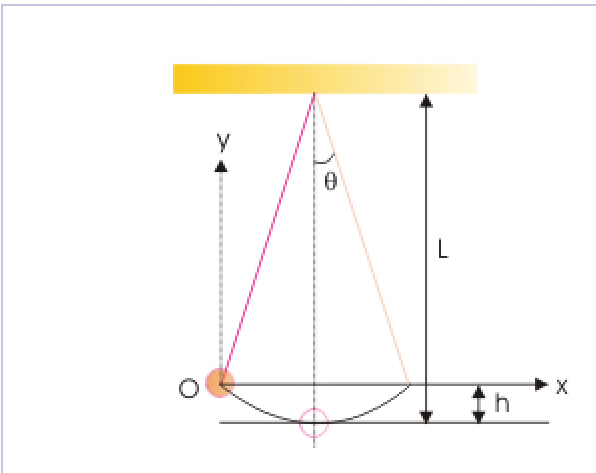
This relationship can be used to interpret equilibrium, if we have values of potential energy function, U(x), with respect to displacement “x”. A plot of

U(x) .vs. x will indicate position of equilibrium, where tangent to the plot is parallel to x-axis so that slope of the curve is zero at that point.

Stable equilibrium

In order to correlate, potential energy with stable equilibrium, we draw an indicative potential energy plot of a pendulum bob, which is displaced through a maximum angle of 15. Let pendulum of length, L = 1 m and mass of pendulum bob (of high density material) = 10 kg. The maximum potential energy corresponds to maximum angular displacement.

Pendulum



A pendulum bob is given angular displacement and released.

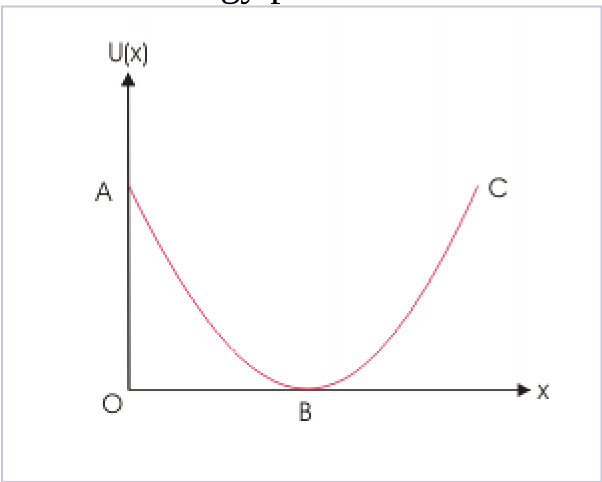
$$U_{\max} = mgh = mgL(1 - \cos\theta)$$

$$\Rightarrow U_{\max} = 10X10X1\left(1 - \cos 15^0\right) = 100X(1 - 0.966) = 3.4 \quad J$$

The maximum potential energy is equal to maximum mechanical energy that the bob can have in the setup. The important thing to note about the

plot is that we measure “x” from point “O” – from the extreme point in the left. The indicative plot is shown here :

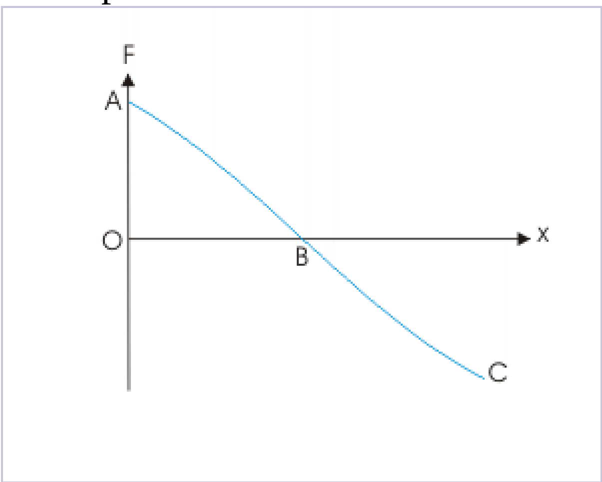
Potential energy plot



Potential energy is plotted against displacement.

Further, as force is negative of the slope of the potential energy curve, it is first positive when slope of potential energy curve is negative; negative when slope is positive. An indicative force - displacement plot corresponding to potential energy curve is shown here.

Force plot



Force on the pendulum bob is

plotted against displacement.

These two pairs of plot let us analyze the equilibrium of pendulum bob. For this, we consider motion of the bob towards left from its mean position. The bob gains potential energy at the expense of kinetic energy. Further, the bob has negative velocity as it is moving in opposite to the reference direction. From "F-x" plot, we see that force is acting in positive x-direction. This means that velocity and force are in opposite direction. As such, pendulum bob is decelerated. Ultimately bob comes to a stop and then reverses direction towards point “B”. In this reverse journey, it acquires kinetic energy at the expense of loosing potential energy.

Once past “B” in opposite direction i.e towards right, it again gains potential energy at the expense of kinetic energy. It has positive velocity as it is moving in the reference direction. From "F-x" plot, we see that force is acting in negative x-direction. The velocity and force are in opposite direction in this side of motion also. As such, pendulum bob is again decelerated. Ultimately bob comes to a stop and then reverses direction towards point “B”. In this reverse journey, it acquires kinetic energy at the expense of loosing potential energy.

For the given set up, maximum mechanical energy corresponds to position $\theta = 15°$. What it means that pendulum never crosses the points “A” and “B”. These points, therefore, are the “turning points” for the given maximum mechanical energy of the set up (=3.4 J).

We, therefore conclude that the equilibrium of pendulum bob is a “stable” equilibrium at “B” within maximum angular displacement. If we give a small disturbance to the bob, its motion is bounded by the energy imparted during the disturbance. As the kinetic energy imparted is less than 3.4 J, it does not escape beyond the turning points specified for the set up. We should note that this situation with respect to pendulum bob is similar to the spherical ball placed inside a spherical shell.

From the discussion so far, we conclude that :

1: First derivative of potential energy function with respect to displacement is zero.

$$\frac{dU}{dx} = 0$$

2: Potential energy of the body is minimum for stable equilibrium for a given potential energy function and maximum allowable mechanical energy.

$$U(x) = U_{\text{min}}$$

For this, the second derivative of potential energy function is positive at equilibrium point,

$$\frac{d^2U}{dx^2} > 0$$

3: The position of stable equilibrium is bounded by two turning points corresponding to maximum allowable mechanical energy.

4: Force acts to restore the original position of the body in stable equilibrium.

Example

Problem 1 : The potential energy of a particle in a conservative system is given by the potential energy function,

$$U(x) = \frac{1}{2}ax^2 - bx$$

where “a” and “b” are two positive constants. Find the equilibrium position and determine the nature of equilibrium.

Solution : The system here is a conservative system. This means that only conservative forces are in operation. In order to determine the equilibrium

position, we make use of two facts (i) negative of first differential of potential energy function gives the net force on the system and (ii) for equilibrium, net external force is zero.

$$\Rightarrow \frac{dU}{dx} = \frac{1}{2} X 2ax - b$$

For equilibrium, external force is zero,

$$F = -\frac{dU}{dx} = -\left(\frac{1}{2} X 2ax - b\right) = 0$$

$$\Rightarrow x = \frac{b}{a}$$

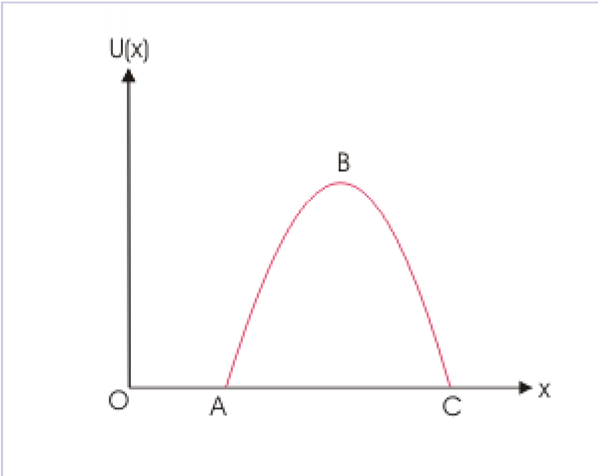
Further, we need to find the second derivative of potential energy function and investigate the resulting value.

$$\Rightarrow \frac{d^2U}{dx^2} = a$$

It is given that “a” is a positive constant. It means that the particle possess minimum potential energy at $x = \frac{b}{a}$. Hence, particle is having stable equilibrium at this position.

Unstable equilibrium

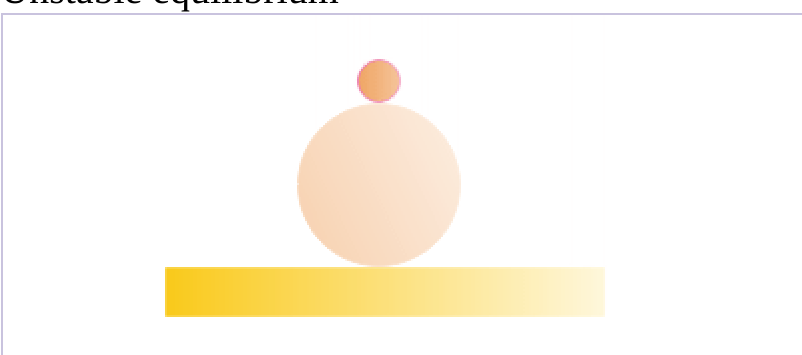
The nature of the potential energy plot for unstable equilibrium is inverse to that of stable equilibrium curve. A typical plot is shown here in the figure.
Unstable equilibrium



Potential energy is plotted against displacement.

The position of equilibrium at point “B”, where slope of the curve is zero corresponds to maximum potential energy, which in turn is equal to maximum mechanical energy allowable for the body. We can refer this plot to the case of a ball placed over a spherical shell as shown below.

Unstable equilibrium

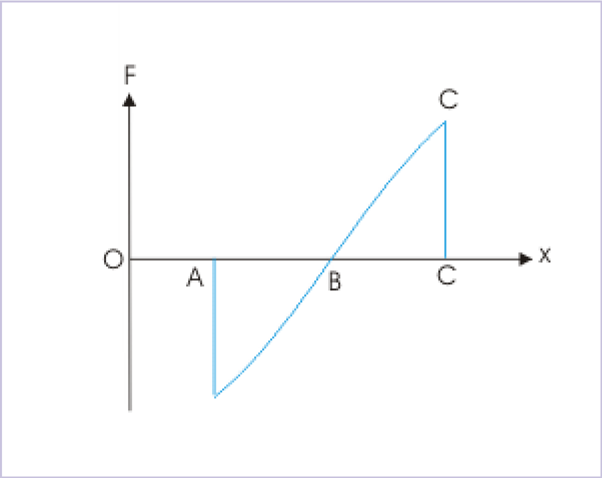


A small spherical ball is placed over a spherical shell.

It is easy to realize that the ball has maximum potential energy at the top as is shown in potential energy plot. Further, as force is negative of the slope

of the potential energy curve, it is first negative, when slope of potential energy curve is negative; negative, when slope is positive. An indicative force - displacement plot corresponding to potential energy curve is shown here.

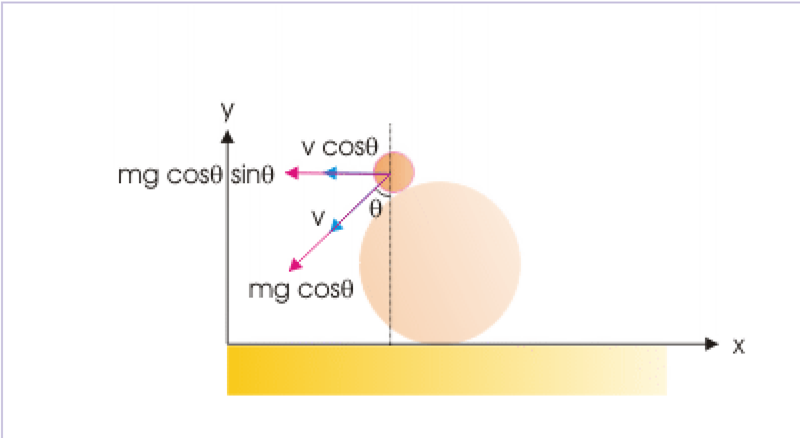
Unstable equilibrium



Force on the spherical ball is plotted against displacement.

The two pairs of plots let us analyze the equilibrium of ball, placed on the shell. For this we consider motion of the ball towards left. The ball gains kinetic energy at the expense of potential energy. The projection of velocity in x-direction is negative. From the figure below, we see that component of gravitational force is constrained to be tangential to sphere. Its component in x-direction is also acting in the negative direction.

Unstable equilibrium



Components of velocity and force in along x - direction.

Thus, both components of velocity and acceleration are in the opposite direction to the reference x-axis direction. As such, the spherical ball is accelerated and keeps gaining kinetic energy without any possibility of restoration to original energy state. Ultimately, the ball lands on the horizontal surface, which we have considered to be at zero reference potential. There is no component of gravitational force in horizontal direction on the surface. As such, it is stopped after some distance due to friction or keeps moving with uniform velocity if surface is smooth.

Similar is the case, when ball moves to the right from its equilibrium position.

From the discussion so far, we conclude that :

1: First derivative of potential energy function with respect to displacement is zero.

$$\frac{dU}{dx} = 0$$

2: Potential energy of the body is maximum for stable equilibrium for a given potential energy function and maximum allowable mechanical

energy.

$$U(x) = U_{\max}$$

For this, the second derivative of potential energy function is negative at equilibrium point,

$$\frac{d^2U}{dx^2} < 0$$

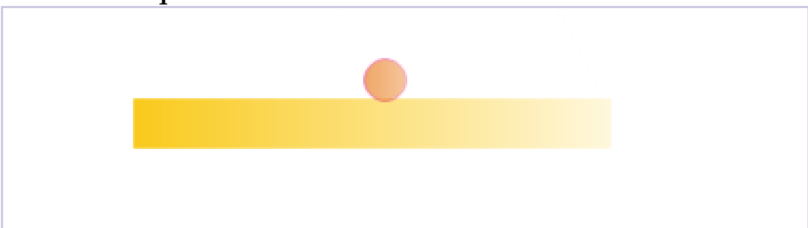
3: There is no bounding pairs of turning points like in the case of stable equilibrium.

4: There is no restoring force the body. External force (gravity) aids in acquiring kinetic energy by the body.

Neutral equilibrium

The nature of the potential energy plot for neutral equilibrium is easy to visualize. Let us consider equilibrium of the ball, which is lying on horizontal surface. If we consider the horizontal surface to be the zero reference potential level, then potential energy plot is simply the x-axis itself; if not, it is a straight line parallel to x-axis.

Neutral equilibrium



A small spherical ball is placed on a horizontal surface.

On the other hand, Force – displacement plot is essentially x-axis as component of gravity in horizontal direction is zero. When ball is disturbed from its position, it merely moves till friction stops it. If the surface is smooth, ball keeps moving with the velocity imparted during disturbance.

From the discussion so far, we conclude that :

1: First derivative of potential energy function with respect to displacement is zero.

$$\frac{dU}{dx} = 0$$

2: The second derivative of potential energy function is equal to zero.

$$\frac{d^2U}{dx^2} = 0$$

3: There is no bounding pairs of turning points like in the case of stable equilibrium.

4: There is no restoring force on the body . External force (gravity) neither acts to restore the body or aid in acquiring kinetic energy by the body.

Gravitation

Gravitation is an inherent property of all matter. Two bodies attract each other by virtue of their mass. This force between two bodies of any size (an atom or a galaxy) signifies existence of matter and is known as gravitational force.

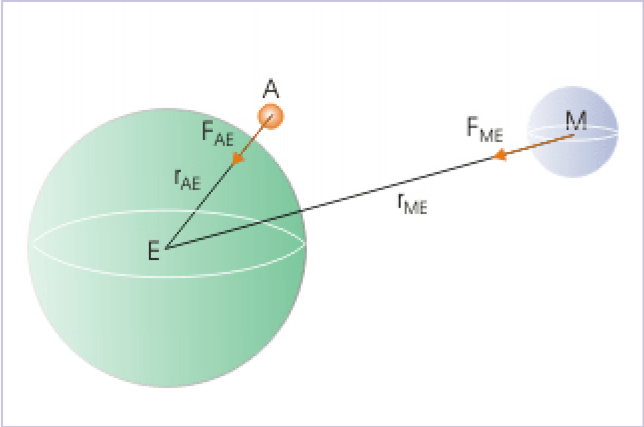
Gravitational force is weakest of four fundamental forces. It is, therefore, experienced only when at least one of two bodies has considerable mass. This presents difficulties in setting up illustrations with terrestrial objects. On the other hand, gravitation is the force that sets up our universe and governs motions of all celestial bodies. Orbital motions of satellites – both natural and artificial – are governed by gravitational force.

Newton derived a law to quantify gravitational force between two "particles". The famous incidence of an apple falling from a tree stimulated Newton's mind to analyze observations and carry out series of calculations that finally led him to propose universal law of gravitation. A possible sequence of reasoning, leading to the postulation is given here :

1: The same force of attraction works between "Earth (E) and an apple (A)" and between "Earth (E) and Moon (M)".

2: From the analysis of data available at that time, he observed that the ratio of forces of attraction for the above two pairs is equal to the ratio of square of distance involved as :

Gravitational force



Earth - moon - apple system

$$\frac{F_{ME}}{F_{AE}} = \frac{r_{AE^2}}{r_{ME^2}}$$

3: From above relation, Newton concluded that the force of attraction between any pair of two bodies is inversely proportional to the square of linear distance between them.

$$F \propto \frac{1}{r^2}$$

4: From second law of motion, force is proportional to mass of the body being subjected to gravitational force. From third law of motion, forces exist in equal and opposite pair. Hence, gravitational force is also proportional to the mass of other body. Newton concluded that force of gravitation is proportional to the product of mass of two bodies.

$$F \propto m_1 m_2$$

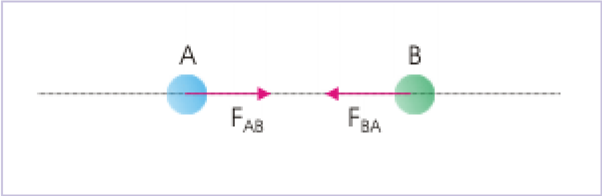
5: Combining two “proportional” equations and introducing a constant of proportionality,"G", Newton proposed the gravitational law as :

$$\Rightarrow F = \frac{G m_1 m_2}{r^2}$$

In order to emphasize the universal character of gravitational force, the constant “G” is known as “Universal gravitational constant”. Its value is :

$$G = 6.67X10^{-11} \quad N - m^2/kg^2$$

This law was formulated based on the observations on real bodies - small (apple) and big (Earth and moon). However, Newton’s law of gravitation is stated strictly for two particles. The force pair acts between two particles, along the line joining their positions as shown in the figure :
Gravitational force



Force between two particles placed at a distance

Extension of this law to real bodies like Earth and Moon pair can be understood as bodies are separated by large distance (about $0.4X10^6$ km), compared to dimension of bodies (in thousands km). Two bodies, therefore, can be treated as particles.

On the other hand, Earth can not be treated as particle for “Earth and Apple” pair, based on the reasoning of large distance. Apple is right on the surface of Earth. Newton proved a theorem that every spherical shell (hollow sphere) behaves like a particle for a particle external to it. Extending this theorem, Earth, being continuous composition of infinite numbers of shells of different radii, behaves - as a whole - like a particle for external object like an apple.

As a matter of fact, we will prove this theorem, employing gravitational field concept for a spherical mass like that of Earth. For the time being, we consider Earth and apple as particles, based on the Newton’s shell theory. In that case, the distance between Apple and center of Earth is equal to the radius of Earth i.e 6400 km.

Magnitude of force

The magnitude of gravitational force between terrestrial objects is too small to experience. A general question that arises in the mind of a beginner is “why do not we experience this force between, say, a book and pencil?” The underlying fact is that gravitational force is indeed a very small force for masses that we deal with in our immediate surrounding - except Earth.

We can appreciate this fact by calculating force of gravitation between two particle masses of 1 kg each, which are 1 m apart :

$$\Rightarrow F = \frac{6.67X10^{-11}X1X1}{1^2} = 6.67X10^{-11} \quad N$$

This is too insignificant a force to manifest against bigger forces like force of gravitation due to Earth, friction, force due to atmospheric pressure, wind etc.

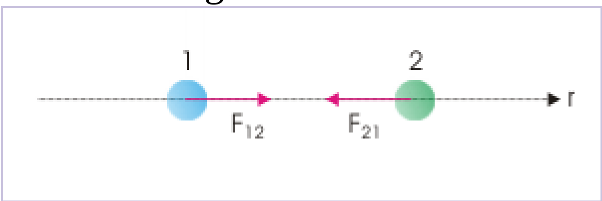
Evidently, this is the small value of "G", which renders force of gravitation so small for terrestrial objects. Gravitation plays visible and significant role, where masses are significant like that of planets including our Earth, stars and such other massive aggregation, including "black holes" with extraordinary gravitational force to hold back even light. This is the reason, we experience gravitational force of Earth, but we do not experience gravitational force due to a building or any such structures on Earth.

Gravitational force vector

Newton's law of gravitation provides with expression of gravitational force between two bodies. Here, gravitational force is a vector. However, force vector is expressed in terms of quantities, which are not vectors. The linear distance between two masses, appearing in the denominator of the expression, can have either of two directions from one to another point mass.

Even if, we refer the linear distance between two particles to a reference direction, the vector appears in the denominator and is, then, squared also. In order to express gravitational force in vector form, therefore, we shall consider a unit vector in the reference direction and use the same to denote the direction of force as:

Direction of gravitational force



Force between two particles placed at a distance

$$\mathbf{F}_{12} = \frac{Gm_1 m_2 \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{F}_{21} = -\frac{Gm_1 m_2 \hat{\mathbf{r}}}{r^2}$$

Note that we need to put a negative sign before the second expression to make the direction consistent with the direction of gravitational force of attraction. We can easily infer that sign in the expression actually depends on the choice of reference direction.

Net gravitational force

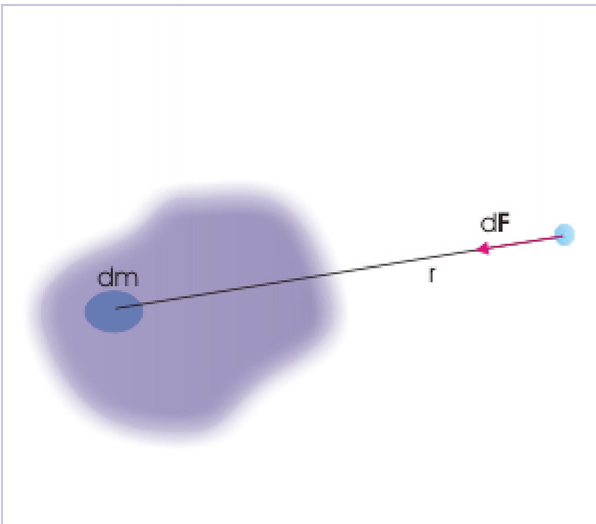
Gravitation force is a vector quantity. The net force of gravitation on a particle is equal to resultant of forces due to all other particles. This is also known as "superposition principle", according to which net effect is sum of individual effects. Mathematically,

$$\Rightarrow \mathbf{F} = \Sigma \mathbf{F}_i$$

Here, **F** is the net force due to other particles 1, 2, 3, and so on.

An extended body is considered to be continuous aggregation of elements, which can be treated as particles. This fact can be represented by an integral of all elemental forces due to all such elements of a body, which are treated as particles. The force on a particle due to an extended body, therefore, can be computed as :

Net gravitational force



Net gravitational force is vector
sum of individual gravitations
due to particle like masses.

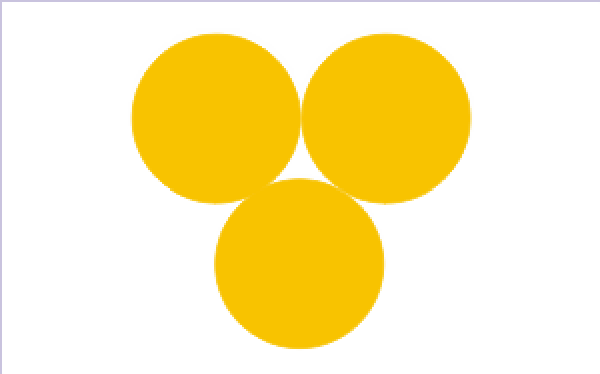
$$\mathbf{F} = \int d\mathbf{F}$$

where integration is evaluated to include all mass of a body.

Examples

Problem 1: Three identical spheres of mass “M” and radius “R” are assembled to be in contact with each other. Find gravitational force on any of the sphere due to remaining two spheres. Consider no external gravitational force exists.

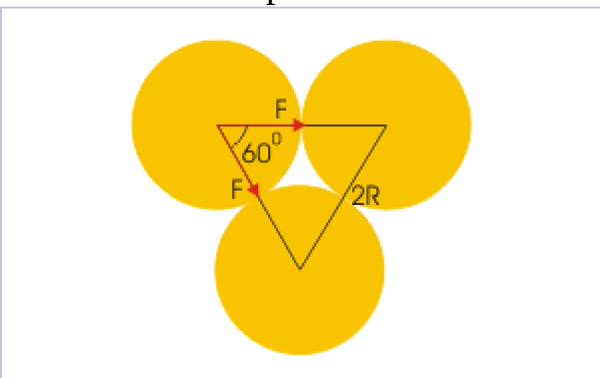
Three identical sphered in contact



Three identical spheres of mass "M" and radius "R" are assembled in contact with each other.

Solution : The gravitational forces due to pairs of any two speres are equal in magnitude, making an angle of 60° with each other. The resultant force is :

Three identical sphered in contact



Each sphere is attracted by other two spheres.

$$\Rightarrow R = \sqrt{\left(F^2 + F^2 + 2F^2 \cos 60^0\right)}$$

$$\Rightarrow R = \sqrt{\left(2F^2 + 2F^2 X \frac{1}{2}\right)}$$

$$\Rightarrow R = \sqrt{3}F$$

Now, the distance between centers of mass of any pair of spheres is “2R”. The gravitational force is :

$$F = \frac{GM^2}{(2R)^2} = \frac{GM^2}{4R^2}$$

Therefore, the resultant force on a sphere is :

$$F = \frac{\sqrt{3}GM^2}{4R^2}$$

Problem 2: Two identical spheres of uniform density are in contact. Show that gravitational force is proportional to the fourth power of radius of either sphere.

Solution : The gravitational force between two spheres is :

$$F = \frac{Gm^2}{(2r)^2}$$

Now, mass of each of the uniform sphere is :

$$m = \frac{4\pi r^3 X\rho}{3}$$

Putting this expression in the expression of force, we have :

$$\Rightarrow F = \frac{GX16\pi^2 r^6 \rho^2}{9X4r^2} = \frac{Gx16\pi^2 r^4 \rho^2}{36}$$

Since all other quantities are constants, including density, we conclude that gravitational force is proportional to the fourth power of radius of either sphere,

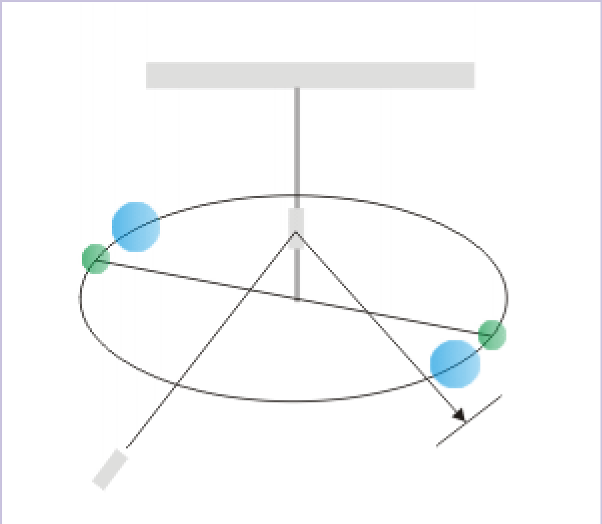
$$\Rightarrow F \propto r^4$$

Measurement of universal gravitational constant

The universal gravitational constant was first measured by Cavendish. The measurement was an important achievement in the sense that it could measure small value of “G” quite accurately.

The arrangement consists of two identical small spheres, each of mass “m”. They are attached to a light rod in the form of a dumb-bell. The rod is suspended by a quartz wire of known coefficient of torsion “k” such that rod lies in horizontal plane. A mirror is attached to quartz wire, which reflects a light beam falling on it. The reflected light beam is read on a scale. The beam, mirror and scale are all arranged in one plane.

Cavendish experiment



Measurement of universal gravitational constant

The rod is first made to suspend freely and stabilize at an equilibrium position. As no net force acts in the horizontal direction, the rod should rest in a position without any torsion in the quartz string. The position of the reflected light on the scale is noted. This reading corresponds to neutral position, when no horizontal force acts on the rod. The component of Earth's gravitation is vertical. Its horizontal component is zero. Therefore, it is important to keep the plane of rotation horizontal to eliminate effect of Earth's gravitation.

Two large and heavier spheres are, then brought across, close to smaller sphere such that centers of all spheres lie on a circle as shown in the figure above. The gravitational forces due to each pair of small and big mass, are perpendicular to the rod and opposite in direction. Two equal and opposite force constitutes a couple, which is given by :

$$\tau_G = F_G L$$

where "L" is the length of the rod.

The couple caused by gravitational force is balanced by the torsion in the quartz string. The torque is proportional to angle "θ" through which the rod rotates about vertical axis.

$$\tau_T = k\theta$$

The position of the reflected light is noted on the scale for the equilibrium. In this condition of equilibrium,

$$\Rightarrow F_G L = k\theta$$

Now, the expression of Newton's law of gravitation for the gravitational force is:

$$F_G = \frac{GMm}{r^2}$$

where "m" and "M" are mass of small and big spheres. Putting this in the equilibrium equation, we have :

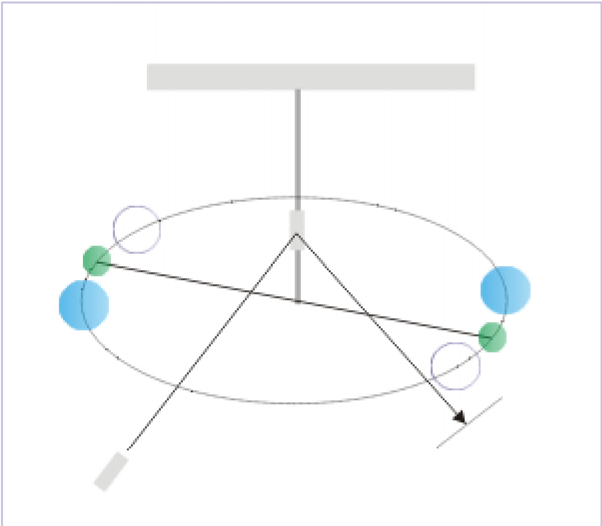
$$\Rightarrow \frac{GMmL}{r^2} = k\theta$$

Solving for “G”, we have :

$$\Rightarrow G = \frac{r^2 k\theta}{MmL}$$

In order to improve accuracy of measurement, the bigger spheres are, then, placed on the opposite sides of the smaller spheres with respect to earlier positions (as shown in the figure below). Again, position of reflected light is noted on the scale for equilibrium position, which should lie opposite to earlier reading about the reading corresponding to neutral position.

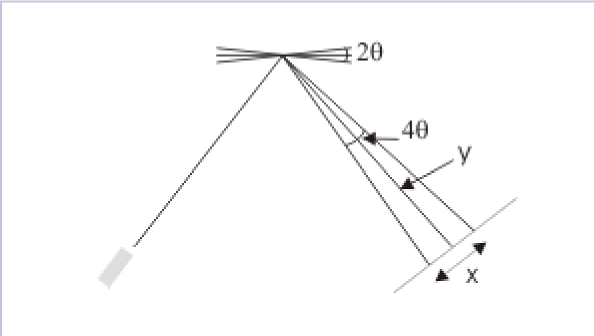
Cavendish experiment



Measurement of universal gravitational constant

The difference in the readings (x) on the scale for two configurations of larger spheres is read. The distance between mirror and scale (y) is also determined. The angle subtended by the arc “x” at the mirror is twice the angle through which mirror rotates between two configurations. Hence,

Cavendish experiment



Measurement of angle

$$\Rightarrow 4\theta = \frac{x}{y}$$

$$\Rightarrow \theta = \frac{x}{4y}$$

We see here that beam, mirror and scale arrangement enables us to read an angle, which is 2 times larger than the actual angle involved. This improves accuracy of the measurement. Putting the expression of angle, we have the final expression for determination of “G”,

$$G = \frac{r^2 kx}{4MmLy}$$

Gravity

The term “gravity” is used for the gravitation between two bodies, one of which is Earth.

Earth is composed of layers, having different densities and as such is not uniform. Its density varies from 2 kg/m^3 for crust to nearly 14 kg/m^3 for the inner core. However, inner differentiation with respect to mass is radial and not directional. This means that there is no preferential direction in which mass is aggregated more than other regions. Applying Newton’s shell theorem, we can see that Earth, if considered as a solid sphere, should behave as a point mass for any point on its surface or above it.

In the nutshell, we can conclude that density difference is not relevant for a point on the surface or above it so long Earth can be considered spherical and density variation is radial and not directional. As this is approximately the case, we can treat Earth, equivalently as a sphere of uniform mass distribution, having an equivalent uniform (constant) density. Thus, force of gravitation on a particle on the surface of Earth is given by :

$$F = \frac{GMm}{R^2}$$

where “M” and “m” represents masses of Earth and particle respectively. For any consideration on Earth’s surface, the linear distance between Earth and particle is constant and is equal to the radius of Earth (R).

Gravitational acceleration (acceleration due to gravity)

In accordance with Newton’s second law of motion, gravity produces acceleration in the particle, which is situated on the surface. The acceleration of a particle mass “m’, on the surface of Earth is obtained as :

$$\Rightarrow a = \frac{F}{m} = \frac{GM}{R^2}$$

The value corresponding to above expression constitutes the reference gravitational acceleration. However, the calculation of gravitational acceleration based on this formula would be idealized. The measured value of gravitational acceleration on the surface is different. The measured value of acceleration

incorporates the effects of factors that we have overlooked in this theoretical derivation of gravitational acceleration on Earth.

We generally distinguish gravitational acceleration as calculated by above formula as " g_0 " to differentiate it from the one, which is actually measured(g) on the surface of Earth. Hence,

$$g_0 = a = \frac{F}{m} = \frac{GM}{R^2}$$

This is a very significant and quite remarkable relationship. The gravitational acceleration does not dependent on the mass of the body on which force is acting! This is a special characteristic of gravitational force. For all other forces, acceleration depends on the mass of the body on which force is acting. We can easily see the reason. The mass of the body appears in both Newton's law of motion and Newton's law of gravitation. Hence, they cancel out, when two equations are equated.

Factors affecting Gravitational acceleration

The formulation for gravitational acceleration considers Earth as (i) uniform (ii) spherical and (iii) stationary body. None of these assumptions is true. As such, measured value of acceleration (g) is different to gravitational acceleration, " g_0 ", on these counts :

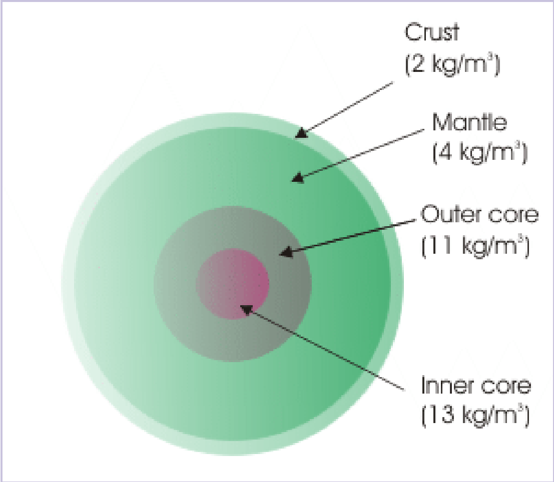
1. Constitution of the Earth
2. Shape of the Earth
3. Rotation of the Earth

In addition to these inherent factors resulting from the consequence of "real" Earth, the measured value of acceleration also depends on the point of measurement in vertical direction with respect to mean surface level or any reference for which gravitational acceleration is averaged. Hence, we add one more additional factor responsible for variation in the gravitational acceleration. The fourth additional factor is relative vertical position of measurement with respect to Earth's surface.

Constitution of the Earth

Earth is not uniform. Its density varies as we move from its center to the surface. In general, Earth can be approximated to be composed of concentric shells of different densities. For all practical purpose, we consider that the density gradation is radial and is approximated to have an equivalent uniform density within these concentric shells in all directions.

Constitution of the Earth



Density of Earth varies radially.

The main reason for this directional uniformity is that bulk of the material constituting Earth is fluid due to high temperature. The material, therefore, has a tendency to maintain uniform density in a given shell so conceived.

We have discussed that the radial density variation has no effect on a point on the surface or above it. This variation of density, however, impacts gravitational acceleration, when the point in the question is at a point below Earth's surface.

In order to understand the effect, let us have a look at the expression of gravitational expression :

$$g_0 = \frac{F}{m} = \frac{GM}{R^2}$$

The impact of moving down below the surface of Earth, therefore, depends on two factors

1. Mass (M) and

2. Distance from the center of Earth (r)

A point inside a deep mine shaft, for example, will result in a change in the value of gravitational acceleration due to above two factors.

We shall know subsequently that gravitational force inside a spherical shell is zero. Therefore, mass of the spherical shell above the given point does not contribute to gravitational force and hence acceleration at that point. Thus, the value of "M" in the expression of gravitational acceleration decreases as we go down from the Earth's surface. This, in turn, decreases gravitational acceleration at a point below Earth's surface. At the same time, the distance to the center of Earth decreases. This factor, in turn, increases gravitational acceleration.

If we assume uniform density, then the impact of "decrease in mass" is greater than that of impact of "decrease in distance". We shall prove this subsequently when we consider the effect of vertical position. As such, acceleration is expected to decrease as we go down from Earth's surface.

In reality the density is not uniform. Crust being relatively light and thin, the impact of first factor i.e. "decrease in mass" is less significant initially and consequently gravitational acceleration actually increases initially for some distance as we go down till it reaches a maximum value at certain point below Earth's surface. For most of depth beyond, however, gravitational acceleration decreases with depth.

Shape of the Earth

Earth is not a sphere. It is an ellipsoid. Its equatorial radius is greater than polar radius by 21 km. A point at pole is closer to the center of Earth. Consequently, gravitational acceleration is greater there than at the equator.

Besides, some part of Earth is protruded and some part is depressed below average level. Once again, factors of mass and distance come into picture. Again, it is the relative impact of two factors that determine the net effect. Consider a point right at the top of Mt. Everest, which is about 8.8 km from the mean sea level. Imagine incrementing radius of Earth's sphere by 8.8 km. Most of the volume so created is not filled. The proportionate increase in mass (mass of Everest mountain range) is less than that in the squared distance from the center of Earth. As such, gravitational acceleration is less than its average value on the surface. It is actually

9.80 m/s^2 as against the average of 9.81 m/s^2 , which is considered to be the accepted value for the Earth's surface.

Rotation of Earth

Earth rotates once about its axis of rotation in 1 day and moves around Sun in 365 days. Since Earth and a particle on Earth both move together with a constant speed around Sun, there is no effect in the measured acceleration due to gravity on the account of Earth's translational motion. The curved path around Sun can be approximated to be linear for distances under consideration. Hence, Earth can serve as inertial frame of reference for the application of Newton's law of motion, irrespective of its translational motion.

However, consideration of rotation of Earth about its axis changes the nature of Earth's reference. It is no more an inertial frame. A particle at a point, "P", is rotating about the axis of rotation. Clearly, a provision for the centripetal force should exist to meet the requirement of circular motion. We should emphasize here that centripetal force is not an additional force by itself, but is a requirement of circular motion, which should be met with by the forces operating on the particle. Here, gravitational force meets this requirement and, therefore, gets modified to that extent.

Here, we shall restrict our consideration specifically to the effect of rotation. We will ignore other factors that affect gravitational acceleration. This means that we consider Earth is a solid uniform sphere. If it is so then, measured value of acceleration is equal to reference gravitational acceleration (g_0) as modified by rotation.

As we have studied earlier, we can apply Newton's law in a non-inertial reference by providing for pseudo force. We should recall that pseudo force is applied in the direction opposite to the direction of acceleration of the frame of reference, which is centripetal acceleration in this case. The magnitude of pseudo force is equal to the product of mass of the particle and centripetal acceleration. Thus,

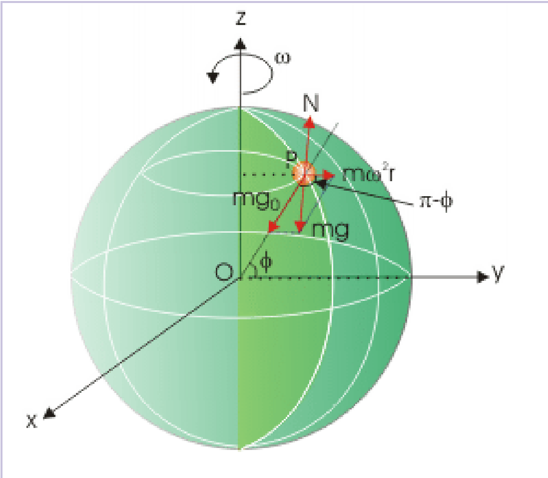
$$F_P = m\omega^2 r$$

After considering pseudo force, we can enumerate forces on the particle at "P" at an latitude "φ" as shown in the figure :

1. Pseudo force ($m\omega^2 r$)

2. Normal force (N)
3. gravitational force (mg_0)

Forces on the particle on the surface of Earth



The particle is at rest under action of three forces.

The particle is subjected to normal force against the net force on the particle or the weight as measured. Two forces are equal in magnitude, but opposite in direction.

$$N = W = mg$$

It is worthwhile to note here that gravitational force and normal force are different quantities. The measured weight of the particle is equal to the product of mass and the measured acceleration. It is given by the expression "mg". On the other hand, Gravitational force is given by the Newton's equation, which considers Earth at rest. It is equal to " mg_0 ".

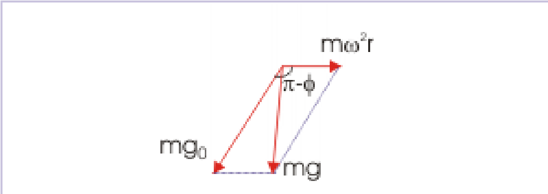
$$\Rightarrow F_G = \frac{GM}{R^2} = mg_0$$

Since particle is stationary on the surface of Earth, three forces as enumerated above constitute a balanced force system. Equivalently, we can say that the resultant of pseudo and gravitational forces is equal in magnitude, but opposite in direction to the normal force.

$$\mathbf{N} = \mathbf{F}_G + \mathbf{F}_P$$

In other words, resultant of gravitational and pseudo forces is equal to the magnitude of measured weight of the particle. Applying parallelogram theorem for vector addition of gravitational and pseudo forces, the resultant of the two forces is :

Forces on the particle on the surface of Earth



The particle is at rest under action of three forces.

$$N^2 = (mg_0)^2 + \left(m\omega^2 R\right)^2 + 2m^2\omega^2 g_0 R\cos(180 - \varphi)$$

Putting N = mg and rearranging, we have:

$$\Rightarrow m^2g^2 = m^2g_0^2 + m^2\omega^4R^2 - 2m^2\omega^2 g_0 R\cos\varphi$$

Angular velocity of Earth is quite a small value. It may be interesting to know the value of the term having higher power of angular velocity. Since Earth completes one revolution in a day i.e an angle of "2π" in 24 hrs, the angular speed of Earth is :

$$\Rightarrow \omega = \frac{2\pi}{24X60X60} = 7.28X10^{-5} \quad \text{rad}/s$$

The fourth power of angular speed is almost a zero value :

$$\Rightarrow \omega^4 = 2.8X10^{-17}$$

We can, therefore, safely neglect the term " $m^2\omega^4r^2$ ". The expression for the measured weight of the particle, therefore, reduces to :

$$\Rightarrow g^2 = g_0^2 - 2\omega^2 g_0 R\cos\varphi$$

$$\Rightarrow g = g_0\left(1 - \frac{2\omega^2 R\cos\varphi}{g_0}\right)^{1/2}$$

Neglecting higher powers of angular velocity and considering only the first term of the binomial expansion,

$$\Rightarrow g = g_0\left(1 - \frac{1}{2}X\frac{2\omega^2 R\cos\varphi}{g_0}\right)$$

$$\Rightarrow g = g_0 - \omega^2 R\cos\varphi$$

This is the final expression that shows the effect of rotation on gravitational acceleration (mg_0). The important point here is that it is not only the magnitude that is affected by rotation, but its direction is also affected as it is no more directed towards the center of Earth.

There is no effect of rotation at pole. Being a point, there is no circular motion involved and hence, there is no reduction in the value of gravitational acceleration. It is also substantiated from the expression as latitude angle is φ = 90° for the pole and corresponding cosine value is zero. Hence,

$$\Rightarrow g = g_0 - \omega^2 R\cos 90^0 = g_0$$

The reduction in gravitational acceleration is most (maximum) at the equator, where latitude angle is φ = 0° and corresponding cosine value is maximum (=1).

$$\Rightarrow g = g_0 - \omega^2 R\cos 0^0 = g_0 - \omega^2 R$$

We can check approximate reduction at the equator, considering R = 6400 km = 6400000 m = $6.4X10^6$ m.

$$\Rightarrow \omega^2 R = \left(7.28X10^{-5}\right)^2 X\left(6.4X10^6\right) = 3.39X10^{-3} \quad m/s^2 = 0.0339 \quad m/s^2$$

This is the maximum reduction possible due to rotation. Indeed, we can neglect this variation for all practical purposes except where very high accuracy is required.

Vertical position

In this section, we shall discuss the effect of the vertical position of the point of measurement. For this, we shall consider Earth as a perfect sphere of radius "R"

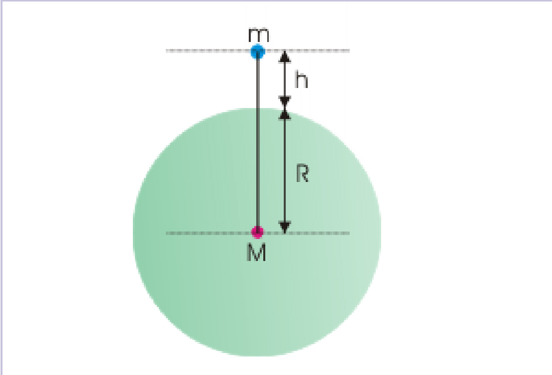
and uniform density, “ρ”. Further, we shall first consider a point at a vertical height “h” from the surface and then a point at a vertical depth “d” from the surface.

Gravitational acceleration at a height

Gravitational acceleration due to Earth on its surface is equal to gravitational force per unit mass and is given by :

$$g_0 = \frac{F}{m} = \frac{GM}{R^2}$$

Gravity at an altitude



Distance between center of Earth and particle changes at an altitude.

where “M” and “R” are the mass and radius of Earth. It is clear that gravitational acceleration will decrease if measured at a height “h” from the Earth’s surface. The mass of Earth remains constant, but the linear distance between particle and the center of Earth increases. The net result is that gravitational acceleration decreases to a value “g’” as given by the equation,

$$\Rightarrow g\prime = \frac{F\prime}{m} = \frac{GM}{(R+h)^2}$$

We can simplify this equation as,

$$\Rightarrow g\prime = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$$

Substituting for the gravitational acceleration at the surface, we have :

$$\Rightarrow g\prime = \frac{g_0}{\left(1+\frac{h}{R}\right)^2}$$

This relation represents the effect of height on gravitational acceleration. We can approximate the expression for situation where h<< R.

$$\Rightarrow g\prime = \frac{g_0}{\left(1+\frac{h}{R}\right)^2} = g_0\left(1+\frac{h}{R}\right)^{-2}$$

As h<<R, we can neglect higher powers of “h/R” in the binomial expansion of the power term,

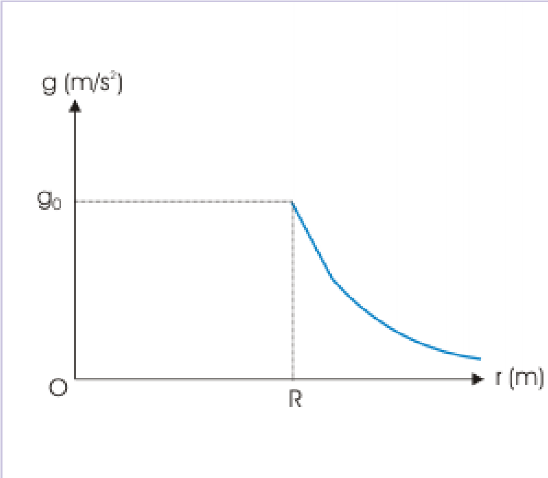
$$\Rightarrow g\prime = g_0\left(1-\frac{2h}{R}\right)$$

We should always keep in mind that this simplified expression holds for the condition, h << R. For small vertical altitude, gravitational acceleration decreases linearly with a slope of “-2/R”. If the altitude is large as in the case of a communication satellite, then we should resort to the original expression,

$$\Rightarrow g\prime = \frac{GM}{(R+h)^2}$$

If we plot gravitational acceleration .vs. altitude, the plot will be about linear for some distance.

Acceleration .vs. linear distance

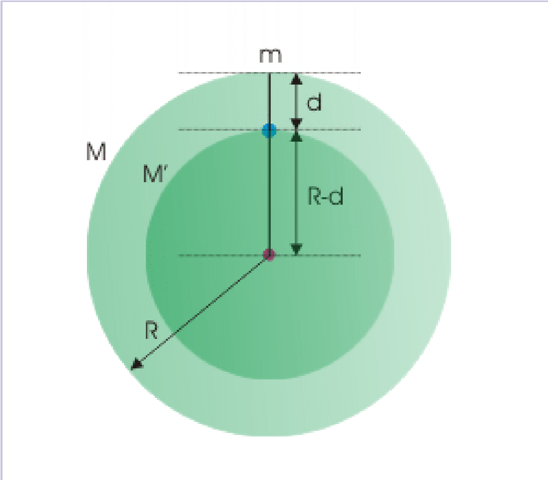


The plot shows variations in gravitational acceleration as we move vertically upwards from center of Earth.

Gravitational acceleration at a depth

In order to calculate gravitational acceleration at a depth "d", we consider a concentric sphere of radius "R-d" as shown in the figure. Here, we shall make use of the fact that gravitational force inside a spherical shell is zero. It means that gravitational force due to the spherical shell above the point is zero. On the other hand, gravitational force due to smaller sphere can be calculated by treating it as point mass. As such, net gravitational acceleration at point "P" is :

Gravity at a depth



Distance between center of Earth and particle changes at an altitude.

$$\Rightarrow g\prime = \frac{F\prime}{m} = \frac{GM\prime}{(R-d)^2}$$

where “M’” is the mass of the smaller sphere. If we consider Earth as a sphere of uniform density, then :

$$M = V\rho$$

$$\Rightarrow \rho = \frac{M}{\frac{4}{3}\pi R^3}$$

Hence, mass of smaller sphere is equal to the product :

$$M\prime = \rho V\prime$$

$$\Rightarrow M\prime = \frac{\frac{4}{3}\pi(R-d)^3 XM}{\frac{4}{3}\pi R^3} = \frac{(R-d)^3 XM}{R^3}$$

Substituting in the expression of gravitational acceleration, we have :

$$\Rightarrow g\prime = \frac{G(R-d)^3 XM}{(R-d)^2 R^3}$$

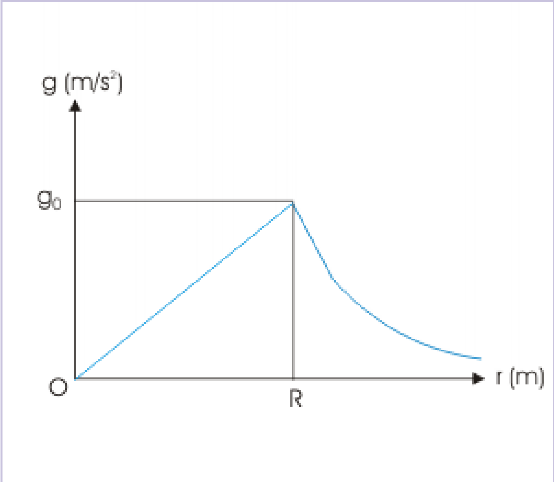
Inserting gravitational acceleration at the surface ($g_0 = GM/R^2$), we have :

$$\Rightarrow g\prime = \frac{g_0 (R-d)^3}{(R-d)^2 R} = \frac{g_0 (R-d)}{R}$$

$$g\prime = g_0 \left(1 - \frac{d}{R}\right)$$

This is also a linear equation. We should note that this expression, unlike earlier case of a point above the surface, makes no approximation . The gravitational acceleration decreases linearly with distance as we go down towards the center of Earth. Conversely, the gravitational acceleration increases linearly with distance as we move from the center of Earth towards the surface.

Acceleration .vs. linear distance



The plot shows variations in gravitational acceleration as we move away from center of Earth.

The plot above combines the effect of altitude and the effect of depth along a straight line, starting from the center of Earth.

Gravitational acceleration .vs. measured acceleration

We have made distinction between these two quantities. Here, we shall discuss the differences once again as their references and uses in problem situations can be confusing.

1: For all theoretical discussion and formulations, the idealized gravitational acceleration (g_0) is considered as a good approximation of actual gravitational acceleration on the surface of Earth, unless otherwise told. The effect of rotation is indeed a small value and hence can be neglected for all practical purposes, unless we deal with situation, requiring higher accuracy.

2: We should emphasize that both these quantities (g_0 and g) are referred to the surface of Earth. For points above or below, we use symbol (g') for effective gravitational acceleration.

3: If context requires, we should distinguish between "g0" and "g". The symbol " g_0 " denotes idealized gravitational acceleration on the surface, considering Earth (i) uniform (ii) spherical and (iii) stationary. On the other hand, "g" denotes actual measurement. We should, however, be careful to note that measured value is also not the actual measurement of gravitational acceleration. This will be clear from the point below.

4: The nature of impact of "rotation" on gravitational acceleration is different than due to other factors. We observed in our discussion in this module that "constitution of Earth" impacts the value of gravitational acceleration for a point below Earth's surface. Similarly, shape and vertical positions of measurements affect gravitational acceleration in different ways. However, these factors only account for the "actual" change in gravitational acceleration. Particularly, they do not modify the gravitational acceleration itself. For example, shape of Earth accounts for actual change in the gravitational acceleration as polar radius is actually smaller than equatorial radius.

Now, think about the change due to rotation. What does it do? It conceals a part of actual gravitational acceleration itself. A part of gravitational force is used to provide for the centripetal acceleration. We measure a different gravitational acceleration than the actual one at that point. We should keep this difference in mind while interpreting acceleration. In the nutshell, rotation alone affects measurement of actual gravitational acceleration, whereas other factors reflect actual change in gravitational acceleration.

5: What is actual gravitational acceleration anyway? From the discussion as above, it is clear that actual gravitational acceleration on the surface of Earth needs to account for the part of the gravitational force, which provides centripetal force. Hence, actual gravitational acceleration is :

$$g_{\text{actual}} = g + \omega^2 R \cos \varphi$$

Note that we have made correction for centripetal force in the measured value (g) – not in the idealized value (g_0). It is so because measured value accounts actual impacts due to all factors. Hence, if we correct for rotation – which alone affects measurement of actual gravitational acceleration, then we get the actual gravitational acceleration at a point on the surface of the Earth.

Gravity (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to gravity. The questions are categorized in terms of the characterizing features of the subject matter :

- Acceleration at a Height
- Acceleration at a Depth
- Comparison of acceleration due to gravity
- Rotation of Earth
- Comparison of gravitational acceleration
- Rate of change of gravity

Acceleration at a Height

Problem 1 : At what height from the surface of Earth will the acceleration due to gravity is reduced by 36 % from the value at the surface. Take, R = 6400 km.

Solution : The acceleration due to gravity decreases as we go vertically up from the surface. The reduction of acceleration by 36 % means that the height involved is significant. As such, we can not use the approximated expression of the effective accelerations for h<< R as given by :

$$g' = g\left(1 - \frac{2h}{R}\right)$$

Instead, we should use the relation,

$$\Rightarrow g\prime = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

Note that we have considered reference gravitational acceleration equal to acceleration on the surface. Now, it is given that :

$$\Rightarrow g\prime = 0.64g$$

Hence,

$$\Rightarrow 0.64g = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

$$\Rightarrow \left(1+\frac{h}{R}\right)^2 X0.64 = 1$$

$$\Rightarrow \left(1+\frac{h}{R}\right) = \frac{10}{8} = \frac{5}{4}$$

$$\Rightarrow \frac{h}{R} = \frac{5}{4} - 1 = \frac{1}{4}$$

$$\Rightarrow h = \frac{R}{4} = \frac{6400}{4} = 1600 \quad km$$

Note : If we calculate, considering h << R, then

$$\Rightarrow 0.64g = g\left(1 - \frac{2h}{R}\right)$$

$$\Rightarrow 0.64R = R - 2h$$

$$\Rightarrow h = \frac{R(1-0.64)}{2} = 0.18R = 0.18X6400 = 1152 \quad km$$

Acceleration at a Depth

Problem 2 : Assuming Earth to be uniform sphere, how much a weight of 200 N would weigh half way from the center of Earth.

Solution : Assuming, $g = g_0$, the accelerations at the surface (g) and at a depth (g') are related as :

$$g\prime = g\left(1 - \frac{d}{R}\right)$$

In this case,

$$\Rightarrow d = R - \frac{R}{2} = \frac{R}{2}$$

Putting in the equation of effective acceleration, we have :

$$\Rightarrow g\prime = g\left(1 - \frac{R}{2R}\right) = \frac{g}{2}$$

The weight on the surface corresponds to "mg" and its weight corresponds to "mg'". Hence,

$$\Rightarrow mg\prime = \frac{mg}{2} = \frac{200}{2} = 100 \quad N$$

Comparison of acceleration due to gravity

Problem 3 : Find the ratio of acceleration due to gravity at a depth "h" and at a height "h" from Earth's surface. Consider h >> R, where "R" is the radius of Earth.

Solution : The acceleration due to gravity at appoint "h" below Earth's surface is given as :

$$g_1 = g_0\left(1 - \frac{h}{R}\right)$$

The acceleration due to gravity at a point “h” above Earth’s surface is given as :

$$g_2 = \frac{g_0}{\left(1 + \frac{h}{R}\right)^2}$$

Note that we have not incorporated approximation for h>>R. We shall affect the same after getting the expression for the ratio .

The required ratio without approximation is :

$$\Rightarrow \frac{g_1}{g_2} = \frac{g_0\left(1 - \frac{h}{R}\right)\left(1 + \frac{h}{R}\right)^2}{g_0}$$

$$\Rightarrow \frac{g_1}{g_2} = \left(1 - \frac{h}{R}\right)\left(1 + \frac{h}{R}\right)^2$$

$$\Rightarrow \frac{g_1}{g_2} = \left(1 - \frac{h}{R}\right)\left(1 + \frac{h^2}{R^2} + \frac{2h}{R}\right)$$

For h >> R, we can neglect terms of higher power than 1,

$$\Rightarrow \frac{g_1}{g_2} = \left(1 - \frac{h}{R}\right)\left(1 + \frac{2h}{R}\right)$$

$$\Rightarrow \frac{g_1}{g_2} = \left(1 - \frac{h}{R} + \frac{2h}{R} - \frac{2h^2}{R^2}\right)$$

Again, neglecting term with higher power,

$$\Rightarrow \frac{g_1}{g_2} = \left(1 + \frac{h}{R}\right)$$

Rotation of Earth

Problem 4 : If “ρ” be the uniform density of a spherical planet, then find the shortest possible period of rotation of the planet about its axis of rotation.

Solution : A planet needs to hold material it is composed. We have seen that centripetal force required for a particle on the surface is maximum at the equator. Therefore, gravitational pull of the planet should be as least sufficient enough to hold the particle at the equator. Corresponding maximum angular speed corresponding to this condition is obtained as :

$$\frac{GMm}{R^2} = m\omega^2 R$$

Time period is related to angular speed as :

$$\omega = \frac{2\pi}{T}$$

Substituting for angular speed in force equation, we get the expression involving shortest time period :

$$\Rightarrow T^2 = \frac{4\pi^2 R^3}{GM}$$

The mass of the spherical planet of uniform density is :

$$\Rightarrow M = \frac{4\pi\rho R^3}{3}$$

Putting in the equation of time period,

$$\Rightarrow T^2 = \frac{4x3\pi^2 R^3}{Gx4\pi\rho R^3} = \frac{3\pi}{G\rho}$$

$$\Rightarrow T = \sqrt{\left(\frac{3\pi}{G\rho}\right)}$$

Problem 5 : Considering Earth to be a sphere of uniform density, what should be the time period of its rotation about its own axis so that acceleration due to gravity at the equator becomes zero. Take g = 10 m/s^2 and R = 6400 km.

Solution : We know that the measurement of gravitational acceleration due to gravity is affected by rotation of Earth. Let g' be the effective acceleration and $g_0 = g$. Then,

$$g\prime = g - R\omega^2 \cos \Phi$$

where Φ is latitude angle.

$$\text{Here,} \quad \Phi = 0^{0,} \quad \cos \Phi = \cos 0^0 = 1,$$

$$g\prime = g - R\omega^2$$

Now, angular velocity is connected to time period as :

$$\omega = \frac{2\pi}{T}$$

Combining two equations, we have :

$$\Rightarrow g\prime = g - \frac{RX4\pi^2}{T^2}$$

$$\Rightarrow 4\pi^2 R = (g - g\prime)T^2$$

$$\Rightarrow T = \sqrt{\frac{4\pi^2 R}{(g - g\prime)}}$$

According to question, effective acceleration is zero,

$$g\prime = 0$$

Hence,

$$\Rightarrow T = 2\pi\sqrt{\frac{6400X10^3}{10}}$$

$$\Rightarrow T = \pi X1600 \quad s$$

$$T = 1.4 \quad hr$$

Comparison of gravitational acceleration

Problem 6 : A planet has 8 times the mass and average density that of Earth. Find acceleration on the surface of planet, considering both bodies spherical in shape. Take acceleration on the surface of Earth as 10 m/s^2.

Solution : Let subscript "1" and "2" denote Earth and planet respectively. Then, ratio of accelerations is :

$$\frac{g_2}{g_1} = \frac{\frac{GM_2}{R_2^2}}{\frac{GM_1}{R_1^2}} = \frac{M_2 R_1^2}{M_1 R_2^2}$$

Here,

$$M_2 = 8M_1$$

$$\Rightarrow \frac{g_2}{g_1} = \frac{8M_1 R_1^2}{M_1 R_2^2} = \frac{8R_1^2}{R_2^2}$$

We need to relate radii in order to evaluate the ratio as above. For this, we shall use given information about density. Here,

$$\Rightarrow \rho_2 = 8\rho_1$$

$$\Rightarrow \frac{M_2}{V_2} = \frac{8M_1}{V_1}$$

$$\text{But}, M_2 = 8M_{1,}$$

$$\Rightarrow \frac{8M_1}{V_2} = \frac{8M_1}{V_1}$$

$$\Rightarrow V_1 = V_2$$

$$\Rightarrow R_1 = R_2$$

Now, evaluating the ratio of accelerations, we have :

$$\Rightarrow g_2 = 8g_1 = 8X10 = 80 \quad m/s^2$$

Rate of change of gravity

Problem 7 : Find the rate of change of weight with respect height “h” near Earth’s surface.

Solution : According to question, we are required to find the rate of change of the weight near Earth’s surface. Hence, we shall use the expression for h<<R/ Also let $g_0 = g$. Then,

$$g\prime = g\left(1 - \frac{2h}{R}\right)$$

Weight at height, “h”, is given by :

$$\Rightarrow W = mg\prime = mg\left(1 - \frac{2h}{R}\right) = mg - \frac{2mgh}{R}$$

The rate of change of acceleration due to gravity at a height “h” is given as :

$$\Rightarrow \frac{dW}{dh} = \frac{d}{dh}\left(mg - \frac{2mgh}{R}\right)$$

$$\Rightarrow \frac{dW}{dh} = -\frac{2mg}{R}$$

Problem 8 : What is fractional change in gravitational acceleration at a height “h” near the surface of Earth.

Solution : The fractional change of a quantity “x” is defined as “Δx/x”. Hence, fractional change in gravitational acceleration is “Δ g/g”. Let $g_0 = g$. Now, effective acceleration at a height “h” near Earth’s surface is given by :

$$g\prime = g\left(1 - \frac{2h}{R}\right)$$

$$\Rightarrow g\prime - g = -\frac{2hg}{R}$$

$$\Rightarrow \frac{g\prime - g}{g} = -\frac{2h}{R}$$

$$\Rightarrow \frac{\Delta g}{g} = -\frac{2h}{R}$$

Gravitational potential energy

The concept of potential energy is linked to a system – not to a single particle or body. So is the case with gravitational potential energy. True nature of this form of energy is often concealed in practical consideration and reference to Earth. Gravitational energy is not limited to Earth, but is applicable to any two masses of any size and at any location. Clearly, we need to expand our understanding of various physical concepts related with gravitational potential energy.

Here, we shall recapitulate earlier discussions on potential energy and apply the same in the context of gravitational force.

Change in gravitational potential energy

The change in the gravitational potential energy of a system is related to work done by the gravitational force. In this section, we shall derive an expression to determine change in potential energy for a system of two particles. For this, we consider an elementary set up, which consists of a stationary particle of mass, " m_1 " and another particle of mass, " m_2 ", which moves from one position to another.

Now, we know that change in potential energy of the system is equal to negative of the work by gravitational force for the displacement of second particle :

$$\Delta U = -W_G$$

On the other hand, work by gravitational force is given as :

$$W_G = \int F_G dr$$

Combining two equations, the mathematical expression for determining change in potential energy of the system is obtained as :

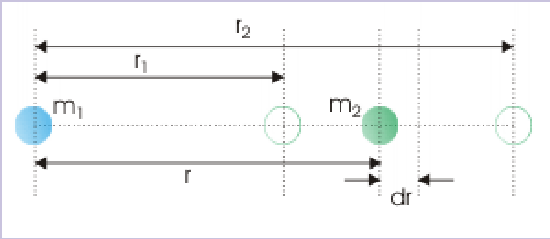
$$\Rightarrow \Delta U = -\int_{r_1}^{r_2} F_G dr$$

In order to evaluate this integral, we need to set up the differential equation first. For this, we assume that stationary particle is situated at the origin of reference. Further, we consider an intermediate position of the particle of mass " m_2 " between two positions through which it is moved along a straight line. The change

in potential energy of the system as the particle moves from position “r” to “r+dr” is :

$$dU = -F_G dr$$

Change in gravitational potential energy



The particle is moved from one position to another.

We get the expression for the change in gravitational potential energy by integrating between initial and final positions of the second particle as :

$$\Delta U = -\int_{r_1}^{r_2} F_G dr$$

We substitute gravitational force with its expression as given by Newton's law of gravitation,

$$F = -\frac{Gm_1m_2}{r^2}$$

Note that the expression for gravitational force is preceded by a negative sign as force is directed opposite to displacement. Now, putting this value in the integral expression, we have :

$$\Rightarrow \Delta U = \int_{r_1}^{r_2} \frac{Gm_1m_2dr}{r^2}$$

Taking out constants from the integral and integrating between the limits, we have :

$$\Rightarrow \Delta U = Gm_1m_2\left[-\frac{1}{r}\right]_{r_1}^{r_2}$$

$$\Rightarrow \Delta U = U_2 - U_1 = Gm_1m_2\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

This is the expression of gravitational potential energy change, when a particle of mass " m_2 " moves from its position from " r_1 " to " r_2 " in the presence of particle of mass " m_1 ". It is important to realize here that " $1/r_1$ " is greater than " $1/r_2$ ". It means that the change in gravitational potential energy is positive in this case. In other words, it means that final value is greater than initial value. Hence, gravitational potential energy of the two particles system is greater for greater linear distance between particles.

Absolute gravitational potential energy

An arrangement of the system is referred to possess zero potential energy with respect to a particular reference. For this we visualize that particles are placed at very large distance. Theoretically, the conservative force like gravitation will not affect bodies which are at infinity. For this reason, zero gravitational reference potential of a system is referred to infinity. The measurement of gravitational potential energy of a system with respect to this theoretical reference is called absolute gravitational potential energy of the system.

$$U(r) = -\int_{\infty}^{r} F_G dr$$

As a matter of fact, this integral can be used to define gravitational potential energy of a system:

Gravitational potential energy
: The gravitational potential energy of a system of particles is equal to "negative" of the work by the gravitational force as a particle is brought from infinity to its position in the presence of other particles of the system.

For practical consideration, we can choose real specific reference (other than infinity) as zero potential reference. Important point is that selection of zero reference is not a limitation as we almost always deal with change in potential

energy – not the absolute potential energy. So long we are consistent with zero potential reference (for example, Earth's surface is considered zero gravitational potential reference), we will get the same value for the difference in potential energy, irrespective of the reference chosen.

We can also define gravitation potential energy in terms of external force as :

Gravitational potential energy
> The gravitational potential energy of a system of particles is equal to the work by the external force as a particle is brought from infinity slowly to its position in the presence of other particles of the system.

Earth systems

We have already formulated expressions for gravitational potential energy for "Earth – body" system in the module on potential energy.

The potential energy of a body raised to a height "h" has been obtained as :

$$U = mgh$$

Generally, we refer gravitational potential energy of "Earth- particle system" to a particle – not to a system. This is justified on the basis of the fact that one member of the system is relatively very large in size.

All terrestrial bodies are very small with respect to massive Earth. A change in potential energy of the system is balanced by a corresponding change in kinetic energy in accordance with conservation of mechanical energy. Do we expect a change in the speed of Earth due to a change in the position of ,say, a tennis ball? All the changes due to change in the position of a tennis ball is reflected as the change in the speed of the ball itself – not in the speed of the Earth. So dropping reference to the Earth is not inconsistent to physical reality.

Gravitational potential energy of two particles system

We can determine potential energy of two particles separated by a distance "r", using the concept of zero potential energy at infinity. According to definition, the integral of potential energy of the particle is evaluated for initial position at infinity to a final position, which is at a distance "r" from the first particle at the origin of reference.

Here,

$$U_1 = 0$$

$$U_2 = U(\text{ say})$$

$$r_1 = \infty$$

$$r_2 = r(\text{say})$$

Putting values in the expression of the change of potential energy, we have :

$$\Rightarrow U - 0 = Gm_1m_2\left[\frac{1}{\infty} - \frac{1}{r}\right]$$

$$\Rightarrow U = -\frac{Gm_1m_2}{r}$$

By definition, this potential energy is equal to the negative of work by gravitational force and equal to the work by an external force, which does not produce kinetic energy while the particle of mass " m_2 " is brought from infinity to a position at a distance "r" from other particle of mass " m_1 ".

We see here that gravitational potential energy is a negative quantity. As the particles are farther apart, "1/r" becomes a smaller fraction. Potential energy, being a negative quantity, increases. But, the magnitude of potential energy becomes smaller. The maximum value of potential energy is zero for r = ∞ i.e. when particles are at very large distance from each other

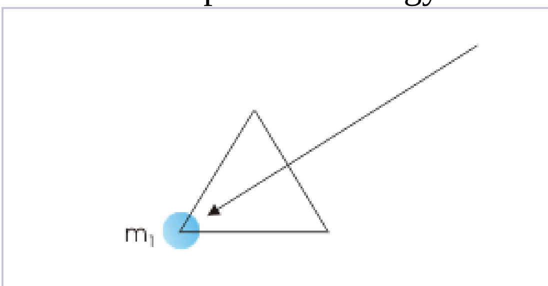
On the other hand, the fraction "1/r" is a bigger fraction when the particles are closer. Gravitational potential energy, being a negative quantity, decreases. The magnitude of potential energy is larger. This is consistent with the fact that particles are attracted by greater force when they are closer. Hence, if a particles are closer, then it is more likely to be moved by the gravitational force. A particle away from the first particle has greater potential energy, but smaller magnitude. It is attracted by smaller gravitational force and is unlikely to be moved by gravitational force as other forces on the particle may prevail.

Gravitational potential energy of a system of particles

We have formulated expression for the gravitational potential energy of two particles system. In this section, we shall find gravitational potential energy of a

system of particles, starting from the beginning. We know that zero gravitational potential energy is referred to infinity. There will no force to work with at an infinite distance. Since no force exists, no work is required for a particle to bring the first particle from infinity to a point in a gravitation free region. So the work by external force in bringing first particle in the region of zero gravitation is "zero".

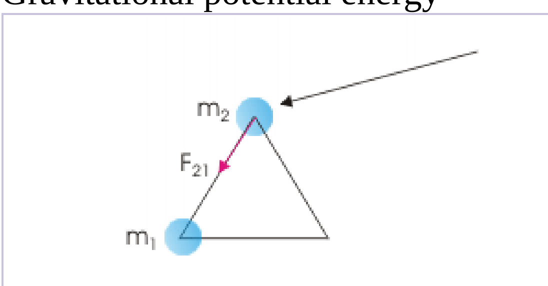
Gravitational potential energy



First particle is brought in a region of zero gravitation.

What about bringing the second particle (2) in the vicinity of the first particle (1)? The second particle is brought in the presence of first particle, which has certain mass. It will exert gravitational attraction on the second particle. The potential energy of two particles system will be given by the negative of work by gravitational force due to particle (1)as the second particle is brought from infinity :

Gravitational potential energy



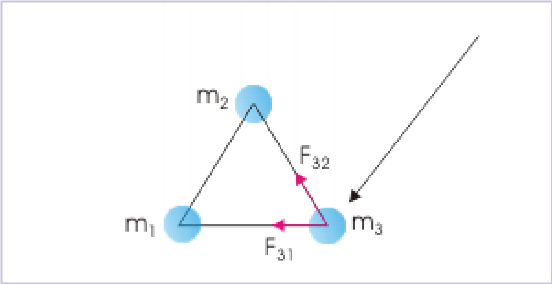
Second particle is brought in the gravitation of first particle.

$$U_{12} = -\frac{Gm_1m_2}{r_{12}}$$

We have subscripted linear distance between first and second particle as " r_{12} ". Also note that work by gravitational force is independent of the path i.e. how force and displacement are oriented along the way second particle is brought near first particle.

Now, what about bringing the third particle of mass, " m_3 ", in the vicinity of the first two particles? The third particle is brought in the presence of first two particles, which have certain mass. They will exert gravitational forces on the third particle. The potential energy due to first particle is equal to the negative of work by gravitational force due to it :

Gravitational potential energy



Third particle is brought in the gravitation of first and second particles.

$$U_{13} = -\frac{Gm_1m_3}{r_{13}}$$

Similarly, the potential energy due to second particle is equal to the negative of work by gravitational force due to it :

$$U_{23} = -\frac{Gm_2m_3}{r_{23}}$$

Thus, potential energy of three particles at given positions is algebraic sum of negative of gravitational work in (i) bringing first particle (ii) bringing second particle in the presence of first particle and (iii) bringing third particle in the presence of first two particles :

$$\Rightarrow U = -G\left(\frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}\right)$$

Induction of forth particle in the system will involve work by gravitation in assembling three particles as given by the above expression plus works by the individual gravitation of three already assembled particles when fourth particle is brought from the infinity.

$$\Rightarrow U = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} + \frac{m_1 m_4}{r_{14}} + \frac{m_2 m_4}{r_{24}} + \frac{m_3 m_4}{r_{34}}\right)$$

Proceeding in this fashion, we can calculate potential energy of a system of particles. We see here that this process resembles the manner in which a system of particles like a rigid body is constituted bit by bit. As such, this potential energy of the system represents the "energy of constitution" and is called "self energy" of the rigid body or system of particles. We shall develop alternative technique (easier) to measure potential energy and hence "self energy" of regular geometric shapes with the concept of gravitational potential in a separate module.

Examples

Problem 1: Find the work done in bringing three particles, each having a mass of 0.1 kg from large distance to the vertices of an equilateral triangle of 10 cm in a gravity free region. Assume that no change of kinetic energy is involved in bringing particles.

Solution : We note here that all three particles have same mass. Hence, product of mass in the expression of gravitational potential energy reduces to square of mass. The gravitational potential energy of three particles at the vertices of the equilateral triangle is :

$$U = -\frac{3Gm^2}{a}$$

where "a" is the side of the equilateral triangle.

Putting values,

$$\Rightarrow U = -\frac{3X6.67X10^{-11}X0.1^2}{0.1} = -3X6.67X10^{-10}X0.01 = -20X10^{-12} \quad J$$

$$\Rightarrow U = -2X10^{-11} \quad J$$

Hence, work done by external force in bringing three particles from large distance is :

$$\Rightarrow W = U = -2X10^{-11} J$$

Work and energy

An external force working on a system brings about changes in the energy of system. If change in energy is limited to mechanical energy, then work by external force will be related to change in mechanical energy as :

$$W_F = \Delta E = \Delta U + \Delta K$$

A change in gravitational potential energy may or may not be accompanied with change in kinetic energy. It depends on the manner external force works on the system. If we work on the system in such a manner that we do not impart kinetic energy to the particles of the system, then there is no change in kinetic energy. In that case, the work by external force is equal to the change in gravitational potential energy alone.

There can be three different situations :

Case 1 : If there is change in kinetic energy, then work by external force is equal to the change in potential and kinetic energy:

$$W_F = \Delta U + \Delta K$$

Case 2 : If there is no change in kinetic energy, then work by external force is equal to the change in potential energy alone :

$$\Delta K = 0$$

Putting in the expression of work,

$$W_F = \Delta U$$

Case 3 : If there is no external force, then work by external force is zero. The change in one form of mechanical energy is compensated by a corresponding negative change in the other form. This means that mechanical energy of the system is conserved. Here,

$$W_F = 0$$

Putting in the expression of work,

$$\Rightarrow \mathit{\Delta} U + \mathit{\Delta} K = 0$$

We shall, now, work with two illustrations corresponding to following situations :

- Change in potential energy without change in kinetic energy
- Change in potential energy without external force

Change in potential energy without change in kinetic energy

Problem 2: Three particles, each having a mass of 0.1 kg are placed at the vertices of an equilateral triangle of 10 cm. Find the work done to change the positions of particles such that side of the triangle is 20 cm. Assume that no change of kinetic energy is involved in changing positions.

Solution : The work done to bring the particles together by external force in gravitational field is equal to potential energy of the system of particles. This means that work done in changing the positions of the particles is equal to change in potential energy due to change in the positions of particles. For work by external force,

$$W_F = \mathit{\Delta} U + \mathit{\Delta} K$$

Here, ΔK = 0

$$W_F = \mathit{\Delta} U$$

Now, we have seen that :

$$U = -\frac{3Gm^2}{a}$$

Hence, change in gravitational potential energy is :

$$\Rightarrow \mathit{\Delta} U = -\frac{3Gm^2}{a_2} - \left(-\frac{3Gm^2}{a_1}\right)$$

$$\Rightarrow \mathit{\Delta} U = 3Gm^2\left[-\frac{1}{a_2} + \frac{1}{a_1}\right]$$

Putting values, we have :

$$\Rightarrow \varDelta U = 3X6.67X10^{-11}X0.1^2\left[-\frac{1}{0.2} + \frac{1}{0.1}\right]$$

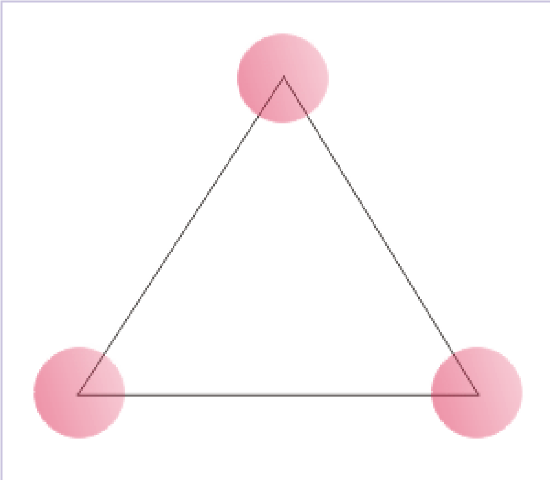
$$\Rightarrow \varDelta U = 3X6.67X10^{-11}X0.12X5$$

$$\Rightarrow \varDelta U = 1.00X10^{-11} \quad J$$

Change in potential energy without external force

Problem 3: Three identical solid spheres each of mass “m” and radius “R” are released from positions as shown in the figure (assume no external gravitation). What would be the speed of any of three spheres just before they collide.

Three particles system



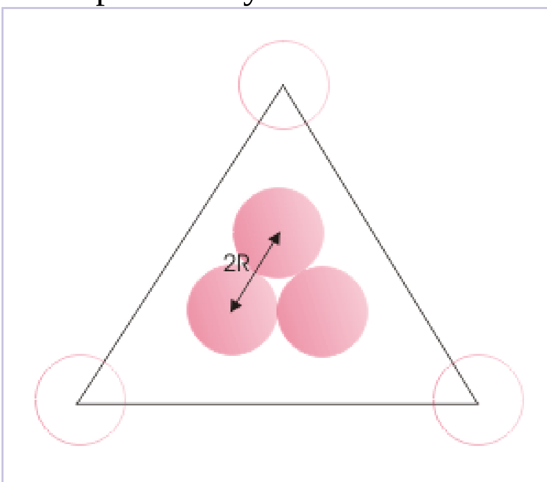
Positions before being released.

Solution : Since no external force is involved, the mechanical energy of the system at the time of release should be equal to mechanical energy just before the collision. In other words, the mechanical energy of the system is conserved. The initial potential energy of system is given by,

$$U_i = -\frac{3Gm^2}{a}$$

Let “v” be the speed of any sphere before collision. The configuration just before the collision is shown in the figure. We can see that linear distance between any two centers of two identical spheres is “2R”. Hence, potential energy of the configuration before collision is,

Three particles system



Positions just before collision.

$$U_f = -\frac{3Gm^2}{2R}$$

Applying conservation of mechanical energy,

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow 0 - \frac{3Gm^2}{a} = \frac{1}{2}mv^2 - \frac{3Gm^2}{2R}$$

$$v = \sqrt{\left\{Gm\left(\frac{1}{R} - \frac{2}{a}\right)\right\}}$$

Gravitational field

We have studied gravitational interaction in two related manners. First, we studied it in terms of force and then in terms of energy. There is yet another way to look at gravitational interactions. We can study it in terms of gravitational field.

In the simplest form, we define a gravitational field as a region in which gravitational force can be experienced. We should, however, be aware that the concept of force field has deeper meaning. Forces like gravitational force and electromagnetic force work with “action at a distance”. As bodies are not in contact, it is conceptualized that force is communicated to bodies through a force field, which operates on the entities brought in its region of influence.

Electromagnetic interaction, which also abides inverse square law like gravitational force, is completely described in terms of field concept. Theoretical conception of gravitational force field, however, is not complete yet. For this reason, we would restrict treatment of gravitational force field to the extent it is in agreement with well established known facts. In particular, we would not conceptualize about physical existence of gravitational field unless we refer “general relativity”.

A body experiences gravitational force in the presence of another mass. This fact can be thought to be the result of a process in which presence of a one mass modifies the characteristics of the region around itself. In other words, it creates a gravitational field around itself. When another mass enters the region of influence, it experiences gravitational force, which is given by Newton’s law of gravitation.

Field strength

Field strength (**E**) is equal to gravitational force experienced by unit mass in a gravitational field. Mathematically,

$$\mathbf{E} = \frac{\mathbf{F}}{m}$$

Its unit is N/kg. Field strength is a vector quantity and abides by the rules of vector algebra, including superposition principle. Hence, if there are number of bodies, then resultant or net gravitational field due to them at a given point is vector sum of individual fields,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \ldots\ldots.$$

$$\Rightarrow \mathbf{E} = \Sigma \mathbf{E}_i$$

Significance of field strength

An inspection of the expression of gravitational field reveals that its expression is exactly same as that of acceleration of a body of mass, “m”, acted upon by an external force, “**F**”. Clearly,

$$\mathbf{E} = \mathbf{a} = \frac{\mathbf{F}}{m}$$

For this reason, gravitational field strength is dimensionally same as acceleration. Now, dropping vector notation for action in a particular direction of force,

$$\Rightarrow E = a = \frac{F}{m}$$

We can test this assertion. For example, Earth’s gravitational field strength can be obtained, by substituting for gravitational force between Earth of mass,"M", and a particle of mass, "m" :

$$\Rightarrow E = \frac{F}{m} = \frac{GMm}{r^2 m} = \frac{GM}{r^2} = g$$

Thus, Earth’s gravitational field strength is equal to gravitational acceleration, “g”.

Field strength, apart from its interpretation for the action at a distance, is a convenient tool to map a region and thereby find the force on a body brought in the field. It is something like knowing “unit rate”. Suppose if we

are selling pens and if we know its unit selling price, then it is easy to calculate price of any numbers of pens that we sale. We need not compute the unit selling price incorporating purchase cost, overheads, profit margins etc every time we make a sale.

Similar is the situation here. Once gravitational field strength in a region is mapped (known), we need not be concerned about the bodies which are responsible for the gravitational field. We can compute gravitational force on any mass that enters the region by simply multiplying the mass with the unit rate of gravitational force i.e. field strength,

$$\mathbf{F} = m\mathbf{E}$$

In accordance with this interpretation, we determine gravitational force on a body brought in the gravitational field of Earth by multiplying the mass with the gravitational field strength,

$$\Rightarrow F = mE = mg$$

This approach has following advantages :

1: We can measure gravitational force on a body without reference to other body responsible for gravitational field. In the context of Earth, for example, we compute gravitational force without any reference to the mass of Earth. The concept of field strength allows us to study gravitational field in terms of the mass of one body and as such relieves us from considering it always in terms of two body system. The effect of one of two bodies is actually represented by its gravitational field strength.

2: It simplifies mathematical calculation for gravitational force. Again referring to the context of Earth's gravity, we see that we hardly ever use Newton's gravitational law. We find gravitational force by just multiplying mass with gravitational field strength (acceleration). Imagine if we have to compute gravitational force every time, making calculation with masses of Earth and the body and the squared distance between them!

Comparison with electrostatic field

There is one very important aspect of gravitational field, which is unique to it. We can appreciate this special feature by comparing gravitational field with electrostatic field. We know that the electrostatic force, like gravitational force, also follows inverse square law. Electrostatic force for two point charges separated by a linear distance, "r", is given by Coulomb's law as :

$$F_E = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$$

The electrostatic field (E_E)is defined as the electrostatic force per unit positive charge and is expressed as :

$$E_E = \frac{F_E}{q} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}q = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

The important point, here, is that electrostatic field is not equal to acceleration. Recall that Newton's second law of motion connects force (any type) with "mass" and "acceleration" as :

$$F = ma$$

This relation is valid for all kinds of force - gravitational or electrostatic or any other type. What we mean to say that there is no corresponding equation like "F=qa". Mass only is the valid argument of this relation. As such, electrostatic field can not be equated with acceleration as in the case of gravitational field.

Thus, equality of "field strength" with "acceleration" is unique and special instance of gravitational field - not a common feature of other fields. As a matter of fact, this instance has a special significance, which is used to state "equivalence of mass" - the building block of general theory of relativity.

We shall discuss this concept in other appropriate context. Here, we only need to underline this important feature of gravitational field.

Example

Problem 1: A charged particle of mass “m” carries a charge “q”. It is projected upward from Earth’s surface in an electric field “E”, which is directed downward. Determine the nature of potential energy of the particle at a given height, “h”.

Solution : The charged particle is acted upon simultaneously by both gravitational and electrostatic fields. Here, gravity works against displacement. The work by gravity is, therefore, negative. Hence, potential energy arising from gravitational field (with reference from surface) is positive as :

$$U_G = -W_G = -(-F_G h) = mE_G h = mgh$$

As given in the question, the electrostatic field is acting downward. Since charge on the particle is positive, electrostatic force acts downward. It means that work by electrostatic force is also negative. Hence, potential energy arising from electrostatic field (with reference from surface) is :

$$U_E = -W_E = -(-F_E h) = qE_E h = qEh$$

Total potential energy of the charged particle at a height “h” is :

$$\Rightarrow U = U_G + U_E = (mgh + qEh) = (mg + qE)h$$

The quantities in the bracket are constant. Clearly, potential energy is a function of height.

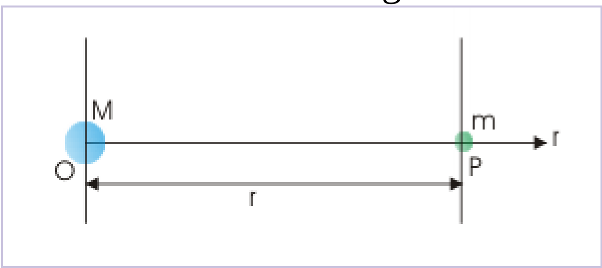
It is important to realize that description in terms of respective fields enables us to calculate forces without referring to either Newton's gravitation law or Coulomb's law of electrostatic force.

Gravitational field due to a point mass

Determination of gravitational force strength due to a point mass is easy. It is so because, Newton's law of gravitation provides the expression for determining force between two particles.

Let us consider a particle of mass, "M", for which we are required to find gravitational field strength at a certain point, "P". For convenience, let us consider that the particle is situated at the origin of the reference system. Let the point, where gravitational field is to be determined, lies at a distance "r" from the origin on the reference line.

Gravitational field strength



Gravitational field at a point "P" due to mass "M"

We should make it a point to understand that the concept of gravitational field is essentially "one" particle/ body/entity concept. We need to measure gravitational force at the point, "P", on a unit mass as required by the definition of field strength. It does not exist there. In order to determine field strength, however, we need to visualize as if unit mass is actually present there.

We can do this two ways. Either we visualize a point mass exactly of unit value or we visualize any mass, "m", and then calculate gravitational force. In the later case, we divide the gravitational force as obtained from Newton's law of gravitation by the mass to get the force per unit mass. In either case, we call this point mass as test mass. If we choose to use a unit mass, then :

$$\Rightarrow E = F = \frac{GMX1}{r^2} = \frac{GM}{r^2}$$

On the other hand, if we choose any arbitrary test mass, "m", then :

$$\Rightarrow E = \frac{F}{m} = \frac{GMm}{r^2 m} = \frac{GM}{r^2}$$

However, there is a small catch here. The test mass has its own gravitational field. This may unduly affect determination of gravitational field due to given particle. In order to completely negate this possibility, we may consider a mathematical expression as given here, which is more exact for defining gravitational field :

$$E = \lim_{m \to 0} \quad \frac{F}{m}$$

Nevertheless, we know that gravitational force is not a very strong force. The field of a particle of unit mass can safely be considered negligible.

The expression for the gravitational field at point "P", as obtained above, is a scalar value. This expression, therefore, measures the magnitude of gravitational field - not its direction. We can realize from the figure shown above that gravitational field is actually directed towards origin, where the first particle is situated. This direction is opposite to the positive reference direction. Hence, gravitational field strength in vector form is preceded by a negative sign :

$$\Rightarrow \mathbf{E} = \frac{\mathbf{F}}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

where " $\hat{\mathbf{r}}$ " is unit vector in the reference direction.

The equation obtained here for the gravitational field due to a particle of mass, "M", is the basic equation for determining gravitational field for any system of particles or rigid body. The general idea is to consider the system being composed of small elements, each of which can be treated at particle. We, then, need to find the net or resultant field, following superposition principle. We shall use this technique to determine gravitational field due to certain regularly shaped geometric bodies in the next module.

Example

Problem 2 : The gravitational field in a region is in xy-plane is given by 3**i** + **j**. A particle moves along a straight line in this field such that work done by gravitation is zero. Find the slope of straight line.

Solution : The given gravitational field is a constant field. Hence, gravitational force on the particle is also constant. Work done by a constant force is given as :

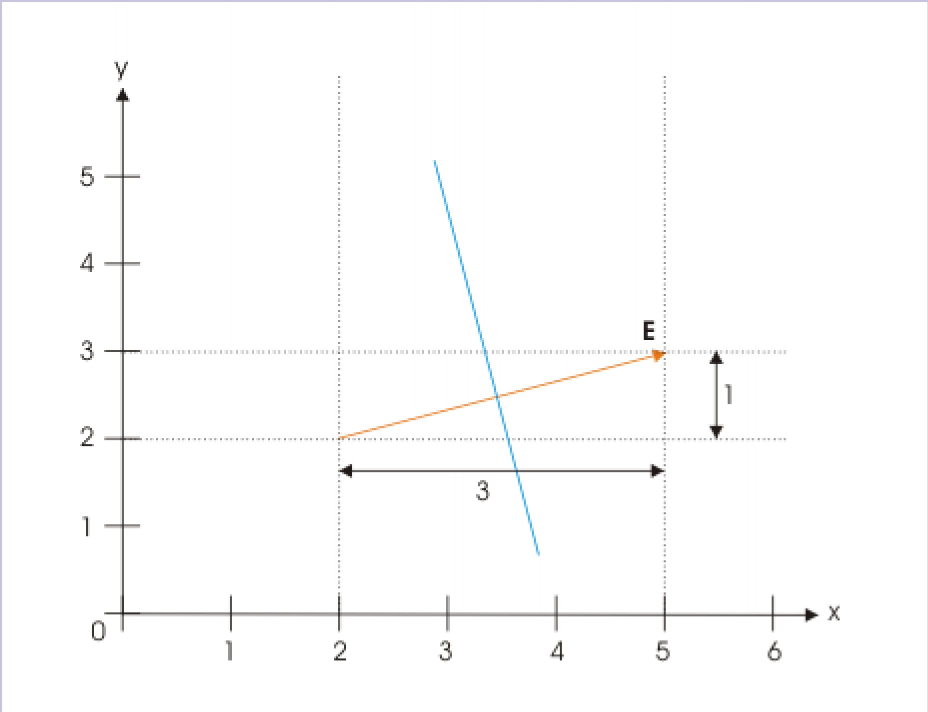
$$W = \mathbf{F} . \mathbf{r}$$

Let "m" be the mass of the particle. Then, work is given in terms of gravitational field as :

$$\Rightarrow W = m\mathbf{E} . \mathbf{r}$$

Work done in the gravitational field is zero, if gravitational field and displacement are perpendicular to each other. If " s_1 " and " s_2 " be the slopes of the direction of gravitational field and that of straight path, then the slopes of two quantities are related for being perpendicular as :

Work by gravitational force



Gravitational field and displacement of particle are perpendicular to each other.

$$s_1 s_2 = -1$$

Note that slope of a straight line is usually denoted by letter “m”. However, we have used letter “s” in this example to distinguish it from mass, which is also represented by letter “m”.

In order to find the slope of displacement, we need to know the slope of the straight line, which is perpendicular to the direction of gravitational field.

Now, the slope of the line of action of gravitational field is :

$$\Rightarrow s_1 = \frac{1}{3}$$

Hence, for gravitational field and displacement to be perpendicular,

$$\Rightarrow s_1 s_2 = \quad \frac{1}{3} \quad s_2 = -1$$

$$s_2 = -3$$

Gravitational field due to rigid bodies

Gravitational field of rigid bodies

We shall develop few relations here for the gravitational field strength of bodies of particular geometric shape without any reference to Earth’s gravitation.

Newton’s law of gravitation is stated strictly in terms of point mass. The expression of gravitational field due to a particle, as derived from this law, serves as starting point for developing expressions of field strength due to rigid bodies. The derivation for field strength for geometric shapes in this module, therefore, is based on developing technique to treat a real body mass as aggregation of small elements and combine individual effects. There is a bit of visualization required as we need to combine vectors, having directional property.

Along these derivations for gravitational field strength, we shall also establish Newton’s shell theory, which has been the important basic consideration for treating spherical mass as point mass.

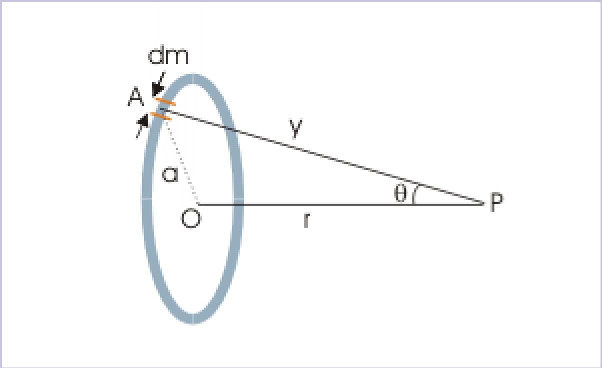
The celestial bodies - whose gravitational field is appreciable and whose motions are subject of great interest - are usually spherical. Our prime interest, therefore, is to derive expression for field strength of solid sphere. Conceptually, a solid sphere can be considered being composed of infinite numbers of closely packed spherical shells. In turn, a spherical shell can be conceptualized to be aggregation of thin circular rings of different diameters.

The process of finding the net effect of these elements fits perfectly well with integration process. Our major task, therefore, is to suitably set up an integral expression for elemental mass and then integrate the elemental integral between appropriate limits. It is clear from the discussion here that we need to begin the process in the sequence starting from ring --> spherical shell --> solid sphere.

Gravitational field due to a uniform circular ring

We need to find gravitational field at a point “P” lying on the central axis of the ring of mass “M” and radius “a”. The arrangement is shown in the figure. We consider a small mass “dm” on the circular ring. The gravitational field due to this elemental mass is along PA. Its magnitude is given by :

Gravitational field due to a ring

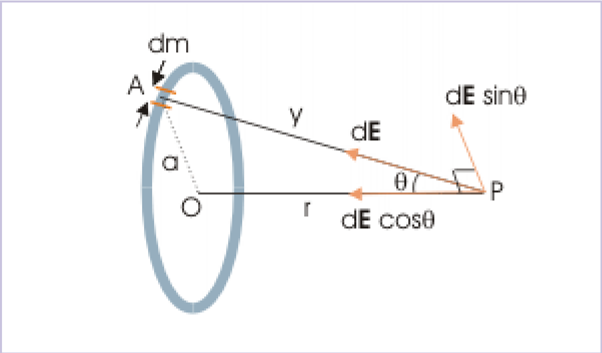


The gravitational field is measured on axial point "P".

$$dE = \frac{G dm}{PA^2} = \frac{G dm}{(a^2 + r^2)}$$

We resolve this gravitational field in the direction parallel and perpendicular to the axis in the plane of OAP.

Gravitational field due to a ring



The net gravitational field is axial.

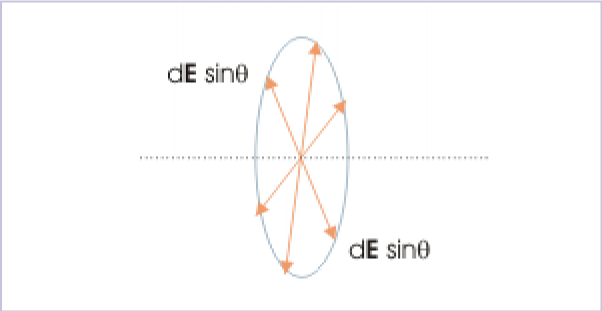
$$dE_{||} = dE \cos\theta$$

$$dE_{\perp} = dE \sin\theta$$

We note two important things. First, we can see from the figure that measures of “y” and “θ” are same for all elemental mass. Further, we are considering equal elemental masses. Therefore, the magnitude of gravitational field due any of the elements of mass “dm” is same, because they are equidistant from point “P”.

Second, perpendicular components of elemental field intensity for pair of elemental masses on diametrically opposite sides of the ring are oppositely directed. On integration, these perpendicular components will add up to zero for the whole of ring. It is clear that we can assume zero field strength perpendicular to axial line, if mass distribution on the ring is uniform. For uniform ring, the net gravitational intensity will be obtained by integrating axial components of elemental field strength only. Hence,

Gravitational field due to a ring



Perpendicular components cancel each other.

$$\Rightarrow E = \int dE \cos\theta$$

$$\Rightarrow E = \int \frac{G\mathbb{d}m \cos\theta}{(a^2 + r^2)}$$

The trigonometric ratio “cosθ” is a constant for all points on the ring. Taking out cosine ratio and other constants from the integral,

$$\Rightarrow E = \frac{G\cos\theta}{(a^2 + r^2)} \int \mathbb{d}m$$

Integrating for m = 0 to m = M, we have :

$$\Rightarrow E = \frac{GM\cos\theta}{(a^2 + r^2)}$$

From triangle OAP,

$$\Rightarrow \cos\theta = \frac{r}{(a^2 + r^2)^{\frac{1}{2}}}$$

Substituting for “cosθ” in the equation ,

$$\Rightarrow E = \frac{GMr}{(a^2 + r^2)^{\frac{3}{2}}}$$

For r = 0, E = 0. The gravitation field at the center of ring is zero. This result is expected also as gravitational fields due to two diametrically opposite equal elemental mass are equal and opposite and hence balances each other.

Position of maximum gravitational field

We can get the maximum value of gravitational field by differentiating its expression w.r.t linear distance and equating the same to zero,

$$\frac{\mathbb{d}E}{dr} = 0$$

This yields,

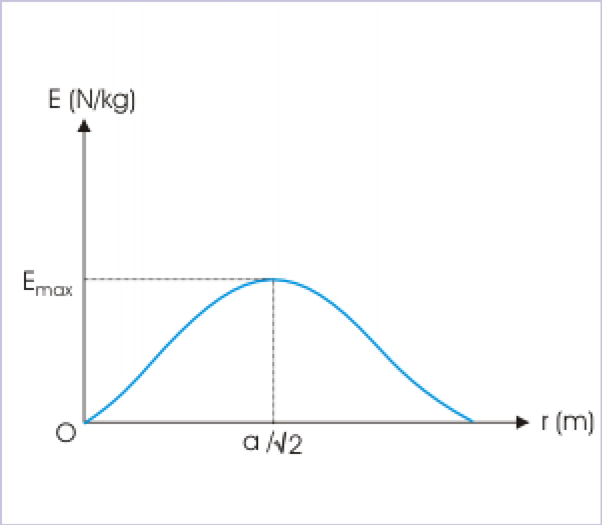
$$\Rightarrow r = \frac{a}{\sqrt{2}}$$

Substituting in the expression of gravitational field, the maximum field strength due to a circular ring is :

$$\Rightarrow E_{\max} = \frac{GMa}{2^{\frac{1}{2}}\left(a^2 + \frac{a^2}{2}\right)^{\frac{3}{2}}} = \frac{GMa}{3\sqrt{3}a^2}$$

The plot of gravitational field with axial distance shows the variation in the magnitude,

Gravitational field due to a ring

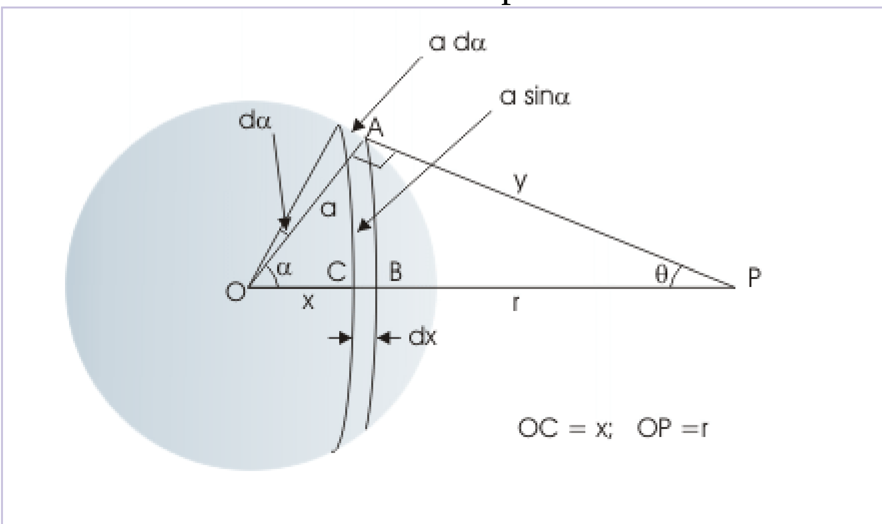


The gravitational field along the axial line.

Gravitational field due to thin spherical shell

The spherical shell of radius “a” and mass “M” can be considered to be composed of infinite numbers of thin rings. We consider one such ring of infinitesimally small thickness “dx” as shown in the figure. We derive the required expression following the sequence of steps as outlined here :

Gravitational field due to thin spherical shell



The gravitational field is measured on axial point "P".

(i) Determine mass of the elemental ring in terms of the mass of shell and its surface area.

$$dm = \frac{M}{4\pi a^2} X 2\pi a \sin\alpha dx = \frac{Ma\sin\alpha dx}{2a^2}$$

From the figure, we see that :

$$dx = a d\alpha$$

Putting these expressions,

$$\Rightarrow dm = \frac{Ma\sin\alpha dx}{2a^2} = \frac{Ma\sin\alpha a d\alpha}{2a^2} = \frac{M\sin\alpha d\alpha}{2}$$

(ii) Write expression for the gravitational field due to the elemental ring. For this, we employ the formulation derived earlier for the ring,

$$\Rightarrow dE = \frac{G dm \cos\theta}{AP^2}$$

Putting expression for elemental mass,

$$\Rightarrow dE = \frac{GM \sin\alpha d\alpha \cos\theta}{2y^2}$$

(v) Set up integral for the whole disc

We see here that gravitational fields due to all concentric rings are directed towards the center of spherical shell along the axis.

$$\Rightarrow E = GM \int \frac{\sin\alpha \cos\theta d\alpha}{2y^2}$$

The integral expression has three varibles "α","θ" and "y".Clearly, we need to express variables in one variable “x”. From triangle, OAP,

$$\Rightarrow y^2 = a^2 + r^2 - 2ar\cos\alpha$$

Differentiating each side of the equation,

$$\Rightarrow 2y dy = 2ar \sin\alpha d\alpha$$

$$\Rightarrow \sin\alpha d\alpha = \frac{y dy}{ar}$$

Again from triangle OAP,

$$\Rightarrow a^2 = y^2 + r^2 - 2yr\cos\theta$$

$$\Rightarrow \cos\theta = \frac{y^2 + r^2 - a^2}{2yr}$$

Putting these values in the integral,

$$\Rightarrow E = GM \int \frac{dy\left(y^2 + r^2 - a^2\right)}{4ar^2y^2}$$

$$\Rightarrow E = GM \int \frac{dy}{4ar^2}\left(1 - \frac{a^2 - r^2}{y^2}\right)$$

We shall decide limits of integration on the basis of the position of point "P" – whether it lies inside or outside the shell. Integrating expression on right side between two general limits, initial (L_1) and final (L_2),

$$\Rightarrow E = GM \int_{L_1}^{L_2} \frac{dy}{4ar^2}\left(1 - \frac{a^2 - r^2}{y^2}\right)$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[y + \frac{a^2 - r^2}{y}\right]_{L_1}^{L_2}$$

Evaluation of integral for the whole shell

Case 1 : The point "P" lies outside the shell. The total gravitational field is obtained by integrating the integral from y = r-a to y = r+a,

$$\Rightarrow E = \frac{GM}{4ar^2}\left[y + \frac{a^2 - r^2}{y}\right]_{r-a}^{r+a}$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[r + a + \frac{a^2 - r^2}{r + a} - r + a - \frac{a^2 - r^2}{r - a}\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[2a + \left(a^2 - r^2\right)\left(\frac{1}{r + a} - \frac{1}{r - a}\right)\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2} X 4a$$

$$\Rightarrow E = \frac{GM}{r^2}$$

This is an important result. We have been using this result by the name of Newton's shell theory. According to this theory, a spherical shell, for a particle outside it, behaves as if all its mass is concentrated at its center. This is how we could calculate gravitational attraction between Earth and an apple. Note that radius of the shell, "a", does not come into picture.

Case 2 : The point "P" lies outside the shell. The total gravitational field is obtained by integrating the integral from x = a-r to x = a+r,

$$\Rightarrow E = \frac{GM}{4ar^2}\left[y + \frac{a^2 - r^2}{y}\right]_{a-r}^{a+r}$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[a + r + \frac{a^2 - r^2}{a + r} - a + r - \frac{a^2 - r^2}{a - r}\right]$$

$$\Rightarrow E = \frac{GM}{4ar^2}\left[2r + \left(a^2 - r^2\right)\left(\frac{1}{a + r} - \frac{1}{a - r}\right)\right]$$

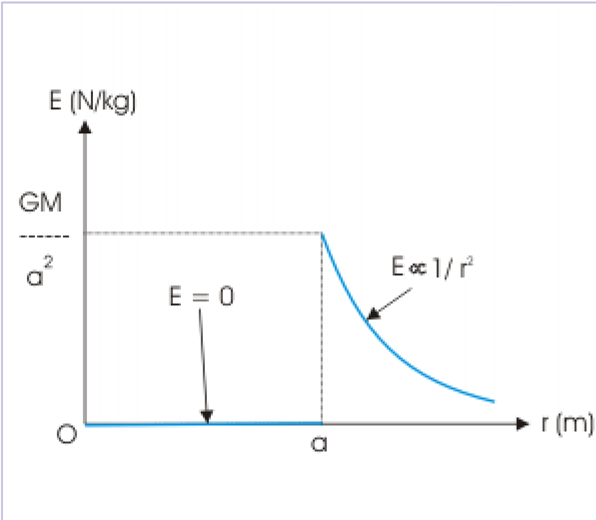
$$\Rightarrow E = \frac{GM}{4ar^2}\left[2r - 2r\right] = 0$$

This is yet another important result, which has been used to determine gravitational acceleration below the surface of Earth. The mass residing outside the sphere drawn to include the point below Earth's surface, does not contribute to gravitational force at that point.

The mass outside the sphere is considered to be composed of infinite numbers of thin shells. The point within the Earth lies inside these larger shells. As gravitational intensity is zero within a shell, the outer shells do not contribute to the gravitational force on the particle at that point.

A plot, showing the gravitational field strength, is shown here for regions both inside and outside spherical shell :

Gravitational field due to thin spherical shell

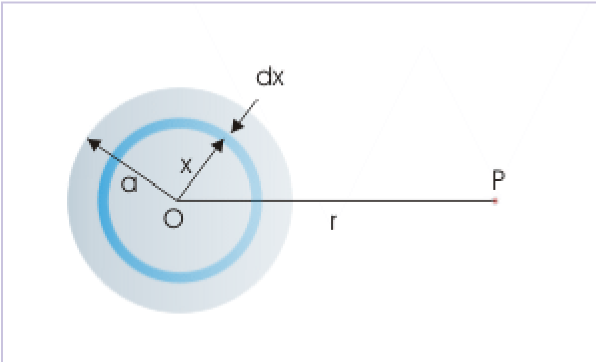


The gravitational field along linear distance from center.

Gravitational field due to uniform solid sphere

The uniform solid sphere of radius “a” and mass “M” can be considered to be composed of infinite numbers of thin spherical shells. We consider one such spherical shell of infinitesimally small thickness “dx” as shown in the figure. The gravitational field strength due to thin spherical shell at a point outside shell, which is at a linear distance “r” from the center, is given by

Gravitational field due to solid sphere



The gravitational field at a distance "r" from the center of sphere.

$$dE = \frac{G dm}{r^2}$$

The gravitational field strength acts along the line towards the center of sphere. As such, we can add gravitational field strengths of individual shells to obtain the field strength of the sphere. In this case, most striking point is that the centers of all spherical shells are coincident at one point. This means that linear distance between centers of spherical shell and the point ob observation is same for all shells. In turn, we can conclude that the term “ r^2 ” is constant for all spherical shells and as such can be taken out of the integral,

$$\Rightarrow E = \int \frac{G dm}{r^2} = \frac{G}{r^2} \int dm = \frac{GM}{r^2}$$

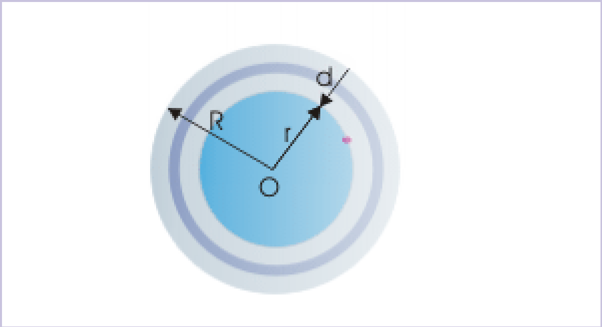
We can see here that a uniform solid sphere behaves similar to a shell. For a point outside, it behaves as if all its mass is concentrated at its center. Note that radius of the sphere, “a”, does not come into picture. Sphere behaves as a point mass for a point outside.

Gravitational field at an inside point

We have already derived this relation in the case of Earth.

For this reason, we will not derive this relation here. Nevertheless, it would be intuitive to interpret the result obtained for the acceleration (field strength) earlier,

Gravitational field inside solid sphere



The gravitational field at a distance "r" from the center of sphere.

$$\Rightarrow g\prime = g_0\left(1 - \frac{d}{R}\right)$$

Putting value of “g0” and simplifying,

$$\Rightarrow g\prime = \frac{GM}{R^2}\left(1 - \frac{d}{R}\right) = \frac{GM}{R^2}\left(\frac{R-d}{R}\right) = \frac{GMr}{R^3}$$

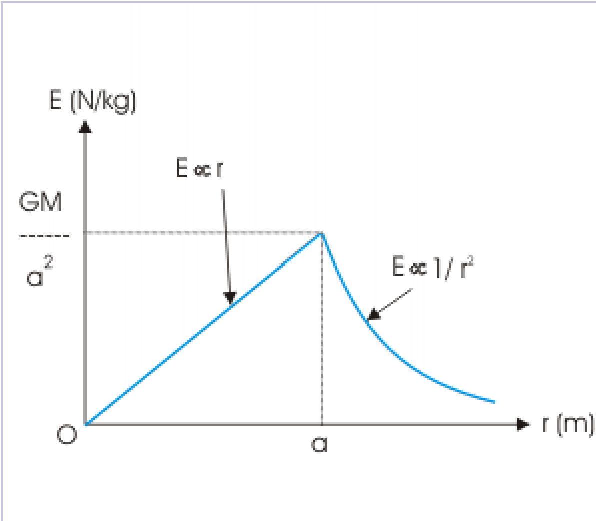
As we have considered “a” as the radius of sphere here – not “R” as in the case of Earth, we have the general expression for the field strength insider a uniform solid sphere as :

$$\Rightarrow E = \frac{GMr}{a^3}$$

The field strength of uniform solid sphere within it decreases linearly within “r” and becomes zero as we reach at the center of the sphere. A plot,

showing the gravitational field strength, is shown here for regions both inside and outside :

Gravitational field due to uniform solid sphere



The gravitational field along linear distance from center.

Gravitational field (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

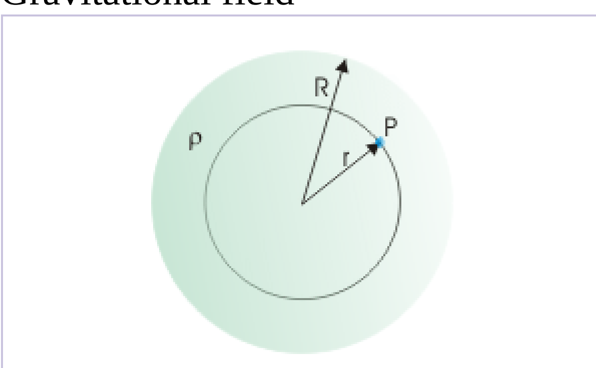
Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to gravitational field. The questions are categorized in terms of the characterizing features of the subject matter :

- Gravitational field
- Gravitational force
- Superposition principle

Gravitational field

Problem 1 : Calculate gravitational field at a distance "r" from the center of a solid sphere of uniform density, "ρ", and radius "R". Given that r < R.
Gravitational field



Gravitational field inside a solid sphere.

Solution : The point is inside the solid sphere of uniform density. We apply the theorem that gravitational field due to mass outside the sphere of radius “r” is zero at the point where field is being calculated. Let the mass of the sphere of radius “r” be “m”, then :

$$m = \frac{4}{3}\pi r^3 \rho$$

The gravitational field due to this sphere on its surface is given by :

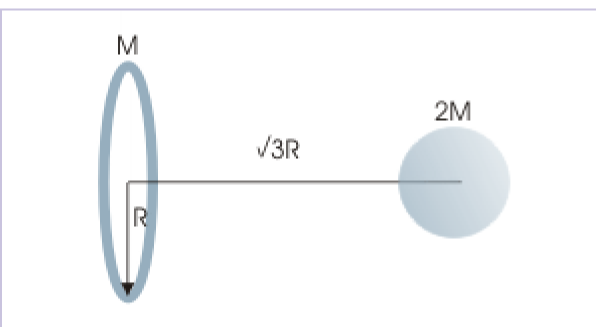
$$E = \frac{Gm}{r^2} = \frac{GX\frac{4}{3}\pi r^3 \rho}{r^2}$$

$$\Rightarrow E = \frac{4G\pi r\rho}{3}$$

Gravitational force

Problem 2 : A sphere of mass “2M” is placed a distance “√3 R” on the axis of a vertical ring of radius “R” and mass “M”. Find the force of gravitation between two bodies.

Gravitational force



The center of sphere lies on the axis of ring.

Solution : Here, we determine gravitational field due to ring at the axial position, where center of sphere lies. Then, we multiply the gravitational field with the mass of the sphere to calculate gravitational force between two bodies.

The gravitational field due to ring on its axis is given as :

$$E = \frac{GMx}{(R^2 + x^2)^{\frac{3}{2}}}$$

Putting values,

$$\Rightarrow E = \frac{GM\sqrt{3}R}{\left\{R^2 + \left(\sqrt{3}R\right)^2\right\}^{\frac{3}{2}}}$$

$$\Rightarrow E = \frac{\sqrt{3}GM}{8R^2}$$

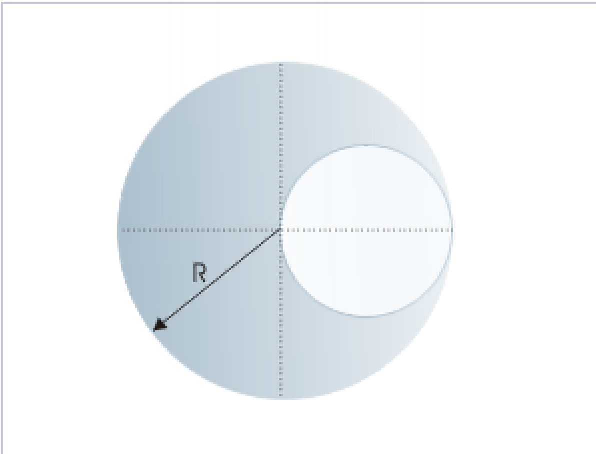
The sphere acts as a point mass. Therefore, the gravitational force between two bodies is :

$$\Rightarrow F = 2ME = \frac{2\sqrt{3}GM^2}{8R^2} = \frac{\sqrt{3}GM^2}{4R^2}$$

Superposition principle

Problem 3 : A spherical cavity is made in a solid sphere of mass "M" and radius "R" as shown in the figure. Find the gravitational field at the center of cavity due to remaining mass.

Superposition principle



The gravitational field at the center of spherical cavity

Solution : According to superposition principle, gravitational field (**E**) due to whole mass is equal to vector sum of gravitational field due to remaining mass ($\mathbf{E}_1$) and removed mass ($\mathbf{E}_2$).

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

The gravitation field due to a uniform solid sphere is zero at its center. Therefore, gravitational field due to removed mass is zero at its center. It means that gravitational field due to solid sphere is equal to gravitational field due to remaining mass. Now, we know that "**E**" at the point acts towards center of sphere. As such both "**E**" and " $\mathbf{E}_1$ " acts along same direction. Hence, we can use scalar form,

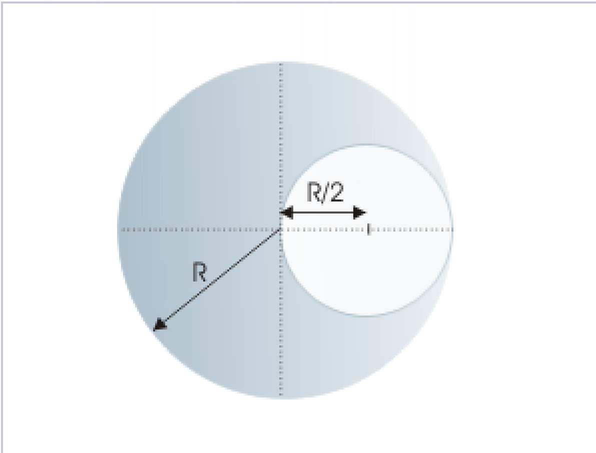
$$E_1 = E$$

Now, gravitational field due to solid sphere of radius "R" at a point "r" within the sphere is given as :

$$E = \frac{GMr}{R^3}$$

Here,

Superposition principle



The gravitational field at the center of spherical cavity

$$r = R - \frac{R}{2} = \frac{R}{2}$$

Thus,

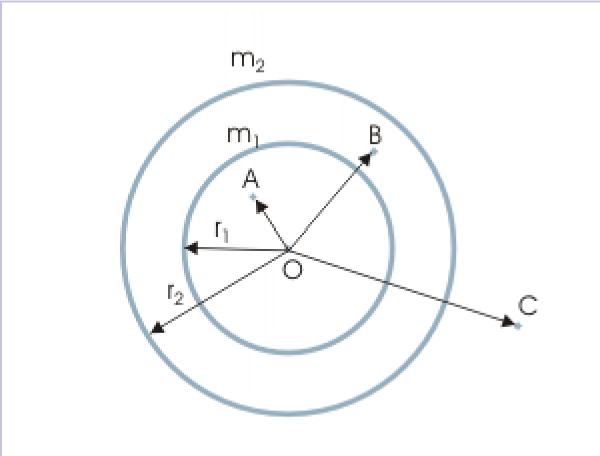
$$\Rightarrow E = \frac{GMR}{2R^3} = \frac{GM}{2R^2}$$

Therefore, gravitational field due to remaining mass, “ E_1 ”, is :

$$\Rightarrow E_1 = E = \frac{GM}{2R^2}$$

Problem 4 : Two concentric spherical shells of mass “ m_1 ” and “ m_2 ” have radii “ r_1 ” and “ r_2 ” respectively, where $r_2 > r_1$. Find gravitational intensity at a point, which is at a distance “r” from the common center for following situations, when it lies (i) inside smaller shell (ii) in between two shells and (iii) outside outer shell.

Solution : Three points "A", "B" and "C" corresponding to three given situations in the question are shown in the figure :
Superposition principle



The gravitational field at three different points

The point inside smaller shell is also inside outer shell. The gravitational field inside a shell is zero. Hence, net gravitational field at a position inside the smaller shell is zero,

$$E_1 = 0$$

The gravitational field strength due to outer shell (E_o) at a point inside is zero. On the other hand, gravitational field strength due to inner shell (E_i) at a point outside is :

$$\Rightarrow E_i = \frac{GM}{r^2}$$

Hence, net gravitational field at position in between two shells is :

$$E_2 = E_i + E_o = \frac{Gm_1}{r^2}$$

A point outside outer shell is also outside inner shell. Hence, net field strength at a position outside outer shell is :

$$E_3 = Ei + Eo = \frac{Gm_1}{r^2} + \frac{Gm_2}{r^2}$$

$$\Rightarrow E_3 = \frac{G(m_1 + m_2)}{r^2}$$

Gravitational potential

Description of force having “action at a distance” is best described in terms of force field. The “per unit” measurement is central idea of a force field. The field strength of a gravitational field is the measure of gravitational force experienced by unit mass. On a similar footing, we can associate energy with the force field. We shall define a quantity of energy that is associated with the position of unit mass in the gravitational field. This quantity is called gravitational potential (V) and is different to potential energy as we have studied earlier. Gravitational potential energy (U) is the potential energy associated with any mass - as against unit mass in the gravitational field.

Two quantities (potential and potential energy) are though different, but are closely related. From the perspective of force field, the gravitational potential energy (U) is the energy associated with the position of a given mass in the gravitational field. Clearly, two quantities are related to each other by the equation,

$$U = mV$$

The unit of gravitational potential is Joule/kg.

There is a striking parallel among various techniques that we have so far used to study force and motion. One of the techniques employs vector analysis, whereas the other technique employs scalar analysis. In general, we study motion in terms of force (vector context), using Newton’s laws of motion or in terms of energy employing “work-kinetic energy” theorem or conservation law (scalar context).

In the study of conservative force like gravitation also, we can study gravitational interactions in terms of either force (Newton’s law of gravitation) or energy (gravitational potential energy). It follows, then, that study of conservative force in terms of “force field” should also have two perspectives, namely that of force and energy. Field strength presents the perspective of force (vector character of the field), whereas gravitational potential presents the perspective of energy (scalar character of field).

Gravitational potential

The definition of gravitational potential energy is extended to unit mass to define gravitational potential.

Gravitational potential
: The gravitational potential at a point is equal to "negative" of the work by the gravitational force as a particle of unit mass is brought from infinity to its position in the gravitational field.

Or

Gravitational potential
: The gravitational potential at a point is equal to the work by the external force as a particle of unit mass is brought from infinity to its position in the gravitational field.

Mathematically,

$$V = -W_G = -\int_{\infty}^{r} \frac{F_G \mathbb{d}r}{m} = -\int_{\infty}^{r} E \mathbb{d}r$$

Here, we can consider gravitational field strength, "E" in place of gravitational force, " F_G " to account for the fact we are calculating work per unit mass.

Change in gravitational potential in a field due to point mass

The change in gravitational potential energy is equal to the negative of work by gravitational force as a particle is brought from one point to another in a gravitational field. Mathematically,

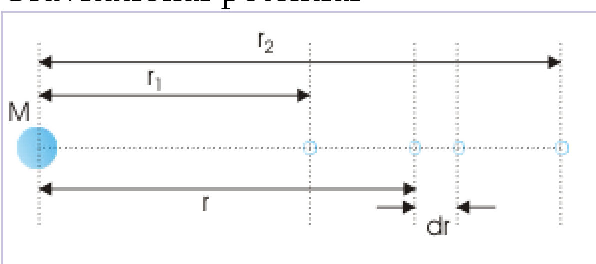
$$\Delta U = -\int_{r_1}^{r_2} F_G \mathbb{d}r$$

Clearly, change in gravitational potential is equal to the negative of work by gravitational force as a particle of unit mass is brought from one point to another in a gravitational field. Mathematically, :

$$\Rightarrow \Delta V = \frac{\Delta U}{m} = -\int_{r_1}^{r_2} E dr$$

We can easily determine change in potential as a particle is moved from one point to another in a gravitational field. In order to find the change in potential difference in a gravitational field due to a point mass, we consider a point mass "M", situated at the origin of reference. Considering motion in the reference direction of "r", the change in potential between two points at a distance "r" and "r+dr" is :

Gravitational potential



Gravitational potential difference in a gravitational field due to a point.

$$\Rightarrow \Delta V = -\int_{r_1}^{r_2} \frac{GM dr}{r^2}$$

$$\Rightarrow \Delta V = -GM\left[-\frac{1}{r}\right]_{r_1}^{r_2}$$

$$\Rightarrow \Delta V = GM\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

In the expression, the ratio " $\frac{1}{r_1}$ " is smaller than " $\frac{1}{r_2}$ ". Hence, change in gravitational potential is positive as we move from a point closer to the mass responsible for gravitational field to a point away from it.

Example

Problem 1: A particle of mass 2 kg is brought from one point to another. The increase in kinetic energy of the mass is 4 J, whereas work done by the external force is -10 J. Find potential difference between two points.

Solution : So far we have considered work by external force as equal to change in potential energy. However, if we recall, then this interpretation of work is restricted to the condition that work is done slowly in such a manner that no kinetic energy is imparted to the particle. Here, this is not the case. In general, we know from the conservation of mechanical energy that work by external force is equal to change in mechanical energy:

$$W_F = \varDelta E_{mech} = \varDelta K + \varDelta U$$

Putting values,

$$\Rightarrow -10 = 4 + \varDelta U$$

$$\Rightarrow \varDelta U = -10 - 4 = -14 \quad J$$

As the change in potential energy is negative, it means that final potential energy is less than initial potential energy. It means that final potential energy is more negative than the initial.

Potential change is equal to potential energy change per unit mass. The change in potential energy per unit mass i.e. change in potential is :

$$\Rightarrow \varDelta V = \frac{\varDelta U}{m} = -\frac{14}{2} = -7 \quad J$$

Absolute gravitational potential in a field due to point mass

The expression for change in gravitational potential is used to find the expression for the potential at a point by putting suitable values. When,

$$V_1 = 0$$

$$V_2 = V(\text{say})$$

$$r_1 = \infty$$

$$r_2 = r(\text{say})$$

$$\Rightarrow v = -\frac{GM}{r}$$

This is the expression for determining potential at a point in the gravitational field of a particle of mass “M”. We see here that gravitational potential is a negative quantity. As we move away from the particle, 1/r becomes a smaller fraction. Therefore, gravitational potential increases being a smaller negative quantity. The magnitude of potential, however, becomes smaller. The maximum value of potential is zero for r = ∞.

This relation has an important deduction. We know that particle of unit mass will move towards the particle responsible for the gravitational field, if no other force exists. This fact underlies the natural tendency of a particle to move from a higher gravitational potential (less negative) to lower gravitational potential (more negative). This deduction, though interpreted in the present context, is not specific to gravitational field, but is a general characteristic of all force fields. This aspect is more emphasized in the electromagnetic field.

Gravitational potential and field strength

A change in gravitational potential (ΔV) is equal to the negative of work by gravity on a unit mass,

$$\varDelta V = -E\varDelta r$$

For infinitesimal change, we can write the equation,

$$\Rightarrow \mathbb{d}V = -E\mathbb{d}r$$

$$\Rightarrow E = -\frac{\mathbb{d}V}{\mathbb{d}r}$$

Thus, if we know potential function, we can find corresponding field strength. In words, gravitational field strength is equal to the negative potential gradient of the gravitational field. We should be slightly careful here. This is a relationship between a vector and scalar quantity. We have taken the advantage by considering field in one direction only and expressed the relation in scalar form, where sign indicates the direction with respect to assumed positive reference direction. In three dimensional region, the relation is written in terms of a special vector operator called “grad”.

Further, we can see here that gravitational field – a vector – is related to gravitational potential (scalar) and position in scalar form. We need to resolve this so that evaluation of the differentiation on the right yields the desired vector force. As a matter of fact, we handle this situation in a very unique way. Here, the differentiation in itself yields a vector. In three dimensions, we define an operator called “grad” as :

$$\text{grad} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)$$

where " $\frac{\partial}{\partial x}$ ” is partial differentiation operator with respect to "x". This is same like normal differentiation except that it considers other dimensions (y,z) constant. In terms of “grad”,

$$\mathbf{E} = -\text{grad } V$$

Gravitational potential and self energy of a rigid body

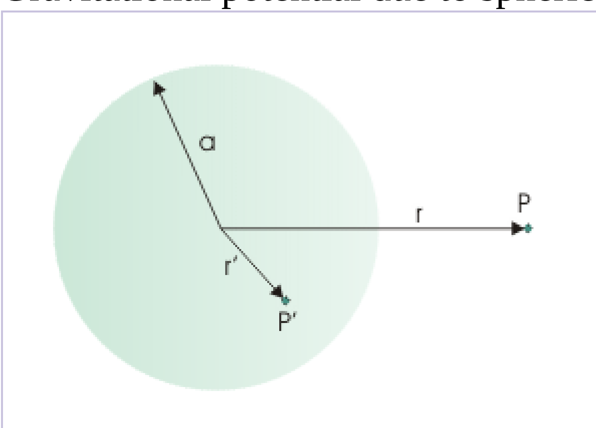
Gravitational potential energy of a particle of mass “m” is related to gravitational potential of the field by the equation,

$$U = mV$$

This relation is quite handy in calculating potential energy and hence “self energy” of a system of particles or a rigid body. If we recall, then we calculated “self energy” of a system of particles by a summation process of work in which particles are brought from infinity one by one. The important point was that the gravitational force working on the particle kept increasing as more and more particles were assembled. This necessitated to calculate work by gravitational forces due to each particle present in the region, where they are assembled.

Now, we can use the “known” expressions of gravitational potential to determine gravitational potential energy of a system, including rigid body. We shall derive expressions of potential energy for few regular geometric bodies in the next module. One of the important rigid body is spherical shell, whose gravitational potential is given as :

Gravitational potential due to spherical shell



Gravitational potential at points inside and outside a spherical shell.

For a point inside or on the shell of radius “a”,

$$V = -\frac{GM}{a}$$

This means that potential inside the shell is constant and is equal to potential at the surface.

For a point outside shell of radius “a” (at a linear distance, “r” from the center of shell) :

$$V = -\frac{GM}{r}$$

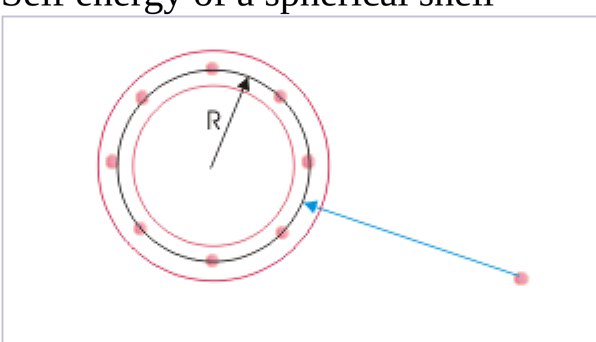
This means that shell behaves as a point mass for potential at a point outside the shell. These known expressions allow us to calculate gravitational potential energy of the spherical shell as explained in the section below.

Self energy of a spherical shell

The self potential energy is equal to work done by external force in assembling the shell bit by bit. Since zero gravitational potential energy is referred to infinity, the work needs to be calculated for a small mass at a time in bringing the same from infinity.

In order to calculate work, we draw a strategy in which we consider that some mass has already been placed symmetrically on the shell. As such, it has certain gravitational potential. When a small mass “dm” is brought, the change in potential energy is given by :

Self energy of a spherical shell



Self energy is equal to work in bringing particles one by one

from the infinity.

$$\mathbb{d}U = V\mathbb{d}m = -\frac{Gm}{R}\mathbb{d}m$$

We can determine total potential energy of the shell by integrating the expressions on either side of the equation,

$$\Rightarrow \int \mathbb{d}U = -\frac{G}{R}\int m\mathbb{d}m$$

Taking constants out from the integral on the right side and taking into account the fact that initial potential energy of the shell is zero, we have :

$$\Rightarrow U = -\frac{G}{R}\left[\frac{m^2}{2}\right]_0^M$$

$$\Rightarrow U = -\frac{GM^2}{2R}$$

This is total potential energy of the shell, which is equal to work done in bringing mass from infinity to form the shell. This expression, therefore, represents the self potential energy of the shell.

In the same manner, we can also find “self energy” of a solid sphere, if we know the expression for the gravitational potential due to a solid sphere.

Gravitational potential due to rigid body

We have derived expression for gravitational potential due to point mass of mass, “M”, as :

$$V = -\frac{GM}{r}$$

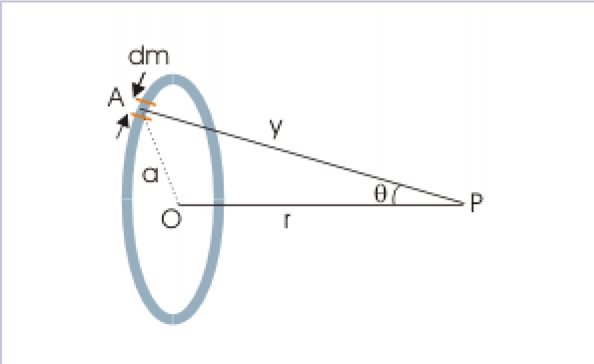
We can find expression of potential energy for real bodies by considering the same as aggregation of small elements, which can be treated as point mass. We can, then, combine the potential algebraically to find the potential due to the body.

The derivation is lot like the derivation of gravitational field strength. There is, however, one important difference. Derivation of potential expression combines elemental potential – a scalar quantity. As such, we can add contributions from elemental parts algebraically without any consideration of direction. Indeed, it is a lot easier proposition.

Again, we are interested in finding gravitational potential due to a solid sphere, which is generally the shape of celestial bodies. As discussed earlier in the course, a solid sphere is composed of spherical shells and spherical shell, in turn, is composed of circular rings of different radii. Thus, we proceed by determining expression of potential from ring --> spherical shell --> solid sphere.

Gravitational potential due to a uniform circular ring

We need to find gravitational potential at a point “P” lying on the central axis of the ring of mass “M” and radius “a”. The arrangement is shown in the figure. We consider a small mass “dm” on the circular ring. The gravitational potential due to this elemental mass is :
Gravitational potential due to a uniform circular ring



Gravitational potential at an
axial point

$$dV = -\frac{G\,dm}{PA} = -\frac{G\,dm}{\left(a^2 + r^2\right)^{\frac{1}{2}}}$$

We can find the sum of the contribution by other elements by integrating above expression. We note that all elements on the ring are equidistant from the point, “P”. Hence, all elements of same mass will contribute equally to the potential. Taking out the constants from the integral,

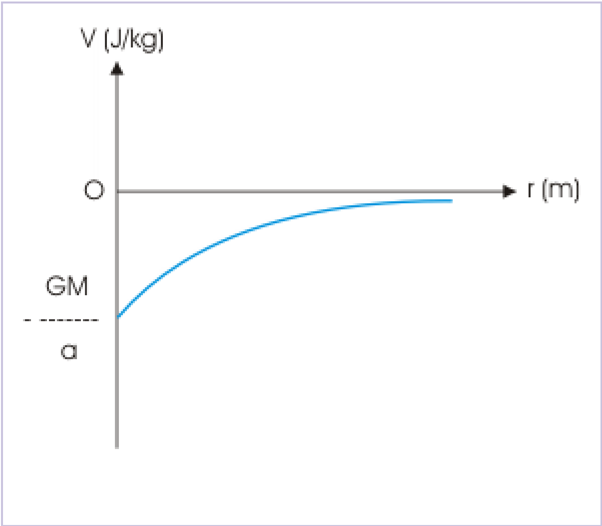
$$\Rightarrow V = -\frac{G}{\left(a^2 + r^2\right)^{\frac{1}{2}}} \int_0^M dm$$

$$\Rightarrow V = -\frac{GM}{\left(a^2 + r^2\right)^{\frac{1}{2}}} = -\frac{GM}{y}$$

This is the expression of gravitational potential due to a circular ring at a point on its axis. It is clear from the scalar summation of potential due to elemental mass that the ring needs not be uniform. As no directional attribute is attached, it is not relevant whether ring is uniform or not? However, we have kept the nomenclature intact in order to correspond to the case of gravitational field, which needs to be uniform for expression as

derived. The plot of gravitational potential for circular ring is shown here as we move away from the center.

Gravitational potential due to a uniform circular ring



The plot of gravitational potential on axial position

Check : We can check the relationship of potential, using differential equation that relates gravitational potential and field strength.

$$\Rightarrow E = -\frac{dV}{dr} = \frac{d}{dr}\left\{\frac{GM}{(a^2+r^2)^{\frac{1}{2}}}\right\}$$

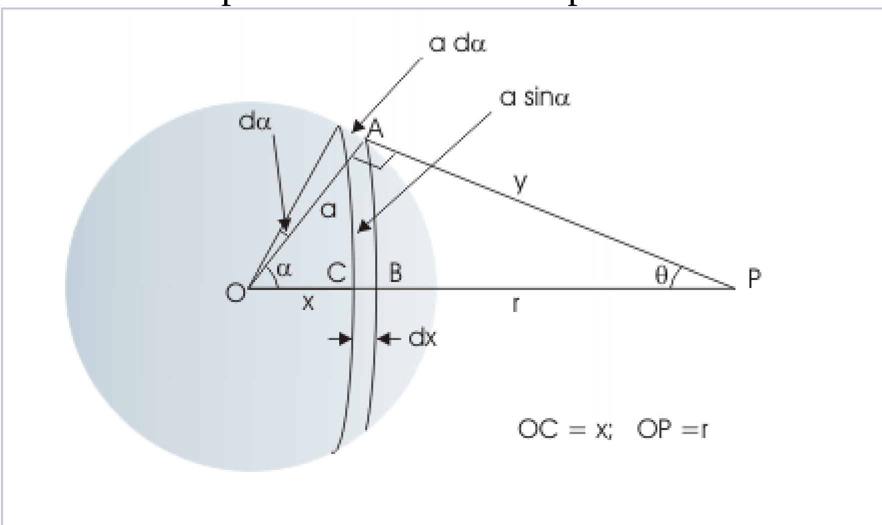
$$\Rightarrow E = GMx - \frac{1}{2}X\left(a^2+r^2\right)^{-\frac{1}{2}-1}X2r$$

$$\Rightarrow E = -\frac{GMr}{(a^2+r^2)^{\frac{3}{2}}}$$

The result is in excellent agreement with the expression derived for gravitational field strength due to a uniform circular ring.

Gravitational potential due to thin spherical shell

The spherical shell of radius “a” and mass “M” can be considered to be composed of infinite numbers of thin rings. We consider one such thin ring of infinitesimally small thickness “dx” as shown in the figure. We derive the required expression following the sequence of steps as outlined here :
Gravitational potential due to thin spherical shell



Gravitational field due to thin spherical shell at a distance "r"

(i) Determine mass of the elemental ring in terms of the mass of shell and its surface area.

$$dm = \frac{M}{4\pi a^2} X 2\pi a \sin\alpha dx = \frac{Ma\sin\alpha dx}{2a^2}$$

From the figure, we see that :

$$dx = a d\alpha$$

Putting these expressions,

$$\Rightarrow dm = \frac{Ma\sin\alpha dx}{2a^2} = \frac{Ma\sin\alpha a d\alpha}{2a^2} = \frac{M\sin\alpha d\alpha}{2}$$

(ii) Write expression for the gravitational potential due to the elemental ring. For this, we employ the formulation derived earlier,

$$dV = -\frac{Gdm}{y}$$

Putting expression for elemental mass,

$$\Rightarrow dV = -\frac{GM\sin\alpha d\alpha}{2y}$$

(i) Set up integral for the whole disc

$$\Rightarrow V = -GM\int\frac{\sin\alpha d\alpha}{2y}$$

Clearly, we need to express variables in one variable “x”. From triangle, OAP,

$$y^2 = a^2 + r^2 - 2ar\cos\alpha$$

Differentiating each side of the equation,

$$\Rightarrow 2ydy = 2ar\sin\alpha d\alpha$$

$$\Rightarrow \sin\alpha d\alpha = \frac{ydy}{ar}$$

Replacing expression in the integral,

$$\Rightarrow V = -GM\int\frac{dy}{2ar}$$

We shall decide limits of integration on the basis of the position of point “P” – whether it lies inside or outside the shell. Integrating expression on

right side between two general limits, initial (L_1) and final (L_2),

$$\Rightarrow V = -\frac{GM}{2ar}\left[y\right]_{L_1}^{L_2}$$

Case 1: Gravitational potential at a point outside

The total gravitational field is obtained by integrating the integral from y = r-a to y = r+a,

$$\Rightarrow V = -\frac{GM}{2ar}\left[y\right]_{r-a}^{r+a}$$

$$\Rightarrow V = -\frac{GM}{2ar}\left[\left[r + a - r + a\right]\right] = -\frac{GM}{2ar}2a$$

$$\Rightarrow V - \frac{GM}{r}$$

This is an important result. It again brings the fact that a spherical shell, for a particle outside it, behaves as if all its mass is concentrated at its center. In other words, a spherical shell can be considered as particle for an external point.

Check : We can check the relationship of potential, using differential equation that relates gravitational potential and field strength.

$$E = -\frac{dV}{dr} = \frac{d}{dr}\frac{GM}{r}$$

$$\Rightarrow E = -GMX - 1Xr^{-2}$$

$$\Rightarrow E = -\frac{GM}{r^2}$$

The result is in excellent agreement with the expression derived for gravitational field strength outside a spherical shell.

Case 2: Gravitational potential at a point inside

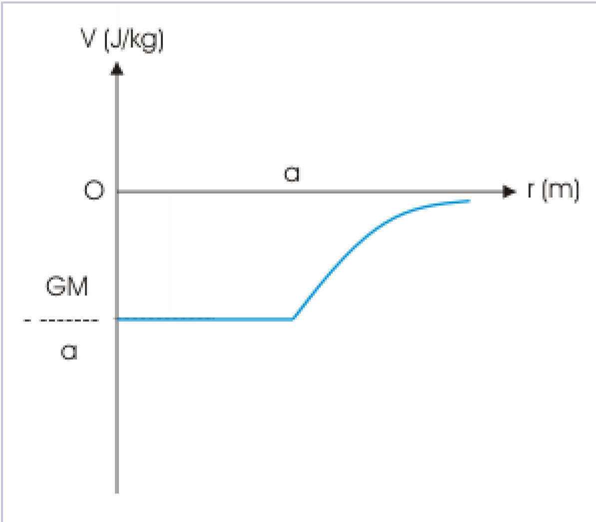
The total gravitational field is obtained by integrating the integral from y = a-r to y = a+r,

$$\Rightarrow V = -\frac{GM}{2ar}\Big[y\Big]_{a-r}^{a+r}$$

We can see here that "a-r" involves mass of the shell to the right of the point under consideration, whereas "a+r" involves mass to the left of it. Thus, total mass of the spherical shell is covered by the limits used. Now,

$$\Rightarrow V = -\frac{GM}{2ar}\Big[a + r - a + r\Big] = -\frac{GM}{2ar}2r = -\frac{GM}{a}$$

The gravitational potential is constant inside the shell and is equal to the potential at its surface. The plot of gravitational potential for spherical shell is shown here as we move away from the center.
Gravitational potential due to thin spherical shell



The plot of gravitational potential inside spherical shell

Check : We can check the relationship of potential, using differential equation that relates gravitational potential and field strength.

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\frac{GM}{a}$$

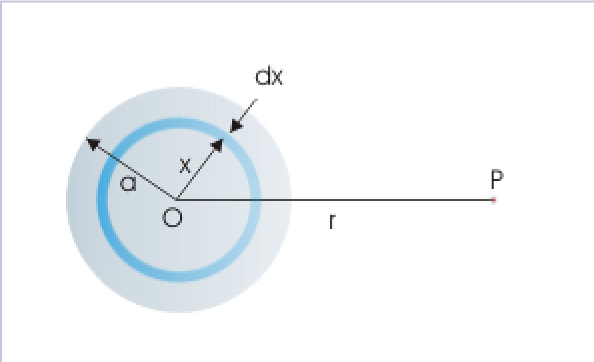
$$\Rightarrow E = 0$$

The result is in excellent agreement with the result obtained for gravitational field strength inside a spherical shell.

Gravitational potential due to uniform solid sphere

The uniform solid sphere of radius "a" and mass "M" can be considered to be composed of infinite numbers of thin spherical shells. We consider one such thin spherical shell of infinitesimally small thickness "dx" as shown in the figure.

Gravitational potential due to solid sphere



Solid sphere is composed of infinite numbers of thin spherical shells

Case 1 : The point lies outside the sphere

In this case, potential due to elemental spherical shell is given by :

$$dV = -\frac{G dm}{r}$$

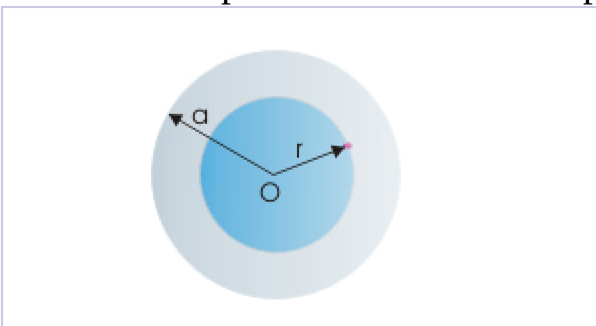
In this case, most striking point is that the centers of all spherical shells are coincident at center of sphere. This means that linear distance between centers of spherical shells and the point of observation is same for all shells. In turn, we conclude that the term "r" is constant for all spherical shells and as such can be taken out of the integral,

$$\Rightarrow V = -\frac{G}{r}\int dm$$

$$\Rightarrow V = -\frac{GM}{r}$$

Case 2 : The point lies inside the sphere

We calculate potential in two parts. For this we consider a concentric smaller sphere of radius “r” such that point “P” lies on the surface of sphere. Now, the potential due to whole sphere is split between two parts :
Gravitational potential due to solid sphere



The solid sphere is split in two parts - a smaller sphere of radius "r" and remaining part of the solid sphere.

$$V = V_S + V_R$$

Where " V_S" denotes potential due to solid sphere of radius “r” and " V_S " denotes potential due to remaining part of the solid sphere between x = r and x = a. The potential due to smaller sphere is :

$$\Rightarrow V_S = -\frac{GM\prime}{r}$$

The mass, “M’” of the smaller solid sphere is :

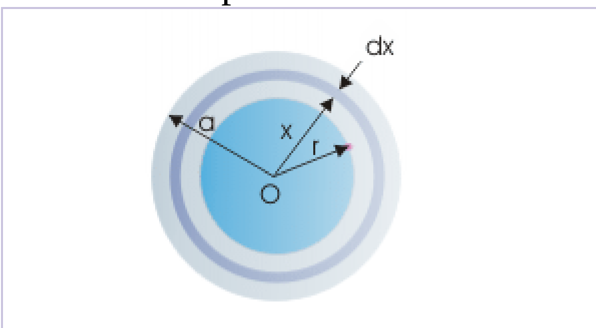
$$M\prime = \frac{3M}{4\pi a^3} X \frac{4\pi r^3}{3} = \frac{Mr^3}{a^3}$$

Putting in the expression of potential, we have :

$$\Rightarrow V_S = -\frac{GMr^2}{a^3}$$

In order to find the potential due to remaining part, we consider a spherical shell of thickness “dx” at a distance “x” from the center of sphere. The shell lies between x = r and x =a. The point “P” is inside this thin shell. As such potential due to the shell at point “P” inside it is constant and is equal to potential at the spherical shell. It is given by :

Gravitational potential due to remaining part



A spherical shell between point and the surface of solid sphere

$$\Rightarrow dV_R = -\frac{Gdm}{x}$$

We need to calculate the mass of the thin shell,

$$\Rightarrow M\prime = \frac{3M}{4\pi a^3} X 4\pi x^2 dx = \frac{3Mx^2 dx}{a^3}$$

Substituting in the expression of potential,

$$\Rightarrow dV_R = -\frac{G3Mx^2 dx}{a^3 x}$$

We integrate the expression for obtaining the potential at “P” between limits x = r and x =a,

$$\Rightarrow V_R = -\frac{3GM}{a^3}\int_r^a x dx$$

$$\Rightarrow V_R = -\frac{3GM}{a^3}\left[\frac{x^2}{2}\right]_r^a$$

$$\Rightarrow V_R = -\frac{3GM}{2a^3}\left(a^2 - r^2\right)$$

Adding two potentials, we get the expression of potential due to sphere at a point within it,

$$\Rightarrow V_R = -\frac{GMr^2}{a^3} - \frac{3GM}{2a^3}\left(a^2 - r^2\right)$$

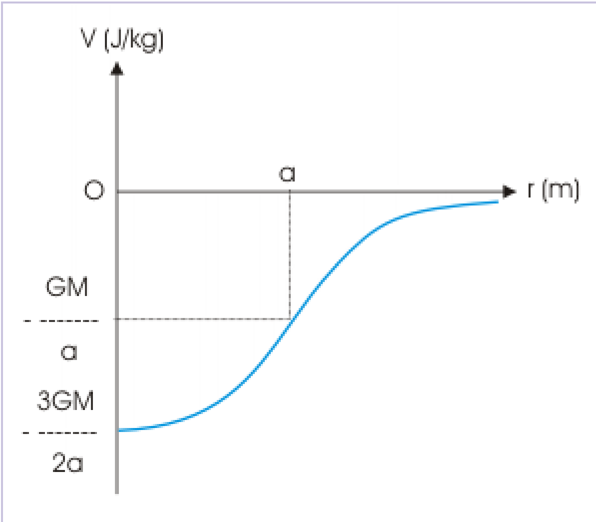
$$\Rightarrow V_R = -\frac{GM}{2a^3}\left(3a^2 - r^2\right)$$

This is the expression of gravitational potential for a point inside solid sphere. The potential at the center of sphere is obtained by putting r = 0,

$$\Rightarrow V_C = -\frac{3GM}{2a}$$

This may be an unexpected result. The gravitational field strength is zero at the center of a solid sphere, but not the gravitational potential. However, it is entirely possible because gravitational field strength is rate of change in potential, which may be zero as in this case.

The plot of gravitational potential for uniform solid sphere is shown here as we move away from the center.
Gravitational potential due to solid sphere



The plot of gravitational potential for uniform solid sphere

Gravitational potential (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to gravitational field. The questions are categorized in terms of the characterizing features of the subject matter :

- Potential
- Gravitational field
- Potential energy
- Conservation of mechanical energy

Potential

Problem 1 : A particle of mass “m” is placed at the center of a uniform spherical shell of equal mass and radius “R”. Find the potential at a distance “R/4” from the center.

Solution : The potential at the point is algebraic sum of potential due to point mass at the center and spherical shell. Hence,

$$V = -\frac{Gm}{\frac{R}{4}} - \frac{Gm}{R}$$

$$\Rightarrow V = -\frac{5Gm}{R}$$

Problem 2 : The gravitational field due to a mass distribution is given by the relation,

$$E = \frac{A}{x^2}$$

Find gravitational potential at "x".

Solution : Gravitational field is equal to negative of first differential with respect to displacement in a given direction.

$$E = -\frac{dV}{dx}$$

Substituting the given expression for "E", we have :

$$\Rightarrow \frac{A}{x^2} = -\frac{dV}{dx}$$

$$\Rightarrow dV = -\frac{Adx}{x^2}$$

Integrating between initial and final values of infinity and "x",

$$\Rightarrow \Delta V = V_f - V_i = -A \int \frac{dx}{x^2}$$

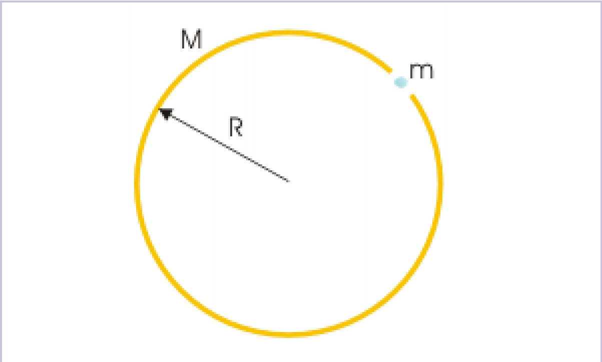
We know that potential at infinity is zero gravitational potential reference. Hence, V_i = 0. Let V_f = V, then:

$$\Rightarrow V = -A \left[-1/x \right]_{\infty}^{x} = -A \left[-\frac{1}{x} + 0 \right] = \frac{A}{x}$$

Gravitational field

Problem 3 : A small hole is created on the surface of a spherical shell of mass, “M” and radius “R”. A particle of small mass “m” is released a bit inside at the mouth of the shell. Describe the motion of particle, considering that this set up is in a region free of any other gravitational force.

Gravitational force



A particle of small mass “m” is released at the mouth of hole.

Solution : The gravitational potential of a shell at any point inside the shell or on the surface of shell is constant and it is given by :

$$V = -\frac{GM}{R}$$

The gravitational field,”E”, is :

$$E = -\frac{dV}{dr}$$

As all quantities in the expression of potential is constant, its differentiation with displacement is zero. Hence, gravitational field is zero inside the shell :

$$E = 0$$

It means that there is no gravitational force on the particle. As such, it will stay where it was released.

Potential energy

Problem 4 : A ring of mass “M” and radius “R” is formed with non-uniform mass distribution. Find the minimum work by an external force to bring a particle of mass “m” from infinity to the center of ring.

Solution : The work done in carrying a particle slowly from infinity to a point in gravitational field is equal to potential energy of the “ring-particle” system. Now, Potential energy of the system is :

$$W_F = U = mV$$

The potential due to ring at its center is independent of mass-distribution. Recall that gravitational potential being a scalar quantity are added algebraically for individual elemental mass. It is given by :

$$V = -\frac{GM}{r}$$

Hence, required work done,

$$\Rightarrow W_F = U = -\frac{GMm}{r}$$

The negative work means that external force and displacement are opposite to each other. Actually, such is the case as the particle is attracted into gravitational field, external force is applied so that particle does not acquire kinetic energy.

Conservation of mechanical energy

Problem 5 : Imagine that a hole is drilled straight through the center of Earth of mass “M” and radius “R”. Find the speed of particle of mass dropped in the hole, when it reaches the center of Earth.

Solution : Here, we apply conservation of mechanical energy to find the required speed. The initial kinetic energy of the particle is zero.

$$K_i = 0$$

On the other hand, the potential energy of the particle at the surface is :

$$Ui = mVi = -\frac{GMm}{R}$$

Let “v” be the speed of the particle at the center of Earth. Its kinetic energy is :

$$K_f = \frac{1}{2}mv^2$$

The potential energy of the particle at center of Earth is :

$$U_f = mV_f = -\frac{3GMm}{2R}$$

Applying conservation of mechanical energy,

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow 0 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{3GMm}{2R}$$

$$v = \sqrt{\left(\frac{GM}{R}\right)}$$

Artificial satellites
Artificial satellites are the backbone of modern communication systems.

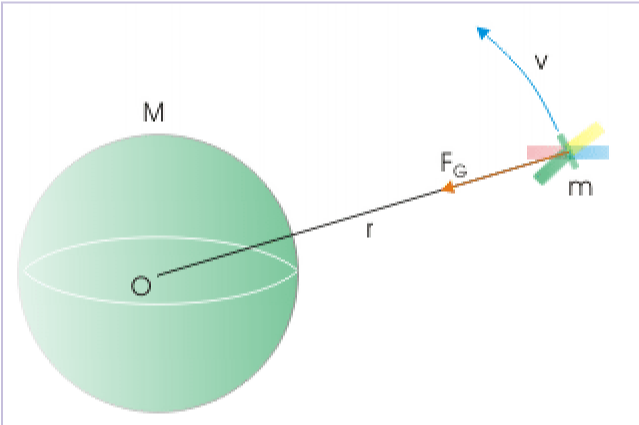
The motion of a satellite or space-station is a direct consequence of Earth’s gravity. Once launched in the appropriate orbit, these man-made crafts orbit around Earth without any propulsion. In this module, we shall study basics of satellite motion without going into details of the technology. Also, we shall develop analysis framework of artificial satellite, which can as well be extended to analysis of natural satellite like our moon. For the analysis here, we shall choose a simple framework of “two – body” system, one of which is Earth.

We should be aware that gravity is not the only force of gravitation working on the satellite, particularly if satellite is far off from Earth’s surface. But, Earth being the closest massive body, its gravitational attraction is dominant to the extent of excluding effect of other bodies. For this reason, our analysis of satellite motion as “isolated two body system” is good first approximation.

Mass of artificial satellite is negligible in comparison to that of Earth. The “center of mass” of the “two body system” is about same as the center of Earth. There is possibility of different orbits, which are essentially elliptical with different eccentricity. A satellite close to the surface up to 2000 km describes nearly a circular trajectory. In this module, we shall confine ourselves to the analysis of satellites having circular trajectory only.

Speed of the satellite

Satellites have specific orbital speed to move around Earth, depending on its distance from the center of Earth. The satellite is launched from the surface with the help of a rocket, which parks it in particular orbit with a tangential speed appropriate for that orbit. Since satellite is orbiting along a circular path, there is requirement for the provision of centripetal force, which is always directed towards the center of orbit. This requirement of centripetal force is met by the force of gravity. Hence,
Satellite



Gravitational attraction provides for the requirement of centripetal force for circular motion of satellite.

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\Rightarrow v = \sqrt{\left(\frac{GM}{r}\right)}$$

where “M” is Earth’s mass and “r” is linear distance of satellite from the "center of mass" of Earth.

The important thing to realize here are : (i) orbital speed of the satellite is independent of the mass of the satellite (ii) a satellite at a greater distance moves with lesser velocity. As the product “GM” appearing in the numerator of the expression is constant, we can see that

$$\Rightarrow v \propto \frac{1}{\sqrt{r}}$$

This conclusion is intuitive in the sense that force of gravitation is lesser as we move away from Earth’s surface and the corresponding centripetal force

as provided by gravity is smaller. As such, orbital speed is lesser.

This fact has compounding effect on the time period of the satellite. In the first place, a satellite at a greater distance has to travel a longer distance in one revolution than the satellite closer to Earth's surface. At the same time, orbital speed is lesser as we move away. It is, then, imperative that time period of revolution increases for satellite at greater distance.

We can write the equation of orbital speed in terms of acceleration due to gravity at the surface (g = g_0), which is given by :

$$g = \frac{GM}{R^2}$$

$$\Rightarrow GM = gR^2$$

Substituting in the equation of orbital velocity, we have :

$$\Rightarrow v = \sqrt{\left(\frac{GM}{r}\right)} = \sqrt{\left(\frac{gR^2}{r}\right)} = \sqrt{\left(\frac{gR^2}{R+h}\right)}$$

where "h" is the vertical height of the satellite above the surface. Rearranging,

$$\Rightarrow v = R\sqrt{\left(\frac{g}{R+h}\right)}$$

Time period of revolution

Time period of revolution is equal to time taken to travel the perimeter of circular path. The time period of rotation is :

$$T = \frac{2\pi r}{v}$$

Substituting expression of "v" as obtained earlier,

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{(GM)}}$$

This is the expression of time period for a satellite revolving in a circular orbit. Like orbital speed, the time period is also independent of the mass of the satellite. Now, squaring both sides, we have :

$$\Rightarrow T^2 = \frac{2\pi r^3}{GM}$$

Clearly, square of time period of a satellite is proportional the cube of the linear distance for the circular orbit,

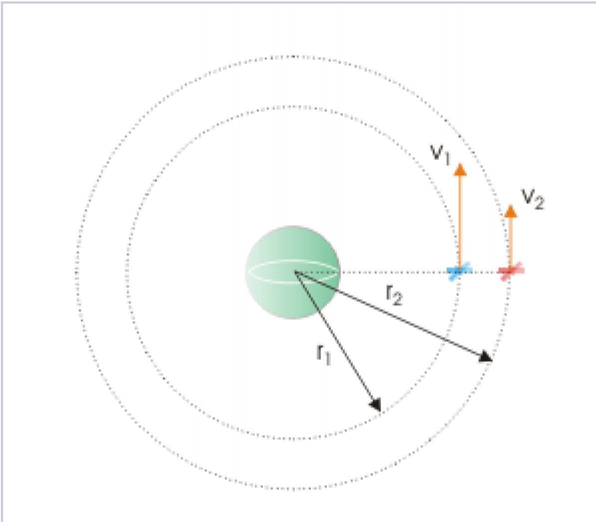
$$\Rightarrow T^2 \propto r^3$$

Example

Problem 1: Two satellites revolve around Earth along a coplanar circular orbit in the plane of equator. They move in the same sense of direction and their periods are 6 hrs and 24 hrs respectively. The satellite having period of 6 hrs is at a distance 10000 km from the center of Earth. When the satellites are at the minimum possible separation between each other, find the magnitude of relative velocity between two satellites.

Solution : Let us denote two satellites with subscripts “1” and “2”. Let the satellite designated with “1” is closer to the Earth. The positions of satellites, corresponding to minimum possible separation, are shown in the figure.

Pair of satellites



Two satellites move around Earth in two concentric circular orbits.

The distance between center of Earth and the satellite “1” is 10000 km, but this data is not available for the other satellite. However, we can evaluate other distance, using the fact that square of time period of a satellite is proportional to the cube of the linear distance for the circular orbit.

$$\frac{r_2^3}{r_1^3} = \frac{T_2^2}{T_1^2} = \left(\frac{24}{6}\right)^2 = 16$$

$$\Rightarrow \frac{r_2}{r_1} = 2$$

$$\Rightarrow r_2 = 2r_1 = 2x10^4 = 2X10^4 \quad km$$

We can now determine velocity of each satellite as :

$$v = \frac{2\pi r}{T}$$

For the first satellite,

$$\Rightarrow v_1 = \frac{2\pi X10000}{6} = \frac{10000\pi}{3}$$

For the second satellite,

$$\Rightarrow v_2 = \frac{2\pi X20000}{24} = \frac{10000\pi}{6}$$

Hence, magnitude of relative velocity is :

$$\Rightarrow v_1 - v_2 = \frac{\pi}{6} X10000 = 5238 \quad km/hr$$

Energy of “Earth-satellite” system

For consideration of energy, Earth can be treated as particle of mass “M”. Thus, potential energy of “Earth – satellite” as two particles system is given by :

$$U = -\frac{GMm}{R}$$

Since expression of orbital speed of the satellite is known, we can also determine kinetic energy of the satellite as :

$$K = \frac{1}{2}mv^2$$

Putting expression of speed, “v”, as determined before,

$$\Rightarrow K = \frac{GMm}{2r}$$

Note that kinetic energy of the satellite is positive, which is consistent with the fact that kinetic energy can not be negative. Now, mechanical energy is algebraic sum of potential and kinetic energy. Hence, mechanical energy of “Earth – satellite” system is :

$$\Rightarrow E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$\Rightarrow E = -\frac{GMm}{2r}$$

Here, total mechanical energy of the system is negative. We shall subsequently see that this is characteristic of a system, in which bodies are bounded together by internal force.

Relation among energy types

The expression of mechanical energy of the "Earth – satellite" system is typical of two body system in which one body revolves around other along a circular path. Particularly note the expression of each of the energy in the equation,

$$E = K + U$$

$$\Rightarrow -\frac{GMm}{2r} = \frac{GMm}{2r} - \frac{GMm}{r}$$

Comparing above two equations, we see that magnitude of total mechanical energy is equal to kinetic energy, but different in sign. Hence,

$$E = -K$$

Also, we note that total mechanical energy is half of potential energy. Hence,

$$E = \frac{U}{2}$$

These relations are very significant. We shall find resemblance of forms of energies in the case of Bohr's orbit as well. In that case, nucleus of hydrogen atom and electron form the two – body system and are held together by the electrostatic force.

Importantly, it provides an unique method to determine other energies, if we know any of them. For example, if the system has mechanical energy of $-200X10^6 \quad J$, then :

$$K = -E = -\left(-200X10^6\right) = 200X10^6 \quad J$$

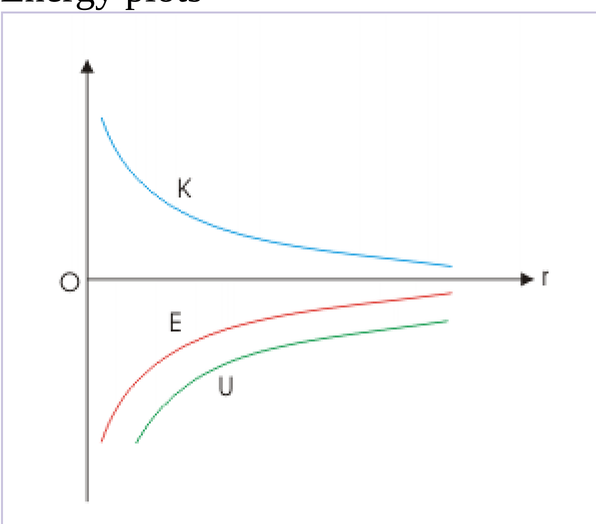
and

$$U = 2E = -400X10^6 \quad J$$

Energy plots of a satellite

An inspection of the expression of energy forms reveals that that linear distance “r” is the only parameter that can be changed for a satellite of given mass, “m”. From these expressions, it is also easy to realize that they have similar structure apart from having different signs. The product “GMm” is divided by “r” or “2r”. This indicates that nature of variation in their values with linear distance “r” should be similar.

Energy plots



Plots of kinetic, potential and mechanical energy .vs. distance

Since kinetic energy is a positive quantity, a plot of kinetic energy .vs. linear distance, “r”, is a hyperbola in the first quadrant. The expression of mechanical energy is exactly same except for the negative sign. Its plot with linear distance, therefore, is an inverted replica of kinetic energy plot in fourth quadrant. Potential energy is also negative like mechanical energy. Its plot also falls in the fourth quadrant. However, magnitude of potential energy is greater than that of mechanical energy as such the plot is displaced further away from the origin as shown in the figure.

From plots, we can conclude one important aspect of zero potential reference at infinity. From the figure, it is clear that as the distance increases and becomes large, not only potential energy, but kinetic energy also tends to become zero. We can, therefore, conclude that an object at infinity possess zero potential and kinetic energy. In other words, mechanical energy of an object at infinity is considered zero.

Gravitational binding energy

A system is bounded when constituents of the system are held together. The “Earth-satellite” system is a bounded system as members of the system are held together by gravitational attraction. Subsequently, we shall study such other bounded systems, which exist in other contexts as well. Bounded system of nucleons in a nucleus is one such example.

The characterizing aspect of a bounded system is that mechanical energy of the system is negative. However, we need to qualify that it is guaranteed to be negative when zero reference potential energy is at infinity.

Let us check out this requirement for the case of “Earth-satellite” system. The mechanical energy of Earth- satellite system is indeed negative :

$$E = -\frac{G\,\mathrm{Mm}}{2r}$$

where “M” and “m” are the mass of Earth and satellite. Hence, "Earth - satellite" system is a bounded system.

We can infer from the discussion of a bounded system that the "binding energy" is the amount of energy required to disintegrate (dismember) a bounded system. For example, we can consider a pebble lying on Earth's surface. What is the energy required to take this pebble far off in the interstellar space, where Earth's gravity ceases to exist? We have seen that infinity serves as a theoretical reference, where gravitational field ceases to exits. Further, if we recall, then potential energy is defined as the amount of work done by external agency to bring a particle slowly from infinity to a position in gravitational field. The work by external force is negative as its acts opposite to the displacement. Clearly, taking pebble to the infinity is reverse action. Work by external force is in the direction of displacement. As such, work done in this case is positive. Therefore, binding energy of the pebble is a positive quantity and is equal to the magnitude of potential energy for the pebble. If its mass is "m", then binding energy of the "Earth-pebble" system is :

$$\Rightarrow E_B = -U = -\left(-\frac{G\,\mathrm{Mm}}{r}\right) = \frac{G\,\mathrm{Mm}}{r}$$

where "M" and "m" are the mass of Earth and pebble respectively and "R" is the radius of Earth.

This is, however, a specific description of dismembering process. In general, a member of the system will have kinetic energy due to its motion. Let us consider the case of "Earth-satellite" system. The satellite has certain kinetic energy. If we want to take this satellite to infinity, we would first require to bring the satellite to a dead stop and then take the same to infinity. Therefore, binding energy of the system is a positive quantity, which is equal to the magnitude of the mechanical energy of the system.

Binding energy

Binding energy is equal to the modulus of mechanical energy.

Going by the definition, the binding energy of the "Earth-satellite" system is :

$$\Rightarrow E_B = -E = -\left(-\frac{G\,\mathrm{Mm}}{2r}\right) = \frac{G\,\mathrm{Mm}}{2r}$$

where “r” is the linear distance between the center of Earth and satellite.

Satellite systems

The satellites are made to specific tasks. One of the most significant applications of artificial satellite is its use in telecast around the world. Earlier it was difficult to relay telecast signals due to spherical shape of Earth. In recent time, advancements in communication have brought about astounding change in the way we live. The backbone of this communication wonder is variety of satellite systems orbiting around Earth.

Satellite systems are classified for different aspects of satellite motion. From the point of physics, it is the orbital classification of satellite systems, which is more interesting. Few of the famous orbits are described here. Almost all orbits generally describe an elliptical orbit. We shall discuss elliptical orbits in the module dedicated to Kepler’s law. For the present, however, we can approximate them to be circular for analysis purpose.

1: Geocentric orbit : It is an orbit around Earth. This is the orbit of artificial satellite, which is launched to revolve around Earth. Geocentric orbit is further classified on the basis of distance from Earth’s surface (i) low Earth orbit up to 2000 km (ii) middle Earth orbit between 2000 and geo-synchronous orbit (36000 km) and (iii) high Earth orbit above geo-synchronous orbit (36000 km).

2: Heliocentric Orbit : It is an orbit around Sun. The orbits of planets and all other celestial bodies in the solar system describe heliocentric orbits.

3: Geosynchronous Orbit : The time period of this orbit is same as the time period of Earth.

4: Geostationary Orbit : The plane of rotation is equatorial plane. The satellite in this orbit has time period equal to that of Earth. Thus, motion of satellite is completely synchronized with the motion of Earth. The sense of

rotation of the satellite is same as that of Earth. The satellite, therefore, is always above a given position on the surface. The orbit is at a distance of 36000 km from Earth’s surface and about 42400 (= 36000 + 6400) km from the center of Earth. The orbit is also known as Clarke’s orbit after the name of author, who suggested this orbit.

5: Molniya Orbit – It is an orbit having inclination of 63.4° with respect to equatorial plane and orbital period equal to half that of Earth.

6: Polar orbit : The orbit has an inclination of 90° with respect to the equatorial plane and as such, passes over Earth’s poles.

Another important classification of satellite runs along the uses of satellites. Few important satellite types under this classification are :

1: Communication satellites : They facilitate communication around the world. The geostationary satellite covers ground locations, which are close to equator. Geostationary satellites appears low from a positions away from equator. For locations at different latitudes away from equator, we need to have suitably designed orbits so that the area can be covered round the clock. Molniya orbit is one such orbit, which is designed to provide satellite coverage through a satellite system, consisting of more than one satellite.

2: Astronomical satellites : They are designed for studying celestial bodies.

3: Navigational satellites : They are used to specify location on Earth and develop services based on navigation.

4: Earth observation satellites : They are designed for studying Earth system, environment and disaster management.

5: Weather satellites : They facilitate to monitor weather and related services.

6: Space station : It is an artificial structure in space for human beings to stay and do assigned experiments/works

As a matter of fact, there is quite an elaborate classification system. We have only named few important satellite systems. In particular, there are varieties of satellite systems, including reconnaissance satellites, to meet military requirement.

Acknowledgment

Author wishes to thank Arunabha guha, Physics dept, Georgian court university, Lakewood, New jersey, USA for pointing out a mistake in the example contained in this module.

Projection in gravitational field

Gravitational force of attraction is a binding force. An object requires certain minimum velocity to break free from this attraction. We are required to impart object with certain kinetic energy to enable it to overcome gravitational pull. As the object moves away, gravitational pull becomes smaller. However, at the same time, speed of the object gets reduced as kinetic energy of the object is continuously transferred into potential energy. Remember, potential energy is maximum at the infinity.

Depending on the initial kinetic energy imparted to the projectile, it will either return to the surface or will move out of the Earth's gravitational field.

The motion of a projectile, away from Earth's surface, is subjected to variable force – not a constant gravity as is the case for motion near Earth's surface. Equivalently, acceleration due to gravity, "g", is no more constant at distances thousands of kilometers away. As such, equations of motion that we developed and used (like v = u+at) for constant acceleration is not valid for motion away from Earth.

We have already seen that analysis using energy concept is suitable for such situation, when acceleration is not constant. We shall, therefore, develop analysis technique based on conservation of energy.

Context of motion

We need to deal with two forces for projectile : air resistance i.e. friction and gravitational force. Air resistance is an external non-conservative force, whereas gravity is an internal conservative force to the "Earth-projectile" system. The energy equation for this set up is :

$$W_F = \Delta K + \Delta U$$

Our treatment in the module, however, will neglect air resistance for mathematical derivation. This is a base consideration for understanding motion of an object in a gravitational field at greater distances. Actual motion will not be same as air resistance at higher velocity generates

tremendous heat and the projectile, as a matter of fact, will either burn up or will not reach the distances as predicted by the analysis. Hence, we should keep this limitation of our analysis in mind.

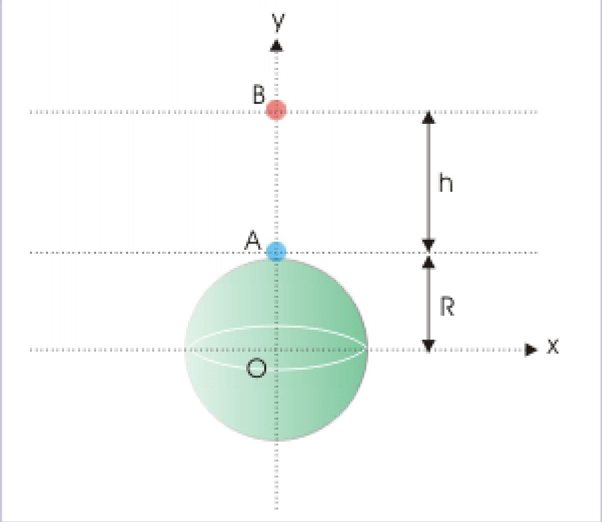
Nevertheless, the situation without friction is an ideal situation to apply law of conservation of energy. There is only conservative force in operation on the object in translation. The immediate consequence is that work by this force is independent of path. As there is no external force on the system, the changes takes place between potential and kinetic energy in such a manner that overall change in mechanical energy always remains zero. In other words, only transfer of energy between kinetic and gravitational potential energy takes place. As such,

$$\Rightarrow \Delta K + \Delta U = 0$$

Change in potential energy

Earlier, we used the expression "mgh" to compute potential energy or change in potential energy. We need to correct this formula for determining change in potential energy by referring calculation of potential energy to infinity. Using formula of potential energy with infinity as reference, we determine the potential difference between Earth's surface and a point above it, as :

Gravitational potential difference



An object at a height "h"

$$\Rightarrow \Delta U = -\frac{GMm}{(R+h)} - \left(-\frac{GMm}{R}\right)$$

$$\Rightarrow \Delta U = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

We can eliminate reference to gravitational constant and mass of Earth by using relation of gravitational acceleration at Earth’s surface ($g = g_0$),

$$g = \frac{GM}{R^2}$$

$$\Rightarrow GM = gR^2$$

Substituting in the equation of change in potential energy, we have :

$$\Rightarrow \Delta U = mgR^2\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$\Rightarrow \Delta U = mgR^2 X \frac{h}{R(R+h)}$$

$$\Rightarrow \Delta U = \frac{mgh}{1+\frac{h}{R}}$$

It is expected that this general formulation for the change in potential energy should be reduced to approximate form. For h<<R, we can neglect “h/R” term and,

$$\Rightarrow \Delta U = mgh$$

Maximum Height

For velocity less than escape velocity (the velocity at which projectile escapes the gravitation field of Earth), the projected particle reaches a maximum height and then returns to the surface of Earth.

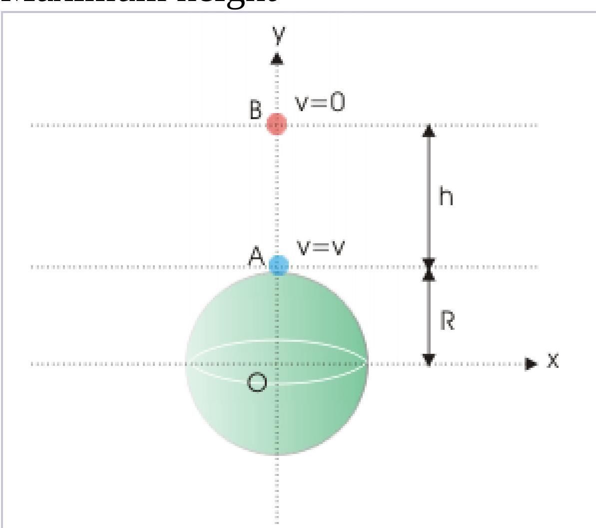
When we consider that acceleration due to gravity is constant near Earth's surface, then applying conservation of mechanical energy yields :

$$\frac{1}{2}mv^2 + 0 = 0 + mgh$$

$$\Rightarrow h = \frac{v^2}{2g}$$

However, we have seen that "mgh" is not true measure of change in potential energy. Like in the case of change in potential energy, we come around the problem of variable acceleration by applying conservation of mechanical energy with reference to infinity.

Maximum height



The velocity is zero at maximum height, "h".

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow -\frac{GMm}{R} + \frac{1}{2}mv^2 = 0 + -\frac{GMm}{R+h}$$

$$\Rightarrow \frac{GM}{R+h} = \frac{GM}{R} - \frac{v^2}{2}$$

$$\Rightarrow R+h = \frac{GM}{\frac{GM}{R} - \frac{v^2}{2}}$$

$$\Rightarrow h = \frac{GM}{\frac{GM}{R} - \frac{v^2}{2}} - R$$

$$\Rightarrow h = \frac{GM - GM + \frac{v^2R}{2}}{\frac{GM}{R} - \frac{v^2}{2}}$$

$$\Rightarrow h = \frac{v^2R}{\frac{2GM}{R} - v^2}$$

We can also write the expression of maximum height in terms of acceleration at Earth's surface using the relation :

$$\Rightarrow GM = gR^2$$

Substituting in the equation and rearranging,

$$\Rightarrow h = \frac{v^2}{2g - \frac{v^2}{R}}$$

This is the maximum height attained by a projection, which is thrown up from the surface of Earth.

Example

Problem 1: A particle is projected vertically at 5 km/s from the surface Earth. Find the maximum height attained by the particle. Given, radius of Earth = 6400 km and g = 10 m/s^2.

Solution : We note here that velocity of projectile is less than escape velocity 11.2 km/s. The maximum height attained by the particle is given by:

$$h = \frac{v^2}{2g - \frac{v^2}{R}}$$

Putting values,

$$\Rightarrow h = \frac{\left(5X10^3\right)^2}{2X10 - \frac{\left(5X10^3\right)^2}{\left(6.4X10^6\right)}}$$

$$\Rightarrow h = \frac{\left(25X10^6\right)}{2X10 - \frac{\left(25X10^6\right)}{\left(6.4X10^6\right)}}$$

$$\Rightarrow h = 1.55x10^6 = 1550000 \quad m = 1550 \quad km$$

It would be interesting to compare the result, if we consider acceleration to be constant. The height attained is :

$$h = \frac{v^2}{2}g = 25X\frac{10^6}{20} = 1.25X10^6 = 1250000 \quad m = 1250 \quad km$$

As we can see, approximation of constant acceleration due to gravity, results in huge discrepancy in the result.

Escape velocity

In general, when a body is projected up, it returns to Earth after achieving a certain height. The height of the vertical flight depends on the speed of

projection. Greater the initial velocity greater is the height attained.

Here, we seek to know the velocity of projection for which body does not return to Earth. In other words, the body escapes the gravitational influence of Earth and moves into interstellar space. We can know this velocity in verities of ways. The methods are equivalent, but intuitive in approach. Hence, we shall present here all such considerations :

1: Binding energy :

Gravitational binding energy represents the energy required to eject a body out of the influence of a gravitational field. It is equal to the energy of the system, but opposite in sign. In the absence of friction, this energy is the mechanical energy (sum of potential and kinetic energy) in gravitational field.

Now, it is clear from the definition of binding energy itself that the initial kinetic energy of the projection should be equal to the binding energy of the body in order that it moves out of the gravitational influence. Now, the body to escape is at rest before being initiated in projection. Thus, its binding energy is equal to potential energy only.

Therefore, kinetic energy of the projection should be equal to the magnitude of potential energy on the surface of Earth,

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

where " v_e " is the escape velocity. Note that we have used "R" to denote Earth's radius, which is the distance between the center of Earth and projectile on the surface. Solving above equation for escape velocity, we have :

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)}$$

2. Conservation of mechanical energy :

The act of putting a body into interstellar space is equivalent to taking the body to infinity i.e. at a very large distance. Infinity, as we know, has been used as zero potential energy reference. The reference is also said to represent zero kinetic energy.

From conservation of mechanical energy, it follows that total mechanical energy on Earth's should be equal to mechanical energy at infinity i.e. should be equal to zero. But, we know that potential energy at the surface is given by :

$$U = -\frac{GMm}{R}$$

On the other hand, for body to escape gravitational field,

$$K + U = 0$$

Therefore, kinetic energy required by the projectile to escape is :

$$\Rightarrow K = -U = \frac{GMm}{R}$$

Now, putting expression for kinetic energy and proceeding as in the earlier derivation :

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)}$$

3: Final velocity is not zero :

We again use conservation of mechanical energy, but with a difference. Let us consider that projected body of mass, "m" has initial velocity "u" and an intermediate velocity, "v", at a height "h". The idea here is to find condition

for which intermediate velocity ,”v”, never becomes zero and hence escape Earth’s influence. Applying conservation of mechanical energy, we have :

$$E_i = E_f$$

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mu^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{R+h}$$

Rearranging,

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{GMm}{R} + \frac{GMm}{R+h}$$

In order that, final velocity (“v”) is positive, the expressions on the right should evaluate to a positive value. For this,

$$\Rightarrow \frac{1}{2}mu^2 \geq \frac{GMm}{R}$$

For the limiting case, u = v_e ,

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)}$$

Interpreting escape velocity

These three approaches to determine escape velocity illustrates how we can analyze a given motion in gravitational field in many different ways. We should be aware that we have determined the minimum velocity required to escape Earth’s gravity. It is so because we have used the expression of potential energy, which is defined for work by external force slowly.

However, it is found that the velocity so calculated is good enough for escaping gravitational field. Once projected body achieves considerable

height, the gravitational attraction due to other celestial bodies also facilitates escape from Earth's gravity.

Further, we can write the expression of escape velocity in terms of gravitational acceleration (consider $g = g_0$),

$$g = \frac{GM}{r^2}$$

$$\Rightarrow \frac{GM}{r} = gr$$

Putting in the expression of escape velocity, we have :

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)} = \sqrt{(2gR)}$$

Escape velocity of Earth

In the case of Earth,

$$M = 5.98X10^{24} \quad kg$$

$$R = 6.37X10^6 \quad m$$

$$\Rightarrow v_e = \sqrt{\left(\frac{2X6.67X10^{-11}X5.98X10^{24}}{6.37X10^6}\right)}$$

$$\Rightarrow v_e = 11.2 \quad km/s$$

We should understand that this small numerical value is deceptive. Actually, it is almost impossible to impart such magnitude of speed. Let us have a look at the magnitude in terms of “km/hr”,

$$v_e = 11.2 \quad km/s = 11.2X60x60 = 40320 \quad km/hr$$

If we compare this value with the speed of modern jet (which moves at 1000 km/hr), this value is nearly 40 times! It would be generally a good idea to project object from an artificial satellite instead, which itself may move at great speed of the order of about 8-9 km/s. The projectile would need only additional 2 or 3 km/hr of speed to escape, if projected in the tangential direction of the motion of the satellite.

This mechanism is actually in operation in multistage rockets. Each stage acquires the speed of previous stage. The object (probe or vehicle) can, then, be let move on its own in the final stage to escape Earth's gravity. The mechanism as outlined here is actually the manner an interstellar probe or vehicle is sent out of the Earth's gravitational field. We can also appreciate that projection, in this manner, has better chance to negotiate friction effectively as air resistance at higher altitudes is significantly less or almost negligible.

This discussion of escape velocity also underlines that the concept of escape velocity is related to object, which is not propelled by any mechanical device. An object, if propelled, can escape gravitational field at any speed.

Escape velocity of Moon :

In the case of Earth's moon,

$$M = 7.4X10^{20} \quad kg$$

$$R = 1.74X10^{6} \quad m$$

$$\Rightarrow v_e = \sqrt{\left(\frac{2X6.67X10^{-11}X7.4X10^{20}}{1.74X10^{6}}\right)}$$

$$\Rightarrow v_e = 2.4 \quad km/s$$

The root mean square velocity of gas is greater than this value. This is the reason, our moon has no atmosphere. Since sound requires a medium to propagate, we are unable to talk directly there as a consequence of the absence of atmosphere.

Direction of projection

It may appear that we may need to fire projectile vertically to let it escape in interstellar space. This is not so. The spherical symmetry of Earth indicates that we can project body in any direction with the velocity as determined such that it clears physical obstructions in its path. From this point of view, the term “velocity” is a misnomer as direction of motion is not involved. It would have been more appropriate to call it “speed”.

The direction, however, makes a difference in escape velocity for some other reason. The Earth rotates in particular direction – it rotates from East to west at a linear speed of 465 m/s. So if we project the body in the tangential direction east-ward, then Earth rotation helps body’s escape. The effective escape velocity is 11200 – 465 = 10735 m/s. On the other hand, if we project west-ward, then escape velocity is 11200 + 465 = 11635 m/s.

Escape velocity and Black hole

Black holes are extremely high density mass. This represents the final stage of evolution of a massive star, which collapses due to its own gravitational force. Since mass remains to be very large while radius is reduced (in few kms), the gravitational force becomes extremely large. Such great is the gravitational force that it does not even allow light to escape.

Lesser massive star becomes neutron star instead of black hole. Even neutron star has very high gravitational field. We can realize this by calculating escape velocity for one such neutron star,

$$M = 3X10^{30} \quad kg$$

$$R = 3X10^{4} \quad m$$

$$\Rightarrow v_e = \sqrt{\left(\frac{2X6.67X10^{-11}X3X10^{30}}{3X10^{4}}\right)}$$

$$\Rightarrow v_e = 1.1X10^{5} \quad m/s$$

It is quite a speed comparable with that of light. Interstellar Black hole is suggested to be 5 times the mass of neutron star and 10 times the mass of sun! On the other hand, its dimension is in few kilometers. For this reason, following is possible :

$$\Rightarrow v_e = \sqrt{\left(\frac{2GM}{R}\right)} > c$$

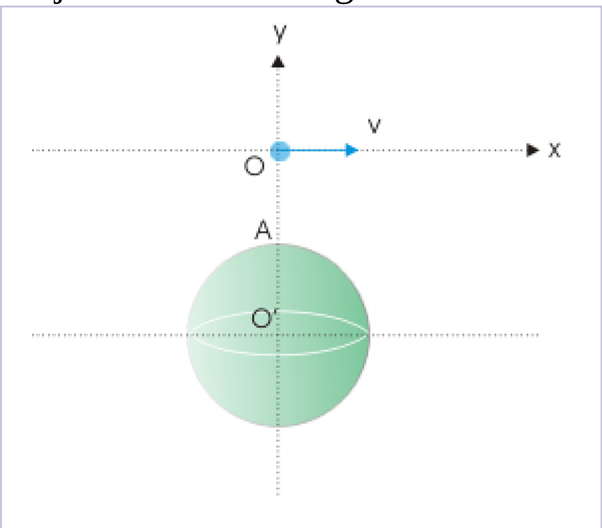
where “c” is the speed of light. Hence even light will not escape the gravitational force of a black hole as the required velocity for escape is greater than speed of light.

Nature of trajectory

In this section, we shall attempt to analyze trajectory of a projectile for different speed range. We shall strive to get the qualitative assessment of the trajectory – not a quantitative one.

In order to have a clear picture of the trajectory of a projectile, let us assume that a projectile is projected from a height, in x-direction direction as shown. The point of projection is, though, close to the surface; but for visualization, we have shown the same at considerable distance in terms of the dimension of Earth.

Projection in Earth's gravitation



Projectile is projected with
certain velocity in x-direction.

Let “ v_O ” be the speed of a satellite near Earth’s surface and “ v_e ” be the escape velocity for Earth’s gravity. Then,

$$v_O = \sqrt{(gR)}$$

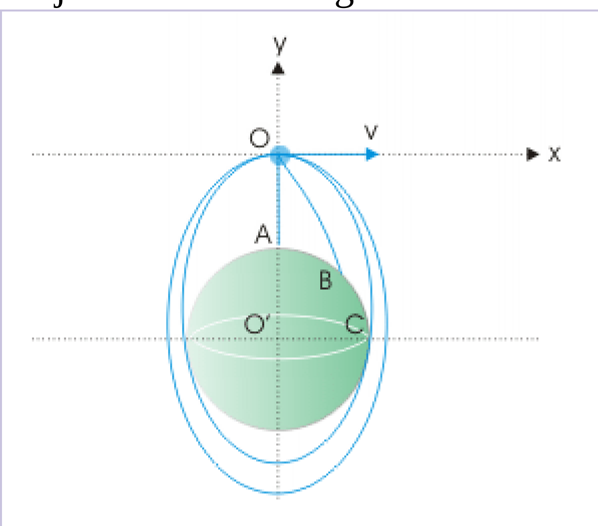
$$v_e = \sqrt{(2gR)}$$

Different possibilities are as following :

1 : $v = 0$: The gravity pulls the projectile back on the surface. The trajectory is a straight line (OA shown in the figure below).

2 : $v < v_C$: We denote a projection velocity “ v_C ” of the projectile such that it always clears Earth’s surface (OC shown in the figure below). A limiting trajectory will just clear Earth’s surface. If the projection velocity is less than this value then the trajectory of the projectile will intersect Earth and projectile will hit the surface (OB shown in the figure below).

Projection in Earth's gravitation



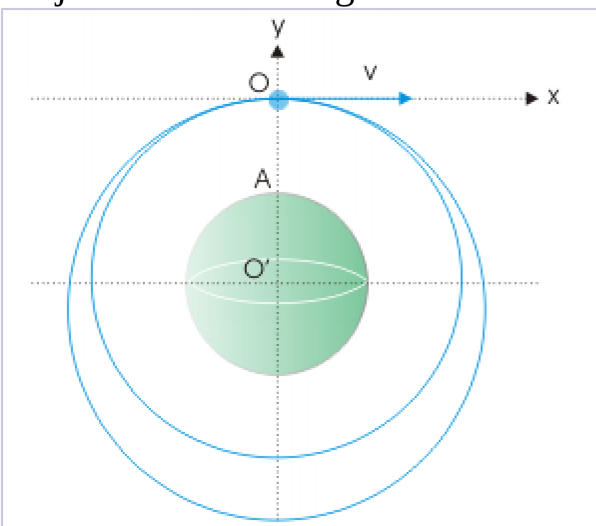
Projectile is projected with

certain velocity in x-direction.

3 : $v_C < v < v_O$: Since projection velocity is greater than limiting velocity to clear Earth and less than the benchmark velocity of a satellite in circular orbit, the projectile will move along an elliptical orbit. The Earth will be at one of the foci of the elliptical trajectory (see figure above).

4 : $v = v_O$: The projectile will move along a circular trajectory (see inner circle in the figure below).

Projection in Earth's gravitation

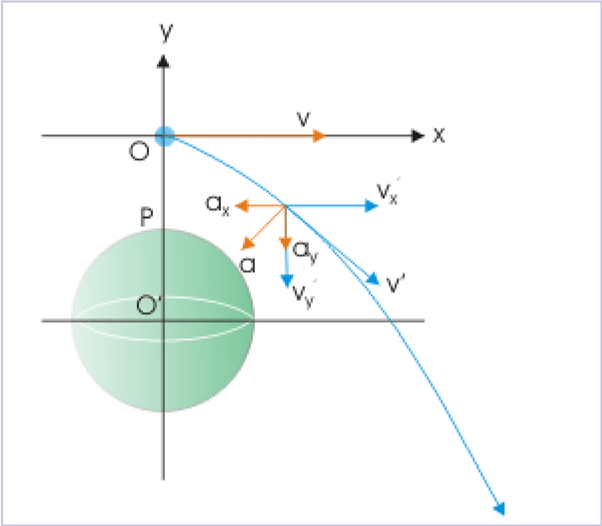


Projectile is projected with certain velocity in x-direction.

5 : $v_O < v < v_e$: The projection velocity is greater than orbital velocity for circular trajectory, the path of the projectile is not circular. On the other hand, since projection velocity is less than escape velocity, the projectile will not escape gravity either. It means that projectile will be bounded to the Earth. Hence, trajectory of the projectile is again elliptical with Earth at one of the foci (see outer ellipse in the figure above).

6 : $v = v_e$: The projectile will escape gravity. In order to understand the nature of trajectory, we can think of force acting on the particle and resulting motion. The gravity pulls the projectile in the radial direction towards the center of Earth. Thus, projectile will have acceleration in radial direction all the time. The component of gravity along x-direction is opposite to the direction of horizontal component of velocity. As such, the particle will be retarded in x-direction. On the other hand, vertical component of gravity will accelerate projectile in the negative y – direction.

Projection in Earth's gravitation

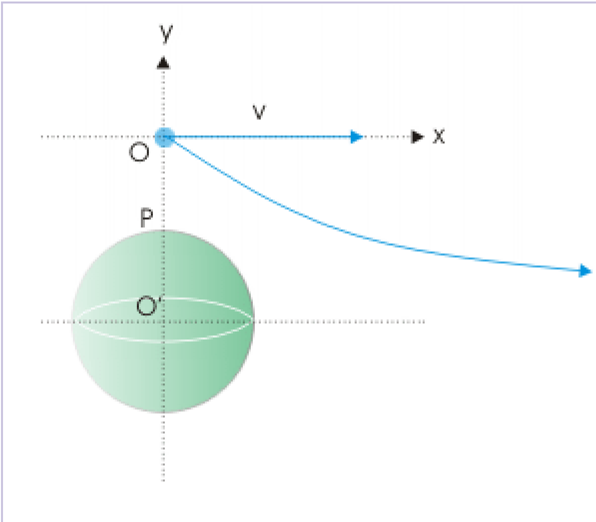


Projectile is projected with certain velocity in x-direction.

However, as the projection speed of the projectile is equal to escape velocity, the projectile will neither be intersected by Earth's surface nor be bounded to the Earth. The resulting trajectory is parabola leading to the infinity. It is an open trajectory.

7 : $v > v_e$: We can infer that projection velocity is just too great. The impact of gravity will be for a very short duration till the projectile is close to Earth. However, as distance increases quickly, the impact of gravitational force becomes almost negligible. The final path is parallel to x-direction.

Projection in Earth's gravitation



Projectile is projected with
certain velocity in x-direction.

Projection in gravitational field (application)

Questions and their answers are presented here in the module text format as if it were an extension of the treatment of the topic. The idea is to provide a verbose explanation, detailing the application of theory. Solution presented is, therefore, treated as the part of the understanding process – not merely a Q/A session. The emphasis is to enforce ideas and concepts, which can not be completely absorbed unless they are put to real time situation.

Representative problems and their solutions

We discuss problems, which highlight certain aspects of the study leading to the projection in gravitational field. The questions are categorized in terms of the characterizing features of the subject matter :

- Satellite
- Vertical projection
- Escape velocity

Satellite

Problem 1: A satellite of mass "m" is to be launched into an orbit around Earth of mass "M" and radius "R" at a distance "2R" from the surface. Find the minimum energy required to launch the satellite in the orbit.

Solution : The energy to launch the satellite should equal to difference of total mechanical energy of the system in the orbit and at Earth's surface. The mechanical energy of the satellite at the surface is only its potential energy. It is given by,

$$E_S = -\frac{GMm}{R}$$

On the other hand, satellite is placed at a total distance of R + 2R = 3R. The total mechanical energy of the satellite in the orbit is,

$$E_O = -\frac{GMm}{2r} = -\frac{GMm}{2X3R} = -\frac{GMm}{6R}$$

Hence, energy required to launch the satellite is :

$$E = E_O - E_S$$

$$\Rightarrow E = -\frac{GMm}{6R} - \left(-\frac{GMm}{R}\right) = \frac{5GMm}{6R}$$

Problem 2: A satellite revolves around Earth of radius "R" with a speed "v". If rockets are fired to stop the satellite to make it standstill, then find the speed with which satellite will strike the Earth. Take g = 10 m/s^2.

Solution : When the seed of satellite is reduced to zero, it starts falling towards center of Earth. Let the velocity with which it strikes the surface be "v'" and distance between center of Earth and satellite be "r".

Applying conservation of energy :

$$\frac{1}{2}mv\prime^2 - \frac{GMm}{R} = 0 - \frac{GMm}{r}$$

For satellite, we know that kinetic energy is :

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$

Also, we can express "GM" in terms of acceleration at the surface,

$$gR^2 = GM$$

Substituting these expressions in the equation of law of conservation of mechanical energy and rearranging,

$$\Rightarrow \frac{1}{2}mv\prime^2 = gRm - mv^2$$

$$\Rightarrow \frac{1}{2}v\prime^2 = gR - v^2$$

$$\Rightarrow v\prime = \sqrt{2(gR - v^2)}$$

$$\Rightarrow v\prime = \sqrt{(20R - 2v^2)}$$

Vertical projection

Problem 3: A particle is projected with initial speed equal to the orbital speed of a satellite near Earth's surface. If the radius of Earth is "R", then find the height to which the particle rises.

Solution : It is given that speed of projection is equal to orbital speed of a satellite near Earth's surface. The orbital speed of the satellite near Earth's surface is given by putting "r = R" in the expression of orbital velocity :

$$v = \sqrt{\left(\frac{GM}{R}\right)}$$

$$v^2 = \frac{GM}{R}$$

Since orbital velocity is less than escape velocity, the particle is returned to the surface after attaining a certain maximum height, "h". Applying conservation of energy, the height attained by projectile is obtained as :

$$h = \frac{v^2}{2g - \frac{v^2}{R}}$$

Substituting for "v" and "g", we have :

$$\Rightarrow h = \frac{\frac{GM}{R}}{\frac{2GM}{R^2} - \frac{GM}{R}}$$

$$\Rightarrow h = R$$

Escape velocity

Problem 4: A particle is fired with a velocity 16 km/s from the surface Earth. Find its velocity with which it moves in the interstellar space. Consider Earth's escape velocity as 11.2 km/s and neglect friction.

Solution : We observe here that initial velocity of the particle is greater than Earth's escape velocity. We can visualize this situation in terms of energy. The kinetic energy of the particle is used to (i) overcome the mechanical energy binding it to the gravitational influence of Earth and (ii) to move into interstellar space with a certain velocity.

Let "v", " v_e " and " v_i " be velocity of projection, escape velocity and velocity in the interstellar space respectively. Then, applying law of conservation of energy :

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mv_i^2$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{GMm}{R} + \frac{1}{2}mv_i^2$$

Here, we have considered gravitational potential energy in the interstellar space as zero. Also, we know that kinetic energy corresponding to escape velocity is equal to the magnitude of gravitational potential energy of the particle on the surface. Hence,

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mv_e^2 + \frac{1}{2}mv_i^2$$

$$\Rightarrow v_i^2 = v^2 - v_e^2$$

The escape velocity for Earth is 11.2 km/s. Putting values in the equation, we have :

$$\Rightarrow v_i^2 = (16)^2 - (11.2)^2$$

$$\Rightarrow v_i^2 = 256 - 125.44 = 130.56$$

$$\Rightarrow v_i = 11.43 \quad km/s$$

Problem 5: A satellite is orbiting near surface with a speed “v”. What additional velocity is required to be imparted to the satellite so that it escapes Earth’s gravitation. Consider, g = 10 m/s^2 and R = 6400 km.

Solution : The orbital speed of the satellite near Earth’s surface is given by :

$$v = \sqrt{\left(\frac{GM}{R}\right)}$$

We can write this expression in terms of acceleration at the surface (g),

$$\Rightarrow v = \sqrt{\left(\frac{GM}{R}\right)} = \sqrt{\left(\frac{gR^2}{R}\right)} = \sqrt{(gR)}$$

On the other hand, escape velocity is given by :

$$v_e = \sqrt{(2gR)}$$

Hence, additional velocity to be imparted is difference of two speeds,

$$\Rightarrow v_e - v = \sqrt{(2gR)} - \sqrt{(gR)}$$

$$\Rightarrow v_e - v = \left(\sqrt{2} - 1\right)\sqrt{(gR)}$$

$$\Rightarrow v_e - v = \left(\sqrt{2} - 1\right)\sqrt{\left(10X6.4X10^6\right)}$$

$$v_e - v = 3.31X10^3 \quad m/s$$

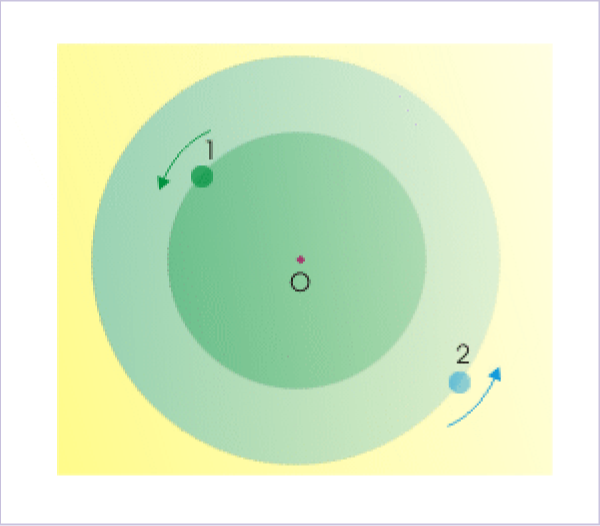
Two body system - linear motion

“Two body” system represents the starting point for studying motion of celestial bodies, including Earth. In general, gravitational force is dominant for a pair of masses in such a manner that influence of all other bodies can be neglected as first approximation. In that case, we are left with an isolated “two body” system. The most important deduction of this simplifying assumption is that isolated system is free of external force. This means that “center of mass” of the isolated system in not accelerated.

In the solar system, one of the massive bodies is Sun and the other is one of the planets. In this case, Sun is relatively much larger than second body. Similarly, in Earth-moon system, Earth is relatively much larger than moon. On the other hand, bodies are of similar mass in a “binary stars” system. There are indeed various possibilities. However, we first need to understand the basics of the motion of isolated two bodies system, which is interacted by internal force of attraction due to gravitation. Specially, how do they hold themselves in space?

In this module, we shall apply laws of mechanics, which are based on Newton’s laws of motion and Newton's law of gravitation. Most characterizing aspect of the motion is that two bodies, in question, move in a single plane, which contains their center of mass. What it means that the motion of two body system is coplanar.

Two body system



The plane of "one body and center of mass" and plane of "other body and center of mass" are in the same plane.

Newtonian mechanics provides a general solution in terms of trajectory of a conic section with different eccentricity. The trajectories like linear, circular, elliptical, parabolic, hyperbolic etc are subsets of this general solution with specific eccentricity. Here, we do not seek mathematical derivation of generalized solution of the motion. Rather, we want to introduce simpler trajectories like that of a straight line, circle etc. first and then interpret elliptical trajectory with simplifying assumptions. In this module, we shall limit ourselves to the motion of “two body” system along a straight line. We shall take up circular motion in the next module.

In a way, the discussion of motion of “two body” system is preparatory before studying Kepler’s laws of planetary motion, which deals with specific case of elliptical trajectory.

Note: The general solution of two bodies system involves polar coordinates (as it suits the situation), vector algebra and calculus. In this module, however, we have retained rectangular coordinates for the most part with scalar derivation and limited our discussion to specific case of linear trajectory.

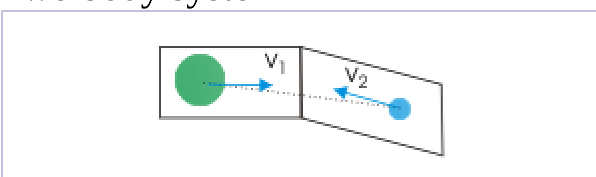
Straight line trajectory

This is simplest motion possible for "two body" system. The bodies under consideration are initially at rest. In this case, center of mass of two bodies is a specific point in the given reference. Also, it is to be noted that center of mass lies always between two bodies and not beyond them.

Since no external force is applied, the subsequent motion due to internal gravitational force does not change the position of center of mass in accordance with second law of motion. The bodies simply move towards each other such that center of mass remains at rest.

The two bodies move along a straight line joining their centers. The line of motion also also passes through center of mass. This has one important implication. The plane containing one body and “center of mass” and the plane containing other body and “center of mass” are same. It means that motions of two bodies are “coplanar” with line joining the centers of bodies and center of mass. The non-planar motions as shown in the figure below are not possible as motion is not along the line joining the centers of two bodies.

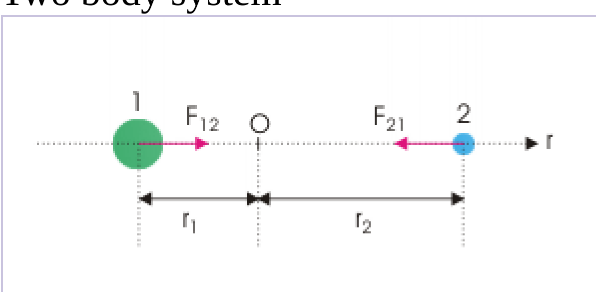
Two body system



The motion of two body system can not be non-planar.

Let suscripts "1" and "2" denote two bodies. Also, let “ r_1 ”, “ v_1 ”, “ a_1 ” and “ r_2 ”, “ v_2”, “ a_2 ” be the magnitudes of linear distance from the center of mass, speeds and magnitudes of accelerations respectively of two bodies under consideration. Also let “center of mass” of the system is the origin of reference frame. Then, by definition of center of mass :

Two body system



Center of mass is the origin of
reference frame.

$$r_{cm} = \frac{-m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

But “center of mass” lies at the origin of the reference frame,

$$\Rightarrow r_{cm} = \frac{-m_1 r_1 + m_2 r_2}{m_1 + m_2} = 0$$

$$\Rightarrow m_1 r_1 = m_2 r_2$$

Taking first differentiation of position with respect to time, we have :

$$\Rightarrow v_{cm} = \frac{-m_1 v_1 + m_2 v_2}{m_1 + m_2} = 0$$

$$\Rightarrow m_1 v_1 = m_2 v_2$$

Taking first differentiation of velocity with respect to time, we have :

$$\Rightarrow a_{cm} = \frac{-m_1 a_1 + m_2 a_2}{m_1 + m_2} = 0$$

$$\Rightarrow m_1 a_1 = m_2 a_2$$

Considering only magnitude and combining with Newton’s law of gravitation,

$$F_{12} = F_{21} = \frac{G m_1 m_2}{(r_1 + r_2)^2}$$

Since distance of bodies from center of mass changes with time, the gravitational force on two bodies is equal in magnitude at a given instant, but varies with time.

Newton’s Second law of motion

We can treat “two body” system equivalent to “one body” system by stating law of motion in appropriate terms. For example, it would be interesting to know how force can be related to the relative acceleration with which two bodies are approaching towards each other. Again, we would avoid vector notation and only consider the magnitudes of accelerations involved. The relative acceleration is sum of the magnitudes of individual accelerations of the bodies approaching towards each other :

$$a_r = a_1 + a_2$$

According to Newton's third law, gravitational force on two bodies are pair of action and reaction and hence are equal in magnitude. The magnitude of force on each of the bodies is related to acceleration as :

$$F = m_1 a_1 = m_2 a_2$$

$$\Rightarrow m_1 a_1 = m_2 a_2$$

We can note here that this relation, as a matter of fact, is same as obtained using concept of center of mass. Now, we can write magnitude of relative acceleration of two bodies, “ a_r ”, in terms of individual accelerations is :

$$\Rightarrow a_r = a_1 + \frac{m_1 a_1}{m_2}$$

$$\Rightarrow a_r = a_1 \left(\frac{m_1 + m_2}{m_2} \right)$$

$$\Rightarrow a_1 = \frac{m_2 a_r}{m_1 + m_2}$$

Substituting in the Newton’s law of motion,

$$\Rightarrow F = m_1 a_1 = \frac{m_1 m_2 a_r}{m_1 + m_2}$$

$$\Rightarrow F = \mu a_r$$

Where,

$$\Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Reduced mass

The quantity given by the expression :

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

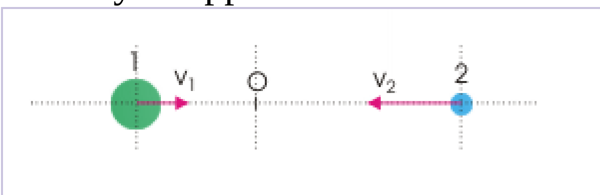
is known as “reduced mass”. It has the same unit as that of “mass”. It represents the “effect” of two bodies, if we want to treat “two body” system as “one body” system.

In the nutshell, we can treat motion of “two body” system along a straight line as “one body” system, which has a mass equal to “μ” and acceleration equal to relative acceleration, “ a_r ”.

Velocity of approach

The two bodies are approaching towards each other. Hence, magnitude of velocity of approach is given by :

Velocity of approach



Two bodies are approaching each other along a straight line.

$$v_r = v_1 + v_2$$

This is the expression of the magnitude of velocity of approach. We can write corresponding vector equation for velocity of approach in relation to reference direction. In this module, however, we will avoid vector notation or vector interpretation to keep the discussion simplified.

Kinetic energy

The kinetic energy of the “two body” system is given as the sum of kinetic energy of individual bodies,

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

We can write this expression of kinetic energy in terms of relative velocity i.e. velocity of approach. For this, we need to express individual speeds in terms of relative speed as :

$$v_r = v_1 + v_2$$

But,

$$m_1 v_1 = m_2 v_2$$

$$\Rightarrow v_2 = \frac{m_1 v_1}{m_2}$$

Substituting in the expression of relative velocity,

$$\Rightarrow v_r = v_1 + \frac{m_1 v_1}{m_2}$$

$$\Rightarrow v_1 = \frac{m_2 v_r}{m_1 + m_2}$$

Similarly,

$$\Rightarrow v_2 = \frac{m_1 v_r}{m_1 + m_2}$$

Now, putting these expressions of individual speed in the equation of kinetic energy :

$$\Rightarrow K = \frac{m_1 m_2^2 v_r^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2 v_r^2}{(m_1 + m_2)^2}$$

$$\Rightarrow K = \frac{m_1 m_2 v_r^2}{(m_1 + m_2)^2}(m_1 + m_2)$$

$$\Rightarrow K = \frac{m_1 m_2 v_r^2}{m_1 + m_2}$$

$$\Rightarrow K = \frac{1}{2}\mu v_r^2$$

This result also indicates that we can treat “two body” system as “one body” system from the point of view of kinetic energy, as if the body has reduced mass of “μ” and speed equal to the magnitude of relative velocity, “ v_r”.

Example

Problem 1: Two masses “ m_1 ” and “ m_2 ” are initially at rest at a great distance. At a certain instant, they start moving towards each other, when released from their positions. Considering absence of any other gravitational field, calculate velocity of approach when they are at a distance “r” apart.

Solution : The bodies are at large distance in the beginning. There is no external gravitational field. Hence, we can consider initial gravitational energy of the system as zero (separated y infinite distance). Also, the bodies are at rest in the beginning. The initial kinetic energy is also zero. In turn, initial mechanical energy of the system is zero.

Let “ v_r ” be the velocity of approach, when the bodies are at a distance “r” apart. Applying conservation of mechanical energy, we have :

$$K_i + U_i = K_f + U_f$$

$$\Rightarrow 0 + 0 = \frac{1}{2}\mu v_r^2 - \frac{Gm_1m_2}{r}$$

$$\Rightarrow v_r^2 = \frac{2Gm_1m_2}{\mu r} = \frac{2Gm_1m_2(m_1+m_2)}{m_1m_2r}$$

$$\Rightarrow v_r = \sqrt{\left\{\frac{2G(m_1+m_2)}{r}\right\}}$$

Conclusions

From the discussion above, we conclude the followings about the motion of “two body” system along a straight line :

1: Each body follows a straight line trajectory.

2: The line joining centers of two bodies pass through center of mass.

3: The planes of two motions are in the same plane. In other words, two motions are coplanar.

4: Magnitude of gravitational force is same for two bodies, but they vary as the distance between them changes.

5: We can treat “two body” system equivalent to “one body” system by using concepts of (i) reduced mass “μ” (ii) relative velocity, “ v_r ” and (iii) relative acceleration “ a_r ”.

Two body system - circular motion

The trajectory of two body system depends on the initial velocities of the bodies and their relative mass. If the mass of the bodies under consideration are comparable, then bodies move around their “center of mass” along two separate circular trajectories. This common point about which two bodies revolve is also known as “barycenter”.

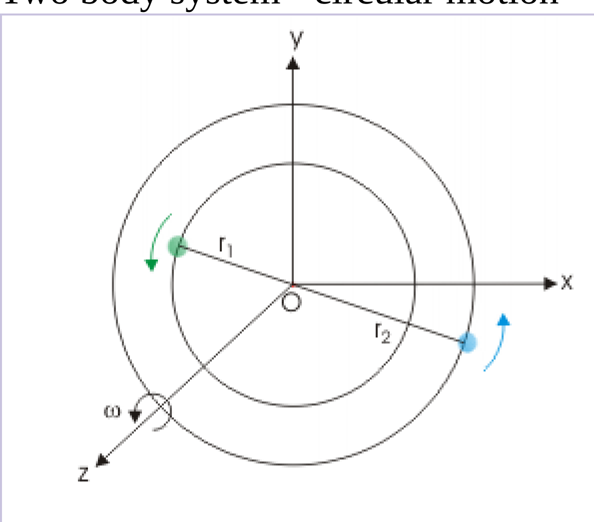
In order to meet the requirement imposed by laws of motion and conservation laws, the motion of two bodies executing circular motion is constrained in certain ways.

Circular trajectory

Since external force is zero, the acceleration of center of mass is zero. This is the first constraint. For easy visualization of this constraint, we consider that center of mass of the system is at rest in a particular reference frame.

Now, since bodies are moving along two circular paths about "center of mass", their motions should be synchronized in a manner so that the length of line, joining their centers, is a constant . This is required; otherwise center of mass will not remain stationary in the chosen reference. Therefore, the linear distance between bodies is a constant and is given by :

Two body system - circular motion



Each body moves around center

of mass.

$$r = r_1 + r_2$$

Now this condition can be met even if two bodies move in different planes. However, there is no external torque on the system. It means that the angular momentum of the system is conserved. This has an important deduction : the plane of two circular trajectories should be same.

Mathematically, we can conclude this, using the concept of angular momentum. We know that torque is equal to time rate of change of angular momentum,

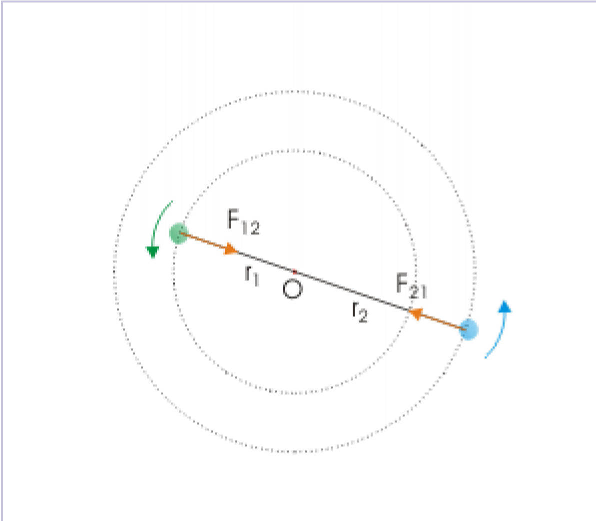
$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}$$

But, external torque is zero. Hence,

$$\Rightarrow \mathbf{r} \times \mathbf{F} = 0$$

It means that "**r**" and "**F**" are always parallel. It is only possible if two planes of circles are same. We, therefore, conclude that motions of two bodies are coplanar. For coplanar circular motion, center of mass is given by definition as :

Two body system - circular motion



Each body moves around center of mass.

$$r_{cm} = \frac{-m_1 r_1 + m_2 r_2}{m_1 + m_2} = 0$$

$$\Rightarrow m_1 r_1 = m_2 r_2$$

Taking first differentiation with respect to time, we have :

$$\Rightarrow m_1 v_1 = m_2 v_2$$

Now dividing second equation by first,

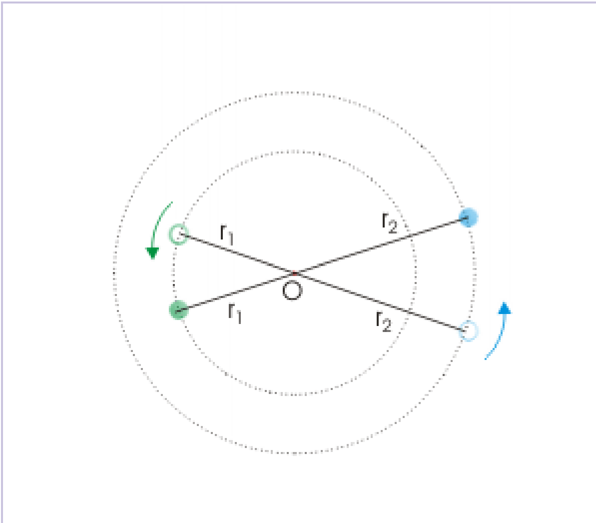
$$\Rightarrow \frac{m_1 v_1}{m_1 r_1} = \frac{m_2 v_2}{m_2 r_2}$$

$$\Rightarrow \frac{v_1}{r_1} = \frac{v_2}{r_2}$$

$$\Rightarrow \omega_1 = \omega_2 = \omega \quad (say)$$

It means that two bodies move in such a manner that their angular velocities are equal.

Two body system - circular motion



Both bodies move with same angular velocity.

Gravitational force

The gravitational force on each of the bodies is constant and is given by :

$$F = \frac{Gm_1m_2}{(r_1 + r_2)^2} = \frac{Gm_1m_2}{r^2}$$

Since gravitational force provides for the requirement of centripetal force in each case, it is also same in two cases. Centripetal force is given by :

$$F_C = m_1r_1\omega^2 = m_2r_2\omega^2 = \frac{Gm_1m_2}{r^2}$$

Angular velocity

Each body moves along a circular path. The gravitational force on either of them provides the centripetal force required for circular motion. Hence, centripetal force is :

$$m_1 r_1 \omega^2 = \frac{G m_1 m_2}{(r_1 + r_2)^2}$$

$$\Rightarrow \omega^2 = \frac{G m_2}{r_1 (r_1 + r_2)^2}$$

Let the combined mass be “M”. Then,

$$M = m_1 + m_2$$

Using relation $m_1 r_1 = m_2 r_2$, we have :

$$\Rightarrow M = \frac{m_2 r_2}{r_1} + m_2 = m_2 \left(\frac{r_1 + r_2}{r_1} \right)$$

$$\Rightarrow m_2 = \frac{M r_1}{r_1 + r_2}$$

Substituting in the equation, involving angular velocity,

$$\Rightarrow \omega^2 = \frac{G M r_1}{r_1 (r_1 + r_2)^3} = \frac{GM}{r^3}$$

$$\Rightarrow \omega = \sqrt{\left(\frac{GM}{r^3} \right)}$$

This expression has identical form as for the case when a body revolves around another body at rest along a circular path (compare with “Earth – satellite” system). Here, combined mass “M” substitutes for the mass of heavier mass at the center and sum of the linear distance replaces the radius of rotation.

The linear velocity is equal to the product of the radius of circle and angular velocity. Hence,

$$v_1 = \omega r_1$$

$$v_2 = \omega r_2$$

Time period

We can easily find the expression for time period of revolution as :

$$T = \frac{2\pi}{\omega} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{(GM)}}$$

This expression also has the same form as for the case when a body revolves around another body at rest along a circular path (compare with "Earth – satellite" system). Further squaring on either side, we have :

$$\Rightarrow T^2 \propto r^3$$

Moment of inertia

Here, we set out to find moment of inertia of the system about the common axis passing through center of mass and perpendicular to the plane of rotation. For this, we consider each of the bodies as point mass. Note that two bodies are rotating about a common axis with same angular velocity. Clearly, MI of the system is :

$$I = m_1 r_1^2 + m_2 r_2^2$$

We can express individual distance in terms of their sum using following two equations,

$$r = r_1 + r_2$$

$$m_1 r_1 = m_2 r_2$$

Substituting for " r_1 " in the equation or "r", we have :

$$\Rightarrow r = \frac{m_2 r_2}{m_1} + r_2 = r_2 \left(\frac{m_1 + m_2}{m_1} \right)$$

$$\Rightarrow r_2 = \frac{rm_1}{m_1 + m_2}$$

Similarly, we can express, " r_1 " as :

$$\Rightarrow r_1 = \frac{rm_2}{m_1 + m_2}$$

Substituting for " r_1 " and " r_2 " in the expression of moment of inertia,

$$\Rightarrow I = \frac{m_1 m_2^2 r^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2 r^2}{(m_1 + m_2)^2}$$

$$\Rightarrow I = \frac{m_1 m_2 r^2}{(m_1 + m_2)^2} X (m_1 + m_2)$$

$$\Rightarrow I = \frac{m_1 m_2 r^2}{m_1 + m_2}$$

$$\Rightarrow I = \mu r^2$$

This expression is similar to the expression of momemnt of inertia of a particle about an axis at a perpendicualr distance, "r". It is, therefore, clear that "Two body" system orbiting around center of mass can be treated as "one body" system by using concepts of net distance "r" and reduced mass "μ".

Angular momentum

The bodies move about the same axis with the same sense of rotation. The angular momentum of the system, therefore, is algebraic sum of individual angular momentums.

$$L = L_1 + L_2 = m_1 r_1^2 \omega + m_2 r_2^2 \omega$$

Substituting for " r_1 " and " r_2 " with expressions as obtained earlier,

$$\Rightarrow L = \frac{m_1 m_2^2 r^2 \omega}{(m_1 + m_2)^2} + \frac{m_2 m_1^2 r^2 \omega}{(m_1 + m_2)^2}$$

$$\Rightarrow L = \frac{m_1 m_2 r^2 \omega}{(m_1 + m_2)^2} X (m_1 + m_2)$$

$$\Rightarrow L = \frac{m_1 m_2 r^2 \omega}{(m_1 + m_2)}$$

$$\Rightarrow L = \mu r^2 \omega$$

This expression is similar to the expression of angular momemntum of a particle about an axis at a perpendicualr distance, "r". Once again, we see that “Two body” system orbiting around center of mass can be treated as “one body” system by using concepts of net distance “r” and reduced mass “μ”.

Kinetic energy

The kinetic energy of the system is equal to the algebraic sum of the kinetic energy of the individual body. We write expression of kinetic energy in terms of angular velocity – not in terms of linear velocity. It is so because angular velocity is same for two bodies and can, therefore, be used to simplify the expression for kinetic energy. Now, kinetic energy of the system is :

$$K = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2$$

Substituting for “ r_1 ” and “ r_2 ” with expressions as obtained earlier,

$$\Rightarrow K = \frac{m_1 m_2^2 r^2 \omega^2}{2(m_1 + m_2)^2} + \frac{m_2 m_1^2 r^2 \omega^2}{2(m_1 + m_2)^2}$$

$$\Rightarrow K = \frac{m_1 m_2 r^2 \omega^2}{2(m_1 + m_2)^2} X (m_1 + m_2)$$

$$\Rightarrow K = \frac{m_1 m_2 r^2 \omega^2}{2(m_1 + m_2)}$$

$$\Rightarrow K = \frac{1}{2}\mu r^2 \omega^2$$

This expression of kinetic energy is also similar to the expression of kinetic energy of a particle rotating about an axis at a perpendicualr distance, "r". Thus, this result also substantiates equivalence of “Two body” system as “one body” system, using concepts of net distance “r” and reduced mass “μ”.

Example

Problem 1 : In a binary star system, two stars of “m” and “M” move along two circular trajectories. If the distance between stars is “r”, then find the total mechanical energy of the system. Consider no other gravitational influence on the system.

Solution : Mechanical energy of the system comprises of potential and kinetic energy. Hence,

$$E = \frac{1}{2}\mu r^2 w^2 - \frac{GMm}{r}$$

We know that angular velocity for “two body” system in circular motion is given by :

$$\Rightarrow \omega = \sqrt{\left\{\frac{G(M+m)}{r^3}\right\}}$$

Also, reduced mass is given by :

$$\mu = \frac{Mm}{M+m}$$

Putting in the expression of mechanical energy,

$$\Rightarrow E = \frac{mMr^2G(m+M)}{2(m+M)r^3} - \frac{GMm}{r}$$

$$\Rightarrow E = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$\Rightarrow E = -\frac{GMm}{2r}$$

Conclusions

Thus, we conclude the following :

1: Each body follows a circular path about center of mass.

2: The line joining centers of two bodies pass through center of mass.

3: The planes of two motions are in the same plane. In other words, two motions are coplanar.

4: The angular velocities of the two bodies are equal.

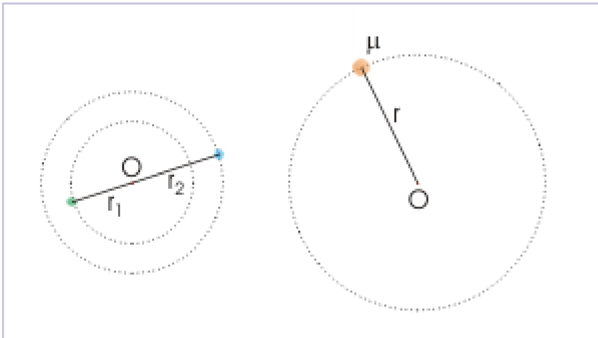
5: The linear distance between two bodies remains constant.

6: Magnitude of gravitational force is constant and same for two bodies.

7: Magnitude of centripetal force required for circular motion is constant and same for two bodies.

8: Since linear velocity is product of angular velocity and distance from the center of revolution, it may be different if the radii of revolutions are different.

9: We can treat two body system with an equivalent one body system by using concepts of (i) combined mass, "M", (ii) net distance "r" and (iii) reduced mass "μ".



Two body system as equivalent
to one body system.

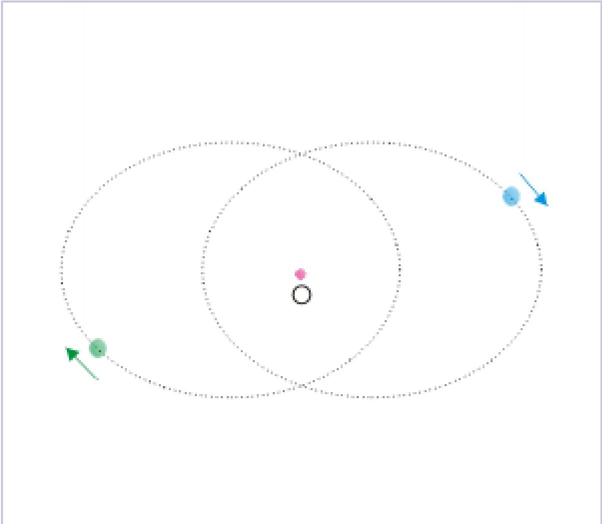
Planetary motion

Kepler’s laws of planetary motion are consistent with Newtonian mechanics.

The trajectory of motion resulting from general solution of “two body” system is a conic section. Subject to initial velocities and relative mass, eccentricity of conic section can have different values. We interpret a conic section for different eccentricity to represent different types of trajectories. We have already discussed straight line and circular trajectories. In this module, we shall discuss elliptical trajectory, which is the trajectory of a planet in the solar system.

In a general scenario of “two body system”, involving elliptical trajectory, each body revolves around the common “center of mass” called “barycenter”. The two elliptical paths intersect, but bodies are not at the point of intersection at the same time and as such there is no collision.

Two body system



Elliptical trajectories

In this module, however, we shall keep our focus on the planetary motion and Kepler’s planetary laws. We are basically seeking to describe planetary motion – particularly that in our solar system. The trajectory of planet is elliptical with one qualification. The Sun, being many times heavier than

the planets, is almost at rest in the reference frame of motion. It lies at one of the foci of the elliptical path of the planets around it. Here, we assume that center of mass is about same as the center of Sun. Clearly, planetary motion is a special case of elliptical motion of “two body system” interacted by mutual attraction.

We should, however, be aware that general solution of planetary motion involves second order differential equation, which is solved using polar coordinates.

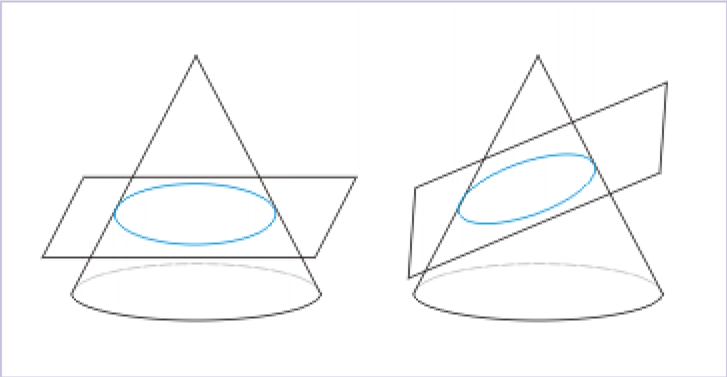
Ellipse

We need to learn about the basics of elliptical trajectory and terminology associated with it. It is important from the point of view of applying laws of Newtonian mechanics. We shall, however, be limited to the basics only.

Conic section

Conic section is obtained by the intersection of a plane with a cone. Two such intersections, one for a circle and one for an ellipse are shown in the figure.

Conic sections



Two conic sections representing a circle and an ellipse are shown.

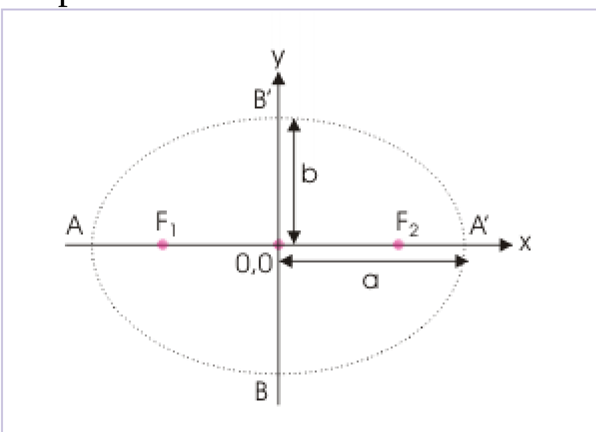
Elliptical trajectory

Here, we recount the elementary geometry of an ellipse in order to understand planetary motion. The equation of an ellipse centered at the origin of a rectangular coordinate (0,0) is :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where "a" is semi-major axis and "b" is semi-minor axis as shown in the figure.

Ellipse



Semi major and minor axes of an ellipse

Note that " F_1 " and " F_2 " are two foci of the ellipse.

Eccentricity

The eccentricity of a conic section is measure of "how different it is from a circle". Higher the eccentricity, greater is deviation. The eccentricity (e) of a conic section is defined in terms of "a" and "b" as :

$$e = \sqrt{\left(1 - \frac{kb^2}{a^2}\right)}$$

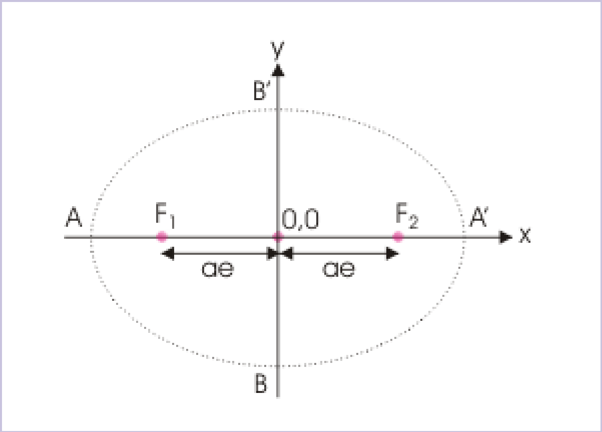
where “k” is 1 for an ellipse, 0 for parabola and -1 for hyperbola. The values of eccentricity for different trajectories are as give here :

1. The eccentricity of a straight line is 1, if we consider b=0 for the straight line.
2. The eccentricity of an ellipse falls between 0 and 1.
3. The eccentricity of a circle is 0
4. The eccentricity of a parabola is 1.
5. The eccentricity of a hyperbola is greater than 1.

Focal points

Focal points (F_1 and F_2) lie on semi major axis at a distance from the origin given by

Focal points



Focal distances from the center of ellipse

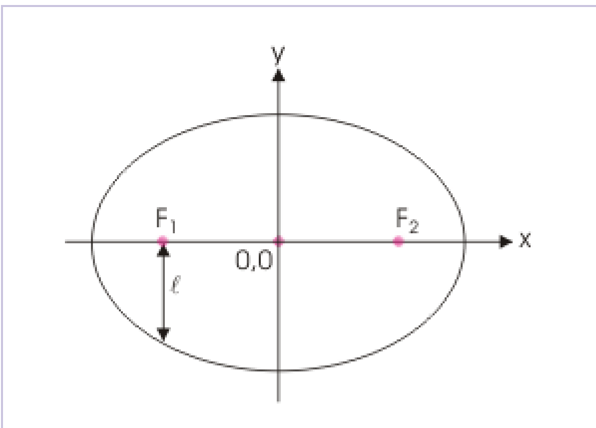
$$f = ae$$

The focus of an ellipse is at a distance “ae” from the center on the semi-major axis. Area of the ellipse is "πab".

Semi latus rectum

Semi latus rectum is equal to distance between one of the foci and ellipse as measured along a line perpendicular to the major axis. This is shown in the figure.

Semi latus rectum



Semi latus rectum is perpendicular distance as shown in the figure.

For an ellipse, Semi latus rectum has the expression in terms of “a” and “b” as :

$$\ell = \frac{b^2}{a}$$

We can also express the same involving eccentricity as :

$$\Rightarrow \ell = a\left(1 - e^2\right)$$

Solar system

The solar system consists of Sun and its planets. The reason they are together is gravitation. The mass of the planet is relatively small with respect to Sun. For example, Earth compares about 10^5 times smaller in mass with respect to Sun :

Mass of Earth :

$$5.98X10^{24} \quad kg$$

Mass of Sun :

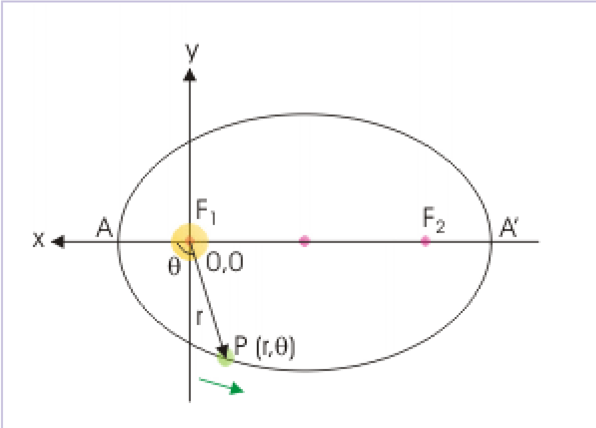
$$1.99X10^{30} \quad kg$$

The planetary motion, therefore, fits nicely with elliptical solution obtained from consideration of mechanics. Sun, being many times heavier, appears to be at the “center of mass” of the system i.e. at one of the foci, while planets revolve around it in elliptical orbits of different eccentricities.

Equation in polar coordinates

Polar coordinates generally suite geometry of ellipse. The figure shows the polar coordinates of a point on the ellipse. It is important to note that one of foci serves as the origin of polar coordinates, whereas the other focus lies on the negative x-axis. For this reason orientation of x-axis is reversed in the figure. The angle is measured anti-clockwise and the equation of ellipse in polar coordinates is :

Equation in polar coordinate



Second focus lies on negative x-axis.

$$r = \frac{\ell}{1 + e\cos\theta}$$

Substituting expression for semi latus rectum

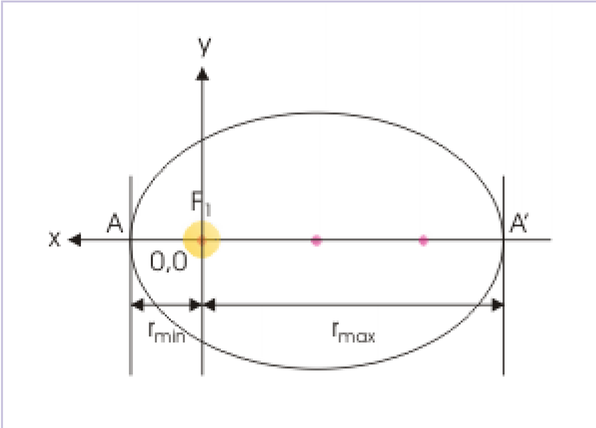
$$\Rightarrow r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

Perihelion distance

Perihelion position corresponds to minimum distance between Sun and planet. If we consider Sun to be at one focus (say F_1), then perihelion distance is " $F_1 A$ " as shown in the figure. We can see that angle θ = 0° for this position.

$$\Rightarrow r_{\min} = \frac{a\left(1 - e^2\right)}{1 + e} = a(1 - e)$$

Minimum and maximum distance



Positions correspond to perihelion and aphelion positions.

From the figure also, it is clear that minimum distance is equal to “a - ae= a(1-e)”.

Aphelion distance

Aphelion position corresponds to maximum distance between Sun and planet. If we consider Sun to be at one focus (say F_1), then perihelion distance is " $F_1 A'$ " as shown in the figure. We can see that angle θ = 180° for this position.

$$\Rightarrow r_{\max} = \frac{a\left(1 - e^2\right)}{1 - e} = a(1 + e)$$

From the figure also, it is clear that maximum distance is equal to “a + ae= a(1+e)”.

We can also prove that the semi-major axis, “a” is arithmetic mean, whereas semi-minor axis, “b”, is geometric mean of “ $r_{\min}$ ” and “ $r_{\max}$”.

Description of planetary motion

We can understand planetary motion by recognizing important aspects of motion like force, velocity, angular momentum, energy etc. The first important difference to motion along circular path is that linear distance between Sun and planet is not constant. The immediate implication is that gravitation force is not constant. It is maximum at perihelion position and minimum at aphelion position.

If “R” is the radius of curvature at a given position on the elliptical trajectory, then centripetal force equals gravitational force as given here :

$$\frac{mv^2}{R} = m\omega^2 R = \frac{GMm}{r^2}$$

Where “M” and “m” are the mass of Sun and Earth; and “r” is the linear distance between Sun and Earth.

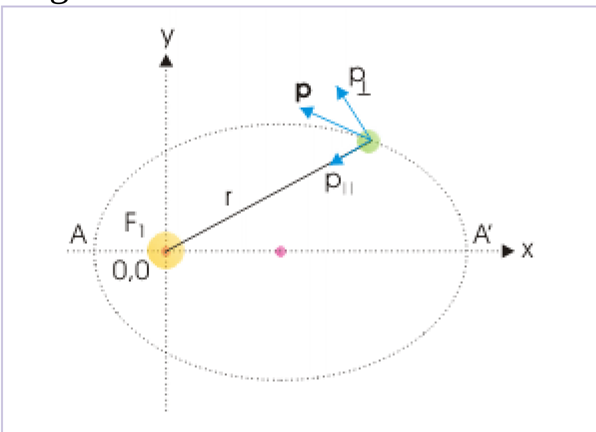
Except for parameters “r” and “R”, others are constant in the equation. We note that radii of curvature at perihelion and aphelion are equal. On the other hand, centripetal force is greatest at perihelion and least at aphelion. From the equation above, we can also infer that both linear and angular velocities of planet are not constant.

Angular momentum

The angular velocity of the planet about Sun is not constant. However, as there is no external torque working on the system, the angular momentum of the system is conserved. Hence, angular momentum of the system is constant unlike angular velocity.

The description of motion in angular coordinates facilitates measurement of angular momentum. In the figure below, linear momentum is shown tangential to the path in the direction of velocity. We resolve the linear momentum along the parallel and perpendicular to radial direction. By definition, the angular momentum is given by :

Angular momentum



Angular momentum of the system is constant.

$$L = rXp_{\perp}$$

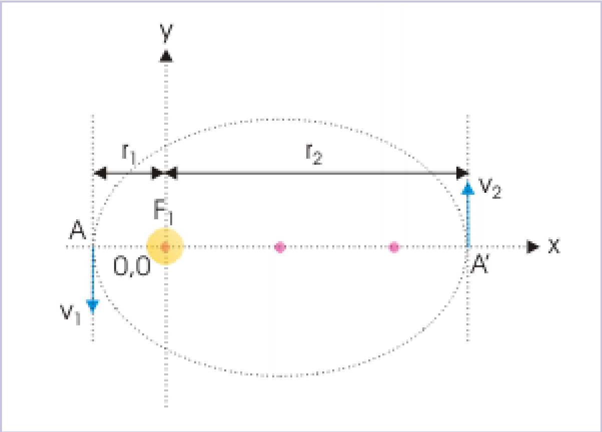
$$L = rXp_{\perp} = rmv_{\perp} = rmX\omega r = m\omega r^2$$

Since mass of the planet "m" is constant, it emerges that the term " ωr^2 " is constant. It clearly shows that angular velocity (read also linear velocity) increases as linear distance between Sun and Earth decreases and vice versa.

Maximum and minimum velocities

Maximum velocity corresponds to perihelion position and minimum to aphelion position in accordance with maximum and minimum centripetal force at these positions. We can find expressions of minimum and maximum velocities, using conservation laws.

Maximum and minimum velocities



Velocities at these positions are perpendicular to semi major axis.

Let " r_1 " and " r_2 " be the minimum and maximum distances, then :

$$r_1 = a(1 - e)$$

$$r_2 = a(1 + e)$$

We see that velocities at these positions are perpendicular to semi major axis. Applying conservation of angular momentum,

$$L = r_1 m v_1 = r_2 m v_2$$

$$r_1 v_1 = r_2 v_2$$

Applying conservation of energy, we have :

$$\frac{1}{2} m v_1^2 - \frac{GMm}{r_1} = \frac{1}{2} m v_2^2 - \frac{GMm}{r_2}$$

Substituting for " v_2 ", " r_1 " and " r_2 ", we have :

$$v_1 = v_{\max} = \sqrt{\left\{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)\right.}$$

$$v_2 = v_{\min} = \sqrt{\left\{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)\right.}$$

Energy of Sun-planet system

As no external force is working on the system and there is no non-conservative force, the mechanical energy of the system is conserved. We have derived expression of linear velocities at perihelion and aphelion positions in the previous section. We can, therefore, find out energy of "Sun-planet" system by determining the same at either of these positions.

Let us consider mechanical energy at perihelion position. Here,

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{r_1}$$

Substituting for velocity and minimum distance, we have :

$$\Rightarrow E = \frac{mGM(1+e)}{2a(1+e)} - \frac{GMm}{a(1-e)}$$

$$\Rightarrow E = \frac{mGM}{a(1-e)}\left(\frac{1+e}{2} - 1\right)$$

$$\Rightarrow E = \frac{mGM}{a(1-e)}\left(\frac{e-1}{2}\right)$$

$$\Rightarrow E = -\frac{GMm}{2a}$$

We see that expression of energy is similar to that of circular trajectory about a center with the exception that semi major axis “a” replaces the radius of circle.

Kepler’s laws

Johannes Kepler analyzed Tycho Brahe’s data and proposed three basic laws that govern planetary motion of solar system. The importance of his laws lies in the fact that he gave these laws long before Newton’s laws of motion and gravitation. The brilliance of the Kepler’s laws is remarkable as his laws are consistent with Newton’s laws and conservation laws.

Kepler proposed three laws for planetary motion. First law tells about the nature of orbit. Second law tells about the speed of the planet. Third law tells about time period of revolution.

- Law of orbits
- Law of velocities
- Law of time periods

Law of orbits

The first law (law of orbits) is stated as :

Law of orbits

> The orbit of every planet is an ellipse with the sun at one of the foci.

This law describes the trajectory of a planet, which is an ellipse – not a circle. We have seen that application of mechanics also provides for elliptical trajectory. Only additional thing is that solution of mechanics yields possibilities of other trajectories as well. Thus, we can conclude that Kepler’s law of orbit is consistent with Newtonian mechanics.

We should, however, note that eccentricity of elliptical path is very small for Earth (0.0167) and large for Mercury (0.206) and Pluto (0.25).

Law of velocities

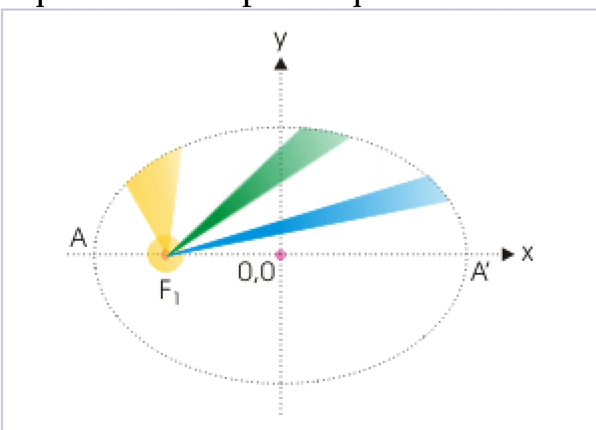
Law of velocities is a statement of comparative velocities of planets at different positions along the elliptical path.

Law of velocities

> The line joining a planet and Sun sweeps equal area in equal times in the planet's orbit.

This law states that the speed of the planet is not constant as generally might have been conjectured from uniform circular motion. Rather it varies along its path. A given area drawn to the focus is wider when it is closer to Sun. From the figure, it is clear that planet covers smaller arc length when it is away and a larger arc length when it is closer for a given orbital area drawn from the position of Sun. It means that speed of the planet is greater at positions closer to the Sun and smaller at positions away from the Sun.

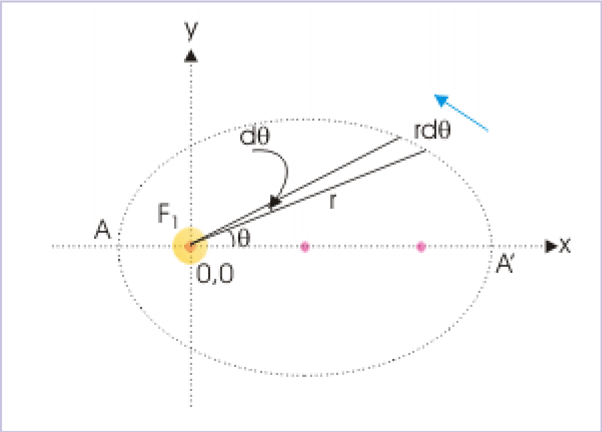
Equal area swept in equal time



Speed of the planet is greater at positions closer to the Sun and smaller at positions away from the Sun.

Further on close examination, we find that Kepler's second law, as a matter of fact, is an statement of the conservations of angular momentum. In order to prove this, let us consider a small orbital area as shown in the figure.

Area swept



Time rate of area is statement of conservation of angular momentum.

$$\Delta A = \frac{1}{2} X \, \text{Base} \, X \, \text{Height} = \frac{1}{2} r \Delta\theta r = \frac{1}{2} r^2 \Delta\theta$$

For infinitesimally small area, the “area speed” of the planet (the time rate at which it sweeps orbital area drawn from the Sun) is :

$$\Rightarrow \frac{đA}{đt} = \frac{1}{2} r^2 \frac{đ\theta}{đt} = \frac{1}{2} r^2 \omega$$

Now, to see the connection of this quantity with angular momentum, let us write the equation of angular momentum :

$$L = r X p_{\perp} = r m v_{\perp} = r m X \omega r = m \omega r^2$$

$$\Rightarrow \omega r^2 = \frac{L}{m}$$

Substituting this expression in the equation of area – speed, we have :

$$\Rightarrow \frac{đA}{đ\theta} = \frac{L}{2m}$$

As no external torque is assumed to exist on the “Sun-planet” system, its angular momentum is conserved. Hence, parameters on the right hand side of the equation i.e. “L” and “m” are constants. This yields that area-speed is constant as proposed by Kepler.

We can interpret the above result other way round also. Kepler’s law says that area-speed of a planet is constant. His observation is based on measured data by Tycho Brahe. It implies that angular momentum of the system remains constant. This means that no external torque applies on the “Sun – planet” system.

Law of time periods

The law relates time period of the planet with semi major axis of the elliptical trajectory.

Law of time periods

> The square of the time period of a planet is proportional to the cube of the semi-major axis.

We have already seen in the case of circular trajectory around a larger mass and also in the case of two body system (see Two body system - circular motion)in which each body is moving along two circular trajectories that time period is given by :

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{(GM)}}$$

The expression of time period for elliptical trajectory is similar except that semi-major axis replaces “r”. We have not proved this in the module, but can be so derived. Squaring each of the side and replacing "r" by "a", we have :

$$\Rightarrow T^2 = \frac{4\pi^2 a^3}{GM}$$

$$\Rightarrow T^2 \propto a^3$$

Conclusions

Thus, we conclude the following :

1: The planet follows a elliptical path about Sun.

2: The Sun lies at one of the foci.

3: Gravitational force, centripetal force, linear and angular velocities are variable with the motion.

4: Velocities are maximum at perihelion and minimum at aphelion.

5: Although angular velocity is variable, the angular momentum of the system is conserved.

6: The expression of total mechanical energy is same as in the case of circular motion with the exception that semi major axis, “a”, replaces radius, “r”.

7: The expression of time period is same as in the case of circular motion with the exception that semi major axis, “a”, replaces radius, “r”.

8: Kepler’s three laws are consistent with Newtonian mechanics.

Acknowldegement

Author wishes to place special thanks to Mr. Mark Prange, Port Isabel, Texas, USA and David F Shively for their valuable editorial suggestions on the subject discussed in this module.